UTMS 2006–14 June 27, 2006

A Limit Theorem on Maximum Value of Hedging with a Homogeneous Filtered Value Measure

by

Yuji Umezawa

UNIVERSITY OF TOKYO

GRADUATE SCHOOL OF MATHEMATICAL SCIENCES

KOMABA, TOKYO, JAPAN

A Limit Theorem on Maximum Value of Hedging with a Homogeneous Filtered Value Measure*

Yuji Umezawa

June 23, 2006

Abstract

The author studies on a hedging problem for an European contingent claim in a certain incomplete market model by using a homogeneous filtered value measure. He considers the minimal hedging risk in discrete time model and its continuous limit. As a result, he shows that this limit is described by a viscosity solution of some Hamilton-Jacobi-Bellman equation.

1 Introduction

Hedging of a contingent claim is an important problem in the mathematical finance. It is well known that every contingent claim can be perfectly hedged in a complete financial market. In incomplete markets, it is still possible to stay on the safe side by "superhedging". However, the cost of superhedging is often too high in many situations. If the investor is unwilling to put up the initial amount of capital requirement by a superhedging strategy, he/she is exposed to some risk and needs a "partial hedging" strategy.

Artzner et al. [1] proposed the concept of "coherent risk measure" to assess the risk of such financial positions by an axiomatic approach. Föllmer and Schied [12] defined the class of "convex risk measures", which are extensions of coherent risk measures, by relaxing the axiom of homogeneity. Cheridito

*This research is supported by the 21 century COE program at Graduate School of Mathematical Sciences, the University of Tokyo.

et al. [5] introduced coherent/convex risk measures for stochastic processes. These papers considered only single-period settings. Many authors such as Artzner et al. [2], Detlefsen, Scandolo [9], and Kusuoka, Morimoto [13] introduced the concept of dynamic risk measure in discrete settings by various ways. Delbaen [8] proposed the concept of dynamic risk measure in a continuous setting.

On partial hedging, Föllmer and Leukert [10],[11] introduced the concept of "quantile hedging" and more generally "Efficient hedging". Also, Roorda [15], Barrieu and El Karoui [4] studied partial hedging by using a coherent/convex risk measure. However their results on hedging with a risk measure still remain in single-period settings.

In this paper, we study on hedging with a risk measure in a multi-period setting. Concretely, we study on a hedging problem for an European contingent claim in incomplete markets by using a homogeneous filtered value measure. We consider the minimal hedging risk in discrete multi-period time market model and its continuous limit. As a result, we prove that this limit is describe by a viscosity solution of some Hamilton-Jacobi-Bellman equation.

This paper is organized as follows. In this section, we introduce the notion of homogeneous filtered value measure, and state our main theorem. In Section 2, we show a result on the representation of a law invariant coherent value measure. In Section 3, we consider a discrete time model and show a kind of Bellman's principle. In Section 4, we give the proof of our main theorem. In Appendix, we remark some brief introduction and results on the viscosity solution theory for readers who are unfamiliar with this theory.

1.1 Definition

Let $\mathcal{L}$ be the set of all probability measures on $(\mathbf{R}, \mathcal{B}(\mathbf{R}))$, $\mathcal{L}^\infty$ the set of $\nu \in \mathcal{L}$ which satisfies $\nu(\mathbf{R} \setminus [-M, M]) = 0$ for some $M > 0$, and $\mathcal{M}$ the set of all probability measures on $([0,1], \mathcal{B}[0,1])$. For $\nu \in \mathcal{L}$, we denote the distribution function of ν by F_ν, i.e., $F_\nu(z) = \nu((-\infty, z]), \quad z \in \mathbf{R}$. We define $Z : [0,1) \times \mathcal{L} \to \mathbf{R}$ by $Z(x, \nu) = \inf\{z;\ F_\nu(z) > x\},\ x \in [0,1),\ \nu \in \mathcal{L}$. For $\alpha \in (0,1]$, We define $\eta_\alpha : \mathcal{L}^\infty \to \mathbf{R}$ by

$$\eta_\alpha(\nu) = \alpha^{-1} \int_0^\alpha Z(x, \nu) dx, \quad \nu \in \mathcal{L}^\infty, \tag{1}$$

and $\eta_0 :\ \mathcal{L}^\infty \to \mathbf{R}$ by

$$\eta_0(\nu) = \inf\{z \in \mathbf{R} \mid \nu((-\infty, z]) > 0\}, \quad \nu \in \mathcal{L}^\infty. \tag{2}$$

Definition 1.1. *We say that the mapping* $\eta : \mathcal{L}^\infty \to \mathbf{R}$ *is a mild value measure if there exists a subset* $\mathcal{M}_0 \subset \mathcal{M}$ *such that*

$$\eta(\nu) = \inf_{m \in \mathcal{M}_0} \int_0^1 \eta_\alpha(\nu) m(d\alpha), \quad \nu \in \mathcal{L}^\infty.$$

Definition 1.2. *Let* $\eta : \mathcal{L}^\infty \to \mathbf{R}$ *be a mild value measure and* $(\Omega, \mathcal{F}, P)$ *a standard probability space.*
(1) *For any* $X \in L^\infty(\Omega, \mathcal{F}, P)$ *and any sub* σ*-algebra* $\mathcal{G} \subset \mathcal{F}$*, we define a* $\mathcal{G}$*-measurable random variable* $\eta(X|\mathcal{G})$ *by* $\eta(X|\mathcal{G}) = \eta(P(X \in dx|\mathcal{G}))$*, where* $P(X \in dx|\mathcal{G})$ *is a regular conditional probability law of* X *given a sub* σ*-algebra* $\mathcal{G}$*.*
(2) *For any* $X \in L^\infty(\Omega, \mathcal{F}, P)$ *and a filtration* $\{\mathcal{F}_k\}_{k=0}^n$*, we inductively define a* $\{\mathcal{F}_k\}$*-adapted process* $\{U_k\}_{k=0}^n$ *by*

$$\begin{aligned} U_n &= \eta(X|\mathcal{F}_n), \\ U_{k-1} &= \eta(U_k|\mathcal{F}_{k-1}), \quad k = n, n-1, \ldots, 1. \end{aligned} \tag{3}$$

We call $U_k,\ k = 0, 1, \ldots, n$ *a homogeneous filtered value measure for* $X \in L^\infty$ *at* k*. Also we denote* U_0 *by* $\eta(X|\{\mathcal{F}_k\}_{k=0}^n)$*.*

The basic properties of a mild value measure and a homogeneous filtered value measure are shown in [13].

