UTMS 2006–18 August 8, 2006

The timelike Cauchy problem and an inverse problem for general hyperbolic equations

by

Arif Amirov and Masahiro Yamamoto

UNIVERSITY OF TOKYO

GRADUATE SCHOOL OF MATHEMATICAL SCIENCES

KOMABA, TOKYO, JAPAN

THE TIMELIKE CAUCHY PROBLEM AND AN INVERSE PROBLEM FOR GENERAL HYPERBOLIC EQUATIONS

Arif Amirov [1] and Masahiro Yamamoto[2]

[1]Department of Mathematics, Zonguldak Karaelmas University
67100 Zonguldak Turkey
e-mail: amirov@karaelmas.edu.tr
[2] Department of Mathematical Sciences, The University of Tokyo
Komaba Meguro Tokyo 153-8914 Japan
e-mail:myama@ms.u-tokyo.ac.jp

Let $x = (x_1, ..., x_n) \in \mathbb{R}^n$ and $x' = (x_2, ..., x_n)$. Let $D_1 \subset \mathbb{R}^{n-1}$ be a bounded domain and let the boundary ∂D_1 be smooth such that $\partial D_1 \subset \{x' \in \mathbb{R}^{n-1};\ 0 < |x'| < \rho\}$. For $\kappa > 0$ and $\delta > 0$, let $\gamma \in C^2(\overline{D_1})$ satisfy $\gamma(0) = 0$ and

$$\begin{cases} \gamma(x') < -\kappa|x'|^2 + \delta, \quad x' \in D_1, \\ \\ \gamma(x') = -\kappa|x'|^2 + \delta, \quad x' \in \partial D_1. \end{cases} \tag{1}$$

We set

$$\begin{cases} Q_\delta = \{(x,t);\ x' \in D_1,\ \gamma(x') < x_1 < -\kappa|x'|^2 - \kappa t^2 + \delta\}, \\ \\ \Omega = Q_\delta \cap \{t = 0\}, \quad \Gamma = \{x \in \mathbb{R}^n;\ x_1 = \gamma(x'),\ x' \in D_1\}, \end{cases} \tag{2}$$

for $0 < \delta \le \delta_0$, and

$$\psi(x,t) = -\frac{1}{2\kappa}x_1 - \frac{1}{2}|x'|^2 - \frac{1}{2}t^2 + \frac{1}{2\kappa}\delta_0. \tag{3}$$

Then $Q_\delta = \{(x,t);\ x' \in D_1,\ x_1 > \gamma(x'), \psi(x,t) > \frac{1}{2\kappa}(\delta_0 - \delta)\}$. We set

$$T \equiv \frac{1}{\sqrt{\kappa}}\sqrt{\max_{x'\in\overline{D_1}}(\delta - \gamma(x') - \kappa|x'|^2)}, \quad \rho_0 = \left(\rho^2 + \max\{|\min_{|x'|\le\rho}\gamma(x')|^2, \delta^2\}\right)^{\frac{1}{2}}. \tag{4}$$

Then $|x| < \rho_0$ if $x \in \Omega$, and $\overline{Q_\delta} \subset \overline{\Omega} \times [-T, T]$.

Typeset by $\mathcal{AMS}$-TEX

We consider

$$P(x,t,\partial)u \equiv p(x,t)\partial_t^2 u - \sum_{\ell,m=1}^{n} a_{\ell m}(x,t)\partial_\ell\partial_m u + \sum_{\ell=1}^{n} a_\ell(x,t)\partial_\ell u + r(x,t)u = f(x,t), \tag{5}$$

where $p, a_{\ell m} = a_{m\ell} \in C^1(\overline{\Omega}\times[-T,T])$, $a_\ell, r \in L^\infty(\Omega\times(-T,T))$, $p > 0$ on $\overline{\Omega}\times[-T,T]$ and $\sum_{\ell,m=1}^n a_{\ell m}(x,t)\xi_\ell\xi_m > 0$ for $(x,t)\in\overline{\Omega}\times[-T,T]$ and $\xi' \in \mathbb{R}^n$. Here we use the following notations: $t = x_{n+1}$, $\partial_{n+1} = \partial_t = \frac{\partial}{\partial t}$, $\partial_\ell = \frac{\partial}{\partial x_\ell}$, $1 \le \ell \le n$, $\xi = (\xi_1, ..., \xi_{n+1}) \in \mathbb{R}^{n+1}$, $\xi' = (\xi_1, ..., \xi_n)$, $\nabla' = (\partial_1, ..., \partial_n)$, $\nabla = (\partial_1, ..., \partial_n, \partial_t)$. Henceforth $\sum_{\ell,m}$ means the sum where the suffixes ℓ, m vary over $1, \ldots n$ if we do not specify, and we omit the (x,t)-dependency if there is no fear of confusion. For example, $\sum_j = \sum_{j=1}^n$ and $p\partial_t^2 u$ means $p(x,t)\partial_t^2 u(x,t)$.

