UTMS 2006–20 August 18, 2006

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BLOW-UP DIRECTIONS FOR QUASILINEAR PARABOLIC EQUATIONS

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Abstract. We consider the Cauchy problem for quasilinear parabolic equations $u_t = \Delta\phi(u) + f(u)$ with the bounded nonnegative initial data $u_0(x)\,(\not\equiv 0)$, where $f(\xi)$ is a positive function in $\xi > 0$ satisfying a blow-up condition $\int_1^\infty 1/f(\xi)\,d\xi < \infty$. We study blow-up nonnegative solutions with the least blow-up time, i.e., the time coinciding with the blow-up time of a solution of the corresponding ordinary differential equation $dv/dt = f(v)$ with the initial data $\|u_0\|_{L^\infty(\mathbf{R}^N)} > 0$. Such a blow-up solution blows up at space infinity in some direction (directional blow-up) and this direction is called a *blow-up direction*. We give a sufficient condition on u_0 for directional blow-up. Moreover, we completely characterize blow-up directions by the profile of the initial data, which gives a sufficient and necessary condition on u_0 for blow-up with the least blow-up time, provided that $f(\xi)$ grows up more rapidly than $\phi(\xi)$.

The first and third authors were supported by The 21st Century COE Program "New Mathematical Development Center to Support Scientific Technology", Japan Society for the Promotion of Science (JSPS).

The second author was partially supported by Grant-in-Aid for Scientific Research (C), No. 17540171, JSPS.

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1. Introduction

In this paper we shall consider the Cauchy problem for quasilinear parabolic equations

$$u_t = \Delta\phi(u) + f(u) \quad \text{in } (x,t) \in \mathbf{R}^N \times (0,T), \tag{1.1}$$

$$u(x,0) = u_0(x) \quad \text{in } x \in \mathbf{R}^N, \tag{1.2}$$

where $u_t = \partial u/\partial t$, Δ is the N-dimensional Laplacian, $\phi(\xi)$, $f(\xi)$ with $\xi \geq 0$ and $u_0(x)$ with $x \in \mathbf{R}^N$ are nonnegative functions. We shall only consider nonnegative solutions $u = u(x,t)$.

Throughout this paper we assume the following conditions:

(A1) $u_0(x) \geq 0, \in BC(\mathbf{R}^N)$ (bounded continuous functions in $\mathbf{R}^N$).

(A2) $\phi(\xi)$, $f(\xi) \in C^1(\bar{\mathbf{R}}_+) \cap C^\infty(\mathbf{R}_+)$ where $\mathbf{R}_+ = (0,\infty)$ and $\bar{\mathbf{R}}_+ = [0,\infty)$; $\phi(\xi) > 0$, $\phi'(\xi) > 0$, $\phi''(\xi) \geq 0$ and $f(\xi) > 0$ for $\xi > 0$; $\phi(0) = 0$.

Under these conditions, it is well known that a unique bounded nonnegative weak solution of (1.1)(1.2) exists locally in time (see [16, 14, 5, 2]). The definition of a weak solution of (1.1)(1.2) is given in §2.

Moreover, we assume the following blow-up condition which is also a "necessary" condition to raise blow-up.

(A3)

$$\int_1^\infty \frac{d\xi}{f(\xi)} < \infty.$$

Remark 1.1. The typical example of (1.1) which satisfies (A2) and (A3) is equation

$$u_t = \Delta u^m + u^p \quad \text{in } \mathbf{R}^N \times (0,T) \tag{1.3}$$

where $m \geq 1$ and $p > 1$.

Under the assumption (A3), the solution u of (1.1)(1.2) blows up in finite time for some initial data. Namely, if we put

$$t_b(u_0) = \sup\{\, T > 0\,;\ \text{the solution } u \text{ of (1.1)(1.2) is bounded in } \mathbf{R}^N \times (0,T)\},$$

then $t_b(u_0) < \infty$ and

$$\lim_{t \uparrow t_b(u_0)} \|u(\cdot,t)\|_{L^\infty(\mathbf{R}^N)} = \infty. \tag{1.4}$$

We say that the time $t_b(u_0)$ is *the blow-up time* of u.

For example, if $u_0(x) \equiv M$ (> 0) then the solution u blows up in finite time, that is,

$$t_b(M) = \int_M^\infty \frac{d\xi}{f(\xi)} < \infty.$$

We note that this solution u coincides with a blow-up solution v_M of the corresponding ordinary differential equation

$$\begin{cases} dv/dt = f(v), & t > 0, \\ v(0) = M, \end{cases} \tag{1.5}$$

with the blow-up time $t_b(M) < \infty$. We also note that $v_M(t)$ is expressed exactly as follows :

$$v_M(t) = G^{-1}(G(M) - t),$$

where $v = G^{-1}(\eta)$ is the inverse function of $\eta = G(v)$ and

$$G(v) = \int_v^\infty \frac{d\xi}{f(\xi)}\,.$$

Let $M = \|u_0\|_{L^\infty(\mathbf{R}^N)} > 0$ and let v_M be a solution to the problem (1.5). Then, all solutions u of (1.1)(1.2) satisfy

$$u(x,t) \le v_M(t) \quad \text{in } (x,t) \in \mathbf{R}^N \times (0, t_b(M)) \tag{1.6}$$

and

$$t_b(u_0) \ge t_b(M). \tag{1.7}$$

Among all solutions u, we are interested in solutions of (1.1)(1.2) whose blow-up times $t_b(u_0)$ coincide with $t_b(M)$. When $t_b(u_0) = t_b(M) < \infty$, we call the time $t_b(u_0)$ *the least blow-up time* and the solution a *blow-up solution with the least blow-up time*. The purpose of the present paper is to study blow-up solutions with the least blow-up time.

Throughout this paper, for $M > 0$ we define T_M as

$$T_M \equiv t_b(M) = \int_M^\infty \frac{d\xi}{f(\xi)}$$

and v_M a solution to the problem (1.5). We also use notations $\|\cdot\|_\infty = \|\cdot\|_{L^\infty(\mathbf{R}^N)}$,

$$B_R(x_0) = \{x \in \mathbf{R}^N\,;\, |x - x_0| < R\} \text{ with } x_0 \in \mathbf{R}^N \quad \text{and} \quad B_R = B_R(0).$$

The next theorem is easily seen from the definition of a blow-up solution with the least blow-up time.

Theorem 1.2. *Assume (A1)-(A3). Let* $u_0 \not\equiv \|u_0\|_\infty$*. Put* $M = \|u_0\|_\infty$ *and let* u *be a blow-up weak solution of* (1.1)(1.2) *with the least blow-up time* T_M*. Then,* u *has the following properties:*

(i)

$$u(x,t) < v_M(t) \quad \text{in } (x,t) \in \mathbf{R}^N \times (0,T_M). \tag{1.8}$$

(ii)

$$\|u(\cdot,t)\|_\infty = \lim_{R\to\infty} \sup_{|x|\ge R} u(x,t) = v_M(t) \quad \text{in } t \in [0,T_M). \tag{1.9}$$

Hence, the initial data u_0 *should satisfy*

$$\lim_{R\to\infty} \sup_{|x|\ge R} u_0(x) = \|u_0\|_\infty. \tag{1.10}$$

(iii) The solution u *blows up at space infinity, that is, there exists a sequence* $\{(x_n,t_n)\} \subset \mathbf{R}^N \times (0,T_M)$ *such that* $|x_n| \to \infty$*,* $t_n \uparrow T_M$ *and* $u(x_n,t_n) \to \infty$ *as* $n \to \infty$*.*

Conversely, if (ii) or (iii) holds, then the solution u blows up at the least blow-up time. We note that (1.10) is not a sufficient condition for blow-up with the least blow-up time. We will give a sufficient and necessary condition on u_0 for such blow-up, when $f(\xi)$ grows up more rapidly than $\phi(\xi)$ (see Theorem 1.11).

By this theorem, we can see that the blow-up solution with the least blow-up time blows up at space infinity in some direction (see Corollary 1.3). We call such a blow-up phenomenon *directional blow-up* and the direction in which directional blow-up occurs a *blow-up direction.* More precisely, a direction $\psi \in \mathbf{S}^{N-1}$, where $\mathbf{S}^{N-1}$ is the $(N-1)$-dimensional unit sphere, is called a *blow-up direction* if there exists a sequence $\{(x_n,t_n)\} \subset \mathbf{R}^N \times (0,T_M)$ such that $|x_n| \to \infty$, $t_n \uparrow T_M$ and $u(x_n,t_n) \to \infty$ as $n \to \infty$ and

$$\frac{x_n}{|x_n|} \to \psi \quad \text{as } n \to \infty. \tag{1.11}$$

When a direction $\psi \in \mathbf{S}^{N-1}$ is not a blow-up direction, we call the direction ψ a *non-blow-up direction* and the phenomenon *directional non-blow-up* in the direction ψ. Some of these notations were introduced in [12]. We get the next corollary.

Corollary 1.3. *Assume (A1)-(A3). Let* $u_0 \not\equiv \|u_0\|_\infty$ *and let* u *be a blow-up weak solution of* (1.1)(1.2) *with the least blow-up time. Then* u *has at least one blow-up direction.*

So, our interests are focussed on characterizing blow-up directions of a blow-up solution with the least blow-up time by the profile of the initial data as well as finding a condition on u_0 for a solution to blow up at the least blow-up time.

These problems have recently been discussed in Giga-Umeda [11, 12] for equation (1.1) with semilinearity

$$u_t = \Delta u + f(u) \quad \text{in } (x,t) \in \mathbf{R}^N \times (0,T), \tag{1.12}$$

where

$$f'(\xi) \geq 0 \quad \text{for } \xi \in \mathbf{R} \tag{1.13}$$

and for some $M > 0$, $p > 1$, $\delta_0 \in (0,1)$ and $\xi_0 > 0$,

$$f(M) > 0 \quad \text{and} \quad f(\delta\xi) \leq \delta^p f(\xi) \text{ for } \delta \in (\delta_0, 1) \text{ and } \xi \geq \xi_0. \tag{1.14}$$

The interesting results concerning with blow-up directions have been obtained there. Giga and Umeda treat in [11] the case where $f(u) = u^p$ and consider in [12] a sign-changing solution (so that the solution may blow up to both ∞ and $-\infty$). But, we mention their results only for nonnegative solutions.

It was shown in [11] that if $\lim_{|x|\to\infty} u_0(x) = \|u_0\|_\infty \equiv M > 0$, then the solution u blows up at the least blow-up time T_M and satisfies that $\lim_{|x|\to\infty} u(x,t) = v_M(t)$ uniformly on compact subsets of $\{0 \leq t < T_M\}$. Namely, all directions $\psi \in \mathbf{S}^{N-1}$ are blow-up directions. It was also shown that when $u_0(x) \not\equiv M$, the solution u never blows up in $\mathbf{R}^N$ at the blow-up time T_M, that is, the solution blows up only at space infinity. To state this result exactly, we introduce the set

$$\begin{aligned} S = \{x \in \mathbf{R}^N;\ &\text{there exists a sequence } \{(x_n, t_n)\} \subset \mathbf{R}^N \times (0, T_M) \\ &\text{such that } x_n \to x,\ t_n \uparrow T_M \text{ and } u(x_n, t_n) \to \infty \text{ as } n \to \infty\}. \end{aligned}$$

S is called *the blow-up set* of u and each x of S a *blow-up point* of u. Then, it was shown in [11] that $S = \emptyset$ (see [12] for general $f(u)$).