1.2 Main Theorem

Let $\mathcal{Z} = \{z_1, z_2, \ldots, z_N\} \subset \mathbf{R}^M$, equipped with the discrete topology, and $\mathcal{B}(\mathcal{Z})$ the Borel algebra with respect to this. We define a probability $\hat{P}$ on $(\mathcal{Z}, \mathcal{B}(\mathcal{Z}))$ by $\hat{P}[\{z_j\}] = p_j,\ j = 1, 2, \ldots, N$, where $p_j > 0,\ j = 1, 2, \ldots, N$ and $\sum_{j=1}^N p_j = 1$. Also we define M-dimensional random variables $\hat{Z}$ and $\hat{Y}^{(n)},\ n \in \mathbf{N}$ on $(\mathcal{Z}, \mathcal{B}(\mathcal{Z}))$ by

$$\hat{Z}(z) = (\hat{Z}_1(z), \hat{Z}_2(z), \ldots, \hat{Z}_M(z)) = z,\ z \in \mathcal{Z}, \tag{4}$$

$$\hat{Y}^{(n)}(z) = (\hat{Y}_1^{(n)}(z), \hat{Y}_2^{(n)}(z), \ \ldots, \hat{Y}_M^{(n)}(z)),$$

$$\hat{Y}_i^{(n)}(z) = \exp(\hat{Z}_i(z)\sqrt{\frac{T}{n}} + b_i\frac{T}{n}), \quad i = 1, 2, \ldots, M,\ n \in \mathbf{N}, \tag{5}$$

where $b_i,\ i = 1, 2, \ldots, M$ and T are positive numbers. We denote by $\hat{\mathcal{P}}$ the set of all probability measures on $(\mathcal{Z}, \mathcal{B}(\mathcal{Z}))$. We identify $\hat{\mathcal{P}}$ as the hyperplane $\{(q_1, q_2, \ldots, q_N)|\ q_j \geq 0,\ j = 1, 2, \ldots, N,\ \sum_{j=1}^N q_j = 1\}$ on $\mathbf{R}^N$.

Let $(\Omega, \mathcal{F}, P)$ be the direct product probability space of countably infinite copies of $(\mathcal{Z}, \mathcal{B}(\mathcal{Z}), \hat{P})$. We define M-dimensional random variables Z_k and $Y_k^{(n)},\ k, n \in \mathbf{N}$ on $(\Omega, \mathcal{F})$ by

$$Z_k(\omega) = \hat{Z}(\omega_k), \tag{6}$$

$$Y_k^{(n)}(\omega) = \hat{Y}^{(n)}(\omega_k), \tag{7}$$

where $\omega = (\omega_1, \omega_2, \dots) \in \Omega$. We give a filtration $\{\mathcal{F}_k\}_{k=0,1,2,\dots}$ on $(\Omega, \mathcal{F})$ by

$$\begin{aligned} &\mathcal{F}_0 = \sigma(\emptyset, \Omega), \\ &\mathcal{F}_k = \sigma(Z_1, Z_2, \dots, Z_k), \quad k = 1, 2, \dots. \end{aligned} \tag{8}$$

Note that the process $(Y_\cdot^{(n)})$ is $\{\mathcal{F}_k\}$-adapted, and $Y_k^{(n)},\ k = 1, 2, \dots$ is independent of $\mathcal{F}_l,\ \ l = 1, 2, \dots, k-1$ for each $n \in \mathbf{N}$. We denote by $\mathcal{P}$ the set of all probability measures which is absolutely continuous with respect to P.

We define an $\{\mathcal{F}_k\}$-adapted M-dimensional process $\{S_k^{(n)}\}_{k=1,2,\dots,n}$ for each $n \in \mathbf{N}$ by

$$\begin{aligned} S_k^{(n)} &= (S_{1,k}^{(n)}, S_{2,k}^{(n)}, \quad \dots, \ S_{M,k}^{(n)}), \quad k = 1, 2, \dots, n \\ S_{i,k}^{(n)} &= S_{i,0} \prod_{l=1}^{k} Y_{i,l}^{(n)}, \quad i = 1, 2, \dots M. \end{aligned} \tag{9}$$

where $S_0 = (S_{1,0}, S_{2,0}, \dots, S_{M,0}) \in (0, \infty)^M$ is a constant vector. $S_k^{(n)}$ is interpreted as the price vector of M risky assets at time k. We call a $\{\mathcal{F}_k\}$-predictable M dimensional process $\xi = (\xi_k)_{k=1,2\dots}$ a self-financing strategy, and denote by $\mathcal{SF}$ the set of all self-financing strategy. Then we define a random variable $V_k^{(n)}(v, \xi)$ for $v \geq 0$, and $\xi \in \mathcal{SF}$ by $V_0^{(n)}(v, \xi) = v$ and

$$V_k^{(n)}(v, \xi) = v + \sum_{l=1}^{k} \xi_l \cdot (S_l^{(n)} - S_{l-1}^{(n)}). \tag{10}$$

$V_k^{(n)}(v, \xi)$ represents the discount value of self-financing portfolio (v, ξ) at time k.

Hereafter we use the following type of mild value measure:

$$\eta(\nu) = \int_0^1 \eta_\alpha(\nu) \mu(d\alpha), \quad \nu \in \mathcal{L}^\infty, \ \mu \in \mathcal{M}, \tag{11}$$

and fix it. We define $\hat{\mathcal{Q}}$ by

$$\hat{\mathcal{Q}} = \{\hat{Q} \in \hat{\mathcal{P}} |\ E^{\hat{Q}}[\hat{X}] \geq 0, \text{ for all random variables } \hat{X} \text{ on } (\mathcal{Z}, \mathcal{B}(\mathcal{Z})) \\ \text{such that } \eta(\nu_{\hat{X}}) \geq 0 \text{ holds.}\},$$

where $\nu_{\hat{X}}$ is the probability distribution on $\hat{X}$.

Let $\hat{C}([0,\infty)^M : \mathbf{R})$ be the set of functions $g : [0,\infty)^M \to \mathbf{R}$ such that

$$\begin{aligned} &|g(x) - g(x')| \leq K|x - x'|, \quad x, x' \in [0,\infty)^M, \\ &|g(x)| \leq K(1 + |x|^{2m}), \quad x \in [0,\infty)^M, \end{aligned} \tag{12}$$

hold for some $K > 0,\ m \in \mathbf{N}$. Take $f \in \hat{C}([0,\infty)^M : \mathbf{R})$ and fix it.

