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A CARLEMAN INEQUALITY FOR THE STATIONARY ANISOTROPIC MAXWELL SYSTEM

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Abstract. A Carleman estimate for the stationary anisotropic Maxwell system is established. Its proof adopts a technique pioneered by Calderón to an overdetermined systems with rough coefficients. As an application, the conditional stability of the Cauchy problem is discussed.

1. Introduction and main result

Let $\Omega \subset \mathbf{R}^3$ be an open set filled with an anisotropic electromagnetic medium and let $E(t,x)$ and $H(t,x)$ be two vector-valued functions $\Omega \to \mathbf{R}^3$, denoting the electric field intensity and the magnetic field intensity, respectively. Furthermore, the electric permittivity $\varepsilon(x)$ and the magnetic permeability $\mu(x)$ are 3×3 positive definite, symmetric matrices with C^1 entries. The stationary (or time-harmonic) Maxwell equations derive from the dynamic Maxwell equations by assuming $E(t,x) = E(x)e^{i\omega t}$ and $H(t,x) = H(x)e^{i\omega t}$ and consist of the following equations

$$\begin{aligned} i\omega\varepsilon(x)E(x) - \nabla \times H(x) &= 0 \\ i\omega\mu(x)H(x) + \nabla \times E(x) &= 0 \\ \nabla \cdot (\varepsilon(x)E(x)) &= 0 \\ \nabla \cdot (\mu(x)H(x)) &= 0 \end{aligned} \tag{1.1}$$

Here $\nabla\times$ denotes the curl operator and $\nabla\cdot$ is the divergence operator. In our case, where the coefficients ε and μ are matrices, we say that the system is anisotropic. If the coefficients are scalars, the system is referred to as isotropic. One of the important applications of the anisotropic Maxwell equations are the equations of crystal optics [KK65]. Our main result is the following Carleman estimate for this system.

Theorem 1.1. *Let Ω be an open set in $\mathbf{R}^3$ and let $\psi \in C^2(\Omega)$ such that $\nabla\psi \neq 0$ for all $x \in \Omega$. Let $(E,H) \in H^1(\Omega)^6$ with compact support in Ω and assume that ε and μ are symmetric, positive definite matrices with entries in $C^1(\overline{\Omega})$.*

Then there exist positive constants λ_0 and C depending only on Ω and ψ such that

(1.2)
$$\frac{1}{s\lambda}\sum_{j=1}^{3}\int_\Omega e^{-\lambda\psi}(|\partial_j E|^2 + |\partial_j H|^2)e^{2s\phi} + s\lambda^2\int_\Omega e^{\lambda\psi}(|E|^2+|H|^2)e^{2s\phi} \leq C\Big[\|e^{s\phi}(i\omega\varepsilon E - \nabla\times H)\|^2_{L_2(\Omega)}$$
$$+\|e^{s\phi}(i\omega\mu H + \nabla\times E)\|^2_{L_2(\Omega)} + \|e^{s\phi}\nabla\cdot(\varepsilon E)\|^2_{L_2(\Omega)} + \|e^{s\phi}\nabla\cdot(\mu H)\|^2_{L_2(\Omega)}\Big]$$

provided $\lambda \geq \lambda_0$ and $s \geq s_0(\lambda)$. Here $\phi = e^{\lambda\psi} - 1$.

This estimate implies the unique continuation of solutions to the homogeneous Maxwell's system (1.1) across every C^2- surface [H83, Chapter XXVIII].

Corollary 1.2. *Let $(E,H) \in H^1(\Omega)$ be a solution to Maxwell's system and let $S = \{\psi(x) = \psi(x_0)\}$ be a level surface of the function $\psi \in C^2(\overline{\Omega})$ near $x_0 \in \Omega$ such that $\psi'(x_0) \neq 0$.*

If $(E,H) = 0$ on one side of S, then $(E,H) \equiv 0$ in a full neighborhood of $x_0 \in \Omega$.

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Carleman estimates for linear partial differential operators and unique continuation for solutions to homogeneous linear partial differential equations with non-analytic coefficients have been extensively studied since Carleman's work [C39]. By now the problem is rather well understood in the case of scalar operators and equations [H83, Chapter XXVIII], [Ta95]. However, only few results pertain to systems of partial differential equations. The only general result for systems is Calderón's Theorem [Ca58] where unique continuation is proved for a first order evolutions system provided certain assumptions on the characteristics are satisfied.

Regarding the most relevant systems of mathematical physics uniqueness theorems and Carleman estimates for isotropic dynamic Maxwell's equations and the isotropic elastic wave equations have been obtained in [EINT02] and [IIY03]. The key observation is that these systems can be reduced into weakly coupled vector wave equations and then the theory for scalar operators mentioned above is applied. This kind of reduction was done first by N. Weck for the stationary elastic equations [W69]. See also Dehman and Robbiano [DR93], Imanuvilov and Yamamoto [IM04], Weck [W01].

There are a few works on the anisotropic Maxwell equations which we like to mention. V. Vogelsang [V01] and T. Okaji [O02] both prove strong unique continuation in the time-harmonic case. However, both works make structural assumptions on the coefficient matrices. Vogelsang requires that the matrices ε and μ are equal to the identity matrix at the point of interest whereas Okaji requires the coefficients to be scalar multiples of each other at that very point.

Note that our Theorem 1.1 makes no structural assumption on the coefficient matrices; moreover the regularity of the coefficients is assumed to be only C^1 whereas the reduction to a weakly coupled second order system as in [EINT02] requires the coefficients to be C^2.

The proof of Theorem 1.1 is based on the observation that the time-harmonic Maxwell system is a weak coupling of two div-curl systems. The estimate (1.2) is a consequence of the following Carleman estimate for the div-curl system.

Theorem 1.3. *Let Ω be an open set in $\mathbf{R}^3$ and let $\psi \in C^2(\Omega)$ such that $\nabla\psi \neq 0$ for all $x \in \Omega$. Assume that $w \in H^1(\Omega)^3$ has compact support in Ω. and that $A(x) = a_{jk}(x)$ is a 3×3 symmetric, positive matrix with C^1 entries.*

Then there exist positive constants λ_0 and C depending only on ψ and Ω such that

$$\frac{1}{s\lambda}\sum_{j=1}^{3}\int_\Omega e^{-\lambda\psi}|\partial_j w|^2 e^{2s\phi} + s\lambda^2\int_\Omega e^{\lambda\psi}|w|^2 e^{2s\phi} \leq C\left[\|e^{s\phi}\nabla\times w\|^2_{L_2(\Omega)} + \|e^{s\phi}\nabla\cdot(Aw)\|^2_{L_2(\Omega)}\right] \tag{1.3}$$

provided $\lambda \geq \lambda_0$ and $s \geq s_0(\lambda)$. Here $\phi = e^{\lambda\psi} - 1$.