We assume that there exists a constant $\theta_0 > 0$ such that

$$\begin{aligned}\mu_0(x,t,\xi') \equiv &\sum_{\ell,m,k}\{2a_{1k}(x,t)(\partial_k a_{\ell m})(x,t) - 4a_{km}(\partial_k a_{1\ell})\}\xi_\ell\xi_m \\ -2\sum_k a_{1k}(\partial_k p)\frac{(A\xi',\xi')}{p} &- 2p\sum_\ell |\partial_t a_{1\ell}|\left\{\xi_\ell^2 + \frac{(A\xi',\xi')}{p}\right\} \ge \theta_0|\xi'|^2 \end{aligned} \tag{6}$$

for any $(x,t)\in\overline{\Omega}\times[-T,T]$ and $\xi'\in\mathbb{R}^n$ satisfying

$$\pm t\sqrt{p(x,t)(A(x,t)\xi',\xi')} = \frac{1}{2\kappa}\sum_j a_{1j}\xi_j + \sum_{k=2}^n\sum_j a_{kj}\xi_j x_k. \tag{7}$$

Here and henceforth $A(x,t) = (a_{\ell m}(x,t))_{1\le\ell,m\le n}$ and $(\cdot,\cdot)$ denotes the scalar product in $\mathbb{R}^n$.

Case 1. We consider a case where $a_{\ell\ell} = 1$, $a_{\ell m} = 0$ if $\ell \ne m$, and p is t-independent. Then (6) is reduced to

$$2\frac{\partial_1 p(x)}{p(x)} \le -\theta_0, \qquad x \in \overline{\Omega}.$$

This means that the wave speed increases in the x_1-direction.

Case 2. Let $\gamma(x')=0$, that is, let the subboundary Γ be flat. Moreover let $p\equiv 1$, $a_{11}=1$ and $a_{12}=\ldots=a_{1n}=0$ and $a_{\ell m}$, $2\le \ell,m\le n$ are t-independent. We set $\rho=\kappa$, $D_1=\{x'\in\mathbb{R}^{n-1};\,|x'|<\kappa\}$ and $\delta=\kappa^3$ in (2) - (4). Then $\Omega=\{x;\,0<x_1<-\kappa|x'|^2+\kappa^3,\,|x'|<\kappa\}$. Let

$$\sum_{\ell,m=2}^{n}(\partial_1 a_{\ell m}(0,0))\xi_\ell\xi_m\ge\theta_0\sum_{j=2}^{n}|\xi_j|^2\quad\text{for }(\xi_2,...,\xi_n)\in\mathbb{R}^{n-1}. \tag{8}$$

Then for small $\kappa>0$, condition (6) is satisfied for $(x,t)\in\overline{\Omega}\times[-T,T]$ and $\xi'\in\mathbb{R}^n$ satisfying (7).

In fact, for small $\kappa>0$, if $(x,t)\in\overline{\Omega}\times[-T,T]$, then $|x|+|t|\le C_1\kappa$ by (2) - (4), so that

$$2\sum_{\ell,m=2}^{n}(\partial_1 a_{\ell m}(x,t))\xi_\ell\xi_m\ge\theta_0\sum_{j=2}^{n}|\xi_j|^2 \tag{9}$$

for $(x,t)\in\overline{\Omega}\times[-T,T]$ and $(\xi_2,...,\xi_n)\in\mathbb{R}^{n-1}$, if $\kappa>0$ is sufficiently small. Here and henceforth C_j denote generic constants which are independent of κ. Let $(x,t)\in\overline{\Omega}\times[-T,T]$ and $\xi'\in\mathbb{R}^n$ satisfy (7). We note that (7) is reduced to

$$\xi_1+2\kappa\sum_{k,j=2}^{n}a_{kj}\xi_j x_k=\pm 2\kappa t\sqrt{p(x,t)(A(x,t)\xi',\xi')}.$$

Hence $|\xi_1|^2\le C_2\kappa\left(\sum_{j=2}^n|\xi_j|^2+|\xi_1|^2\right)$, and we see that $\sum_{j=2}^n|\xi_j|^2\ge C_3|\xi'|^2$ if $\kappa>0$ is small. Therefore, since the left hand sides of (6) and (9) are same, condition (9) yields (6) if $\kappa>0$ is sufficiently small.