Especially, in [12], they obtained a sufficient and necessary condition on u_0 for directional blow-up in a direction with the least blow-up time (see condition $(\mathrm{A8})_\psi$ below), and this was done for general $f(u)$ by using the mean value of the initial data over a ball centered at $x_0 \in \mathbf{R}^N$:

$$\tilde{A}_R(x_0; u_0) = \int_{B_R(x_0)} u_0(x)\, dx / |B_R(x_0)| \quad \text{for } R > 0, \tag{1.15}$$

where $|B_R(x_0)|$ is the Lebesgue measure of $B_R(x_0)$. In other words, a blow-up direction of the blow-up solution with the least blow-up time is completely characterized by the profile of the initial data.

The main purpose of the present paper is to extend these results of [11, 12] to the quasilinear case $\phi(u)$ for nonnegative solutions. However, we can not apply their methods to the quasilinear case $\phi(u)$, since their methods strongly depend on the semilinearity of the equation ($\phi(u) = u$) and use heavily the expression of a solution by the heat kernel.

We note that in the one-dimensional case, similar results were obtained in Lacey [13] for the initial boundary value problem in a half line

$$\begin{cases} u_t = u_{xx} + f(u) & \text{for } x > 0,\ t > 0, \\ u(0,t) = 1 & \text{for } t > 0, \\ u(x,0) = u_0(x) \geq 1 & \text{for } x > 0. \end{cases}$$

To characterize blow-up directions, we introduce the next mean value of the initial data u_0 with the weight function $e^{-|x|}$ and with the center at $x_0 \in \mathbf{R}^N$, which is different from that of [12]:

$$A_\rho(x_0; u_0) = \int_{\mathbf{R}^N} \rho(x - x_0) u_0(x)\, dx, \tag{1.16}$$

where

$$\rho(x) = \frac{e^{-|x|}}{\int_{\mathbf{R}^N} e^{-|x|}\, dx} \quad \text{in } x \in \mathbf{R}^N, \quad \left(\text{that is, } \int_{\mathbf{R}^N} \rho(x)\, dx = 1 \right). \tag{1.17}$$

Of course, our sufficient and necessary condition for a direction $\psi \in \mathbf{S}^{N-1}$ to be a blow-up direction, which is given below by using (1.16) (see (A5)$_\psi$), is equivalent to that of Giga-Umeda [12] (see Remark 1.9).

Furthermore, we need the next condition on $f(\xi)$ which expresses that $f(\xi)$ grows up more rapidly than $\phi(\xi)$:

(A4) There exist a function $\Psi(\eta)$ and constants $c > 0$ and $\eta_1 > 0$ such that

$$\begin{aligned} &\Psi(\eta) > 0,\ \Psi'(\eta) \geq 0 \ \text{ and } \Psi''(\eta) \geq 0 \quad \text{for } \eta > \eta_1; \\ &\int_{\eta_1+1}^\infty \frac{d\eta}{\Psi(\eta)} < \infty; \\ &\{f(\phi^{-1}(\eta))\}'\Psi(\eta) - f(\phi^{-1}(\eta))\Psi'(\eta) \geq c\Psi(\eta)\Psi'(\eta) \quad \text{for } \eta > \eta_1, \end{aligned}$$

where $\xi = \phi^{-1}(\eta)$ is the inverse function of $\eta = \phi(\xi)$.

Condition (A4) leads to condition (A3). The original condition of (A4) was introduced by Friedman-Mcload [7] to obtain single point blow-up when $\phi(\xi) = \xi$ and was re-formulated into weaker version (A4) with $\phi(\xi) = \xi$ by Fujita-Chen [8] and Chen [4] (see also [15] for general $\phi(\xi)$). We note that the condition (A4) is weaker than the condition (1.14) which is assumed in [12] for the semilinear case (see Remark 1.10).

Remark 1.4. If $p > m$ then equation (1.3) satisfies (A4).

The main results are summarized by the next two theorems. Theorem 1.5 gives a sufficient condition on u_0 for directional blow-up with the least blow-up time (see condition $(\mathrm{A5})_\psi$ below) only under the assumption (A3). Theorem 1.8 shows that such a sufficient condition on u_0 is also a necessary condition for directional blow-up, when $f(\xi)$ grows up more rapidly than $\phi(\xi)$ (see condition (A4)). Namely, in Theorem 1.8, we can completely characterize blow-up directions of a blow-up solution with the least blow-up time by using the profile of the initial data. We note here that partial results were obtained by Seki [18] for the special equation (1.3) with the restricted case $p > m \geq 1$. He extended the results of [12] to this case and gave some sufficient (not necessary) condition on u_0 for directional blow-up (or directional non-blow-up) with the least blow-up time.

Let us introduce the condition on u_0 for a direction $\psi \in \mathbf{S}^{N-1}$ to be a blow-up direction:

$(\mathrm{A5})_\psi$ There exists a sequence $\{x_n\} \subset \mathbf{R}^N$ satisfying $\lim_{n\to\infty} |x_n| = \infty$ and $\lim_{n\to\infty} x_n/|x_n| = \psi$ such that

$$\lim_{n\to\infty} A_\rho(x_n; u_0) = \|u_0\|_\infty, \tag{1.18}$$

where $A_\rho(x; u_0)$ is defined by (1.16).

Theorem 1.5. *Assume (A1)-(A3). Suppose that u_0 satisfies condition $(\mathrm{A5})_\psi$ for some $\psi \in \mathbf{S}^{N-1}$. Then the solution u of (1.1)(1.2) blows up at the least blow-up time T_M with $M = \|u_0\|_\infty$ and ψ is a blow-up direction of u. Furthermore, u satisfies that for each $R > 0$,*

$$\lim_{n\to\infty} \sup_{x\in B_R(x_n)} |u(x,t) - v_M(t)| = 0 \tag{1.19}$$

uniformly on compact subsets of $\{0 < t < T_M\}$, where the sequence $\{x_n\}$ is as in condition $(\mathrm{A5})_\psi$.

Remark 1.6. Even if the initial data u_0 satisfies condition $(\mathrm{A5})_\psi$ without the conditions $\lim_{n\to\infty}|x_n| = \infty$ and $\lim_{n\to\infty} x_n/|x_n| = \psi$, the solution u in Theorem 1.5 satisfies (1.19) and hence it blows up at the least blow-up time.

Remark 1.7. Using Theorem 1.5 (see also Remark 1.6), we can easily see the next result which was obtained in [11] (for the case where $\phi(u) = u$ and $f(u) = u^p$ with $p > 1$) and in [18] (for the case where $\phi(u) = u^m$ and $f(u) = u^p$ with $p \geq m > 1$): If $\lim_{|x|\to\infty} u_0(x) = \|u_0\|_\infty \equiv M$, then the solution u satisfies that $\lim_{|x|\to\infty} u(x,t) = v_M(t)$ uniformly on compact subsets of $\{0 < t < T_M\}$. In fact, assume contrary that $u(x,t)$ does not converge to $v_M(t)$ uniformly on some compact set K of $\{0 < t < T_M\}$ as $|x| \to \infty$. Then, there exist a constant $\varepsilon_0 > 0$ and a sequence $\{(x_n, t_n)\} \subset \mathbf{R}^N \times K$ satisfying $\lim_{n\to\infty}|x_n| = \infty$ such that $u(x_n, t_n) - v_M(t_n) \leq -\varepsilon_0$. On the other hand, since u_0 satisfies $\lim_{n\to\infty} A_\rho(x_n; u_0) = \|u_0\|_\infty$, we see (1.19) by Remark 1.6. This is a contradiction and so we get the assertion.

Theorem 1.8. *Assume (A1)(A2)(A4). Let $u_0 \not\equiv \|u_0\|_\infty$ and let u be a blow-up weak solution of* (1.1)(1.2) *with the least blow-up time T_M where $M = \|u_0\|_\infty$. Then, the following hold:*

(i) u blows up only at space infinity, that is, the blow-up set S is empty: $S = \emptyset$.

(ii) A direction $\psi \in \mathbf{S}^{N-1}$ is a blow-up direction if and only if u_0 satisfies condition $(\mathrm{A5})_\psi$ *for ψ. Furthermore, the solution u satisfies* (1.19) *for each $R > 0$, where the sequence $\{x_n\}$ is as in the condition $(A5)_\psi$.*

Remark 1.9. Let $\psi \in \mathbf{S}^{N-1}$. Let $\{R_n\}$ ($R_n > 1$) be a sequence of numbers diverging to ∞ as $n \to \infty$. Then, condition $(\mathrm{A5})_\psi$ is equivalent to each of the following three conditions:

$(\mathrm{A6})_\psi$ There exists a sequence $\{x_n\} \subset \mathbf{R}^N$ satisfying $\lim_{n\to\infty}|x_n| = \infty$ and $\lim_{n\to\infty} x_n/|x_n| = \psi$ such that

$$u_0(x + x_n) \to \|u_0\|_\infty \quad \text{as } n \to \infty \quad \text{a. e. in } \mathbf{R}^N. \tag{1.20}$$

$(\mathrm{A7})_\psi$ There exists a sequence $\{x_n\} \subset \mathbf{R}^N$ satisfying $\lim_{n\to\infty}|x_n| = \infty$ and $\lim_{n\to\infty} x_n/|x_n| = \psi$ such that for any $R > 1$,

$$\lim_{n\to\infty} \tilde{A}_R(x_n; u_0) = \|u_0\|_\infty. \tag{1.21}$$

$(\mathrm{A8})_\psi$ There exists a sequence $\{x_n\} \subset \mathbf{R}^N$ satisfying $\lim_{n\to\infty}|x_n| = \infty$ and $\lim_{n\to\infty} x_n/|x_n| = \psi$ such that

$$\lim_{n\to\infty} \inf_{r\in[1,R_n]} \tilde{A}_r(x_n; u_0) = \|u_0\|_\infty. \tag{1.22}$$

Here, $\tilde{A}_r(x_0; u_0)$ is defined by (1.15).

Condition $(\mathrm{A8})_\psi$ appears in Theorem 3 (i) of [12] as a sufficient and necessary condition for directional blow-up. This equivalence will be shown in Appendix B (see Proposition 7.1). So, in Theorem 1.5, Remark 1.6 and (ii) of Theorem 1.8, condition $(A5)_\psi$ can be replaced by each of these conditions.

Remark 1.10. As above-mentioned, a similar result to Theorem 1.8 was obtained in [12] in the semilinear case $\phi(\xi) = \xi$. However, under the assumption (A2) with $\phi(\xi) = \xi$, we can treat in Theorem 1.8, a wider class of functions $f(\xi)$ than Giga and Umeda treat in [12], for example, $f(\xi) = (\xi + 1)\{\log(\xi + 1)\}^b$ ($b > 2$), since condition (A4) is weaker than condition (1.14). In fact, the condition (1.14) implies that $f(\xi)/\xi^p$ is nondecreasing in $\xi > \xi_0$ and hence

$$f'(\xi) \geq \frac{pf(\xi)}{\xi} \quad \text{for } \xi > \xi_0.$$

So, if (1.14) holds, then f satisfies (A4) by taking $\Psi(\xi) = \xi^{p'}$ ($1 < p' < p$). Unfortunately, we have no results in Theorem 1.8 for the case where $f(\xi)$ does not grow up more rapidly than $\phi(\xi)$, for example, $\phi(\xi) = \xi$ and $f(\xi) = (\xi + 1)\{\log(\xi + 1)\}^b$ ($b \in (1, 2]$). In Theorem 1.5, we require only conditions (A2) and (A3) so that we can treat the case where $f(\xi)$ grows up more slowly than $\phi(\xi)$, for example $f(\xi) = \xi^p$ and $\phi(\xi) = \xi^m$ with $1 < p \leq m$.