Indeed, the proof of Theorem 1.1 follows by adding the Carleman estimate of Theorem 1.3 applied to the functions E and H,

$$\begin{aligned}\frac{1}{s\lambda}\sum_{j=1}^{3}\int_\Omega e^{-\lambda\psi}(|\partial_j E|^2 + |\partial_j H|^2)e^{2s\phi} + s\lambda^2\int_\Omega e^{\lambda\psi}(|E|^2+|H|^2)e^{2s\phi}\\ \leq C\left[\|e^{s\phi}\nabla\times H\|^2_{L_2(\Omega)} + \|e^{s\phi}\nabla\times E\|^2_{L_2(\Omega)} + \|e^{s\phi}\nabla\cdot(\varepsilon E)\|^2_{L_2(\Omega)} + \|e^{s\phi}\nabla\cdot(\mu H)\|^2_{L_2(\Omega)}\right]\end{aligned}$$

for s and λ sufficiently large. The use of the triangle inequality gives (with a larger C)

$$\begin{aligned}\frac{1}{s\lambda}\sum_{j=1}^{3}\int_\Omega e^{-\lambda\psi}(|\partial_j E|^2 + |\partial_j H|^2)e^{2s\phi} + s\lambda^2\int_\Omega e^{\lambda\psi}(|E|^2+|H|^2)e^{2s\phi}\\ \leq C\Big[\|e^{s\phi}(i\omega\varepsilon E - \nabla\times H)\|^2_{L_2(\Omega)} + \|e^{s\phi}(i\omega\mu H + \nabla\times E)\|^2_{L_2(\Omega)} + \|e^{s\phi}\nabla\cdot(\varepsilon E)\|^2_{L_2(\Omega)}\\ + \|e^{s\phi}\nabla\cdot(\mu H)\|^2_{L_2(\Omega)} + \|e^{2s\phi}H\|^2_{L_2(\Omega)} + \|e^{2s\phi}E\|^2_{L_2(\Omega)}\Big]\end{aligned}$$

Now the last two terms can be moved into the right hand side, provided s and λ are sufficiently large. This yields (1.2).

This paper is structured as follows. Section 2 is dedicated to the proof of Theorem 1.3. We follow Calderón's approach as explained in [Ni73]. There are certain obstacles to be overcome: The div-curl system is overdetermined which makes its diagonalization more difficult. Furthermore, the estimate for the first-order derivatives in (1.3) requires extra attention. Moreover, we show that the Caleronón's approach can be adopted to operators with C^1 coefficients.

Section 3 contains applications of the Carleman estimate: The conditional stability of the Cauchy problem and a boundary estimate for the Cauchy problem.

2. Proof of Theorem1.3

It will suffice to prove Theorem 1.3 locally, i.e. for a function $w \in H^1(\Omega)$ compactly supported in a small open neighborhood W of some point $x_0 \in \Omega$ [H63, Chapter 8].

The essence of Calderón's approach is to consider the system as an evolution in direction normal to the level surfaces of ψ. Hence we will introduce new coordinates in which the level surfaces of ψ become the surfaces y_3 = constant. Then we will delete one equation to obtain a 3×3 system which is then diagonalized by the means of pseudo-differential operators, see equation (2.11) below. This diagonal system allows certain integral estimates which then can be returned to the original variables.

2.1. **Change of coordinates.** Consider the level surface $S = \{x \in W \,;\, \psi(x) = \psi(x_0)\}$. Assuming that W is sufficiently small we introduce geodesic local coordinates in W with respect to the level surface S. We denote these coordinates by $\{y_1, y_2, y_3\}$ and assume that $\{y_1, y_2\}$ are orthogonal coordinates in S and that $y_3 = \psi(x) - \psi(x_0)$ is the normal coordinate. The corresponding coordinate mapping is denoted by $x = \Phi(y)$ and $\Phi'(y) > 0$ for all $y \in \Phi^{-1}(W)$. We note that $\Phi^{-1}(S) = \{y \in \Phi^{-1}(W) \;:\; y_3 = 0\} \subset \mathbf{R}^2$ and assume that $\Phi^{-1}(W)$ is a cylinder $\Phi^{-1}(S) \times (-h, h)$ for some $h > 0$.

The standard Euclidean metric in $\mathbf{R}^3$ induces the Riemannian metric with metric tensor

$$G(y) = {}^t\Phi'(y)\Phi'(y) = \begin{pmatrix} g_1 & 0 & 0 \\ 0 & g_2 & 0 \\ 0 & 0 & g_3 \end{pmatrix}$$

in $\Phi^{-1}(W)$. For future reference we set $g(y) = \det G(y)$. The differential basis of vector fields will be denoted by $\{\partial/\partial y_1, \partial/\partial y_2, \partial/\partial y_3\}$ or by $\{\partial_1, \partial_2, \partial_3\}$ and the corresponding orthonormal basis by $\{f_1, f_2, f_3\}$ where $f_1 = \partial_1/\sqrt{g_1}$, $f_2 = \partial_2/\sqrt{g_2}$ and $f_3 = \partial_3/\sqrt{g_3}$. Since $\psi \in C^2(W)$ the metric tensor $G(y) \in C^1(W)$.

Given a vector field $w(x)$ with respect to the standard Euclidean basis $\{e_1, e_2, e_3\}$ we find a representation u with respect to the new basis vectors $\{f_1, f_2, f_3\}$ by

$$u(y) = {}^t\Psi(y)w(\Phi(y)) \text{ where } \Psi(y) = \Phi'(y)G^{-1/2}(y) \tag{2.4}$$

Every vector field in $\Omega \cap W$ can be represented in the form $u = u_1 f_1 + u_2 f_2 + u_3 f_3$. We represent the operators curl and div with respect to the coordinates $\{y_1, y_2, y_3\}$ [C96, p. 362].

$$\mathrm{div} u = \frac{1}{\sqrt{g_1}}\frac{\partial u_1}{\partial y_1} + \frac{1}{\sqrt{g_2}}\frac{\partial u_2}{\partial y_2} + \frac{1}{\sqrt{g_3}}\frac{\partial u_3}{\partial y_3} + L_1 \cdot u$$

where $L_1 = L_1(y)$ is a vector with three components and

$$\mathrm{curl} u = \left(\frac{1}{\sqrt{g_2}}\partial_2 u_3 - \frac{1}{\sqrt{g_3}}\partial_3 u_2\right) f_1 + \left(\frac{1}{\sqrt{g_3}}\partial_3 u_1 - \frac{1}{\sqrt{g_1}}\partial_1 u_3\right) f_2 + \left(\frac{1}{\sqrt{g_1}}\partial_1 u_2 - \frac{1}{\sqrt{g_2}}\partial_2 u_1\right) f_3 + L_2 u$$

where $L_2 = L_2(y)$ is matrix function. The system

$$P(x, D)w = (\nabla \times w, \nabla \cdot (Aw)) = F(x) \tag{2.5}$$

becomes, after the change of coordinates

$$\tilde{P}(y, D)u = L(y)u + \tilde{F}(y) \tag{2.6}$$

where the symbol of $\tilde{P}$ is

$$\tilde{p}(y, \xi) = \begin{pmatrix} 0 & -g^3\xi_3 & g^2\xi_2 \\ g^3\xi_3 & 0 & -g^1\xi_1 \\ -g^2\xi_2 & g^1\xi_1 & 0 \\ \sum_{j=1}^3 \tilde{a}^{1j} g^j \xi_j & \sum_{j=1}^3 \tilde{a}^{2j} g^j \xi_j & \sum_{j=1}^3 \tilde{a}^{3j} g^j \xi_j \end{pmatrix}$$

Here $g^j = 1/\sqrt{g_j}$ for $j = 1, 2, 3$, and $\tilde{F}$ and $\tilde{A} = (\tilde{a}^{jk})$ are derived from the functions F and A in (2.5). More precisely, using (2.4) we have

$$\begin{aligned}\tilde{A}(y) &= {}^t\Psi(y)A(\Phi(y))\Psi(y)\\ \tilde{F}(y) &= \left({}^t\Psi(y)(F_1(\Phi(y)), F_2(\Phi(y)), F_3(\Phi(y))), F_4(\Phi(y))\right)\end{aligned}$$

The function $L(y)$ is a 4×3 matrix function and depends on L_1, L_2 as well as the first derivatives of the entries of $\tilde{A}$. One verifies that the matrix $\tilde{A}(y)$ is positive definite and symmetric and has C^1 entries since A has these properties.