We note that (8) implies that the wave speed increases inward in the x_1-direction near $(0,0)\in\Gamma$. By a suitable rotation, we can replace (8) by

$$\sum_{\ell,m=2}^{n}(\nabla' a_{\ell m}(0,0),\nu)\xi_\ell\xi_m\ge\theta_0\sum_{j=2}^{n}|\xi_j|^2\quad\text{for }(\xi_2,...,\xi_n),\in\mathbb{R}^{n-1}.$$

where $\nu \in \mathbb{R}^n$ is an arbitrarily fixed unit vector such that $\varepsilon\nu \in \Omega$ with small $\varepsilon > 0$.

We set

$$\mu_1(x,t,\xi') = \sum_{j=2}^{n}\left[\sum_{k,\ell,m}\{4a_{km}(\partial_k a_{j\ell}) - 2a_{jk}(\partial_k a_{\ell m})\}\xi_\ell\xi_m + 2\sum_k a_{jk}(\partial_k p)\frac{(A\xi',\xi')}{p}\right]x_j$$
$$+t\left\{2p\sum_{\ell,m}(\partial_t a_{\ell m})\xi_\ell\xi_m + 2(\partial_t p)(A\xi',\xi')\right\}$$
$$+2|t|\sum_{k,\ell}|\partial_k p||a_{k\ell}|\left(\xi_\ell^2 + \frac{(A\xi',\xi')}{p}\right) + 2p\sum_{j=2}^{n}\sum_\ell|\partial_t a_{j\ell}||x_j|\left(\xi_\ell^2 + \frac{(A\xi',\xi')}{p}\right)$$
$$+4\left(\sum_{k=2}^{n}|[A\xi']_k|^2 + p(A\xi',\xi')\right) \tag{10}$$

and we assume that

$$|\mu_1(x,t,\xi')| \le \theta_1|\xi'|^2 \text{ for } (x,t) \in \overline{\Omega}\times[-T,T] \text{ and } \xi' \in \mathbb{R}^n \text{ satisfying (7).} \tag{11}$$

Here $[A\xi']_k$ denotes the kth component of $A\xi' \in \mathbb{R}^n$. We set

$$\alpha_0 = \min_{(x,t)\in\overline{\Omega}\times[-T,T],|\xi'|=1}(A(x,t)\xi',\xi').$$

We state our first main first result.

Theorem 1. *We assume (1), (6) with (7), (11) and*

$$\frac{1}{2\kappa} > \max\left\{T\left(\frac{\|p\|_{C^1(\overline{\Omega}\times[-T,T])}}{\alpha_0}\right)^{\frac{1}{2}}, \frac{\theta_1}{\theta_0}\right\}. \tag{12}$$

Then there exist $C = C(P,\delta) > 0$, $\eta = \eta(P,\delta) > 0$ and $s_0 = s_0(P,\delta) > 0$ such that

$$\int_{Q_\delta}(s|\nabla u|^2 + s^3|u|^2)\exp(2se^{\eta\psi})dxdt \le C\int_{Q_\delta}|Pu|^2\exp(2se^{\eta\psi})dxdt$$

for all $s \ge s_0$ and $u \in H_0^2(Q_\delta)$.

The proof is given in Appendix and done by verifying the pseudoconvexity (e.g., [6], Theorem 3.2.1' (p.52) in [8]), and the weight $\frac{1}{2\kappa}$ for x_1 in (3) is essential in order to take advantage of the increasing wave speed condition (6) in the x_1-direction. A similar weight function was firstly introduced in [1] for a hyperbolic equation (see also [3]), whose weight function is same as a Carleman estimate in §1 of Chapter IV in [13] for a parabolic equation. As for Carleman estimates for $p\partial_t^2 - \Delta$, see [7], [8], [11]. We refer to [9] which proves a Carleman estimate for a hyperbolic equation (2) with $p \equiv 1$, but the used weight function prevents us from proving the unique continuation across non-convex surface (see function (1.5) in [9]). As for related recent papers, see [14], [17] and [19].