As an immediate consequence of Corollary 1.3 and Theorem 1.8, we can get a necessary and sufficient condition on u_0 for blow-up with the least blow-up time.

Theorem 1.11. *Assume (A1)(A2)(A4). Let $u_0 \not\equiv 0$ and let u be a weak solution of* (1.1)(1.2)*. Then u blows up at the least blow-up time if and only if u_0 satisfies*

$$\sup_{x \in \mathbf{R}^N} A_\rho(x; u_0) = \|u_0\|_\infty. \tag{1.23}$$

Remark 1.12. Similarly as in Remark 1.9, the condition (1.23) in Theorem 1.11 can be replaced by each of the following two conditions:

(A9) There exists a sequence $\{x_n\} \subset \mathbf{R}^N$ satisfying (1.20).

(A10)

$$\sup_{x \in \mathbf{R}^N} \tilde{A}_R(x; u_0) = \|u_0\|_\infty \quad \text{for each } R > 1.$$

Here, we mention the case where $f(\xi)$ does not grow up more rapidly than $\phi(\xi)$. Let the initial data $u_0(x) = u_0(r)$ ($r = |x|$) be a radially symmetric function in x, which is nonincreasing in $r \geq 0$ and has a compact support. Consider a blow-up solution u of the special equation (1.3) with the blow-up time $T > 0$; the case corresponds to $p \leq m$. It is well known that the blow-up phenomena are much different from each other in the cases $p > m$ and $p < m$: Roughly speaking, the solution blows up only at the origin and the support of $u(\cdot, t)$ remains bounded as $t \uparrow T$ if $p > m$, whereas the solution blows up in whole $\mathbf{R}^N$ and the support of $u(\cdot, t)$

spreads out to the whole $\mathbf{R}^N$ as $t \uparrow T$ if $p < m$ (see [9, 10, 15]). We note that when $p = m$, different phenomena from them also occur (see e.g. [10]). In our problem, Theorem 1.5 holds for all cases $p > m$, $p = m$ and $p < m$, but in Theorem 1.8 we have no results for the case $p \le m$, as said in Remark 1.10. So, in Theorem 1.8, different phenomena might occur for $p \le m$.

The methods of the proofs of Theorem 1.5 and 1.8 are quite different from those of Giga-Umeda [11, 12]. As above said, the methods of [11, 12] strongly depend on the semilinearity of the equation ($\phi(u) = u$) and use heavily the expression of a solution by the heat kernel, and so we can not apply their methods to the quasilinear case $\phi(u)$. Our methods of the proofs are based on the comparison principle (or the maximum principle) and the smoothing effect. Especially, the equicontinuity of solutions (a kind of the smoothing effect) due to [5], which depends only on the supremum of solutions, plays an important role. Furthermore, in the proof of Theorem 1.5, the estimate for solutions which arises in the comparison theorem due to [2] (see Proposition 2.3 and Lemma 4.1) plays a crucial role. As to Theorem 1.8, our method relies on construction of a suitable supersolution which has no blow-up points in the interior of the considered domain, and the construction is done by using the result of [15] (see Lemma 2.8) which leads to the existence of a single point blow-up solution.

The rest of the paper is organized as follows. In the next section §2, we define a weak solution of (1.1) and give several preliminary propositions. In §3, we show Theorem 1.2 and Corollary 1.3. We show in §4 Theorem 1.5 and in §5 Theorems 1.8 and 1.11. In Appendix A, for the convenience of readers we prove the comparison theorem for solutions (Proposition 2.3), which leads to Lemma 4.1 for the proof of Theorem 1.5. Finally, in Appendix B, we show that conditions $(\mathrm{A5})_\psi \sim (\mathrm{A8})_\psi$ are equivalent.

2. Definitions and preliminaries

In this section, we define a weak solution of (1.1) and give preliminary propositions. We begin with the definition of a weak solution of (1.1). Let G be a domain in $\mathbf{R}^{\mathbf{N}}$ with smooth boundary ∂G.

Definition 2.1. By a weak solution of equation (1.1) in $G \times (0, T)$, we mean a function $u(x,t)$ in $\bar{G} \times [0, T)$ such that

(i) $u(x,t) \geq 0$ in $\bar{G} \times [0, T)$ and $\in BC(\bar{G} \times [0, \tau])$ (bounded continuous) for each $0 < \tau < T$,

(ii) For any bounded domain $\Omega \subset G$ with smooth boundary $\partial\Omega$, $0 < \tau < T$ and nonnegative $\varphi(x,t) \in C^{2,1}(\bar{\Omega} \times [0, T))$ which vanishes on the boundary $\partial\Omega$,

$$\int_\Omega u(x,\tau)\varphi(x,\tau)\,dx - \int_\Omega u(x,0)\varphi(x,0)\,dx \tag{2.1}$$

$$= \int_0^\tau \int_\Omega \{u\partial_t\varphi + \phi(u)\triangle\varphi + f(u)\varphi\}\,dx\,dt - \int_0^\tau \int_{\partial\Omega} \phi(u)\partial_\nu\varphi\,dSdt,$$

where ν denote the outer unit normal to the boundary.

A supersolution [or subsolution] is similarly defined with the equality of (2.1) replaced by $\geq$ [or $\leq$].

Comparison theorems are given in the case where the domain G is bounded and in the case where $G = \mathbf{R}^N$, as follows. The first comparison theorem (Proposition 2.2) is for a bounded domain G and was already proved by Aronson-Crandall-Peletier [1] when the boundary ∂G is smooth. The second comparison theorem (Proposition 2.3) is for $G = \mathbf{R}^N$. This comparison theorem was shown by Bertsch-Kersner-Peletier [2] and leads to an important lemma (see Lemma 4.1), which plays a crucial role in the proof of Theorem 1.5 as well as the result about the equicontinuity of solutions (see Proposition 2.5). Proposition 2.3 will be shown in Appendix A for the convenience of readers.

Proposition 2.2 (the comparison theorem in the case of a bounded domain)**.** *Assume (A1)(A2). Let G be a bounded domain in $\mathbf{R^N}$ with smooth boundary ∂G. Let u (or v) be a supersolution (or a subsolution) of* (1.1) *in $G \times (0, T)$. If $u \geq v$ on the parabolic boundary of $G \times (0, T)$, then we have $u \geq v$ in the whole $\bar{G} \times [0, T)$.*

Proof. See [1]. □

Proposition 2.3 (the comparison theorem in the case of $G = \mathbf{R}^N$)**.** *Assume (A1)(A2). Let u (or v) be a supersolution (or a subsolution) of* (1.1) *in $\mathbf{R}^N \times (0, T)$. Assume for some $M > 0$,*

$$u(x,t),\ v(x,t) \leq M \quad \textit{for } (x,t) \in \mathbf{R}^N \times [0, T). \tag{2.2}$$

Put

$$K = \sup\left\{\frac{\phi(\xi) - \phi(\eta)}{\xi - \eta} + \frac{|f(\xi) - f(\eta)|}{|\xi - \eta|} + 1\,;\ \xi \neq \eta \textit{ and } 0 \leq \xi, \eta \leq M\right\}. \tag{2.3}$$

Then

$$\int_{\mathbf{R}^N}[v(x,t)-u(x,t)]_+e^{-|x|}\,dx \le e^{2Kt}\int_{\mathbf{R}^N}[v(x,0)-u(x,0)]_+e^{-|x|}\,dx, \tag{2.4}$$

where $[a]_+ = \max\{a, 0\}$.

Proof. See Appendix A. □

The next proposition follows from Proposition 2.3 and the maximum principle.

Proposition 2.4. *Assume (A1)(A2) and assume* $u_0 \not\equiv 0$. *Let* u *be a weak solution of* (1.1)(1.2). *Put* $M = \|u_0\|_{L^\infty(\mathbf{R}^N)} > 0$. *Then,*

$$u(x,t) \le v_M(t) \quad \text{for } (x,t) \in \mathbf{R}^N \times (0, T_M). \tag{2.5}$$

If $u_0 \not\equiv M$, *then*

$$u(x,t) < v_M(t) \quad \text{for } (x,t) \in \mathbf{R}^N \times (0, T_M). \tag{2.6}$$

Proof. Since $u_0(x) \le \|u_0\|_\infty = M$ in $\mathbf{R}^N$ and $v_M(t)$ is a solution of (1.1)(1.2) with the initial data $u_0 \equiv M$, (2.5) holds by the comparison theorem (Proposition 2.3).

Next, assuming $u_0 \not\equiv M$, we prove (2.6). We note that $u(x,t)$ is a C^∞-function in the region of $(x,t) \in \mathbf{R}^N \times (0,T)$ where $u(x,t) > 0$ by virtue of the usual regularization method (see e.g. Ladyzenskaja et al. [14]).

Assume contrary that for some $(x_1\, t_1) \in \mathbf{R}^N \times (0, T_M)$, $u(x_1, t_1) = v_M(t_1) > 0$. Then, $u(x,t) > 0$ in $D = B_d(x_1) \times (t_1 - d', t_1 + d') \subset \mathbf{R}^N \times (0, T_M)$ for small $d > 0$ and $d' > 0$, and so $u \in C^\infty(D)$.

Put $w = \phi(u)$ and $w_M = \phi(v_M)$. Then, $w \le w_M$ in $\mathbf{R}^N \times (0, T_M)$, and w and w_M satisfy the same equation

$$\partial_t \beta(w) = \Delta w + g(w) \quad \text{in } D, \tag{2.7}$$

where $\beta(\eta) = \phi^{-1}(\eta)$ (the inverse function of $\eta = \phi(\xi)$) and $g(\eta) = f(\phi^{-1}(\eta))$. We note that $\beta'(\eta) = 1/\phi'(\phi^{-1}(\eta)) > 0$ for $\eta > 0$. So, putting $z = w - w_M$ we have

$$\beta'(w)z_t - \Delta z + \{\tilde{\beta}' \times (w_M)_t - \tilde{g}\}z = 0 \quad \text{in } D,$$

where

$$\tilde{g} = \int_0^1 g'(\theta w + (1-\theta)w_M)\,d\theta$$

and $\tilde{\beta}'$ is similarly defined. Hence, putting $h(x,t) = z(x,t)e^{\gamma t}$ ($\gamma > 0$) further, we obtain

$$h_t - \frac{1}{\beta'(w)}\Delta h = (\gamma - C_1(x,t))h \quad \text{in } D$$

where $C_1(x,t) = \{\tilde{\beta}' \times (w_M)_t - \tilde{g}\}/\beta'(w)$.

We choose $\gamma > 0$ large to satisfy $\gamma > \sup_{(x,t)\in D}|C_1(x,t)|$. Since $h \le 0$ in D and $h(x_1, t_1) = 0$, the strong maximum principle (Theorem 5, p173 of [17]) implies that $h = 0$ in $B_d(x_1) \times (t_1 - d', t_1]$. Thus, repeating this operation, we get $h = ze^{\gamma t} = 0$ in $\mathbf{R}^N \times [0, t_1]$, that is, $w = w_M$ in $\mathbf{R}^N \times [0, t_1]$ and so $u = v_M$ in $\mathbf{R}^N \times [0, t_1]$. Indeed, this result is shown by using a contradiction argument. Therefore $u_0 \equiv M$. This contradicts the assumption $u_0 \not\equiv M$ and so we get $u < v_M$ in $\mathbf{R}^N \times (0, T_M)$. The proof is complete. $\square$

The next proposition due to DiBenedetto [5] is the result concerned with the equicontinuity of solutions, which plays an important role in the proofs of Theorem 1.5 and 1.8.