2.2. **The diagonalization.** We will write equation (2.6) as an evolution equation in y_3 direction. For that purpose the third curl equation can be dropped since it does not involve any derivatives in normal direction. Moreover, leaving that equation out of $\tilde{p}$ results in a square matrix. Set

$$\underline{p}(y,\xi) = \begin{pmatrix} g^3\xi_3 & 0 & -g^1\xi_1 \\ 0 & g^3\xi_3 & -g^2\xi_2 \\ \tilde{a}^{1j}g^j\xi_j & \tilde{a}^{2j}g^j\xi_j & \tilde{a}^{3j}g^j\xi_j \end{pmatrix}$$

and observe that

$$\det\underline{p}(y,\xi) = g^3\xi_3\left(\xi_3\tilde{a}^{3j}g^j\xi_j + g^2\xi_2\tilde{a}^{2j}g^j\xi_j + g^1\xi_1\tilde{a}^{1j}g^j\xi_j\right) = g^3\xi_3 g^k\xi_k a^{kj}g^j\xi_j = g^3\xi_3({}^t\xi\underline{A}\xi)$$

where the Einstein summation convention is used and $\underline{A}$ is the matrix with the entries $\underline{a}^{jk} = \tilde{a}^{jk}g^jg^k \in C^1(W)$. The vector ${}^t\xi$ is the transpose of the column vector ξ and ${}^t\xi\underline{A}\xi = \underline{a}^{jk}\xi_j\xi_k$.

We write the principal part of equation (2.6) - without the third curl equation - in the form $\underline{p}(y,e_3)D_3u + \underline{P}(y,D')u$ where

$$\underline{p}(y,e_3) = \begin{pmatrix} g^3 & 0 & 0 \\ 0 & g^3 & 0 \\ \underline{a}^{31} & \underline{a}^{32} & \underline{a}^{33} \end{pmatrix}$$

is invertible and $\underline{P}(y,D')$ does not contain any derivatives with respect to y_3.

In what follows $\xi' = (\xi_1, \xi_2, 0)$. Hence we can write equation (2.5) without the third curl equation as

$$D_3u + [\underline{p}(y,e_3)]^{-1}\underline{P}(y,D')u = \underline{F} + \underline{L}u \tag{2.7}$$

with obvious definitions for $\underline{F}$ and $\underline{L}$.

We denote the symbol of $[\underline{p}(y,e_3)]^{-1}\underline{P}(y,D')$ by $-m(y,\xi')$. Next we will find the eigenvalues and eigenvectors of $m(y,\xi')$ which will let us diagonalize equation (2.7). In the following all summations will be from j or/and $k = 1$ to 2.

$$\begin{aligned}\det(\alpha I - m(y,\xi')) &= \det[\alpha I + [\underline{p}(y,e_3)]^{-1}\underline{p}(y,\xi')] = \det[\underline{p}(y,e_3)]^{-1}\det\underline{p}(y;\xi',\alpha)\\ &= \frac{1}{\underline{a}^{33}}\alpha\left(\underline{a}^{jk}\xi_j\xi_k + 2\alpha\underline{a}^{3j}\xi_j + \underline{a}^{33}\alpha^2\right)\end{aligned}$$

Hence the three eigenvalues of $m(y,\xi')$ are

$$\alpha_1 = 0 \text{ and } \alpha_{2,3} = -\frac{\underline{a}^{3j}\xi_j}{\underline{a}^{33}} \pm \sqrt{\left(\frac{\underline{a}^{3j}\xi_j}{\underline{a}^{33}}\right)^2 - \frac{\underline{a}^{jk}\xi_j\xi_k}{\underline{a}^{33}}} \tag{2.8}$$

Note that the eigenvalues α_2 and α_3 are non-real since by the Cauchy-Schwartz inequality $(\underline{a}^{3j}\xi_j)^2 < \underline{a}^{33}\underline{a}^{jk}\xi_j\xi_k$ since the vectors $e_3 = (0,0,1)$ and $\xi' = (\xi_1,\xi_2,0)$ are not collinear. Here the assumption that A is positive definite is critical.

Next we compute the eigenvectors of $m(y,\xi')$. Note that $m(y,\xi')b_j = \alpha_j b_j$ results in $\underline{p}(y;\xi',\alpha_j)b_j = 0$. Using $\alpha_1 = 0$ we have

$$\begin{pmatrix} 0 & 0 & -g^1\xi_1 \\ 0 & 0 & -g^2\xi_2 \\ \tilde{a}^{1j}g^j\xi_j & \tilde{a}^{2j}g^j\xi_j & \tilde{a}^{3j}g^j\xi_j \end{pmatrix}\begin{pmatrix} b_{11} \\ b_{12} \\ b_{13} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

which gives $b_{13} = 0$ and $b_{1k}\tilde{a}^{kj}g^j\xi_j = 0$. Thus $b_1 = {}^t(-\tilde{a}^{2j}g^j\xi_j, \tilde{a}^{1j}g^j\xi_j, 0)$. For $k = 2, 3$ we obtain $b_k = {}^t(g^1\xi_1, g^2\xi_2, g^3\alpha_k)$ since

$$\begin{aligned}
&g^1\xi_1(\tilde{a}^{1j}g^j\xi_j + \tilde{a}^{13}g^3\alpha_k) + g^2\xi_2(\tilde{a}^{2j}g^j\xi_j + \tilde{a}^{23}g^3\alpha_k) + g^3\alpha_k(\tilde{a}^{3j}g^j\xi_j + \tilde{a}^{33}g^3\alpha_k)\\
&= \alpha_k^2\underline{a}^{33} + 2\alpha_k\underline{a}^{3j}\xi_j + \underline{a}^{jk}\xi_j\xi_k = 0
\end{aligned}$$

because of (2.8). In the following we will work with eigenvectors of unit length, i.e. $q_j = b_j/|b_j|$ where $|b_j|$ denotes the Euclidian length of the vector b_j and introduce the symbol $q(y, \xi') = (q_1, q_2, q_3)$. The matrix q diagonalizes the symbol m

$$q^{-1}(y, \xi')m(y, \xi')q(y, \xi') = \begin{pmatrix} \alpha_1 & 0 & 0 \\ 0 & \alpha_2 & 0 \\ 0 & 0 & \alpha_3 \end{pmatrix} = j(y, \xi')$$

We point out that q,q^{-1},m,j are essentially classical symbols with C^1 coefficients, i.e. $q, q^{-1} \in C^1S^0_{cl}$ and $m, j \in C^1S^1_{cl}$. Strictly speaking, in order to obtain a classical symbol the singularity at $\xi' = 0$ has to be removed by a cutoff function. Moreover, all four symbols are C^1 functions in y_3 as well.