If the coefficients $a_{\ell m}$ are sufficiently smooth, then Carleman estimates are constructed by assuming the existence of a function $\varphi(x,t)$ whose level surfaces are pseudoconvex with respect to operator P (see [6]). The pseudoconvexity is easily checked in the case when the coefficients $a_{\ell m}$ and p are sufficiently close to constants in the C^1 norm. In [14, 19], the pseudoconvexty condition is replaced with the assumption of existence of some positive function $d(x)$ whose Hessian with respect to the Riemannian metric associated to P is uniformly positive in Ω. In this case, φ has form $\varphi(x,t) = d(x) - \mu t^2$. In [17], it was shown that under some conditions on the curvature of the Riemannian space, one can take $d(x)$ to be the function $s^2(x, x^0)$ which is the square of the Riemannian distance from some fixed point $x^0 \in \mathbb{R}^n$ to a point $x \in \Omega$ and it was found the connection between the Hessian of this function and the sectional curvatures of the Riemannian space. Although the condition on the curvature in [17] has a certain geometric sense, it is difficult to check when this condition holds. On the other hand, our condition (6) is directly verified and in some cases, it can be interpreted physically by means of the classical Snell law on the refraction: the inward increase of the wave speed is one sufficient condition for the non-existence of closed geodesics, which implies the unique continuation across Γ as a direct consequence of Theorem 1. Moreover the essential difference of our paper is that [17] is not applicable to the unique continuation or inverse problems when observation data are restricted on a non-convex part of the domain. Other difference of this paper from [14, 17, 19] is that we can treat t-dependent principal parts.

Next we apply Theorem 1 to the unique continuation and an inverse problem.

Problem 1. Find a function $u = u(x,t)$ satisfying (5) in $\Omega \times (-T,T)$ and Cauchy conditions

$$u = g, \qquad \frac{\partial u}{\partial \nu} = h \quad \text{on } \Gamma \times (-T,T). \tag{13}$$

If all the coefficients in (5) are analytic and the surface Γ is non-characteristic, then the uniqueness of a solution to Problem 1 follows from the Holmgren theorem (e.g., [6]). In [8, 9], the uniqueness of a solution to Problem 1 in the small for convex Γ was proved by using the technique of Carleman estimates. The uniqueness in Problem 1 in the case where the coefficients of (5) do not depend on t or are analytic in t (and Ω is not necessarily convex near Γ) was studied in [15, 18].

In this work, unlike in the existing works, in the case where the coefficients of (5) are not assumed to be analytic in any of the variables (x,t) and the domain Ω may be concave near Γ, we discuss the uniqueness and the conditional stability for Problem 1.

Theorem 2. *Suppose (1), (6) and (12). Suppose also that $u \in H^2(\Omega \times (-T,T))$ satisfies the equation $Pu = 0$ in $\Omega \times (-T,T)$ and $u = \frac{\partial u}{\partial \nu} = 0$ on $\Gamma \times (-T,T)$. Then there exists a neighbourhood V of the surface Γ and $T_1 \in (0,T)$ such that $u = 0$ on*

$(V \cap \Omega) \times (-T_1, T_1)$.

The examples for the non-uniqueness in Cauchy problems in [9, 12] show that condition (6) is essential for Theorem 2.

Theorem 3. *Suppose (1), (6) and (12). Moreover assume that* $u \in H^2(\Omega \times (-T, T))$ *satisfies (13) and the equation* $Pu = f$ *in* $\Omega \times (-T, T)$. *Then there exist a neighbourhood* V *of the surface* Γ, *a number* $T_1 \in (0, T)$, *and constants* $C > 0$, $\theta \in (0, 1)$ *such that*

$$\|u\|_{H^1((V\cap\Omega)\times(-T_1,T_1))} \le CE^\theta (E^{1-\theta} + \|u\|^{1-\theta}_{H^1(\Omega\times(-T,T))}),$$

where

$$E = \|f\|_{L^2(\Omega\times(-T,T))} + \|g\|_{H^{\frac{3}{2}}(\Gamma\times(-T,T))} + \|g\|_{H^2(-T,T;L^2(\Gamma))}$$
$$+\|h\|_{H^2(-T,T;L^2(\Gamma))} + \|h\|_{L^2(-T,T;H^{\frac{1}{2}}(\Gamma))}.$$

Now consider the following inverse problem.

Problem 2. Suppose that the coefficients of P do not depend on t. Let u satisfy (5), (13) and

$$u(x, 0) = a(x), \qquad x \in \Omega. \tag{14}$$

Then determine a pair of functions (u, r).