We assume the following in what follows.

Assumption 1.3. $M + 1 < N$.

Remark . *This assumption indicates that the market model which we consider here is incomplete.*

Assumption 1.4. *the set* $\hat{\mathcal{Q}} \cap \bigcap_{i=1}^M \{\hat{Q} \in \hat{\mathcal{P}} \mid E^{\hat{Q}}[\hat{Z}_i] = 0\}$ *contains at least one inner point, where we consider the relative topology of the usual topology on* $\mathbf{R}^N$ *to the hyperplane* $\hat{\mathcal{P}}$.

Let us define $\Gamma \in \mathbf{R}^{M \times M}$ by

$$\Gamma = \{\gamma = (\gamma_{ii'})_{i,i'=1,2,\ldots,M} \mid \gamma_{ii'} = E^{\hat{Q}}[\hat{Z}_i \hat{Z}_{i'}], \\ \text{for some } \hat{Q} \in \hat{\mathcal{Q}} \cap \bigcap_{i=1}^M \{\hat{Q} \in \hat{\mathcal{P}} \mid E^{\hat{Q}}[\hat{Z}_i] = 0\}.\}.$$

Note that $\gamma \in \Gamma$ is nonnegative definite. Also we can easily see that Γ is compact with respect to the usual topology on $\mathbf{R}^{M \times M}$.

Our main theorem is the following.

Theorem 1.5.
We have $\lim_{n\to\infty} \sup_{\xi \in \mathcal{SF}} \eta(V_n^{(n)}(v,\xi) + f(S_n^{(n)}) | \{\mathcal{F}_k\}_{k=0}^n) = v + U(0, S_0)$, *where* $U : [0,T] \times [0,\infty)^M \to \mathbf{R}$ *is the unique viscosity solution of the following Hamilton-Jacobi-Bellman equation*:

$$\begin{aligned} &\frac{\partial U}{\partial t} + \inf_{\gamma = (\gamma_{ii'})_{i,i'} \in \Gamma} \sum_{i,i'=1}^M \frac{1}{2} \gamma_{ii'} x_i x_{i'} \frac{\partial^2 U}{\partial x_i x_{i'}} = 0, \\ &U(T, x) = f(x), \quad x \in [0,\infty)^M, \end{aligned} \tag{13}$$

satisfying $U(t,\cdot) \in \hat{C}([0,\infty)^M : \mathbf{R})$ *for each* $t \in [0,T]$.

We give a proof of Theorem 1.5 in Section 4.

Example 1.6. *We consider the case where* $M = 1, N = 3$, *and* $\eta = \eta_\alpha,\ \alpha \in (0,1)$. *We also assume that* $E^{\hat{P}}[\hat{Z}] = \sum_{j=1}^{3} p_j z_j = 0$. *In this case, we have* $\hat{\mathcal{Q}} = \{\hat{Q} \in \hat{\mathcal{P}} \mid \hat{q}_j = \hat{Q}[\hat{Z} = z_j] \le p_j/\alpha,\ j = 1,2,3\}$.

We see that the condition " $\hat{Q} \in \hat{\mathcal{Q}} \cap \{\hat{Q} \in \hat{\mathcal{P}} \mid E^{\hat{Q}}[\hat{Z}] = 0\}$ *" is equivalent to the following*:

$$\hat{q}_1 \in [\underline{q}, \overline{q}], \quad \hat{q}_2 = -\frac{z_3}{z_2 - z_3} - \frac{z_1 - z_3}{z_2 - z_3}\hat{q}_1, \quad \hat{q}_3 = \frac{z_2}{z_2 - z_3} + \frac{z_1 - z_2}{z_2 - z_3}\hat{q}_1,$$

where

$$\underline{q} = 0 \vee \frac{-\alpha z_3 - p_2(z_2 - z_3)}{\alpha(z_1 - z_3)} \vee \frac{-z_2}{z_1 - z_2},$$
$$\overline{q} = \frac{p_1}{\alpha} \wedge \frac{-z_3}{z_1 - z_3} \wedge \frac{-\alpha z_2 + p_3(z_2 - z_3)}{\alpha(z_1 - z_2)}.$$

Also we have $\Gamma = [\underline{q}(z_1 - z_2)(z_1 - z_3) - z_2 z_3,\ \overline{q}(z_1 - z_2)(z_1 - z_3) - z_2 z_3]$. *We can easily see that* $\underline{q} < \overline{q}$. *Hence Assumption* 1.4 *holds.*

2 Result on a Law invariant coherent value measure

We consider a general probability space in this section. Let $(\Omega, \mathcal{F}, P)$ be a probability space, and $\mathcal{P}$ be the set of probability measures on $(\Omega, \mathcal{F})$ that are absolutely continuous with respect to P. We denote by $\mathcal{M}$ the set of all probability measures on $([0,1], \mathcal{B}[0,1])$.

We consider a coherent value measure $\eta : L^\infty(\Omega) \to \mathbf{R}$ that has the following form:

$$\eta(X) = \inf_{\mu \in \mathcal{M}_0} \int_0^1 \eta_\alpha(X)\mu(d\alpha), \quad X \in L^\infty,\ \mathcal{M}_0 \subset \mathcal{M}, \tag{14}$$

where

$$\begin{aligned}
\eta_\alpha(X) = &\ \frac{1}{\alpha}\{E[X 1_{\{X \le q^\alpha(X)\}}] + q^\alpha(X)(\alpha - P[X \le q^\alpha(X)])\}, \\
\eta_0(X) &= \text{ess.inf}(X), \\
q^\alpha(X) &= \inf\{x \in \mathbf{R} \mid P[X \le x] > \alpha\}, \quad \alpha \in [0,1), \quad \alpha \in [0,1), \\
q^1(X) &= \text{ess.sup}(X).
\end{aligned} \tag{15}$$

$\eta_\alpha(X)$ has various definitions. We can also define $\eta_\alpha(X)$ by

$$\eta_\alpha(X) = \frac{1}{\alpha} E[X \wedge q^\alpha(X)] + (1 - \frac{1}{\alpha}) q^\alpha(X). \tag{16}$$

We can easily see that the both definitions are equivalent by direct calculation (see also [3]).