Given $u \in H^1_0(\Phi^{-1}(W))$ we set $v = Q^{-1}u \in H^1(\Phi^{-1}(W))$ where Q^{-1} is the operator with the symbol q^{-1}. Note that $v(-h) = v(h) = 0$ since Q^{-1} is a tangential operator, this is an operator with symbol independent of ξ_3. Now we make use of the operator algebra for operators with classical symbols with C^1 coefficients as discussed in Proposition 4.2A [T91]. This yields

$$u(y) = Q(y, D')v + K(y, D')u \tag{2.9}$$

where K is a continuous linear operator from $H^m(\Phi^{-1}(S)) \to H^{m+1}(\Phi^{-1}(S))$ for $-1 \le m \le 0$ which is also continuously differentiable in y_3. Going back to equation (2.7) we have

$$D_3(Qv + Ku) - M(Qv + Ku) = \underline{F} + \underline{L}u$$

Applying the operator Q^{-1} to both sides of the this equation gives

$$Q^{-1}D_3Qv + Q^{-1}D_3Ku - Q^{-1}MQv - Q^{-1}MKu = Q^{-1}\underline{F} + Q^{-1}\underline{L}u$$

Using the operator algebra for classical symbols with C^1 coefficients [T91, Proposition 4.2A] we obtain

$$D_3v + R_0v + R_{-1}D_3v + R_{-1}D_3u - Jv = R_0\underline{F} + R_0u$$

where $R_0 : L_2(\Phi^{-1}(W)) \to L_2(\Phi^{-1}(S))$ and $R_{-1} : L_2(\Phi^{-1}(S)) \to H^1(\Phi^{-1}(S))$ are continuous mappings. Since $D_3u = Mu + \underline{F} + \underline{L}u$ and $v = Q^{-1}u$ we obtain

$$D_3v - Jv = R_0\underline{F} + R_0u \tag{2.10}$$

Now we introduce the function $z = ve^{s\phi} = v\exp\left(se^{\lambda(y_3 - \psi(x_0))} - 1\right)$. Then z will satisfy the equation

$$D_3z + is\lambda e^{\lambda\psi}z - Jz = e^{s\phi}(R_0\underline{F} + R_0u)$$

or

$$\partial_3z - s\lambda e^{\lambda\psi}z - iJz = ie^{s\phi}(R_0\underline{F} + R_0u) =: G$$

where the last equation defines G. For the components of z we obtain the following equation

$$\begin{aligned}
\partial_3z_1 - s\lambda e^{\lambda\psi}z_1 &= G_1\\
\partial_3z_2 - s\lambda e^{\lambda\psi}z_2 + Tz_2 + iSz_2 &= G_2\\
\partial_3z_3 - s\lambda e^{\lambda\psi}z_3 - Tz_3 + iSz_3 &= G_3
\end{aligned} \tag{2.11}$$

Here $T(y, D')$ is the operator with the symbol $t(y, \xi') = \Im\alpha_2 \in C^1_{cl}S^1$ and S is the operator with the symbol $s(y, \xi') = -\Re\alpha_2 \in C^1_{cl}S^1$. Both operators have real symbols. These three equations are now used to obtain estimates.

2.3. **The integration.** Given two square integrable scalar functions $\varphi_1(y), \varphi_2(y)$ we introduce the L_2 norm and the scalar product on the surfaces parallel to $\Phi^{-1}(S)$ by

$$|\varphi_1(y_3)|^2 = \int |\varphi_1(y)|^2 \sqrt{g(y)}dy_1dy_2 \text{ and } \langle\varphi_1, \varphi_2\rangle(y_3) = \int \varphi_1(y)\overline{\varphi_2(y)}\sqrt{g(y)}dy_1dy_2$$

where $g(y)$ is the determinant of the metric tensor $G(y)$. In the original coordinate system these integrals are surfaces integrals over the level surfaces of ψ. The corresponding L_2 norm and scalar product in $\Phi^{-1}(W)$ by

$$\|\varphi_1\|^2 = \int_{-h}^{h} |\varphi_1(y_3)|^2 dy_3 \text{ and } (\varphi_1, \varphi_2) = \int_{-h}^{h} \langle\varphi_1, \varphi_2\rangle(y_3)dy_3$$

Note that $\int_W |w(x)|^2dx = \|u\|^2$, see (2.4). Based on these L_2 norms one can also introduce Sobolev norms. These norms will be introduced by subscripts, e.g. $|v(y_3)|_1$ is the norm on the space $H^1(\Phi^{-1}(S))$.

Using integration by parts in the y_3 variable the first equation in (2.11) gives

$$\begin{aligned}(2.12)\quad \|G_1\|^2 &= \|\partial_3 z_1 - s\lambda e^{\lambda\psi}z_1\|^2 = \|\partial_3 z_1\|^2 + (s\lambda)^2\|e^{\lambda\psi}z_1\|^2 - 2s\lambda\Re(\partial_3 z_1, e^{\lambda\psi}z_1)\\ &= \|\partial_3 z_1\|^2 + (s\lambda)^2\|e^{\lambda\psi}z_1\|^2 + s\lambda^2(e^{\lambda\psi}z_1, z_1) + \frac{1}{2}s\lambda\left(\frac{\partial_3 g}{g}e^{\lambda\psi}z_1, z_1\right) \geq \frac{1}{2}s\lambda^2(e^{\lambda\psi}z_1, z_1) + \|\partial_3 z_1\|^2\end{aligned}$$

provided λ is sufficiently large. From the second equation we get

$$\begin{aligned}\|G_2\|^2 &= \|\partial_3 z_2 - s\lambda e^{\lambda\psi}z_2 + Tz_2 + iSz_2\|^2\\ &= \|\partial_3 z_2 + iSz_2\|^2 + \|Tz_2 - s\lambda e^{\lambda\psi}z_2\|^2 + 2\Re(\partial_3 z_2 + iSz_2, -s\lambda e^{\lambda\psi}z_2 + Tz_2)\end{aligned} \tag{2.13}$$

Here we consider the last term and observe that

$$-2\Re s\lambda(\partial_3 z_2, e^{\lambda\psi}z_2) \geq \frac{1}{2}s\lambda^2(e^{\lambda\psi}z_2, z_2) \tag{2.14}$$

for λ sufficiently large by (2.12) and
(2.15)

$$2s\lambda\Re(iSz_2, -e^{\lambda\psi}z_2) = is\lambda\left[(e^{\lambda\psi}z_2, Sz_2) - (Sz_2, e^{\lambda\psi}z_2)\right] = is\lambda\left((S^* - S)z_2, e^{\lambda\psi}z_2\right) \geq -C_1 s\lambda(e^{\lambda\psi}z_2, z_2)$$

since S is a differential operator and $S^* - S$ is an operator of order 0. Next we compute

$$\begin{aligned}(2.16)\quad 2\Re(\partial_3 z_2 + iSz_2, Tz_2) &= (\partial_3 z_2, Tz_2) + (Tz_2, \partial_3 z_2) + i(Sz_2, Tz_2) - i(Tz_2, Sz_2)\\ &= (T^*\partial_3 z_2, z_2) - (\partial_3 T z_2, z_2) + i((T^*S - S^*T)z_2, z_2)\\ &= -((\partial_3 T)z_2, z_2) + ((T^* - T)\partial_3 z_2, z_2) + i((T^*S - S^*T)z_2, z_2)\\ &\geq -C_2\left[\int_{-h}^{h}|\Lambda z_2(y_3)||z_2(y_3)|dy_3 + \int_{-h}^{h}|\partial_3 z_2(y_3)||z_2(y_3)|dy_3\right]\end{aligned}$$