Inverse problems similar to Problem 2 were firstly studied in [5] by a method of Carleman estimates. After [5], there have appeared many works where similar methods were used [1, 2, 4, 7-11, 20]. See also [16]. In all of these works, except in [1, 2], inverse problems for hyperbolic equations were studied under the assumption that Ω is convex near Γ.

Theorem 4. *Suppose (1), (6) and (12). Let* $u_j \in H^2(\Omega \times (-T,T))$*,* $j = 1,2$*, satisfy (14) and*

$$p(x)\partial_t^2 u_j - \sum_{\ell,m} a_{\ell m}(x)\partial_\ell \partial_m u_j + \sum_\ell a_\ell(x)\partial_\ell u_j + r_j(x)u_j = 0$$

in $\Omega \times (-T,T)$*. Let*

$$\partial_t u_j \in H^2(\Omega \times (-T,T)) \cap L^\infty(\Omega \times (-T,T)),$$

and $\|\partial_t u_j\|_{L^\infty(\Omega\times(-T,T))}$*,* $\|u_j\|_{H^2(\Omega\times(-T,T))}$*,* $\|\partial_t u_j\|_{H^2(\Omega\times(-T,T))}$*,* $\|r_j\|_{L^\infty(\Omega)} \le M_1$*,* $j = 1,2$*. Suppose also that* $|a(x)| > 0$ *on* $\overline{\Omega}$*. Then there exist a neighbourhood* V *of the surface* Γ*, and constants* $C > 0$ *and* $\theta \in (0,1)$*, which depend on* M_1*,* p*,* $a_{\ell m}$*,* a_ℓ *such that*

$$\|r_1 - r_2\|_{L^2(V\cap\Omega)} \le C\Bigg\{\sum_{k=0}^{1}(\|\partial_t^k(u_1-u_2)\|_{H^{\frac{3}{2}}((-T,T);L^2(\Gamma))} + \|\partial_t^k(u_1-u_2)\|_{H^2((-T,T);L^2(\Gamma))}$$
$$+ \left\|\partial_t^k\left(\frac{\partial}{\partial\nu}(u_1-u_2)\right)\right\|_{H^2((-T,T);L^2(\Gamma))} + \left\|\partial_t^k\left(\frac{\partial}{\partial\nu}(u_1-u_2)\right)\right\|_{L^2((-T,T);H^{\frac{1}{2}}(\Gamma))}\Bigg\}^\theta.$$

Theorems 2 and 3 are proved by the method of Carleman estimates, and Theorem 4 is proved by the method in [5] with a modification by [7]. Although our Carleman estimate requires the compact supports for u, we note that Ω can be concave near Γ and on the rest of $\partial\Omega$, functions under consideration need not to vanish thanks to a usual cutoff function.

Similar results formulated above were obtained in [3] for simpler hyperbolic equation.

Appendix. Proof of Theorem 1.

Let

$$p(x,t,\xi) = \sum_{\ell,m} a_{\ell m}(x,t)\xi_\ell\xi_m - p(x,t)\xi_{n+1}^2$$

for $\xi = (\xi_1, ..., \xi_{n+1}) \in \mathbb{R}^{n+1}$.

First we will verify

$$(A\nabla'\psi, \nabla'\psi) - p|\partial_t\psi|^2 > 0, \qquad (x,t) \in \overline{\Omega} \times [-T,T]. \tag{A1}$$

In fact,

$$(A\nabla'\psi, \nabla'\psi) - p|\partial_t\psi|^2 \ge \alpha_0|\nabla'\psi|^2 - \|p\|_{C(\overline{\Omega}\times[-T,T])}T^2$$
$$\ge \alpha_0\left(\left(\frac{1}{2\kappa}\right)^2 + |x'|^2\right) - \|p\|_{C(\overline{\Omega}\times[-T,T])}T^2 \ge \alpha_0\left(\frac{1}{2\kappa}\right)^2 - \|p\|_{C(\overline{\Omega}\times[-T,T])}T^2 > 0$$

by (12).