Let $\Phi_\mu : [0,1] \to [0,1],\ \mu \in \mathcal{M}$ be a mapping defined by

$$\begin{aligned} &\Phi_\mu(x) = \int_{[1-x,1]} (1 - \frac{1-x}{\alpha}) \mu(d\alpha), \quad x \in [0,1), \\ &\Phi(1) = 1. \end{aligned} \tag{17}$$

We can easily see that $0 \leq \Phi(x) \leq x$, since $\{1-(1-x/\alpha)\} \leq \{1-(1-x)\} = x$ holds for any $\alpha \in (0,1]$.

Our main result in this section is the following.

Proposition 2.1. *we have* $\eta(X) = \inf_{Q \in \mathcal{Q}_0} E^Q[X],\ X \in L^\infty$, *where*

$$\begin{aligned} &\mathcal{Q}_0 = \overline{\text{conv}\ (\bigcup_{\mu \in \mathcal{M}_0} \mathcal{Q}_\mu)}, \\ &\mathcal{Q}_\mu = \{Q \in \mathcal{P} \mid Q[A] \geq \Phi_\mu(P[A]),\ A \in \mathcal{F}\}, \quad \mu \in \mathcal{M}. \end{aligned} \tag{18}$$

Here $\overline{\text{conv}}$ means the closed convex hull in $L^1(\Omega, \mathcal{F}, P)$. We show a brief lemma to prove Proposition 2.1.

Lemma 2.2. *Let* $X \in L^\infty,\ a \geq \text{ess.inf}\, X$, *and* $Q \in \mathcal{P}$. *Then we have*

$$\int_{\text{ess.inf}\, X}^a Q[X > x] dx = E^Q[X \wedge a] - \text{ess.inf}\, X.$$

Proof. Using Fubini's theorem, we have

$$\begin{aligned} &\int_{\text{ess.inf}\, X}^a Q[X > x] dx = \int_{\text{ess.inf}\, X}^a (\int_\Omega 1_{\{X > x\}}(\omega) Q(d\omega)) dx \\ &= \int_\Omega (\int_{\text{ess.inf}\, X}^{X(\omega) \wedge a} dx) Q(d\omega) = E^Q[X \wedge a] - \text{ess.inf}\, X. \end{aligned}$$

□

Now we prove Proposition 2.1. Let $\mathcal{A} = \{X \in L^\infty \mid \eta(X) \geq 0\}$. Theorem 6 in [7] shows that $\eta(X) = \inf_{Q \in \mathcal{Q}} E^Q[X]$ holds for $X \in L^\infty$, where $\mathcal{Q} = \{Q \in \mathcal{P} \mid E^Q[X] \geq 0,\ X \in \mathcal{A}\}$. So it is sufficient to show that $\mathcal{Q} = \mathcal{Q}_0$. Also it is sufficient to show the assertion in the case where $\eta(X) = \int_0^1 \eta_\alpha(X)\mu(d\alpha),\ X \in L^\infty,\ m \in \mathcal{M}$. Indeed, if the claim holds in this case, we see that

$$\begin{aligned}\mathcal{A} &= \bigcap_{\mu \in \mathcal{M}_0} \{X \in L^\infty \mid \int_0^1 \eta_\alpha(X)\mu(d\alpha) \geq 0\} \\ &= \bigcap_{\mu \in \mathcal{M}_0} \{X \in L^\infty \mid E^Q[X] \geq 0, \quad Q \in \mathcal{Q}_\mu\} \\ &= \{X \in L^\infty \mid E^Q[X] \geq 0, \quad Q \in \mathcal{Q}_0\}.\end{aligned}$$

by bipolar theorem. Then using bipolar theorem again, we have $\mathcal{Q} = \mathcal{Q}_0$.

So we consider the case where $\eta(X) = \int_0^1 \eta_\alpha(X)\mu(d\alpha), \quad X \in L^\infty,\ \mu \in \mathcal{M}$. We show that $\mathcal{Q} \subset \mathcal{Q}_\mu$. Take $Q \in \mathcal{Q}$. Since $1_A \in \mathcal{A}$ for $A \in \mathcal{F}$, we have $Q[A] \geq \eta(1_A) = \int_0^1 \eta_\alpha(1_A)\mu(d\alpha)$. We see that

$$\eta_\alpha(1_A) = \begin{cases} 0, & 0 \leq \alpha < P[A^c] \\ 1 - (P[A^c]/\alpha), & P[A^c] \leq \alpha \leq 1, \end{cases}$$

by direct calculations. Hence we have $Q \in \mathcal{Q}_\mu$.

Next we show that $\mathcal{Q}_\mu \subset \mathcal{Q}$. Take $Q \in \mathcal{Q}_\mu$. Then

$$\int_{\text{ess.inf}\, X}^{\text{ess.sup}\, X} Q[X > x]dx \geq \int_{\text{ess.inf}\, X}^{\text{ess.sup}\, X} \{\int_{[P[X \leq x],1]} (1 - \frac{P[X \leq x]}{\alpha})\mu(d\alpha)\}dx,$$

holds for $X \in L^\infty$. We can easily see that the left term equals $E^Q[X] - \text{ess.inf}\, X$ by Lemma 2.2. We calculate the right term. Note that $P[X \leq x] > 0$, for any $x > \text{ess.inf}\, X$. We see that

$$\begin{aligned}&\int_{(\text{ess.inf}\, X,\ \text{ess.sup}\, X]} \{\int_{[P[X \leq x],1]} (1 - \frac{P[X \leq x]}{\alpha})\mu(d\alpha)\}dx \\ &= \int_{(0,1]} \{\int_{\text{ess.inf}\, X}^{q^\alpha(X)} (1 - \frac{P[X \leq x]}{\alpha})dx\}\mu(d\alpha) \\ &= \int_{(0,1]} \{\frac{1}{\alpha}E[X \wedge q^\alpha(X)] + (1 - \frac{1}{\alpha})q^\alpha(X)\}\mu(d\alpha) \\ &\quad - (1 - \mu[\{0\}])\, \text{ess.inf}\, X \\ &= \int_{[0,1]} \eta_\alpha(X)\mu(d\alpha) - \text{ess.inf}\, X,\end{aligned}$$

by Fubini's theorem and Lemma 2.2. Hence we have

$$E^Q[X] \geq \int_{[0,1]} \eta_\alpha(X)\mu(d\alpha) = \eta(X) \geq 0, \quad X \in \mathcal{A},$$

and this implies that $Q \in \mathcal{Q}$. This completes the proof.