Here Λ is the (tangential) elliptic operator with the symbol $(1 + |\xi'|^2)^{1/2}$. In order to justify these estimates for operators with classical symbols with C^1 coefficients we rely once more on Propositions 4.2A and 4.2B [T91]. Combining (2.14)-(2.16) with (2.13) gives

$$\begin{aligned}(2.17)\quad \|G_2\|^2 &\geq \|\partial_3 z_2 + iSz_2\|^2 + \|Tz_2 - s\lambda e^{\lambda\psi}z_2\|^2 + \frac{1}{2}s\lambda^2(e^{\lambda\psi}z_2, z_2)\\ &\quad - C_1 s\lambda(e^{\lambda\psi}z_2, z_2) - C_2\left[\int_{-h}^{h}|\Lambda z_2(y_3)||z_2(y_3)|dy_3 + \int_{-h}^{h}|\partial_3 z_2(y_3)||z_2(y_3)|dy_3\right]\end{aligned}$$

Since $|\partial_3 z_2(y_3)| \leq |\partial_3 z_2(y_3) + iSz_2(y_3)| + |Sz_2(y_3)|$ we have

(2.18)
$$\begin{aligned} C_2 \int_{-h}^{h} |\partial_3 z_2(y_3)||z_2(y_3)|dy_3 &\leq C_2 \int_{-h}^{h} |\partial_3 z_2(y_3) + iSz_2(y_3)||z_2(y_3)|dy_3 + C_2 \int_{-h}^{h} |Sz_2(y_3)||z_2(y_3)|dy_3 \\ &\leq C_2\|\partial_3 z_2 + iSz_2\|\|z_2\| + C_2 \int_{-h}^{h} |\Lambda z_2(y_3)||z_2(y_3)|dy_3 \\ &\leq \|\partial_3 z_2 + iSz_2\|^2 + \frac{1}{4}C_2^2\|z_2\|^2 + C_2 \int_{-h}^{h} |\Lambda z_2(y_3)||z_2(y_3)|dy_3 \end{aligned}$$

which modifies (2.17) into

$$\begin{aligned} \|G_2\|^2 \geq \|Tz_2 - s\lambda e^{\lambda\psi} z_2\|^2 + \frac{1}{2}s\lambda^2(e^{\lambda\psi} z_2, z_2) \\ - C_1 s\lambda(e^{\lambda\psi} z_2, z_2) - \frac{1}{4}C_2^2\|z_2\|^2 - 2C_2 \int_{-h}^{h} |\Lambda z_2(y_3)||z_2(y_3)|dy_3 \end{aligned} \tag{2.19}$$

Since T is uniformly elliptic in y_3 we have

$$2C_2|\Lambda z_2(y_3)| \leq C_3[|Tz_2(y_3)| + |z_2(y_3)|] \leq C_3[|Tz_2(y_3) - s\lambda e^{\lambda\psi} z_2(y_3)| + s\lambda|e^{\lambda\psi} z_2(y_3)| + |z_2(y_3)|] \tag{2.20}$$

with the constant C_3 independent of y_3, which after integration yields

$$\begin{aligned} 2C_2 \int_{-h}^{h} |\Lambda z_2||z_2|dy_3 &\leq C_3 \left[\int_{-h}^{h} |Tz_2 - s\lambda e^{\lambda\psi} z_2||z_2|dy_3 + s\lambda \int_{-h}^{h} e^{\lambda\psi}|z_2|^2 dy_3 + \int_{-h}^{h} |z_2|^2 dy_3\right] \\ &\leq C_3 \left[\|Tz_2 - s\lambda z_2\|\|z_2\| + s\lambda(e^{\lambda\psi} z_2, z_2) + \|z_2\|^2\right] \\ &\leq \frac{C_3^2}{2}\|z_2\|^2 + \frac{1}{2}\|Tz_2 - s\lambda e^{\lambda\psi} z_2\|^2 + C_3[s\lambda(e^{\lambda\psi} z_2, z_2) + \|z_2\|^2] \\ &\leq \frac{1}{2}\|Tz_2 - s\lambda e^{\lambda\psi} z_2\|^2 + C_3 s\lambda(e^{\lambda\psi} z_2, z_2) + (C_3^2/2 + C_3)\|z_2\|^2 \end{aligned}$$

Using this last formula in (2.19), we obtain

$$\|G_2\|^2 \geq \frac{1}{2}s\lambda^2(e^{\lambda\psi} z_2, z_2) - C_1 s\lambda(e^{\lambda\psi} z_2, z_2) - C_4\|z_2\|^2 + \frac{1}{2}\|Tz_2 - s\lambda e^{\lambda\psi} z_2\|^2 \tag{2.21}$$

where $C_4 = C_2^2/4 + C_3^2/2 + C_3$. The uniform ellipticity of T, i.e. formula (2.20) leads after squaring and integrating in y_3 to

$$\frac{1}{\lambda s} \int_{-h}^{h} \frac{1}{e^{\lambda\psi}}|\Lambda z_2|^2 dy_3 \leq C_5 \left[\frac{1}{2}\|Tz_2 - s\lambda e^{\lambda\psi} z_2\|^2 + s\lambda(e^{\lambda\psi} z_2, z_2) + \frac{1}{s\lambda}\|z_2\|^2\right] \tag{2.22}$$

where we choose $s = s(\lambda)$ large enough to guarantee $s\lambda e^{\lambda\psi} \geq 1$. Equation (2.11) provides the estimate

$$\frac{1}{s\lambda e^{\lambda\psi}}|\partial_3 z_2(y_3)|^2 \leq C_6 \left[s\lambda e^{\lambda\psi}|z_2(y_3)|^2 + \frac{1}{s\lambda e^{\lambda\psi}}|\Lambda z_2(y_3)|^2 + \frac{1}{s\lambda e^{\lambda\psi}}|G_2(y_3)|^2\right]$$

where we again choose s sufficiently large. This estimate is then integrated with respect to y_3

$$\frac{1}{s\lambda}(e^{-\lambda\psi}\partial_3 z_2, \partial_3 z_2) \leq C_6 \left[s\lambda(e^{\lambda\psi} z_2, z_2) + \frac{1}{s\lambda}(e^{-\lambda\psi}\Lambda z_2, \Lambda z_2) + \|G_2\|^2\right] \tag{2.23}$$

Finally we combine (2.21), (2.22) and (2.23) into

$$s\lambda^2(e^{\lambda\psi} z_2, z_2) + \frac{1}{s\lambda}[(e^{-\lambda\psi}\Lambda z_2, \Lambda z_2) + (e^{-\lambda\psi}\partial_3 z_2, \partial_3 z_2)] \leq C_7\|G_2\|^2 \tag{2.24}$$

The estimate for z_3 is done in the same manner as the estimate for z_2. The only difference is that T has to be replaced by $-T$. Hence we obtain

$$s\lambda^2(e^{\lambda\psi} z_3, z_3) + \frac{1}{s\lambda}[(e^{-\lambda\psi}\Lambda z_3, \Lambda z_3) + (e^{-\lambda\psi}\partial_3 z_3, \partial_3 z_3)] \leq C_7\|G_3\|^2 \tag{2.25}$$

Combining now the inequalities (2.12),(2.24) and (2.25) we obtain

$$s\lambda^2(e^{\lambda\psi}z,z)+\frac{1}{s\lambda}(e^{-\lambda\psi}\partial_3 z,\partial_3 z)+\frac{1}{s\lambda}(e^{-\lambda\psi}\Lambda z_2,\Lambda z_2)+\frac{1}{s\lambda}(e^{-\lambda\psi}\Lambda z_3,\Lambda z_3)\le C_8\|G\|^2$$

for some positive constant C_8 and sufficiently large λ and $s=s(\lambda)$. Note that this estimate does not contain the tangential derivatives of z_1.