Now, for the proof of the theorem, it is sufficient to verify the positivity of $J(x,t,\xi)$ for $(x,t,\xi) \in \overline{\Omega} \times [-T,T] \times (\mathbb{R}^{n+1} \setminus \{0\})$ satisfying (A2) and (A3) (e.g., Theorem 3.2.1' (p.52) in Isakov [8]):

$$J(x,t,\xi) = \sum_{j,k=1}^{n+1} \left\{ \left(\partial_k \frac{\partial P}{\partial \xi_j}\right) \frac{\partial P}{\partial \xi_k} - (\partial_k P)\frac{\partial^2 P}{\partial \xi_j \partial \xi_k} \right\} \partial_j\psi + \sum_{j,k=1}^{n+1} (\partial_j\partial_k\psi)\frac{\partial P}{\partial \xi_j}\frac{\partial P}{\partial \xi_k}$$
$$\equiv J_1 + J_2.$$

$$p\xi_{n+1}\partial_t\psi = \sum_{j,k} a_{kj}(\partial_k\psi)\xi_j. \tag{A2}$$

$$p\xi_{n+1}^2 = (A\xi', \xi'). \tag{A3}$$

We calculate J_1 and J_2. First we have

$$\partial_k P = \sum_{\ell,m}(\partial_k a_{\ell m})\xi_\ell\xi_m - (\partial_k p)\xi_{n+1}^2, \qquad 1 \le k \le n$$

$$\partial_{n+1}P = \partial_t P = \sum_{\ell,m}(\partial_t a_{\ell m})\xi_\ell\xi_m - (\partial_t p)\xi_{n+1}^2,$$

$$\frac{\partial P}{\partial \xi_j} = 2\sum_\ell a_{j\ell}\xi_\ell, \qquad \partial_k\left(\frac{\partial P}{\partial \xi_j}\right) = 2\sum_\ell(\partial_k a_{j\ell})\xi_\ell, \quad 1 \le j \le n,\ 1 \le k \le n+1,$$

$$\frac{\partial P}{\partial \xi_{n+1}} = -2p\xi_{n+1}, \qquad \partial_k\left(\frac{\partial P}{\partial \xi_{n+1}}\right) = -2(\partial_k p)\xi_{n+1}, \qquad 1 \le k \le n+1,$$
$$\frac{\partial^2 P}{\partial \xi_j \partial \xi_k} = 2a_{jk}, \qquad 1 \le j, k \le n,$$
$$\frac{\partial^2 P}{\partial \xi_j \partial \xi_{n+1}} = 0, \quad 1 \le j \le n, \qquad \frac{\partial^2 P}{\partial \xi_{n+1}^2} = -2p.$$

Then we have

$$\begin{aligned} J_1 = &\sum_{j,k}\left\{\left(\partial_k \frac{\partial P}{\partial \xi_j}\right)\frac{\partial P}{\partial \xi_k} - (\partial_k P)\frac{\partial^2 P}{\partial \xi_j \partial \xi_k}\right\}\partial_j\psi \\ &+ \sum_k \left\{\left(\partial_k \frac{\partial P}{\partial \xi_{n+1}}\right)\frac{\partial P}{\partial \xi_k} - (\partial_k P)\frac{\partial^2 P}{\partial \xi_{n+1}\partial \xi_k}\right\}\partial_{n+1}\psi \\ &+ \sum_j \left\{\left(\partial_{n+1}\frac{\partial P}{\partial \xi_j}\right)\frac{\partial P}{\partial \xi_{n+1}} - (\partial_{n+1}P)\frac{\partial^2 P}{\partial \xi_j \partial \xi_{n+1}}\right\}\partial_j\psi \\ &+ \left\{\partial_{n+1}\left(\frac{\partial P}{\partial \xi_{n+1}}\right)\frac{\partial P}{\partial \xi_{n+1}} - (\partial_{n+1}P)\frac{\partial^2 P}{\partial \xi_{n+1}^2}\right\}\partial_{n+1}\psi \\ &\equiv J_{11} + J_{12} + J_{13} + J_{14}. \end{aligned}$$

First we obtain

$$J_{11} = \sum_{j,k}\left\{4\sum_{\ell,m} a_{km}(\partial_k a_{j\ell})\xi_\ell\xi_m - 2\sum_{\ell,m} a_{jk}(\partial_k a_{\ell m})\xi_\ell\xi_m + 2a_{jk}(\partial_k p)\xi_{n+1}^2\right\}\partial_j\psi$$
$$= -\left[\sum_{k,\ell,m}\{4a_{km}(\partial_k a_{1\ell}) - 2a_{1k}(\partial_k a_{\ell m})\}\xi_\ell\xi_m + 2\sum_k a_{1k}(\partial_k p)\xi_{n+1}^2\right]\frac{1}{2\kappa}$$
$$-\sum_{j=2}^n\left[\sum_{k,\ell,m}\{4a_{km}(\partial_k a_{j\ell}) - 2a_{jk}(\partial_k a_{\ell m})\}\xi_\ell\xi_m + 2\sum_k a_{jk}(\partial_k p)\xi_{n+1}^2\right]x_j,$$