3 Discrete Model

From now to the end of this paper, we use the following notation:

$$xy = (x_1y_1, x_2y_2, \ldots, x_My_M) \in \mathbf{R}^M, \tag{19}$$

for $x = (x_1, x_2, \ldots, x_M),\ y = (y_1, y_2, \ldots, y_M) \in \mathbf{R}^M$.

We consider some maximization problem on hedging with a homogeneous filtered value measure in a discrete time market model. The following setting is parallel to that of Section 1. Let $(\Omega, \mathcal{F}, P)$ be a standard probability space with a filtration $\{\mathcal{F}_k\}_{k=0,1,\ldots,n}$. Also, let be $Y_k = (Y_{1,k}, Y_{2,k},\ \ldots,\ Y_{M,k}),\ \ k = 1, 2, \ldots, n$ be identically distributed M-dimensional random variables such that Y_k is $\{\mathcal{F}_k\}$-measurable and independent of $\mathcal{F}_l,\ \ l = 1, 2, \ldots, k-1$. We assume that $Y_{i,k}(\omega) > 0,\ \ \omega \in \Omega,\ \underline{Y} = \min_{i=1,2,\ldots,M} \operatorname{ess.inf}_\omega Y_{i,1}(\omega) > 0$, and $\overline{Y} = \max_{i=1,2,\ldots,M} \operatorname{ess.sup}_\omega Y_{i,1}(\omega) < \infty$. We denote by $\mathcal{P}$ the set of all probability measures on $(\Omega, \mathcal{F})$ which are absolutely continuous with respect to P.

We define M dimensional $\{\mathcal{F}_k\}$-adapted process $\{S_k\}_{k=1,2,\ldots,n}$ by

$$\begin{aligned} S_k &= (S_{1,k}, S_{2,k},\quad \ldots,\ S_{M,k}), \\ S_{i,k} &= S_{i,0} \prod_{l=1}^{k} Y_{i,l}, \quad i = 1, 2, \ldots M, \end{aligned} \tag{20}$$

where $S_0 = (S_{1,0}, S_{2,0}, \ldots, S_{M,0}) \in (0, \infty)^M$ is a constant vector.

We call an M dimensional $\{\mathcal{F}_k\}$-predictable process $\xi = (\xi_k)_{k=1,2,\ldots}$ a self-financing strategy, and denote by $\mathcal{SF}$ the set of all self-financing strategy. Then we define a random variable $V_k(v, \xi)$ for $v \in (0, \infty),\ \xi \in \mathcal{SF}$ by $V_0(v, \xi) = v$ and $V_k(v, \xi) = v + \sum_{l=1}^{k} \xi_l \cdot (S_l - S_{l-1}),\ k \in \mathbf{N}$.

As in section 1, we use the following type of mild value measure:

$$\eta(\nu) = \int_0^1 \eta_\alpha(\nu)\mu(d\alpha), \quad \nu \in \mathcal{L}^\infty,\ \mu \in \mathcal{M}, \tag{21}$$

and fix it. Let $\Phi_\mu : [0,1] \to [0,1],\ \mu \in \mathcal{M}$ be a mapping defined by (17). We also define

$$\mathcal{Q} = \{Q \in \mathcal{P} \mid E^Q[X] \geq 0 \text{ for all } X \in L^\infty(\Omega, \mathcal{F}_1, P) \\ \text{such that } \eta(\nu_X) \geq 0 \text{ hold } \}.$$

We assume the following.

Assumption 3.1. $\mathcal{Q} \cap \bigcap_{i=1}^M \{Q \in \mathcal{P} \mid E^Q[Y_{i,1}] = 1\} \neq \emptyset$.

Let us denote $\eta(\nu_X)$ by $\bar{\eta}(X)$ for $X \in L^\infty(\Omega, \mathcal{F}, P)$, where ν_X is the distribution of X. Obviously we have $\bar{\eta}(X) = \inf_{Q \in \mathcal{Q}} E^Q[X],\ X \in L^\infty(\Omega, \mathcal{F}, P)$. Also we see that $\mathcal{Q} = \{Q \in \mathcal{P} \mid Q[A] \geq \Phi_\mu(P[A]),\ A \in \mathcal{F}\}$ by Proposition 2.1. We denote

$$\psi_g(x,y) = \bar{\eta}((xy) \cdot (Y_1 - \mathbf{1}) + g(xY_1)),\ \mathbf{1} = (1,1,\ldots,1) \in \mathbf{R}^M, \tag{22}$$

for $x \in [0,\infty)^M,\ y \in \mathbf{R}^M$, and $g \in \hat{C}([0,\infty)^M : \mathbf{R})$. Since

$$\begin{aligned}&|\{(xy) \cdot (Y_1 - \mathbf{1}) + g(xY_1)\} - \{(x'y') \cdot (Y_1 - \mathbf{1}) + g(x'Y_1)\}| \\ &\leq K\{(\overline{Y} + 1)|xy - x'y'| + \overline{Y}|x - x'|\},\end{aligned}$$

$x, x' \in [0,\infty)^M,\ y, y' \in \mathbf{R}^M$, for some $K > 0$, and $\bar{\eta}$ is monotone, i.e., $\bar{\eta}(X) \leq \bar{\eta}(X')$, for $X, X' \in L^\infty(\Omega)$ such that $X \leq X'$, then we have

$$|\psi_g(x,y) - \psi_g(x',y')| \leq K\{(\overline{Y} + 1)|xy - x'y'| + \overline{Y}|x - x'|\}.$$

Hence $(x,y) \mapsto \psi_g(x,y)$ is continuous. Next we define

$$\phi_g(x) = \sup_{y \in \mathbf{R}^M} \psi_g(x,y). \tag{23}$$

Note that $\phi_g(x) < +\infty,\ x \in [0,\infty)^M$. Indeed, there exists $\bar{Q} \in \mathcal{Q} \cap \bigcap_{i=1}^M \{Q \in \mathcal{P} \mid E^Q[Y_{i,1}] = 1\}$ by Assumption 3.1, and then we see that $\phi_g(x) \leq E^{\bar{Q}}[g(xY_1)] \leq K(1 + |x|^{2m})$, for some $K > 0$. Also we have

$$\begin{aligned}\phi_g(x) &= \sup_{y \in \mathbf{R}^M} \inf_{Q \in \mathcal{Q}} E^Q[(xy) \cdot (Y_1 - \mathbf{1}) + g(xY_1)] \\ &= \inf_{Q \in \mathcal{Q}} \sup_{y \in \mathbf{R}^M} E^Q[(xy) \cdot (Y_1 - \mathbf{1}) + g(xY_1)] \\ &= \inf_{Q \in \mathcal{Q} \cap \bigcap_{i=1}^M \{Q \in \mathcal{P} \mid E^Q[Y_{i,1}] = 1\}} E^Q[g(xY_1)],\end{aligned}$$