2.4. **Return to the original variable.** Now we will return to the original variable u. Using $z=ve^{s\phi}$ we get

$$s\lambda^2(e^{\lambda\psi}e^{s\phi}v,e^{s\phi}v)+\frac{1}{s\lambda}(e^{-\lambda\psi}e^{s\phi}\partial_3 v,e^{s\phi}\partial_3 v)$$
$$+\frac{1}{s\lambda}(e^{-\lambda\psi}e^{s\phi}\Lambda v_2,e^{s\phi}\Lambda v_2)+\frac{1}{s\lambda}(e^{-\lambda\psi}e^{s\phi}\Lambda v_3,e^{s\phi}\Lambda v_3)\le C_8\|G\|^2$$

By (2.9) we have

$$|u(y_3)|^2\le C_9\left[|v(y_3)|^2+|u(y_3)|^2_{-1}\right]$$

uniformly in y_3. Assuming that the support of u is small (this can be accomplished by choosing W sufficiently small) we obtain

$$|u(y_3)|_{-1}\le\frac{1}{\sqrt{2C_9}}|u(y_3)|$$

by Poincaré's inequality. The last two inequalities give

$$|u(y_3)|^2\le\frac{C_9}{2}|v(y_3)|^2$$

The same inequality holds for $\partial_3 u$ and $\partial_3 v$, respectively. Hence

$$s\lambda^2(e^{\lambda\psi}e^{s\phi}u,e^{s\phi}u)+\frac{1}{s\lambda}(e^{-\lambda\psi}e^{s\phi}\partial_3 u,e^{s\phi}\partial_3 u)+\frac{1}{s\lambda}(e^{-\lambda\psi}e^{s\phi}\Lambda v_2,e^{s\phi}\Lambda v_2)$$
$$+\frac{1}{s\lambda}(e^{-\lambda\psi}e^{s\phi}\Lambda v_3,e^{s\phi}\Lambda v_3)\le C_{10}\|\underline{F}e^{s\phi}\|^2 \tag{2.26}$$

for λ and s sufficiently large where we also used $G=ie^{s\phi}(R_0\underline{F}+R_0u)$.

Next we will perform a rather detailed analysis of the terms involving v_2 and v_3. Since $v=Q^{-1}u$ we need to work with the components of the matrix operator Q^{-1}. Remember that

$$q(y,\xi')=\begin{pmatrix}-\frac{\tilde a^{2j}g^j\xi_j}{|b_1|} & \frac{\xi_1 g^1}{|b_2|} & \frac{\xi_1 g^1}{|b_2|}\\ \frac{\tilde a^{1j}g^j\xi_j}{|b_1|} & \frac{\xi_2 g^2}{|b_2|} & \frac{\xi_2 g^2}{|b_2|}\\ 0 & \frac{g^3\alpha_2}{|b_2|} & \frac{g^3\alpha_3}{|b_2|}\end{pmatrix}$$

A lengthy but straightforward calculation yields

$$q^{-1}(y,\xi')=\begin{pmatrix}-\frac{|b_1|\xi_2 g^2}{\underline{a}^{jk}\xi_j\xi_k} & \frac{|b_1|\xi_1 g^1}{\underline{a}^{jk}\xi_j\xi_k} & 0\\ -\frac{|b_2|\alpha_3\tilde a^{1j}g^j\xi_j}{2i\Im\alpha_2(\underline{a}^{jk}\xi_j\xi_k)} & -\frac{|b_2|\alpha_3\tilde a^{2j}g^j\xi_j}{2i\Im\alpha_2(\underline{a}^{jk}\xi_j\xi_k)} & \frac{|b_2|}{2g^3 i\Im\alpha_2}\\ \frac{|b_2|\alpha_2\tilde a^{1j}g^j\xi_j}{2i\Im\alpha_2(\underline{a}^{jk}\xi_j\xi_k)} & \frac{|b_2|\alpha_2\tilde a^{2j}g^j\xi_j}{2i\Im\alpha_2(\underline{a}^{jk}\xi_j\xi_k)} & -\frac{|b_2|}{2g^3 i\Im\alpha_2}\end{pmatrix}$$

All the entries in these two matrices are operators of order 0. Setting

$$l_1(y,\xi')=\left(\frac{|b_2|\tilde a^{1j}g^j\xi_j}{2\underline{a}^{jk}\xi_j\xi_k},\frac{|b_2|\tilde a^{2j}g^j\xi_j}{2\underline{a}^{jk}\xi_j\xi_k},0\right)$$
$$l_2(y,\xi')=\left(\frac{|b_2|\Re\alpha_2\tilde a^{1j}g^j\xi_2}{2\Im\alpha_2(\underline{a}^{jk}\xi_j\xi_k)},\frac{|b_2|\Re\alpha_2\tilde a^{2j}g^j\xi_j}{2\Im\alpha_2(\underline{a}^{jk}\xi_j\xi_k)},-\frac{|b_2|}{2g^3\Im\alpha_2}\right)$$

we see that

$$v_2(y_3)=L_1(y,D')\cdot u(y_3)+iL_2(y,D')\cdot u(y_3)\text{ and }v_3(y_3)=L_1(y,D')\cdot u(y_3)-iL_2(y,D')\cdot u(y_3)$$

where L_1 and L_2 are the operators with the symbols $l_1 \in C^1 S^0_{cl}$ and $l_2 \in C^1 S^0_{cl}$, respectively. Properties of the scalar product give

$$|\Lambda v_2(y_3)|^2 + |\Lambda v_3(y_3)|^2 = 2|\Lambda L_1 \cdot u(y_3)|^2 + 2|\Lambda L_2 \cdot u(y_3)|^2 \tag{2.27}$$

Note that the operator $L_1(y, D')$ acts only on u_1 and u_2. Combining $\Lambda L_2(y, D')$ with the third curl equation (the one which was left off the analysis until now) we obtain a first order system with principal symbol

$$\begin{pmatrix} (1+|\xi'|^2)^{1/2}\dfrac{|b_2|\tilde{a}^{1j}g^j\xi_j}{2\underline{a}^{jk}\xi_j\xi_k} & (1+|\xi'|^2)^{1/2}\dfrac{|b_2|\tilde{a}^{2j}g^j\xi_j}{2\underline{a}^{jk}\xi_j\xi_k} \\ -g^2\xi_2 & g^1\xi_1 \end{pmatrix}$$

Its determinant is $(1+|\xi'|^2)|b_2|^2 > 0$ which proves that this system is uniformly elliptic in y_3. Elliptic regularity [WRL95, Chapter 9] yields

$$|\Lambda u_1(y_3)|^2 + |\Lambda u_2(y_3)|^2 \le C_{11}\left[|\underline{F}(y_3)|^2 + |\Lambda L_1(y, D') \cdot u(y_3)|^2 + |u(y_3)|^2\right]$$

and the operator L_2 provides the estimate

$$|\Lambda u_3(y_3)|^2 \le C_{12}\left[|\Lambda L_1(y, D') \cdot u(y_3)|^2 + |\Lambda u_1(y_3)|^2 + |\Lambda u_2(y_3)|^2\right]$$

The last two equations add up to

$$|\Lambda u(y_3)|^2 \le C_{12}\left[|\underline{F}(y_3)|^2 + |\Lambda L_1(y, D') \cdot u(y_3)|^2 + |\Lambda L_2(y, D') \cdot u(y_3)|^2 + |u(y_3)|^2\right]$$

and in connection with (2.27) we obtain

$$|\Lambda u(y_3)|^2 \le C_{13}\left[|\underline{F}(y_3)|^2 + |\Lambda v_2)|^2 + |\Lambda v_3|^2 + |u(y_3)|^2\right] \tag{2.28}$$

Using this formula in (2.26) after multiplication by $e^{s\phi}$ and integration in y_3 yields

$$\frac{1}{s\lambda}\sum_{j=1}^{3}(e^{-\lambda\psi}e^{s\phi}\partial_j u, e^{s\phi}\partial_j u) + s\lambda^2(e^{\lambda\psi}e^{s\phi}u, e^{s\phi}u) \le C_{14}\|\underline{F}e^{s\phi}\|^2$$

and returning to the original coordinates finishes the proof.