Proposition 2.5 (the equicontinuity of solutions). *Assume (A1)-(A3) and assume $u_0 \not\equiv 0$. Let $M = \|u_0\|_\infty > 0$ and let u be a weak solution of* (1.1)(1.2). *Then, for any $\varepsilon > 0$ and $R > 0$, there exists a continuous nondecreasing function $\omega = \omega_{\varepsilon,R,M}$: $\bar{\mathbf{R}}_+ \to \bar{\mathbf{R}}_+$ with $\omega_{\epsilon,R,M}(0) = 0$ depending only on ε, R and M such that*

$$|u(x_1,t_1) - u(x_2,t_2)| \le \omega(|x_1 - x_2| + |t_1 - t_2|^{1/2}) \quad \text{for } (x_1,t_1),\ (x_2,t_2) \in \bar{B}_R \times [\varepsilon, T_M - \varepsilon]. \tag{2.8}$$

Proof. We note that $u(x,t) \le v_M(t)$ in $\mathbf{R}^N \times (0, T_M)$. So, this proposition follows from Lemma 5.2 of [5]. $\square$

Finally, we assume condition (A4) and show the next important proposition for the proof of Theorem 1.8(ii). This proposition will follow from the result of [15] (see Lemma 2.8 below).

Let $R > 0$ and $M > 0$. Let $w_0 \in C^2(\bar{B}_R)$ be a radially symmetric positive function in x, which satisfies that $w_0(r) = w_0(x)$ $(r = |x|)$ is nondecreasing in $r \ge 0$, $w_0(R) = M$ and $0 < w_0(0) < M$. Further, we assume

$$\Delta\phi(w_0) + f(w_0) \ge 0 \quad \text{in } \bar{B}_R. \tag{2.9}$$

Proposition 2.6. *Assume (A2)(A4). Let w be a weak solution to the problem*

$$\begin{cases} w_t = \Delta\phi(w) + f(w) & \text{in } B_R \times (0, T_M), \\ w(x,t) = v_M(t) & \text{on } |x| = R,\ t > 0, \\ w(x,0) = w_0(x) & \text{in } B_R. \end{cases} \tag{2.10}$$

Then,

$$\sup_{(x,t) \in K \times (0,T_M)} w(x,t) < \infty \tag{2.11}$$

for each compact subset K of B_R.

We need two lemmas.

Lemma 2.7. *Assume (A2). Let w be a weak solution of* (2.10). *Then,*

$$0 < w(x,t) \le v_M(t) \quad \text{in } \bar{B}_R \times [0, T_M) \tag{2.12}$$

and $w \in C^\infty(B_R \times (0, T_M))$. Furthermore, for each $t \in (0, T_M)$, $w(x,t) = w(r,t)$ ($r = |x|$) is radially symmetric in $x \in B_R$ and satisfies $\partial w/\partial r > 0$ in $(r,t) \in (0,R) \times (0,T_M)$, and for each $x \in B_R$, $w(x,t)$ is nondecreasing in $t \geq 0$.

Proof. Since $w_0(x) > 0$ in B_R, we see that $w > 0$ in $B_R \times (0, T_M)$ by the positivity of solutions (see Lemma 2.1 of [15]) and so $w \in C^\infty(B_R \times (0, T_M))$ (see the proof of Proposition 2.4). Since v_M is a solution of (1.1)(1.2) with the initial data $u_0 \equiv M$ and $w \leq v_M$ on the parabolic boundary of $B_R \times (0, T_M)$, the comparison theorem (Proposition 2.2) implies that $w(x,t) \leq v_M(t)$ in the whole $\bar{B}_R \times [0, T_M)$.

Next, we show the monotonicity of the solution w with respect to t. The method of the proof is the same as that of [3]. We note that $w_0(x)$ is a subsolution of (1.1). Since $w_0(x) \leq w$ on the parabolic boundary of $B_R \times (0, T_M)$, we see by Proposition 2.2 that $w_0(x) \leq w(x,t)$ in $(x,t) \in B_R \times (0, T_M)$. Hence, applying Proposition 2.2 to $w(x,t)$ and $w(x, t+t_1)$ for each $t_1 \in (0, T_M)$, we have $w(x,t) \leq w(x, t+t_1)$ in $B_R \times (0, T_M - t_1)$ and so $w(x,t)$ is nondecreasing in $t \geq 0$ for each $x \in \mathbf{R}^N$.

Since equation (1.1) is invariant under the rotation in x, we see by the uniqueness of solutions (Proposition 2.2) that for each $t \in (0, T_M)$, the solution $w(x,t) = w(r,t)$ ($r = |x|$) is also radially symmetric in x.

Finally, we shall show that $w_r(r,t) > 0$ in $r \in (0,R)$ for each $t \in (0, T_M)$. Let $\ell \in (0,R)$. For any $x = (x_1, x') \in \mathbf{R} \times \mathbf{R}^{N-1}$, the reflection of x in the hyperplain $\{x_1 = \ell\}$ is denoted by σ_ℓ, that is,

$$\sigma_\ell x = (2\ell - x_1, x').$$

Set $\Omega_\ell = \{x = (x_1, x') \in \mathbf{R} \times \mathbf{R}^{N-1}\,;\, x \in B_R,\ \ell < x_1 < R\}$. We note that $\sigma_\ell \Omega_\ell = \{\sigma_\ell x\,;\, x \in \Omega_\ell\} \subset B_R$. Let $T \in (0, T_M)$. For the above aim, we will prove that $\sigma_\ell w \leq w$ in $\Omega_\ell \times (0,T)$, where $\sigma_\ell w$ is the reflection of w in the hyperplain $\{x_1 = \ell\}$, that is,

$$\sigma_\ell w(x,t) = w(\sigma_\ell x, t).$$

Clearly, $\sigma_\ell w$ is also a solution of the equation of (2.10) in $\Omega_\ell \times (0, T_M)$. We note that the comparison theorem (Proposition 2.2) can not be applied directly, since the boundary $\partial\Omega_\ell$ is not smooth at $x = (\ell, x')$ with $|x| = R$. So, we consider a solution w_n ($n \geq 1$) to the problem

$$\begin{cases} w_t = \Delta\phi(w) + f(w) & \text{in } B_R \times (0, T_M), \\ w(x,t) = v_M(t) + 1/n & \text{on } |x| = R,\ t > 0, \\ w(x,0) = w_0(x) + 1/n & \text{in } B_R \end{cases} \tag{2.13}$$

and compare w_n and $\sigma_\ell w$ in $\Omega_\ell \times (0,T)$ for large $n \geq 1$. Then, we see that $w < w_n$ in $\bar{B}_R \times [0, T_M)$ as in the proof of Proposition 2.4, and $w_n \downarrow w$ as $n \to \infty$ locally uniformly in $\bar{B}_R \times [0, T_M)$ (see [6]). We note that $\sigma_\ell w(x,t) \leq v_M(t) < v_M(t) + 1/n = w_n(x,t)$ on $\{x\,;\, |x| = R,\ x_1 > \ell\} \times (0,T)$ and $\sigma_\ell w(x,t) = w(x,t) < w_n(x,t)$ on $\{x\,;\, |x| \leq R,\ x_1 = \ell\} \times (0,T)$. We also note that $\sigma_\ell w(x,0) \leq w(x,0) < w_n(x,0)$ in

Ω_ℓ by the assumption on w_0. Hence, there exists a domain $G_n \subset \Omega_\ell$ with smooth boundary ∂G_n such that $\sigma_\ell w < w_n$ in $(\Omega_\ell \backslash G_n) \times (0,T)$. Applying Proposition 2.2 to $\sigma_\ell w$ and w_n in $G_n \times (0,T)$, we have $\sigma_\ell w \le w_n$ in $G_n \times (0,T)$ and so $\sigma_\ell w \le w_n$ in $\Omega_\ell \times (0,T)$. Letting $n \to \infty$, we get $\sigma_\ell w \le w$ in $\Omega_\ell \times (0,T)$.

Thus, as in the proof of Lemma 3.1 of [19] (see also [7]), we get $\partial w/\partial r > 0$ in $(r,t) \in (0,R) \times (0,T)$ and so $\partial w/\partial r > 0$ in $(r,t) \in (0,R) \times (0,T_M)$. The proof is complete. □

The next lemma is due to [15].

Lemma 2.8. *Assume (A2)(A4). Let G be a domain in $\mathbf{R}^N$ with smooth boundary ∂G and let $u > 0$ be a weak solution of* (1.1) *in $G \times (0,T)$. Let $\Omega \subset G$ be a domain. If*

$$\partial_t u(x,t) \ge 0 \quad \text{in } \Omega \times (0,T), \tag{2.14}$$

and if there exist $\nu \in \mathbf{S}^{N-1}$ and $\delta > 0$ such that

$$\nu \cdot \nabla u(x,t) \le -\delta |\nabla u(x,t)| < 0 \quad \text{in } \Omega \times (0,T), \tag{2.15}$$

then u does not uniformly blow-up in Ω:

$$\inf_{x\in\Omega} u(x,t) \le L < \infty \quad \text{in } t \in (0,T). \tag{2.16}$$

Proof. This lemma is proved in [15] (see Lemma 4.1 of [15]). □

Proof of Proposition 2.6. Let $0 < r_1 < R$ and put $\Omega_\gamma = \{x = (x_1, \cdots, x_N) \in \mathbf{R}^N; r_1 < x_1 < r_1 + \gamma, -\gamma < x_j < \gamma, j = 2, \cdots, N\}$ for $\gamma > 0$. Choose $\gamma > 0$ small enough to satisfy $\Omega_\gamma \subset B_R$. Put $\nu = (-1, 0, \cdots, 0) \in \mathbf{R}^N$. Then, we have by Lemma 2.7,

$$\nu \cdot \nabla w(x,t) = -\frac{\partial w}{\partial x_1} = -\frac{x_1}{r}|\nabla w| \le -\frac{r_1}{R}|\nabla w| < 0 \text{ in } \Omega_\gamma \times (0,T_M)$$

where $r = |x|$. Also, from Lemma 2.7, we get $\partial_t w(x,t) \ge 0$ in $\Omega_\gamma \times (0,T_M)$. Thus, applying Lemma 2.8 we obtain

$$w(x,t) \le w(r_1,t) = \inf_{x\in\Omega_\gamma} w(x,t) \le L' < \infty \quad \text{in } |x| \le r_1,\ t \in (0,T_M).$$

Since $r_1 \in (0,R)$ can be chosen arbitrarily, we get (2.11). The proof is complete. □

3. Blow-up with the least blow-up time

In this section, we prove Theorem 1.2 and Corollary 1.3, which present the property of a blow-up solution with the least blow-up time.

Proof of Theorem 1.2. (i) (i) is already shown in Proposition 2.4.