by Takahashi's Minimax Theorem in [16]. Since $|g(xY_1)-g(x'Y_1)| \leq K\overline{Y}|x-x'|$ and $|g(xY_1)| \leq K(1+\overline{Y}^{2m}|x|^{2m})$ holds for $x, x' \in [0,\infty)^M$ and $g \in \hat{C}([0,\infty)^M : \mathbf{R})$, where $K>0$ is a constant which is independent of x, x', Q, the mapping $\phi_g : [0,\infty)^M \to \mathbf{R}$ belongs to $\hat{C}([0,\infty)^M : \mathbf{R})$. Then we can define an operator H on $\hat{C}([0,\infty)^M : \mathbf{R})$ by $Hg = \phi_g$. Also we inductively define $H^k g = H(H^{k-1}g),\ k=1,2,\dots$ and $H^0 g = g$ for $g \in \hat{C}([0,\infty)^M : \mathbf{R})$.

Now we fix $n \in \mathbf{N}$. We define random variables $L_k(v,\xi),\ k=0,1,\dots,n$, $v \in (0,\infty)$, and $\xi \in \mathcal{SF}$ inductively by $L_n(v,\xi) = V_n(v,\xi)+f(S_n)$ and

$$L_{k-1}(v,\xi) = \eta(L_k(v,\xi)|\mathcal{F}_{k-1}), \quad k=n,n-1,\dots,1, \tag{24}$$

where $f \in \hat{C}([0,\infty)^M ; \mathbf{R})$. Obviously $L_k(v,\xi),\ k=0,1,\dots,n$ equals the homogeneous value measure for $V_n(v,\xi)+f(S_n)$ at k.

We prove the following theorem, which is our main result in this section.

Theorem 3.2 (Bellman's Principle)**.** *We have*

$$\begin{aligned}\sup_{\xi' \in \mathcal{SF}} & L_k(v,(\xi_1,\xi_2,\dots,\xi_k,{\xi'}_{k+1},{\xi'}_{k+2},\dots,{\xi'}_n))) \\ &= V_k(v,\xi)+H^{n-k}f(S_k), \quad k=0,1,\dots,n,\end{aligned}$$

for any $v \in (0,\infty)$ and $\xi \in \mathcal{SF}$.

We show a lemma to prove Theorem 3.2.

Lemma 3.3. *For any $g \in \hat{C}([0,\infty)^M : \mathbf{R})$ and $\varepsilon > 0$, there exists a Borel measurable function $\gamma^\varepsilon(x) = \gamma_g^\varepsilon(x) = (\gamma_1^\varepsilon(x),\gamma_2^\varepsilon(x),\dots,\gamma_M^\varepsilon(x))$ on $[0,\infty)^M$ such that $Hg(x)-\varepsilon \leq \bar{\eta}((x\gamma^\varepsilon(x))\cdot(Y_1-\mathbf{1})+g(xY_1))$, $x \in [0,\infty)^M$ holds.*

Proof. First we define a multivalued mapping $\Gamma^\varepsilon : ([0,\infty)^M, \mathcal{B}[0,\infty)^M) \Rightarrow (\mathbf{R}^M, \mathcal{B}(\mathbf{R}^M))$ by $\Gamma^\varepsilon(x) = \{y \in \mathbf{R}^M \mid Hg(x)-\varepsilon \leq \psi_g(x,y)\}$. Obviously the set on the right side is nonempty, so it is sufficient to show that this multivalued mapping is measurable, i.e.,

$$\Gamma^{-w,\varepsilon}(A) = \{x \in [0,\infty)^M \mid \Gamma^\epsilon(x)\cap A \neq \emptyset\} \in \mathcal{B}[0,\infty)^M, \tag{25}$$

for any closed set $A \subset \mathbf{R}^M$. If Γ^ε is measurable, there exists a measurable selection $\gamma^\varepsilon(x) \in \Gamma^\varepsilon(x)$ and this mapping satisfies the condition. Also, we may assume that A is compact. Indeed, if $\Gamma^{-w,\varepsilon}(A') \in \mathcal{B}[0,\infty)^M$ for any compact set A', we see that $\Gamma^{-w,\varepsilon}(A) = \bigcup_{m=1}^\infty \Gamma^{-w,\varepsilon}(A\cap[-m,m]^M) \in \mathcal{B}[0,\infty)^M$.

We show that $\Gamma^{-w,\varepsilon}(A)$ is closed for any compact set A. Take a sequence $(x_m)_{m\in\mathbf{N}}$ of $\Gamma^{-w,\varepsilon}(A)$ such that $\lim_{m\to\infty} x_m = x \in [0,\infty)^M$. Then there exists a $y_m \in \Gamma^\varepsilon(x_m)\cap A$ for each m. Since A is compact, we can choose a subsequence $(y_{m(l)})_{l\in\mathbf{N}}$ of $(y_m)_{m\in\mathbf{N}}$ such that $y_{m(l)}$ converges to some $y \in A$ as $l\to\infty$. Taking $\lim_{l\to\infty}$ in both sides of the equation $Hg(x_{m(l)}) - \varepsilon \leq \psi_g(x_{m(l)}, y_{m(l)})$, we have $Hg(x)-\varepsilon \leq \psi_g(x,y)$ because ψ_g and Hg are continuous. This implies that $y\in\Gamma^\varepsilon(x)\cap A$ and $x\in\Gamma^{-w,\varepsilon}(A)$. $\square$

Now we give the proof of Theorem 3.2. First we show that

$$L_k(v,\xi) \leq V_k(v,\xi) + H^{n-k}f(S_k),$$

by mathematical induction on k. Obviously, the claim holds when $k=n$. Suppose that the claim holds for some k, Then we have

$$\begin{aligned}
&L_{k-1}(v,\xi) = \eta(L_k(v,\xi)|\mathcal{F}_{k-1})\\
&\leq \eta(V_k(v,\xi) + H^{n-k}f(S_k)|\mathcal{F}_{k-1})\\
&= V_{k-1}(v,\xi) + \eta((S_{k-1}\xi_k)\cdot(Y_k-\mathbf{1}) + H^{n-k}f(S_{k-1}Y_k)|\mathcal{F}_{k-1})\\
&\leq V_{k-1}(v,\xi) + H^{n-(k-1)}f(S_{k-1}).
\end{aligned}$$

Hence we have the claim.