3. Conditional Stability in the Cauchy Problem

In this section we assume Ω to be a connected domain with a C^2 boundary. Let E and H satisfy (1.1) and

$$E = f, \quad H = g \qquad \text{on } \Gamma, \tag{3.1}$$

where Γ is an arbitrary relatively open subset of $\partial\Omega$. We are interested in estimating E and H in a neighborhood of Γ by means of boundary data f and g.

We set

$$\Omega_\delta = \{x \in \Omega \;:\; \phi(x) > \delta\}, \quad \Gamma_\delta = \{x \in \Omega \;:\; \phi(x) = \delta\}$$

for $\delta > 0$. Let $\nabla\phi(x) \neq 0$ for all $x \in \Omega$. Then, by Theorem 1.1, we can argue similarly to e.g., Theorem 3.2.2 in [I98] to obtain

Theorem 3.1. *Assume that $\overline{\Omega}_0 \subset \Omega \cap \Gamma$. Then, for a solution (E, H) to (1.1) and (3.1), we have*

$$\|E\|_{H^1(\Omega_\delta)} + \|H\|_{H^1(\Omega_\delta)} \le C\left(\|f\|_{H^{\frac{1}{2}}(\Gamma)} + \|g\|_{H^{\frac{1}{2}}(\Gamma)}\right)^{\kappa}\left(1 + \|E\|_{L^2(\Omega)} + \|H\|_{L^2(\Omega)}\right)^{1-\kappa}. \tag{3.2}$$

Here $C > 0$ and $\kappa \in (0, 1)$ are constants which are dependent on ε, μ, ϕ, Γ, δ and independent of choices of f and g.

This is an estimate of E and H in the interior of Ω by means of boundary data on Γ, and does not imply any estimates outside Γ. Next we show such a boundary estimate in the Cauchy problem. Henceforth we fix $\lambda > 0$ sufficiently large.

Theorem 3.2. *We assume that* $\varepsilon, \mu \in \{C^3(\overline{\Omega})\}^9$, *and*

(i) The hypersurface Γ_δ *has a unique apex* z_δ *for each* $\delta \ge 0$.

(ii) $\Gamma_0 \setminus \{z_0\} \subset \Omega$, $z_0 \in \partial\Omega$, Γ_0 *is tangential to* Γ *at* z_0.

(iii) There exist $\gamma > 0$, $C_1 > 0$ *and* $\nu \in \mathbf{R}^3$ *such that* $|\nu| = 1$ *and* $z_t - z_0 = C_1 t^\gamma \nu$ *for* $t > 0$.

We set

$$M = \|E\|_{H^3(\Omega)} + \|H\|_{H^3(\Omega)}. \tag{3.3}$$

Then, for any $\kappa \in (0, \gamma)$, *there exists a constant* $C_2 = C_2(\kappa) > 0$ *such that*

$$|E(z_0)| + |H(z_0)| \le C_2 M \left| \log \frac{1}{\sum_{j=0}^2 \left(\|\partial_\nu^j E\|_{H^{\frac{5}{2}-j}(\Gamma)} + \|\partial_\nu^j H\|_{H^{\frac{5}{2}-j}(\Gamma)} \right)} \right|^{-\kappa}. \tag{3.4}$$

Here ∂_ν denotes the outward normal derivative and we note that $\lim_{\kappa \uparrow \gamma} C_2(\kappa) = \infty$. The stability at the boundary point z_0 is of logarithmic rate and is much worse than (3.2). The exponent $\kappa > 0$ relies on the radius of curvature of $\partial\Omega$ at z_0.

Example Let $z_0 = (0,0,0)$ and for some $r > 0$, let $\Omega \cap \{|x| < r\} \subset \{(x_1, x_2, x_3); x_3 > (x_1^2 + x_2^2)^\gamma\}$, and $\Omega \subset \{x_3 > 0\}$. For $\gamma > 0$, we set $\phi(x) = x_3^{\frac{1}{\gamma}} - (x_1^2 + x_2^2)$. Then $\nabla\phi(x) = (-2x_1, -2x_2, \frac{1}{\gamma} x_3^{\frac{1}{\gamma}-1}) \ne 0$ in Ω by $\Omega \subset \{x_3 > 0\}$. Since $z_t = (0,0,t^\gamma)$, the assumptions in the theorem are satisfied. When the radius of the curvature is larger, also γ is larger, so that estimate (3.4) is improved.

Proof. We set

$$D = \sum_{j=0}^2 \left(\|\partial_\nu^j E\|_{H^{\frac{5}{2}-j}(\Gamma)} + \|\partial_\nu^j H\|_{H^{\frac{5}{2}-j}(\Gamma)} \right).$$

Without loss of generality, we can assume that $M > 1$ and $0 < D < 1$.

By the Sobolev extension theorem, we have $E^*, H^* \in H^3(\Omega)$ such that

$$\partial_\nu^j E = \partial_\nu^j E^*, \quad \partial_\nu^j H = \partial_\nu^j H^* \qquad \text{on } \Gamma \tag{3.5}$$

and

$$\|E^*\|_{H^3(\Omega)} + \|H^*\|_{H^3(\Omega)} \le D. \tag{3.6}$$

We can take $\chi = \chi_\delta \in C^\infty(\mathbb{R}^3)$ such that $0 \le \chi \le 1$ and

$$\chi(x) = \begin{cases} 1, & x \in \Omega_{2\delta}, \\ 0, & x \in \Omega_0 \setminus \Omega_\delta, \end{cases} \tag{3.7}$$

and

$$\|\chi\|_{C^3(\mathbf{R}^3)} \le \frac{C_3}{\delta^3}. \tag{3.8}$$

In fact, we choose a function $\widetilde{\chi} \in C^\infty(\mathbf{R})$ such that $0 \le \widetilde{\chi} \le 1$ and

$$\widetilde{\chi}(t) = \begin{cases} 1, & t \ge 1, \\ 0, & t \le 0. \end{cases}$$

Setting

$$\chi_\delta(x) = \widetilde{\chi}\left(\frac{\phi(x) - \delta}{\delta} \right),$$

we see that this χ_δ satisfies (3.7) and (3.8).