(ii) Let $u_0 \not\equiv \|u_0\|_\infty$ and let u be a blow-up solution of (1.1)(1.2) with the least blow-up time T_M where $M = \|u_0\|_\infty$ (> 0). Assume contrary that for some $t_1 \in [0, T_M)$,

$$\lim_{R\to\infty} \sup_{|x|\geq R} u(x, t_1) < v_M(t_1). \tag{3.1}$$

We first consider the case $t_1 \in (0, T_M)$. Combining (3.1) and (1.8) we get

$$L = \sup_{x\in\mathbf{R}^N} u(x, t_1) < v_M(t_1). \tag{3.2}$$

Hence, by the comparison theorem we have $u(x, t) \leq v_L(t - t_1)$ in $\mathbf{R}^N \times (t_1, t_1 + T_L)$ and so $T_M \geq t_1 + T_L$, where v_L is a solution of (1.5) with M replaced by L, and T_L is the blow-up time of v_L:

$$T_L = \int_L^\infty \frac{d\xi}{f(\xi)}.$$

On the other hand, clearly $T_M = t_1 + T_{v_M(t_1)} < t_1 + T_L$. This is a contradiction to $T_M \geq t_1 + T_L$.

Next, we consider the case $t_1 = 0$. Putting $w_0(r) = \sup_{|y|\geq r} u_0(y)$ for $r \geq 0$, we see that $w_0(x) = w_0(r)$ $(r = |x|)$ is a radially symmetric continuous function in $x \in \mathbf{R}^N$ and is a nonincreasing function in $r \geq 0$. Further, it satisfies

$$u_0(x) \leq w_0(x) \leq M \quad \text{in } x \in \mathbf{R}^N$$

and

$$\lim_{r\to\infty} w_0(r) < M.$$

Hence, letting $w(x, t)$ be a solution of (1.1) with the initial data $w_0(x)$, we also see that for each $t \in (0, T_M)$, $w(x, t) = w(r, t)$ $(r = |x|)$ is a radially symmetric continuous function in $x \in \mathbf{R}^N$ and is a nonincreasing function in $r \geq 0$. We further see that $w(x, t)$ is a blow-up solution with the least blow-up time T_M, since $u(x, t) \leq w(x, t) \leq v_M(t)$ in $(x, t) \in \mathbf{R}^N \times (0, T_M)$ by the comparison theorem. Applying Proposition 2.4 to $w(x, t)$, we have for each $t \in (0, T_M)$,

$$w(x, t) \leq w(0, t) < v_M(t) \quad \text{in } x \in \mathbf{R}^N.$$

So, similarly as in the case $t_1 \in (0, T_M)$, we can lead to a contradiction.

Thus, for any case we lead to a contradiction and hence we get (1.9).

(iii) Let u be a solution of (1.1)(1.2) satisfying (1.9). Then, for any $t \in (0, T_M)$ with $M = \|u_0\|_\infty$, there exists $x_t \in \mathbf{R}^N$ such that $|x_t| \geq 1/(T_M - t)$ and $u(x_t, t) \geq v_M(t) - 1$. Since $\lim_{t\uparrow T_M} v_M(t) = \infty$, we see that the solution u blows up at space infinity at the time $t = T_M$. □

Proof of Corollary 1.3. Let u be a blow-up solution of (1.1)(1.2) with the least blow-up time T_M where $M = \|u_0\|_\infty$ (> 0). Then, because of Theorem 1.2, there exists a sequence $\{(x_n, t_n)\} \subset \mathbf{R}^N \times (0, T_M)$ such that $|x_n| \to \infty$, $t_n \uparrow T_M$ and $u(x_n, t_n) \to \infty$

as $n \to \infty$. Since $x_n/|x_n| \in \mathbf{S}^{N-1}$, there exists a subsequence $\{x_{n_j}\} \subset \{x_n\}$ such that

$$\frac{x_{n_j}}{|x_{n_j}|} \to \psi \qquad \text{as } n \to \infty$$

for some $\psi \in \mathbf{S}^{N-1}$. Thus, we see that ψ is a blow-up direction. The proof is complete. $\square$

4. Directional Blow-up

Assume (A1)-(A3). Let $u_0 \not\equiv 0$. In this section we prove Theorem 1.5 in which a sufficient condition on u_0 for directional blow-up with the least blow-up time is given. The next lemma which immediately follows from Proposition 2.3 plays a crucial role in the proof.

Lemma 4.1. *Assume (A1)-(A3). Let $u_0 \not\equiv 0$ and let u be a weak solution of (1.1)(1.2). Put $M = \|u_0\|_\infty$. Then,*

$$\int_{\mathbf{R}^N} |v_M(t) - u(x,t)|e^{-|x|}\,dx \le C_M(t)\int_{\mathbf{R}^N} |M - u_0(x)|e^{-|x|}\,dx \qquad \text{for } t \in (0, T_M), \tag{4.1}$$

where $C_M(t) > 0$ is an increasing function of t which depends only on M, and goes to ∞ as $t \uparrow T_M$.

Proof. We note that $v_M(t)$ is a solution of (1.1) with the initial data M. (4.1) follows from Proposition 2.3 and (2.5). $\square$

Proof of Theorem 1.5. Let $\psi \in \mathbf{S}^{N-1}$ and suppose that u_0 satisfies $(\mathrm{A5})_\psi$ for ψ. Then, there exists a sequence $\{x_n\} \subset \mathbf{R}^N$ satisfying $\lim_{n\to\infty} |x_n| = \infty$ and $\lim_{n\to\infty} x_n/|x_n| = \psi$ such that

$$\lim_{n\to\infty} A_\rho(x_n; u_0) = \|u_0\|_\infty. \tag{4.2}$$

Let u be a weak solution of (1.1)(1.2). We first show (1.19). Put $u_n(x,t) = u(x_n + x, t)$. Since u_n is a weak solution of (1.1), Lemma 4.1 implies that

$$\begin{aligned} &\int_{\mathbf{R}^N} |v_M(t) - u_n(x,t)|\rho(x)\,dx \\ &\le C_M(t)\int_{\mathbf{R}^N} |\|u_0\|_\infty - u_0(x + x_n)|\rho(x)\,dx \\ &= C_M(t)(\|u_0\|_\infty - A_\rho(x_n; u_0)) \qquad \text{for } t \in (0, T_M). \end{aligned} \tag{4.3}$$

On the other hand, we note that $u_n(x,t) \le v_M(t)$ in $\mathbf{R}^N \times (0, T_M)$. Let $R > 0$ and $\varepsilon > 0$ and consider u_n in $\bar{B}_R \times [\varepsilon, T_M - \varepsilon]$. Then, by virtue of Proposition 2.5, the sequence of solutions $\{u_n\}$ is equicontinuous and uniformly bounded in $\bar{B}_R \times [\varepsilon, T_M - \varepsilon]$, whence there exists a subsequence of $\{u_{n_j}\} \subset \{u_n\}$ such that

$u_{n_j} \to w$ uniformly in $\bar{B}_R \times [\varepsilon, T_M - \varepsilon]$ as $n_j \to \infty$ for some $w \in C(\bar{B}_R \times [\varepsilon, T_M - \varepsilon])$. Letting $n = n_j \to \infty$ in (4.3), we have by the condition (4.2),

$$\int_{B_R} |v_M(t) - w(x,t)|\rho(x)\,dx = 0, \tag{4.4}$$

that is, $w(x,t) = v_M(t)$ in $(x,t) \in \bar{B}_R \times [\varepsilon, T_M - \varepsilon]$. Since the limit $w = v_M$ is independent of choice of a subsequence $\{n_j\}$, we see that $u_n \to w = v_M$ uniformly in $\bar{B}_R \times [\varepsilon, T_M - \varepsilon]$ as $n \to \infty$, which shows (1.19).

From (1.19), the solution u of (1.1)(1.2) blows up at the least blow-up time and ψ is a blow-up direction of u. The proof is complete. $\square$

5. Directional non-blow-up

Assume (A1)(A2)(A4). Let $u_0 \not\equiv \|u_0\|_\infty$ and consider a blow-up solution with the least blow-up time. In this section, we shall prove Theorem 1.8, in which a blow-up direction $\psi \in \mathbf{S}^{N-1}$ is completely characterized by the profile of the initial data (see condition $(\mathrm{A5})_\psi$), when $f(\xi)$ grows up more rapidly than $\phi(\xi)$ (see condition (A4)). Since it is shown in Theorem 1.5 that condition $(\mathrm{A5})_\psi$ is a sufficient condition for directional blow-up in the direction $\psi \in \mathbf{S}^{N-1}$, it is enough to show that the condition $(\mathrm{A5})_\psi$ is also a necessary condition for directional blow-up in ψ. In fact, we prove its contraposition : "If u_0 does not satisfy $(\mathrm{A5})_\psi$ for some $\psi \in \mathbf{S}^{N-1}$, then ψ is a non-blow-up direction", that is, directional non-blow-up in ψ is obtained there. At the same time, it is shown that the blow-up set S is empty. Theorem 1.11 immediately follows from Theorem 1.8.

In order to get directional non-blow-up, the next proposition is a key proposition.

Proposition 5.1. *Assume (A1)(A2)(A4) and assume $u_0 \not\equiv 0$. Let u be a blow-up solution of* (1.1)(1.2) *with the least blow-up time T_M where $M = \|u_0\|_\infty$. Let $0 < L < \|u_0\|_\infty = M$. If*

$$A_\rho(0; u_0) \le L \tag{5.1}$$

where $A_\rho(x; u_0)$ is defined by (1.16)*, then there exists a constant $C_{M,L} > 0$ depending only on M and L such that*

$$u(0,t) \le C_{M,L} \quad \textit{for } t \in (0, T_M). \tag{5.2}$$

We need several lemmas.

Lemma 5.2. *Assume (A1)(A2)(A4) and assume $u_0 \not\equiv 0$. Let $M = \|u_0\|_\infty$ and let u be a blow-up solution of* (1.1)(1.2) *with the least blow-up time T_M. Let $0 < L < M$*

and assume (5.1). *Then, there exists a* $t_1 = t_1(M, L) \in (0, T_M/2)$ *depending only on* M *and* L *such that*

$$A_\rho(0; u(t_1)) = \int_{\mathbf{R}^N} \rho(x)u(x, t_1)\, dx \le \frac{M + L}{2} = L_1\,(< M). \tag{5.3}$$

Hence, there exists a constant $R_0 = R_0(M, L) > 0$ *such that*

$$\frac{1}{\int_{B_{R_0}} \rho(x)\, dx} \int_{B_{R_0}} \rho(x)u(x, t_1)\, dx \le \frac{M + L_1}{2} = L_2\,(< M). \tag{5.4}$$

Proof. We first show (5.3). Choose $t_0 = t_0(M) \in (0, T_M/2)$ small to satisfy $v_M(t_0) \le M + 1$. Then, by Proposition 2.4, we have

$$u(x, t) \le v_M(t) \le v_M(t_0) \le M + 1 \quad \text{for } (x, t) \in \mathbf{R}^N \times (0, t_0]. \tag{5.5}$$

Since $\rho \in L^1(\mathbf{R}^N)$, there exists a sequence of nonnegative functions $\{\rho_n\} \subset C_0^\infty(\mathbf{R}^N)$ such that $\rho_n \to \rho$ in $L^1(\mathbf{R}^N)$ as $n \to \infty$. Hence, there exists $n_0 \ge 1$ depending only on M, L such that

$$\|\rho - \rho_{n_0}\|_{L^1(\mathbf{R}^N)} < \frac{M - L}{4(2M + 1)}. \tag{5.6}$$