$$J_{12} = \sum_k -2(\partial_k p)\xi_{n+1}\left(2\sum_\ell a_{k\ell}\xi_\ell\right)(-t) = 4t\sum_{k,\ell}(\partial_k p)a_{k\ell}\xi_\ell\xi_{n+1},$$

$$J_{13} = \sum_j\left(2\sum_\ell(\partial_t a_{j\ell})\xi_\ell\right)(-2p\xi_{n+1})\partial_j\psi = -4p\sum_{j,\ell}(\partial_t a_{j\ell})\xi_\ell\xi_{n+1}\partial_j\psi$$
$$= \frac{4p}{2\kappa}\sum_\ell(\partial_t a_{1\ell})\xi_\ell\xi_{n+1} + 4p\sum_{j=2}^n\sum_\ell(\partial_t a_{j\ell})\xi_\ell\xi_{n+1}x_j$$

and

$$J_{14} = \left\{ 2p \sum_{\ell,m} (\partial_t a_{\ell m}) \xi_\ell \xi_m - 2p(\partial_t p)\xi_{n+1}^2 + 4p(\partial_t p)\xi_{n+1}^2 \right\} (-t).$$

Hence

$$J_1 = \Bigg[\sum_{k,\ell,m} \{2a_{1k}(\partial_k a_{\ell m}) - 4a_{km}(\partial_k a_{1\ell})\}\xi_\ell\xi_m - 2\sum_k a_{1k}(\partial_k p)\xi_{n+1}^2$$

$$+4p\sum_\ell (\partial_t a_{1\ell})\xi_\ell\xi_{n+1}\Bigg] \frac{1}{2\kappa}$$

$$+\sum_{j=2}^n \left[\sum_{k,\ell,m} \{2a_{jk}(\partial_k a_{\ell m}) - 4a_{km}(\partial_k a_{j\ell})\}\xi_\ell\xi_m - \sum_k 2a_{jk}(\partial_k p)\xi_{n+1}^2 \right] x_j$$

$$+4t\sum_{k,\ell}(\partial_k p)a_{k\ell}\xi_\ell\xi_{n+1} + 4p\sum_{j=2}^n \sum_\ell (\partial_t a_{j\ell})\xi_\ell\xi_{n+1}x_j - t\left\{ 2p\sum_{\ell,m}(\partial_t a_{\ell m})\xi_\ell\xi_m + 2p(\partial_t p)\xi_{n+1}^2 \right\}.$$

Moreover we obtain

$$J_2 = \sum_{j,k=1}^{n+1} (\partial_j\partial_k\psi)\frac{\partial P}{\partial \xi_j}\frac{\partial P}{\partial \xi_k} = \sum_{k=2}^{n+1} (\partial_k^2\psi)\left(\frac{\partial P}{\partial \xi_k}\right)^2$$

$$= -4\sum_{k=2}^n \left(\sum_j a_{kj}\xi_j \right)^2 - 4p^2\xi_{n+1}^2 = -4\sum_{k=2}^n |[A\xi']_k|^2 - 4p^2\xi_{n+1}^2.$$

By (A3) we have

$$\xi_{n+1}^2 = \frac{(A\xi', \xi')}{p}$$

and

$$2|\xi_\ell\xi_{n+1}| \le \xi_\ell^2 + \xi_{n+1}^2 = \xi_\ell^2 + \frac{(A\xi', \xi')}{p}.$$

Consequently

$$J \ge \mu_0(x,t,\xi')\frac{1}{2\kappa} - \mu_1(x,t,\xi').$$

Recall that μ_0 and μ_1 are defined by (6) and (10).

By (6), (11) and (12), we have

$$J \ge \left(\theta_0 \frac{1}{2\kappa} - \theta_1 \right) |\xi'|^2 > 0.$$

Finally we note that (A2) is equivalent to (7). Thus the proof of Theorem 1 is complete.