Furthermore we set $u = \chi(E - E^*)$ and $v = \chi(H - H^*)$. Then $u, v \in H_0^3(\Omega_0)$ and

$$\begin{aligned} \nabla \times v - i\omega\varepsilon(x)u(x) &= (\nabla\chi) \times H - \nabla \times (\chi H^*) + i\omega\varepsilon\chi E^* \\ \nabla \times u + i\omega\mu(x)v(x) &= (\nabla\chi) \times E - \nabla \times (\chi E^*) - i\omega\mu\chi H^* \\ \nabla \cdot (\varepsilon u) &= \nabla\chi \cdot \varepsilon E - \nabla \cdot (\varepsilon\chi E^*) \\ \nabla \cdot (\mu v) &= \nabla\chi \cdot \mu H - \nabla \cdot (\mu\chi H^*) \end{aligned} \tag{3.9}$$

Applying Theorem 1.1 to (3.9), we have

$$\frac{1}{s}\int_{\Omega_0}(|\nabla u|^2+|\nabla v|^2)e^{2s\phi}dx + s\int_{\Omega_0}(|u|^2+|v|^2)e^{2s\phi}dx$$
$$\le C_4\int_{\Omega_0}|\nabla\chi|^2(|E|^2+|H|^2)e^{2s\phi}dx + C_4\int_{\Omega_0}(|E^*|^2+|H^*|^2+|\nabla E^*|^2+|\nabla H^*|^2)e^{2s\phi}dx$$

for large $s>0$. By (3.7) and (3.8), we obtain

$$e^{6s\delta}s^{-1}\int_{\Omega_{3\delta}}(|E|^2+|H|^2+|\nabla E|^2+|\nabla H|^2)dx$$
$$\le \frac{1}{s}\int_{\Omega_{3\delta}}(|u|^2+|v|^2+|\nabla u|^2+|\nabla v|^2)e^{2s\phi}dx + C_5e^{C_5's}D^2$$
$$\le \frac{C_5e^{4s\delta}M^2}{\delta^6}+C_5e^{C_5's}D^2.$$

Therefore

$$\int_{\Omega_{3\delta}}(|E|^2+|H|^2+|\nabla E|^2+|\nabla H|^2)dx \le \frac{C_6e^{-2s\delta}M^2s}{\delta^6}+C_6e^{C_7s}D^2 \le \frac{C_6e^{-s\delta}M^2}{\delta^7}+C_6e^{C_7s}D^2 \tag{3.10}$$

for any $s\ge s_1$: a constant, where we noted that $se^{-s\delta}\le\delta^{-1}$ for $s\ge 0$.

Here and henceforth the constants C_j are independent of s and $\delta\in(0,1)$, γ. Replacing C_6 by $C_6e^{C_7s_0}$, we have (3.10) for any $s>0$. Setting $e^{-s\delta}M^2=e^{C_7s}D^2$, that is, $s=\frac{2}{C_7+\delta}\log\frac{M}{D}$, we have

$$\|E\|_{H^1(\Omega_{3\delta})}+\|H\|_{H^1(\Omega_{3\delta})}\le\frac{C_8}{\delta^{7/2}}M^{\frac{C_7}{C_7+\delta}}D^{\frac{\delta}{C_7+\delta}}.$$

Taking $\partial_i,\partial_i\partial_j$, $1\le i,j\le 3$ in (3.9) and applying the above argument successively, we obtain

$$\|E\|_{H^3(\Omega_{3\delta})}+\|H\|_{H^3(\Omega_{3\delta})}\le\frac{C_8}{\delta^{7/2}}M^{\frac{C_7}{C_7+\delta}}D^{\frac{\delta}{C_7+\delta}}. \tag{3.11}$$

Here we note that the constants C_7 and C_8 are independent of $\delta\in(0,1)$.

By the Sobolev embedding, we have

$$\|E\|_{C^1(\overline{\Omega}_{3\delta})}+\|H\|_{C^1(\overline{\Omega}_{3\delta})}\le\frac{C_8}{\delta^{7/2}}M^{\frac{C_7}{C_7+\delta}}D^{\frac{\delta}{C_7+\delta}}.$$

Replacing 3δ by $3\delta t$ with $t\in[0,1]$, we have

$$\|E\|_{C^1(\overline{\Omega}_{3\delta t})}+\|H\|_{C^1(\overline{\Omega}_{3\delta t})}\le\frac{C_8}{\delta^{7/2}t^{7/2}}M^{\frac{C_7}{C_7+\delta t}}D^{\frac{\delta t}{C_7+\delta t}}. \tag{3.12}$$

Henceforth we fix $\delta>0$ sufficiently small and C_j denotes constants which are further independent of $\theta, t\in(0,1)$ and dependent on δ,γ. We set $h_1(t)=E(z_0+C_1(3\delta t)^\gamma\nu)$ and $h_2(t)=H(z_0+C_1(3\delta t)^\gamma\nu)$. By assumption (ii), we see that $h_1(1)=E(z_{3\delta})$ and $h_2(1)=H(z_{3\delta})$. Therefore

$$E(z_0)=\int_1^0\frac{dh_1(t)}{dt}dt+E(z_{3\delta}). \tag{3.13}$$

On the other hand,

$$\frac{dh_1(t)}{dt}=\nabla E(z_0+C_1(3\delta t)^\gamma\nu)\cdot C_1\gamma 3^\gamma\delta^\gamma t^{\gamma-1}\nu,$$

and, for $\theta\in(0,1)$, inequality (3.13) and the Sobolev embedding yield

$$\left|\frac{dh_1(t)}{dt}\right|\le\|\nabla E\|_{C(\overline{\Omega}_{3\delta t})}C_1\gamma 3^\gamma\delta^\gamma t^{\gamma-1}$$
$$\le C_9\|E\|^{1-\theta}_{C^1(\overline{\Omega}_{3\delta t})}\|E\|^{\theta}_{C^1(\overline{\Omega}_{3\delta t})}t^{\gamma-1}\le C_{10}M^{1-\theta}t^{(\gamma-\frac{7}{2}\theta)-1}M^{\frac{C_7\theta}{C_7+\delta t}}D^{\frac{\delta\theta t}{C_7+\delta t}}.$$

We choose θ such that $\gamma-\frac{7}{2}\theta>0$. Since $0<D<1$, $0<\delta\le 1$ and $M>1$, we have

$$M^{1-\theta}M^{\frac{C_7\theta}{C_7+\delta t}}\le M$$

and we can choose $C_{11} = C_{11}(\delta) > 0$ such that

$$D^{\frac{\delta\theta t}{C_7+\delta t}} \le D^{C_{11}\theta t}, \qquad 0 \le t \le 1.$$

Thus

$$\left|\frac{dh_1(t)}{dt}\right| \le C_{12} M t^{(\gamma-\frac{7}{2}\theta)-1} D^{C_{11}\theta t},$$

so that

$$\begin{aligned}|E(z_0)| &\le C_{13} M \int_0^1 t^{(\gamma-\frac{7}{2}\theta)-1} D^{C_{13}\theta t} dt + C_{13}\|E\|_{C(\overline{\Omega}_{3\delta})} \\ &\le C_{13} M \int_0^\infty t^{(\gamma-\frac{7}{2}\theta)-1} \exp\left(-C_{13}\left(\log\frac{1}{D}\right)\theta t\right) dt + C_{13} M D^{\frac{\delta}{C_7+\delta}} \\ &\le C_{14} M \Gamma(\gamma-\frac{7}{2}\theta)(C_{13}\theta)^{\frac{7}{2}\theta-\gamma}\left(\log\frac{1}{D}\right)^{-(\gamma-\frac{7}{2}\theta)} + C_{14} M D^{\frac{\delta}{C_7+\delta}}.\end{aligned}$$

Since

$$C_{15}\left(\log\frac{1}{D}\right)^{-(\gamma-\frac{7}{2}\theta)} \ge D^{\frac{\delta}{C_7+\delta}},$$

and we can estimate $|H(z_0)|$ in the same way, the proof of Theorem 3.2 is complete. □

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