We further choose $t_1 = t_1(M, L) \in (0, t_0)$ small such that

$$t_1 \left\{ \phi(M + 1) \int_{\mathbf{R}^N} |\Delta \rho_{n_0}(x)|\, dx + \sup_{0 \le \xi \le M+1} f(\xi) \int_{\mathbf{R}^N} \rho_{n_0}(x)\, dx \right\} < \frac{M - L}{4}. \tag{5.7}$$

Considering $\varphi = \rho_{n_0}(x)$ as a test function in (2.1) with $\tau = t_1$, we have

$$\begin{aligned} &\int_{\mathbf{R}^N} u(x, t_1)\rho_{n_0}(x)\, dx \\ &= \int_0^{t_1} \int_{\mathbf{R}^N} \{\phi(u) \triangle \rho_{n_0}(x) + f(u)\rho_{n_0}(x)\}\, dx\, dt + \int_{\mathbf{R}^N} u_0(x)\rho_{n_0}(x)\, dx. \end{aligned} \tag{5.8}$$

Hence, by (5.1), (5.5), (5.6) and (5.7) we get

$$\begin{aligned} A_\rho(0; u(t_1)) &\le \int_{\mathbf{R}^N} u(x, t_1)\rho_{n_0}(x)\, dx + (M + 1)\|\rho - \rho_{n_0}\|_{L^1(\mathbf{R}^N)} \\ &\le \int_0^{t_1} \int_{\mathbf{R}^N} \{\phi(M + 1)|\Delta \rho_{n_0}(x)| + \sup_{0 \le \xi \le M+1} f(\xi) \times \rho_{n_0}(x)\}\, dxdt \\ &\qquad + (2M + 1)\|\rho - \rho_{n_0}\|_{L^1(\mathbf{R}^N)} + A_\rho(0; u_0) \\ &< \frac{M - L}{4} + \frac{M - L}{4} + L = \frac{M + L}{2} = L_1. \end{aligned}$$

So we obtain (5.3).

Next, we show (5.4). Putting

$$\varepsilon(R) = \int_{|x|>R} \rho(x)\, dx$$

we get by (5.3), (5.5) and (1.17),

$$
\begin{aligned}
(5.9)\qquad & \frac{1}{\int_{B_R}\rho(x)\,dx}\int_{B_R}\rho(x)u(x,t_1)\,dx \\
&= \frac{1}{\int_{B_R}\rho(x)\,dx}\int_{B_R}\rho(x)u(x,t_1)\,dx - A_\rho(0;u(t_1)) + A_\rho(0;u(t_1)) \\
&\le \frac{\varepsilon(R)\int_{B_R}\rho(x)u(x,t_1)\,dx - \int_{B_R}\rho(x)\,dx\int_{|x|>R}\rho(x)u(x,t_1)\,dx}{\int_{B_R}\rho(x)\,dx} + L_1 \\
&\le (M+1)\varepsilon(R) + L_1.
\end{aligned}
$$

Thus, if we choose $R > 0$ large to satisfy

$$\varepsilon(R) < \frac{(M - L_1)}{2(M+1)},$$

then

$$\frac{1}{\int_{B_R}\rho(x)\,dx}\int_{B_R}\rho(x)u(x,t_1)\,dx < L_1 + \frac{M-L_1}{2} = \frac{M+L_1}{2} = L_2.$$

□

Hence, we have

Lemma 5.3. *Let u be as in Lemma 5.2. Then, there exists $x_0 \in B_{R_0}$ such that*

$$u(x_0,t_1) \le L_2, \tag{5.10}$$

where t_1, L_2 and R_0 are as in Lemma 5.2. Furthermore, there exists a $r_0 = r_0(t_1, M, L_2, R_0) \in (0, R_0)$ such that

$$u(x,t_1) < \frac{M+L_2}{2} = L_3\,(< M) \quad \text{for } |x - x_0| \le r_0. \tag{5.11}$$

Proof. (5.10) follows from (5.4).

For the proof of (5.11), we use the equicontinuity of solutions (Proposition 2.5). Let $\omega = \omega_{t_1,2R_0,M}$ be as in Proposition 2.5 with $\varepsilon = t_1$ and $R = 2R_0$. We choose $r_0 = r_0(M, L_2, \omega) \in (0, R_0)$ small to satisfy

$$\omega(r) < \frac{M - L_2}{2} \quad \text{for } 0 \le r \le r_0.$$

Then, we have

$$
\begin{aligned}
u(x,t_1) &= u(x,t_1) - u(x_0,t_1) + u(x_0,t_1) \\
&\le \omega(|x-x_0|) + u(x_0,t_1) \le \frac{M-L_2}{2} + L_2 = L_3 \\
&\qquad \text{for } x \in \bar{B}_{r_0}(x_0) \subset B_{2R_0}.
\end{aligned}
$$

The proof is complete. □

Now, let $t_1,\ r_0,\ M,\ R_0$ and L_3 be constants as in Lemma 5.3 and let $w_0 \in C(\bar{B}_{R_0+2})$ be a radially symmetric function in x satisfying that $w_0(r) = w_0(x)$ $(r = |x|)$ is nondecreasing in $r \geq 0$ and

$$w_0(x) \begin{cases} = v_M(t_1) & \text{if } r_0 \leq |x| \leq R_0+2, \\ \geq L_3 & \text{if } r_0/2 \leq |x| < r_0, \\ = L_3 & \text{if } 0 \leq |x| < r_0/2. \end{cases} \tag{5.12}$$

We note that $u(x+x_0,t_1) \leq w_0(x)$ in B_{R_0+2}. Consider the initial boundary value problem

$$\begin{cases} w_t = \Delta\phi(w) + f(w) & \text{in } (x,t) \in B_{R_0+2} \times (t_1, T_M), \\ w(x,t) = v_M(t) & \text{on } |x| = R_0+2,\ t_1 < t < T_M, \\ w(x,t_1) = w_0(x) & \text{in } x \in B_{R_0+2}. \end{cases} \tag{5.13}$$

Lemma 5.4. *Let w be a solution of* (5.13). *Then,*

$$\sup_{(x,t)\in B_{R_0}\times(t_1,T_M)} w(x,t) < \infty. \tag{5.14}$$

Proof. We will use Proposition 2.6. Let w be a solution to the problem (5.13). Then, as in the proof of (2.6),

$$w(x,t) < v_M(t) \quad \text{ in } (x,t) \in B_{R_0+2} \times (t_1, T_M).$$

Let $t_2 \in (t_1, T_M)$. Then, because of the continuity of w,

$$\sup_{x\in B_{R_0+1}} w(x,t_2) \equiv M' < v_M(t_2). \tag{5.15}$$

Now, put

$$\tilde{w}_0(x) = \phi^{-1}(b|x|^2 + \phi(M')) \quad \text{ with } b = \frac{\phi(v_M(t_2)) - \phi(M')}{(R_0+1)^2},$$

where $\xi = \phi^{-1}(\eta)$ is the inverse function of $\eta = \phi(\xi)$. Then, $\Delta\phi(\tilde{w}_0) + f(\tilde{w}_0) \geq 0$, $w(x,t_2) \leq M' \leq \tilde{w}_0(x) \leq v_M(t_2)$ in $x \in B_{R_0+1}$ and $\tilde{w}_0(x) = v_M(t_2)$ on $|x| = R_0+1$. Let $\tilde{w}$ be a solution to the problem

$$\begin{cases} \tilde{w}_t = \Delta\phi(\tilde{w}) + f(\tilde{w}) & \text{in } B_{R_0+1} \times (t_2, T_M), \\ \tilde{w}(x,t) = v_M(t) & \text{on } |x| = R_0+1,\ t_2 < t < T_M, \\ \tilde{w}(x,t_1) = \tilde{w}_0(x) & \text{in } x \in B_{R_0+1}. \end{cases} \tag{5.16}$$

The comparison theorem implies $w(x,t) \leq \tilde{w}(x,t)$ in $(x,t) \in B_{R_0+1} \times (t_2, T_M)$. Therefore, applying Proposition 2.6 to $\tilde{w}$ in $B_{R_0+1} \times (t_2, T_M)$, we have

$$\sup_{(x,t)\in B_{R_0}\times(t_2,T_M)} w(x,t) \leq \sup_{(x,t)\in B_{R_0}\times(t_2,T_M)} \tilde{w}(x,t) < \infty,$$

which leads to (5.14). $\square$

Proof of Proposition 5.1. Let u, x_0, t_1, r_0, M, R_0 and L_3 be as in Lemma 5.3 and let w and w_0 be as in Lemma 5.4. We note that w depends only on t_1, r_0, M, R_0 and L_3 .

Since $u_{x_0}(x,t) = u(x+x_0,t)$ is also a solution of (1.1), $u_{x_0}(x,t_1) \le w_0(x)$ in $x \in B_{R_0+2}$ and $u_{x_0}(x,t) \le v_M(t)$ on $|x| = R_0+2$, $t > t_1$, the comparison theorem implies that

$$u_{x_0}(x,t) = u(x+x_0,t) \le w(x,t) \quad \text{in } (x,t) \in B_{R_0+2} \times (t_1, T_M).$$

It follows from Lemma 5.4 and $0 \in B_{R_0}(x_0)$ that

$$u(0,t) \le \sup_{(x,t)\in B_{R_0}(x_0)\times(t_1,T_M)} u(x,t) \le \sup_{(x,t)\in B_{R_0}\times(t_1,T_M)} w(x,t) = C_{M,L} < \infty \text{ in } (t_1,T_M).$$

The proof is complete. $\square$

Proof of Theorem 1.8. Assume $u_0 \not\equiv \|u_0\|_\infty$. Let u be a blow-up solution of (1.1)(1.2) with the least blow-up time T_M where $M = \|u_0\|_\infty$. We first show (ii). It is enough to show that if a direction $\psi \in \mathbf{S}^{N-1}$ is a blow-up direction, then the initial data u_0 satisfies $(A5)_\psi$ for ψ. For this aim, we prove its contraposition.

Let $\psi \in \mathbf{S}^{N-1}$. Suppose that the initial data u_0 does not satisfy condition $(A5)_\psi$ for ψ. Then, there exists an open neighborhood D of ψ in $\mathbf{S}^{N-1}$ such that

$$\sup_{x/|x|\in D} A_\rho(x;u_0) = L < M = \|u_0\|_\infty. \tag{5.17}$$

Let $x_0 \in \mathbf{R}^N$ satisfy $x_0/|x_0| \in D$. We note that $u_{x_0}(x,t) = u(x+x_0,t)$ is a solution of (1.1) with the initial data $u_0(x+x_0)$. We further note that by (5.17),

$$A_\rho(0; u_{x_0}(x,0)) = A_\rho(x_0;u_0) \le L.$$

Hence, applying Proposition 5.1 to u_{x_0} we have

$$u(x_0,t) = u_{x_0}(0,t) \le C_{M,L} \quad \text{for } t \in (0,T_M)$$

where $C_{M,L} > 0$ is a constant depending only on M and L. Thus, we obtain

$$\sup_{x/|x|\in D} u(x,t) \le C_{M,L} \quad \text{for } t \in (0,T_M),$$

that is, ψ is a non-blow-up direction.