Next we show that there exists $\xi^\varepsilon \in \mathcal{SF}$ for $\varepsilon>0$ such that

$$\begin{aligned}
&V_k(v,\xi) + H^{n-k}f(S_k) - \frac{k\varepsilon}{n}\\
&\leq L_k(v,(\xi_1,\xi_2,\dots,\xi_k,\xi^\varepsilon_{k+1},\xi^\varepsilon_{k+2},\dots,\xi^\varepsilon_n)), \quad k=n,n-1,\dots,1
\end{aligned}$$

for any $\xi\in\mathcal{SF}$. Applying Lemma 3.3 for $g=f$, we see that there exists some Borel measurable function γ^ε on $[0,\infty)^M$ such that

$$(Hf)(x) - \frac{\varepsilon}{n} \leq \bar{\eta}((x\gamma^\varepsilon(x))\cdot(Y_1-\mathbf{1}) + f(xY_1)),$$

for $x\in[0,\infty)^M$. Then, $\xi^\varepsilon_n = \gamma^\varepsilon(S_{n-1})$ is an $\{\mathcal{F}_{n-1}\}$-measurable random variable such that

$$\begin{aligned}
&V_{n-1}(v,\xi) + (Hf)(S_{n-1}) - \frac{\varepsilon}{n}\\
&\leq V_{n-1}(v,\xi) + \bar{\eta}((x\gamma^\varepsilon(x))\cdot(Y_1-\mathbf{1}) + f(xY_1))|_{x=S_{n-1}}\\
&\leq V_{n-1}(v,\xi) + \eta((S_{n-1}\xi^\varepsilon_n)\cdot(Y_n-\mathbf{1}) + f(S_{n-1}Y_n)|\mathcal{F}_{n-1})\\
&\leq \eta(V_n(v,\xi) + f(S_n)|\mathcal{F}_{n-1})\\
&\leq L_{n-1}(v,\xi_1,\xi_2,\dots,\xi_{n-1},\xi^\varepsilon_n).
\end{aligned}$$

Using induction, we can construct ξ^ε such that the condition holds by the same way. This completes the proof of Theorem 3.2.

4 Proof of Theorem 1.5

4.1 Preparations

Let $(\Omega, \mathcal{F}, P)$ be the probability space defined in Section 1.2. For $k \in \mathbf{N}$, we denote by $\mathcal{L}_k$ the set of measurable mappings $\rho : \Omega \times \mathcal{Z} \to \mathbf{R}$ which satisfy the following:

$$\rho(\cdot, z) \text{ is } \{\mathcal{F}_{k-1}\} \text{ measurable for each } z \in \mathcal{Z},$$
$$\sum_{j=1}^{N} \rho(\omega, z_j) p_j = 1, \ \omega \in \Omega,$$
$$\sum_{j \in J} \rho(\cdot, z_j) p_j \geq \Phi_\mu(\sum_{j \in J} p_j), \ J \subset \{1, 2, \ldots, M\}, \ z_j \in \mathcal{Z}, \ j \in J.$$

Also we denote by $\mathcal{Q}^{(n)}, \ n \in \mathbf{N}$ the set of probability measures $Q \in \mathcal{P}$ which satisfy the following:

$$E[\frac{dQ}{dP}|\mathcal{F}_n] = \prod_{k=1}^{n} \rho_k(\cdot, Z_k), \ \rho_k \in \mathcal{L}_k,$$
$$\int_\Omega \rho_k(\omega', Z_k(\omega)) Y_{i,k}^{(n)}(\omega) P(d\omega) = 1, \quad \omega' \in \Omega, \ k = 1, 2, \ldots, n.$$

Let us denote $X_{i,k}^{(n)} = \prod_{l=1}^{k} Y_{i,l}^{(n)}$ for $n \in \mathbf{N}, \ i = 1, 2, \ldots, M$, and $k = 1, 2, \ldots, n$. We also denote $X_k^{(n)} = (X_{1,k}^{(n)}, X_{2,k}^{(n)} \ldots X_{M,k}^{(n)}), \ k = 1, 2, \ldots, n$. We define $a_{ij}^{(n)} = \exp(z_{ij}\sqrt{T/n} + b_i T/n), \quad i = 1, 2, \ldots, M, \ j = 1, 2, \ldots, N,$ where $z_j = (z_{1j}, z_{2j}, \ldots, z_{Mj}) \in \mathcal{Z}$.

Our purpose in this subsection is to prove the following.

Lemma 4.1.
(1) $(H^{(n)})^k g(x) = \inf_{Q \in \mathcal{Q}^{(n)}} E^Q[g(xX_k^{(n)})], \quad n \in \mathbf{N}, \ g \in \hat{C}([0, \infty)^M : \mathbf{R})$,
where $H^{(n)}, \ n \in \mathbf{N}$ *are operators on* $\hat{C}([0, \infty)^M : \mathbf{R})$ *that correspond to* H *in Section* 3.
(2) $\sup_{n \in \mathbf{N}} \sup_{Q \in \mathcal{Q}^{(n)}} E^Q[\max_{k=1,2,\ldots,n} |X_k^{(n)}|^{2m}] < \infty, \quad m \in \mathbf{N}$.
(3) *There exists a positive number* L*, which only depends on* M*, such that* $E^Q[|X_{k+l}^{(n)} - X_k^{(n)}|^4] \leq L(lT/n)^2$ *holds for any* $k, l = 0, 1, \ldots, n, \ k + l \leq n$*, and* $Q \in \mathcal{Q}^{(n)}$.