References

1. A. Kh. Amirov, Doctoral Dissertation in Mathematics and Physics (1988), Novosibirsk State University.
2. A. Kh. Amirov, *Integral Geometry and Inverse Problems for Kinetic Equations*, VSP, Utrecht, 2001.
3. A. Kh. Amirov and M. Yamamoto, Dokl. Akad. Nauk **402** (2005), 1-3.
4. A. L. Bukhgeim, *Introduction to the Theory of Inverse Problems*, VSP, Utretch, 2000.
5. A. L. Bukhgeim and M. V. Klibanov, Dokl. Akad. Nauk **260** (1981), 269-271.
6. L. Hörmander, *Linear Partial Differential Operators*, Springer-Verlag, Berlin, 1963.
7. O. Yu. Imanuvilov and M. Yamamoto, Commun. Partial Differ. Equations **26** (2001), 1409-1425.
8. V. Isakov, *Inverse Problems for Partial Differential Equations*, Springer-Verlag, Berlin, 2005.
9. A. Khaĭdarov, Mat. Sb. **58** (1987), 267-277.
10. M. V. Klibanov, Inverse Probl. **8** (1992), 575-596.
11. M. V. Klibanov and A. Timonov, *Carleman Estimates for Coefficient Inverse Problems and Numerical Applications*, VSP, Utretch, 2004.
12. H. Kumano-go, Proc. Japan Acad. **39** (1963), 578–582.
13. M. M. Lavrent'ev, V. G. Romanov, and S. P. Shishat·skiĭ, *Ill-Posed Problems of Mathematical Physics*, (Nauka, Moscow, 1980; American Mathematical Society, Providence, R.I., 1986).
14. I. Lasiecka, R. Triggiani and P.F. Yao, J. Math. Anal. and Appl. **235** (1999), 13-57.
15. L. Robbiano, Commun. Partial Differ. Equations **16** (1991), 789-800.
16. V. G. Romanov, *Inverse Problems of Mathematical Physics*, (Nauka, Moscow, 1984; VNU Science Press, Utrecht,1987).
17. V. G. Romanov, Dokl. Akad. Nauk **405** (2005), 730-734.
18. D. Tataru, Commun. Partial Differ. Equations **20** (1995), 855-884.
19. R. Triggiani and P.F. Yao, Appl. Math. and Optim. **46** (2002), 334-375.
20. M. Yamamoto, J. Math. Pures Appl. **78** (1999), 65-98.

UTMS

2006–7 Takehiro Fujiwara: *Sixth order methods of Kusuoka approximation.*

2006–8 Takeshi Ohtsuka: *Motion of interfaces by the Allen-Cahn type equation with multiple-well potentials.*

2006–9 Yoshihiro Sawano and Tsuyoshi Yoneda: *On the Young theorem for amalgams and Besov spaces.*

2006–10 Takahiko Yoshida: *Twisted toric structures.*

2006–11 Yoshihiro Sawano, Takuya Sobukawa and Hitoshi Tanaka: *Limiting case of the boundedness of fractional integral operators on non-homogeneous space.*

2006–12 Yoshihiro Sawano and Hitoshi Tanaka: *Equivalent norms for the (vector-valued) Morrey spaces with non-doubling measures.*

2006–13 Shigeo Kusuoka and Song Liang: *A mechanical model of diffusion process for multi-particles.*

2006–14 Yuji Umezawa: *A Limit Theorem on Maximum Value of Hedging with a Homogeneous Filtered Value Measure.*

2006–15 Toshio Oshima: *Commuting differential operators with regular singularities.*

2006–16 Miki hirano, Taku Ishii, and Takayuki Oda: *Whittaker functions for P_J-principal series representations of $Sp(3, R)$.*

2006–17 Fumio Kikuchi and Xuefeng Liu: *Estimation of interpolation error constants for the P_0 and P_1 triangular finite elements.*

2006–18 Arif Amirov and Masahiro Yamamoto: *The timelike Cauchy problem and an inverse problem for general hyperbolic equations.*

The Graduate School of Mathematical Sciences was established in the University of Tokyo in April, 1992. Formerly there were two departments of mathematics in the University of Tokyo: one in the Faculty of Science and the other in the College of Arts and Sciences. All faculty members of these two departments have moved to the new graduate school, as well as several members of the Department of Pure and Applied Sciences in the College of Arts and Sciences. In January, 1993, the preprint series of the former two departments of mathematics were unified as the Preprint Series of the Graduate School of Mathematical Sciences, The University of Tokyo. For the information about the preprint series, please write to the preprint series office.

ADDRESS:
Graduate School of Mathematical Sciences, The University of Tokyo
3–8–1 Komaba Meguro-ku, Tokyo 153-8914, JAPAN
TEL +81-3-5465-7001 FAX +81-3-5465-7012