Next, we show (i). Let $x_0 \in \mathbf{R}^N$ and $R > 0$. Then, since $u_0(x) \not\equiv \|u_0\|_\infty$, we see

$$\sup_{y\in B_R(x_0)} A_\rho(0; u_y(x,0)) = \sup_{y\in B_R(x_0)} A_\rho(y;u_0) \equiv \tilde{L} < \|u_0\|_\infty = M.$$

Hence, similarly as above, we have

$$u(y,t) = u_y(0,t) \le C_{M,\tilde{L}} \quad \text{for } (y,t) \in B_R(x_0)\times(0,T_M)$$

and so $x_0 \notin S$. Since $x_0 \in \mathbf{R}^N$ can be chosen arbitrarily, we get $S = \emptyset$. The proof is complete. $\square$

Proof of Theorem 1.11. Theorem 1.11 follows from Corollary 1.3 and Theorem 1.8. $\square$

6. Appendix A

In this section, we prove Proposition 2.3. As is mentioned in §2, the proposition was shown by Bertsch-Kersner-Peletier [2]. However, for the convenience of readers, we present the details of the proof.

Proof of Proposition 2.3. Let u (or v) be a supersolution (or a subsolution) of (1.1) in $\mathbf{R}^N \times (0,T)$. Assume (2.2) for some $M > 0$. Let $R > 0$. Then, we have for any nonnegative function $\varphi \in C^{2,1}(\bar{B}_R \times [0,T))$ which vanishes on the boundary ∂B_R,

$$
\begin{aligned}
\int_{B_R} (v(x,\tau) - u(x,\tau))\varphi(x,\tau)\,dx - \int\int_{Q_\tau} (v-u)(\varphi_t + \tilde{\phi}\triangle\varphi)\,dxdt \\
\leq \int_{B_R} (v(x,0) - u(x,0))\varphi(x,0)\,dx + \int\int_{Q_\tau} (v-u)\tilde{f}\varphi\,dxdt \\
- \int_0^\tau \int_{\partial B_R} \{\phi(v) - \phi(u)\}\,\partial_\nu \varphi\,dSdt,
\end{aligned}
\tag{6.1}
$$

where $Q_\tau = B_R \times (0,\tau)$ $(\tau < T)$, ν denotes the outer unit normal to the boundary and

$$\tilde{g}(x,t) = \frac{g(v) - g(u)}{v - u} = \int_0^1 g'(\theta v + (1-\theta)u)\,d\theta.$$

Here, we note that

$$0 \leq \tilde{\phi} + 1 \leq K \quad \text{ in } Q_\tau \tag{6.2}$$

and

$$\|\tilde{f}\|_{L^\infty(Q_\tau)} \leq K, \tag{6.3}$$

where K is defined by (2.3).

We take a sequence of smooth positive functions $\{\phi_n\} \subset C^\infty(\mathbf{R}^N \times (-\infty,\infty))$ satisfying the following conditions (see [1]):

$$\frac{1}{n} \leq \phi_n \leq \|\tilde{\phi}\|_{L^\infty(Q_\tau)} + \frac{1}{n} \quad \text{ in } Q_\tau, \tag{6.4}$$

$$\frac{\phi_n - \tilde{\phi}}{\sqrt{\phi_n}} \to 0 \quad \text{ in } L^2(Q_\tau) \quad \text{ as } n \to \infty. \tag{6.5}$$

For example, consider

$$J_\varepsilon(x,t) = (\tilde{\rho}_\varepsilon * \Phi)(x,t) = \varepsilon^{-N-1} \int_{-\infty}^\infty \int_{\mathbf{R}^N} \tilde{\rho}((x-y)/\varepsilon, (t-\tau)/\varepsilon)\Phi(y,\tau)\,dyd\tau,$$

where $\tilde{\rho}(x,t) = \pi^{-(N+1)/2} e^{-|x|^2 - t^2}$ and

$$\Phi = \begin{cases} \tilde{\phi} & \text{ in } Q_\tau \\ 0 & \text{ in } (\mathbf{R}^N \times \mathbf{R}) \backslash Q_\tau, \end{cases}$$

and choose $\varepsilon = \varepsilon_n > 0$ to satisfy $\|J_\epsilon - \tilde{\phi}\|_{L^2(Q_\tau)} < 1/n$. One can choose $\phi_n = J_{\varepsilon_n} + 1/n$ $(n = 1, 2, \cdots)$ as desired functions.

Let $\chi \in C_0^\infty(\mathbf{R}^N)$ satisfy $0 \le \chi \le 1$ and choose $R_0 > 0$ large to satisfy $\operatorname{supp}\chi \subset B_{R_0}$. Let $R > 2R_0$ and $\varepsilon > 0$, and let $\psi_{n,\varepsilon,R} \in C^\infty(\bar{B}_R \times [0,\tau])$ $(n \ge 1)$ be a solution of

$$\begin{cases} \psi_t + \phi_n \Delta\psi = K\psi & \text{in } B_R \times [0,\tau), \\ \psi = 0 & \text{on } \partial B_R \times [0,\tau), \\ \psi(x,\tau) = \chi e^{-\sqrt{|x|^2+\varepsilon}} & \text{in } B_R. \end{cases} \tag{6.6}$$

We need the following lemma. $\square$

Lemma 6.1. *The following hold when $n \ge 1$, $\varepsilon > 0$ and $R > 2R_0$:*
(i) $0 \le \psi_{n,\varepsilon,R} \le e^{-|x|}$ *in* $\bar{Q}_\tau$;
(ii) $\int\int_{Q_\tau} \phi_n(\triangle\psi_{n,\varepsilon,R})^2\,dxdt < C$;
(iii) $\sup_{0\le t\le\tau}\int_{B_R} |\nabla\psi_{n,\varepsilon,R}|^2(t)\,dx < C$;
(iv) $0 \le -\partial_\nu\psi_{n,\varepsilon,R} \le \frac{C}{R}e^{-R/2}$ *on* $\partial B_R \times [0,\tau]$,
where $C > 0$ is a constant depending only on χ (independent of n, ε and R).

Proof. We first show (i) and (iv). We change variables of $\psi_{n,\varepsilon,R}$ as $\zeta(x,t) = \psi_{n,\varepsilon,R}(x,\tau - t)$. Then ζ is a solution to the problem

$$\begin{cases} \zeta_t - \hat{\phi}_n \Delta\zeta = -K\zeta & \text{in } B_R \times (0,\tau], \\ \zeta = 0 & \text{on } \partial B_R \times (0,\tau], \\ \zeta(x,0) = \chi e^{-\sqrt{|x|^2+\varepsilon}} & \text{in } B_R, \end{cases} \tag{6.7}$$

where $\hat{\phi}_n(x,t) = \phi_n(x,\tau - t)$. Let $w(x) = e^{-|x|}$. We compare w and ζ in $\bar{B}_R \times [0,\tau]$. Note that

$$\hat{\phi}_n \le \|\tilde{\phi}\|_{L^\infty(Q_\tau)} + \frac{1}{n} \le K \quad \text{in } Q_\tau.$$

Hence, w is a supersolution of the equation of (6.7) in $B_R\backslash\{0\} \times [0,\tau]$, since

$$-\hat{\phi}_n\Delta w = -\hat{\phi}_n\left(1 - \frac{N-1}{r}\right)w \ge -\hat{\phi}_n w \ge -Kw \quad \text{in } B_R\backslash\{0\},$$

where $r = |x|$.
Thus, putting $z = \zeta - w$ we have

$$\begin{cases} z_t - \hat{\phi}_n\Delta z \le -Kz & \text{in } B_R\backslash\{0\} \times (0,\tau], \\ z = -e^{-R} < 0 & \text{on } \partial B_R \times (0,\tau], \\ z(x,0) \le -(1-\chi)e^{-|x|} \le 0 & \text{in } B_R. \end{cases} \tag{6.8}$$

We shall show $z \le 0$ in $\bar{B}_R \times [0,\tau]$.

We first show that for each $t \in (0,\tau]$, $z(x,t)$ in $\bar{B}_R$ never attains the maximum value at $x = 0$. Assume contrary that for some $t \in (0,\tau]$ $z(x,t)$ in $\bar{B}_R$ attains the maximum value at $x = 0$. Then, $z(x,t) \le z(0,t)$, that is, $\zeta(x,t) - \zeta(0,t) \le w(x) - w(0) = e^{-|x|} - 1$ in B_R. Hence,

$$\frac{\partial\zeta}{\partial x_1}(0,t) = \lim_{h\to+0}\frac{\zeta(h,0,\cdots,0,t) - \zeta(0,t)}{h} \le \lim_{h\to+0}\frac{e^{-h}-1}{h} = -1$$

and

$$\frac{\partial \zeta}{\partial x_1}(0,t)=\lim_{h\to -0}\frac{\zeta(h,0,\cdots,0,t)-\zeta(0,t)}{h}\ge \lim_{h\to -0}\frac{e^h-1}{h}=1.$$

This is a contradiction and so we see that for each $t\in(0,\tau]$, $z(x,t)$ in $\bar{B}_R$ never attains the maximum value at $x=0$.

We now prove that $z\le 0$ in $\bar{B}_R\times[0,\tau]$. Assume contrary that z in $\bar{B}_R\times[0,\tau]$ attains the maximum value at $(x,t)=(x_1,t_1)\in \bar{B}_R\times[0,\tau]$ and $z(x_1.t_1)>0$. Then, $(x_1,t_1)\in B_R\backslash\{0\}\times(0,\tau]$ and so $z_t(x_1,t_1)\ge 0$ and $\Delta z(x_1,t_1)\le 0$. Hence, $z_t(x_1,t_1)-\hat{\phi}_n(x_1,t_1)\Delta z(x_1,t_1)\ge 0>-Kz(x_1,t_1)$. This is a contradiction to (6.8), and so we get $z\le 0$ in $\bar{B}_R\times[0,\tau]$, that is, $\psi_{n,\varepsilon,R}\le w=e^{-|x|}$ in $\bar{B}_R\times[0,\tau]$. Thus, we get (i).

In order to prove (iv), we use a solution $\tilde{w}$ to the problem

$$\begin{cases}\Delta\tilde{w}=\tilde{w} & \text{in } B_R\backslash\bar{B}_{R_0},\\ \tilde{w}=e^{-R_0} & \text{on } |x|=R_0,\\ \tilde{w}=0 & \text{on } |x|=R.\end{cases}\tag{6.9}$$

Then, the comparison theorem implies that $0\le\tilde{w}\le w=e^{-|x|}$ in $B_R\backslash\bar{B}_{R_0}$ and $\tilde{w}(x)=\tilde{w}(r)$ ($r=|x|$) is a radially symmetric function in $x\in B_R\backslash\bar{B}_{R_0}$. Furthermore, $-\hat{\phi}_n\Delta\tilde{w}\ge -K\tilde{w}$ by inequalities $\tilde{w}\ge 0$ and $\hat{\phi}_n\le K$. Hence, as in the proof of (i), it is not difficult to see that $\psi_{n,\varepsilon,R}(x,\tau-t)=\zeta(x,t)\le\tilde{w}(x)$ in $\bar{B}_R\backslash B_{R_0}\times[0,\tau]$, since $\zeta(x,0)=0\le\tilde{w}(x)$ in $\bar{B}_R\backslash B_{R_0}$ and $\zeta(x,t)=\psi_{n,\varepsilon,R}(x,\tau-t)\le\tilde{w}(x)$ on $\partial(B_R\backslash\bar{B}_{R_0})\times[0,\tau]$. Therefore, we have

$$0\le -\partial_\nu\psi_{n,\varepsilon,R}(x,t)\le -\partial_\nu\tilde{w}(x)\quad \text{on } |x|=R \text{ and } t\in[0,\tau].\tag{6.10}$$

On the other hand, let $\xi(x)$ be a nonnegative C^∞-function in $\mathbf{R}^N$ satisfying that $\xi(x)=0$ in $|x|\le 1$ and $\xi(x)=1$ in $|x|\ge 2$. Put $\xi_R(x)=\xi(2x/R)$. Multiplying the both sides of the equation of (6.9) by ξ_R and integrating by parts over $B_R\backslash\bar{B}_{R/2}$, we get

$$\int_{|x|=R}\partial_\nu\tilde{w}\,dS+\int_{R/2\le|x|\le R}\tilde{w}\Delta\xi_R\,dx=\int_{R/2\le|x|\le R}\tilde{w}\xi_R\,dx.\tag{6.11}$$

Hence, noting $0\le\tilde{w}(x)\le e^{-|x|}$ in $B_R\backslash\bar{B}_{R_0}$ and $\tilde{w}(x)=\tilde{w}(r)$ ($r=|x|$) we have

$$\begin{aligned}-\partial_\nu\tilde{w}(R)\int_{|x|=R}dS&\le\int_{R/2\le|x|\le R}\tilde{w}\Delta\xi_R\,dx\\ &\le\frac{4}{R^2}\sup_{1\le|x|\le 2}|\Delta\xi|\int_{R/2\le|x|\le R}dx\times e^{-R/2},\end{aligned}$$

that is,

$$-\partial_\nu\tilde{w}(R)\le\sup_{1\le|x|\le 2}|\Delta\xi|\times\frac{4}{R^2}\frac{\int_{R/2\le|x|\le R}dx}{\int_{|x|=R}dS}e^{-R/2}\le\frac{C}{R}e^{-R/2}\quad\text{for } R>2R_0.$$

Thus, we get (iv).

Finally, we prove (ii) and (iii). Multiply the both sides of the equation of (6.6) by $\Delta\psi_{n,\varepsilon,R}$ and integrate by parts over $B_R \times (t,\tau)$ $(t<\tau)$. Then

$$\begin{aligned}\frac{1}{2}\int_{B_R}|\nabla\psi_{n,\varepsilon,R}|^2(t)\,dx+&\int_t^\tau\int_{B_R}\phi_n(\triangle\psi_{n,\varepsilon,R})^2\,dxdt+K\int_t^\tau\int_{B_R}|\nabla\psi_{n,\varepsilon,R}|^2\,dxdt\\&=\frac{1}{2}\int_{B_R}|\nabla\psi_{n,\varepsilon,R}|^2(\tau)\,dx\le\int_{\mathbf{R}^N}\left(|\nabla\chi|^2+|\chi|^2\right)\,dx,\end{aligned}$$

which is reduced to (ii) and (iii). $\square$

Proof of Proposition 2.3 (continued). Put $\varphi(x,t)=\psi_{n,\varepsilon,R}(x,t)$ as a test function in (6.1). Then for $\tau\in(0,T_M)$,

(6.12)

$$\begin{aligned}&\int_{B_R}(v(x,\tau)-u(x,\tau))\chi e^{-\sqrt{|x|^2+\varepsilon}}\,dx\\&\le\int\int_{Q_\tau}(v-u)\{(\tilde\phi-\phi_n)\Delta\psi_{n,\varepsilon,R}\}\,dxdt\\&\quad+\int_{B_R}[v(x,0)-u(x,0)]_+\psi_{n,\varepsilon,R}(x,0)\,dx+\int\int_{Q_\tau}[(K+\tilde f)(v-u)]_+\psi_{n,\varepsilon,R}\,dxdt\\&\qquad\qquad-\int_0^\tau\int_{\partial B_R}[\phi(v)-\phi(u)]_+\partial_\nu\psi_{n,\varepsilon,R}\,dSdt,\end{aligned}$$

where $[u]_+=\max\{u,0\}$, since $\partial_\nu\psi_{n,\varepsilon,R}\le 0$ on $\partial B_R\times(0,\tau)$.
We note by Lemma 6.1 and (6.5) that

$$\begin{aligned}&\|(\tilde\phi-\phi_n)\triangle\psi_{n,\varepsilon,R}\|_{L^1(Q_\tau)}\\&\le\|(\tilde\phi-\phi_n)/\sqrt{\phi_n}\|_{L^2(Q_\tau)}\|\sqrt{\phi_n}\Delta\psi_{n,\varepsilon,R}\|_{L^2(Q_\tau)}\to 0\quad(\text{ as } n\to\infty).\end{aligned}$$

Hence, if $n\to\infty$ in (6.12), we obtain by Lemma 6.1, (2.2) and (6.3),

$$\begin{aligned}\int_{B_R}&(v(x,\tau)-u(x,\tau))\chi e^{-\sqrt{|x|^2+\varepsilon}}\,dx\\&\le\int_{B_R}[v(x,0)-u(x,0)]_+e^{-|x|}\,dx+2K\int\int_{Q_\tau}[v-u]_+e^{-|x|}\,dxdt\\&\qquad+A\phi(M)\tau R^{N-2}e^{-R/2},\end{aligned}\tag{6.13}$$

where A is a positive constant independent of ε and R. So, letting $R\to\infty$ and $\varepsilon\downarrow 0$ in (6.13) we have

$$\begin{aligned}&\int_{\mathbf{R}^N}(v(x,\tau)-u(x,\tau))\chi e^{-|x|}\,dx\\&\quad\le\int_{\mathbf{R}^N}[v(x,0)-u(x,0)]_+e^{-|x|}\,dx+2K\int_0^\tau\int_{\mathbf{R}^N}[v-u]_+e^{-|x|}\,dxdt\end{aligned}$$

for any $\chi\in C_0^\infty(\mathbf{R}^N)$ satisfying $0\le\chi\le 1$, whence,

$$\int_{\mathbf{R}^N}[v(x,\tau)-u(x,\tau)]_+e^{-|x|}\,dx$$
$$\le \int_{\mathbf{R}^N}[v(x,0)-u(x,0)]_+e^{-|x|}\,dx+2K\int_0^\tau\int_{\mathbf{R}^N}[v-u]_+e^{-|x|}\,dxdt$$
$$\text{for } \tau\in(0,T).$$

Thus, by Gronwall's lemma we obtain (2.4). The proof is complete. □

7. Appendix B

Let $\psi\in\mathbf{S}^{N-1}$. Let $\{R_n\}$ $(R_n>1)$ be a sequence of numbers diverging to ∞ as $n\to\infty$. Assume that u_0 satisfies condition (A1). In this section, we shall show the next proposition.

Proposition 7.1. *Conditions (A5)$_\psi$, (A6)$_\psi$, (A7)$_\psi$ and (A8)$_\psi$ are equivalent.*

We first get the next lemma.

Lemma 7.2. *Conditions (A5)$_\psi$, (A6)$_\psi$ and (A7)$_\psi$ are equivalent.*

Proof. We first show that (A6)$_\psi$ is equivalent to (A7)$_\psi$. Clearly, (1.20) implies (1.21). So, for the proof, it is enough to show that if for some sequence $\{x_n\}\subset\mathbf{R}^N$, (1.21) holds for each $R>1$, then there exists a subsequence $\{x_{n_j}\}\subset\{x_n\}$ satisfying

$$u_0(x+x_{n_j})\to\|u_0\|_\infty \quad \text{as } n_j\to\infty \quad \text{a. e. in } \mathbf{R}^N. \tag{7.1}$$

Assume that for some sequence $\{x_n\}\subset\mathbf{R}^N$, (1.21) holds for each $R>1$. Then, we see that for each $R>1$, $u_0(x+x_n)\to\|u_0\|_\infty$ in $L^1(B_R)$ as $n\to\infty$. Hence, for each $R>1$, there exists a subsequence $\{x_{n_j}\}\subset\{x_n\}$ such that

$$u_0(x+x_{n_j})\to\|u_0\|_\infty \quad \text{as } n_j\to\infty \quad \text{a. e. in } B_R.$$

Therefore, by the diagonal method we can choose a subsequence $\{x_{n_j}\}\subset\{x_n\}$ satisfying (7.1). Thus, we see that (A6)$_\psi$ is equivalent to (A7)$_\psi$.

Similarly, we also see that (A5)$_\psi$ is equivalent to (A6)$_\psi$.

The proof is complete. □

Hence, for the proof of Proposition 7.1, it is enough to show that (A7)$_\psi$ is equivalent to (A8)$_\psi$. For this aim, we need the next lemma.

Lemma 7.3. *Let $R>1$. If for some sequence $\{x_n\}\subset\mathbf{R}^N$,*

$$\lim_{n\to\infty}\tilde{A}_R(x_n;u_0)=\|u_0\|_\infty, \tag{7.2}$$

then

$$\lim_{n\to\infty}\inf_{r\in[1,R]}\tilde{A}_r(x_n;u_0)=\|u_0\|_\infty. \tag{7.3}$$

Proof. Let $R > 1$ and assume (7.2) for some sequence $\{x_n\} \subset \mathbf{R}^N$. Assume contrary that (7.3) does not hold. Then, there exist a subsequence $\{x_{n_j}\} \subset \{x_n\}$, a sequence $\{r_j\} \subset [1, R]$ and a number $L \in (0, \|u_0\|_\infty)$ such that

$$\tilde{A}_{r_j}(x_{n_j}; u_0) \le L \quad \text{for all } j \ge 1.$$

Hence,

$$\begin{aligned}\tilde{A}_R(x_{n_j}; u_0) &= \frac{\int_{|x-x_{n_j}|<r_j} u_0(x)\,dx + \int_{r_j<|x-x_{n_j}|<R} u_0(x)\,dx}{|B_R(x_{n_j})|} \\ &\le \frac{L|B_{r_j}(x_{n_j})| + \|u_0\|_\infty \int_{r_j<|x-x_{n_j}|<R} dx}{|B_R(x_{n_j})|} \\ &\le \|u_0\|_\infty - (\|u_0\|_\infty - L)\frac{|B_1|}{|B_R|} \quad \text{for all } j \ge 1.\end{aligned}$$

This is a contradiction to (7.2) and so we get (7.3). □

Proof of Proposition 7.1. We only prove that $(A7)_\psi$ is equivalent to $(A8)_\psi$.

Clearly, condition $(A8)_\psi$ leads to condition $(A7)_\psi$.

Conversely, we assume $(A7)_\psi$. Then, there exists a sequence $\{x_n\} \subset \mathbf{R}^N$ satisfying $\lim_{n\to\infty} |x_n| = \infty$ and $\lim_{n\to\infty} x_n/|x_n| = \psi$ such that for any $R > 1$, (1.21) holds. Hence, by Lemma 7.3 we get for each $R > 1$,

$$\lim_{n\to\infty} \inf_{r\in[1,R]} \tilde{A}_r(x_n; u_0) = \|u_0\|_\infty.$$

So, for any $n \ge 1$, we can choose $m_n \ge n$ such that $m_n > m_{n-1}$ $(n \ge 2)$ and

$$\|u_0\|_\infty - \frac{1}{n} \le \inf_{r\in[1,R_n]} \tilde{A}_r(x_{m_n}; u_0) \le \|u_0\|_\infty,$$

from which,

$$\lim_{n\to\infty} \inf_{r\in[1,R_n]} \tilde{A}_r(x_{m_n}; u_0) = \|u_0\|_\infty.$$

Therefore, we get (1.22) with $\{x_n\}$ replaced by $\{x_{m_n}\}$. The proof is complete. □

Acknowledgments

The authors would like to thank Prof. Kiyoshi Mochizuki for his suggestion as to studying this topic and for giving them helpful advice and kind encouragement.

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