Proof. We show assertion (1). We see by Proposition 2.1 that $\bar{\eta}(X) = \inf_{Q \in \overline{\mathcal{Q}}_1^{(n)}} E^Q[X]$, for $X \in L^\infty(\Omega, \mathcal{F}_1, P)$, where

$$\begin{aligned}\overline{\mathcal{Q}}_1^{(n)} &= \{Q \in \mathcal{P} \mid Q[A] \geq \Phi_\mu(P[A]),\ A \in \mathcal{F}_1\} \\ &= \{Q \in \mathcal{P} \mid E[\frac{dQ}{dP}|\mathcal{F}_1] = \rho_1(\cdot, Z_1) \in \mathcal{L}_1\}\end{aligned} \tag{26}$$

Then we have $H^{(n)}g(x) = \inf_{Q \in \mathcal{Q}_1^{(n)}} E^Q[g(xY_1^{(n)})]$, where $\mathcal{Q}_1^{(n)} = \overline{\mathcal{Q}}_1^{(n)} \cap \bigcap_{i=1}^M \{Q \in \mathcal{P} \mid E[\rho_1(\omega', Z_1) Y_{i,1}^{(n)}] = 1,\ \omega' \in \Omega\}$, by a way similar to that in Section 3. We can easily see that $\mathcal{Q}^{(n)} \subset \mathcal{Q}_1^{(n)}$. We show the inverse implement. Take $Q \in \mathcal{Q}_1^{(n)}$ and define $\tilde{Q} \in \mathcal{P}^{(n)}$ by $E[d\tilde{Q}/dP|\mathcal{F}_n] = \prod_{k=1}^n \tilde{\rho}_k(\cdot, Z_k)$, where $\tilde{\rho}_k(\omega, z) = E[\frac{dQ}{dP}|\mathcal{F}_1](\omega),\ \omega \in \Omega,\ z \in \mathcal{Z},\ k = 1, 2, \ldots, n$. Then we see that $\tilde{Q} \in \mathcal{Q}^{(n)}$ and $E^Q[g(xY_1^{(n)})] = E^{\tilde{Q}}[g(xY_1^{(n)})]$. This implies that $H^{(n)}g(x) = \inf_{Q \in \mathcal{Q}^{(n)}} E^Q[g(xY_1^{(n)})]$.

Suppose that $(H^{(n)})^l g(x) = \inf_{Q \in \mathcal{Q}^{(n)}} E^Q[g(xX_l^{(n)})],\ \ l \leq k,\ g \in \hat{C}([0,\infty)^M)$ holds for some $k \in \{1, 2, \ldots, n-1\}$. First we show that

$$(H^{(n)})^{k+1} g(x) \leq \inf_{Q \in \mathcal{Q}^{(n)}} E^Q[g(xX_{k+1}^{(n)})]. \tag{27}$$

Take $Q \in \mathcal{Q}^{(n)}$ and $\rho_l \in \mathcal{L}_l,\ l \leq n$ such that $E[dQ/dP|\mathcal{F}_n] = \prod_{l=1}^n \rho_l(\cdot, Z_l)$ holds. We define $Q^{\tilde{\omega}} \in \mathcal{P}$ for each $\tilde{\omega} \in \Omega$ by $E[dQ^{\tilde{\omega}}/dP|\mathcal{F}_n] = \prod_{l=1}^n \rho'_l(\cdot, Z_l)$, where $\rho'_l(\omega, z) = \rho'_{k+1}(\tilde{\omega}, z),\ \omega \in \Omega,\ z \in \mathcal{Z}$. Then we see that $Q^{\tilde{\omega}} \in \mathcal{Q}^{(n)}$ and

$$\begin{aligned}&E^Q[g(xX_{k+1}^{(n)})|\mathcal{F}_k](\tilde{\omega}) = E^Q[g(yY_{k+1}^{(n)})|\mathcal{F}_k](\tilde{\omega})\big|_{y = xX_k^{(n)}} \\ &= E^{Q^{\tilde{\omega}}}[g(yY_1^{(n)})]\big|_{y = xX_k^{(n)}} \geq H^{(n)} g(xX_k^{(n)}).\end{aligned}$$

Also we have

$$\begin{aligned}&E^Q[g(xX_{k+1}^{(n)})] = E^Q[E^Q[g(xX_{k+1}^{(n)})|\mathcal{F}_k]] \\ &\geq E^Q[H^{(n)} g(xX_k^{(n)})] \geq (H^{(n)})^k H^{(n)} g(x) = (H^{(n)})^{k+1} g(x).\end{aligned}$$

This implies that (27) holds.

Next we show that

$$(H^{(n)})^{k+1} g(x) \geq \inf_{Q \in \mathcal{Q}^{(n)}} E^Q[g(xX_{k+1}^{(n)})]. \tag{28}$$

Take $\varepsilon > 0$. Then we see by assumption that there exist $\bar{Q}^0, \bar{Q}^j \in \mathcal{Q}^{(n)}$ $j = 1, 2, \ldots, N$ such that

$$(H^{(n)})^{k+1} g(x) \geq E^{\bar{Q}^0}[(H^{(n)})^k g(xY_1^{(n)})] - \frac{\varepsilon}{2},$$
$$(H^{(n)})^k g(xa_j^{(n)}) \geq E^{\bar{Q}^j}[g(xa_j^{(n)} X_k^{(n)})] - \frac{\varepsilon}{2}. \tag{29}$$

Take $\rho_l^0, \rho_l^j \in \mathcal{L}_l,\ l = 1, 2, \ldots, n,\ j = 1, 2, \ldots, M$, such that $E[\frac{d\bar{Q}^0}{dP}|\mathcal{F}_n] = \prod_{l=1}^n \bar{\rho}_l^0(\cdot, Z_l)$, $E[\frac{d\bar{Q}^j}{dP}|\mathcal{F}_n] = \prod_{l=1}^n \bar{\rho}_l^j(\cdot, Z_l)$ hold. Then we define $\bar{Q} \in \mathcal{P}$ by $E[\frac{d\bar{Q}}{dP}|\mathcal{F}_n] = \prod_{l=1}^n \bar{\rho}_k(\cdot, Z_k)$, where

$$\bar{\rho}_1 = \bar{\rho}_1^0,$$
$$\bar{\rho}_l = \sum_{j=1}^N \bar{\rho}_{l-1}^j 1_{\{Z_1 = z_j\}},\ l = 2, 3, \ldots, n. \tag{30}$$

We see that $\bar{Q} \in \mathcal{Q}^{(n)}$ and

$$(H^{(n)})^{k+1} g(x) \geq E^{\bar{Q}}[g(xX_{k+1}^{(n)})] - \varepsilon \geq \inf_{Q \in \mathcal{Q}^{(n)}} E^Q[g(xX_{k+1}^{(n)})] - \varepsilon. \tag{31}$$

Since $\varepsilon > 0$ is arbitrary, we have (28). This shows the assertion (1).

Next we show the assertion (2). Since $X_{i,n}^{(n)}$ is martingale under $Q \in \mathcal{Q}^{(n)}$, it is sufficient to show that $\sup_{n \in \mathbf{N}} \sup_{Q \in \mathcal{Q}^{(n)}} E^Q[|X_{i,n}^{(n)}|^{2m}] < \infty$, for each $i = 1, 2, \ldots, M$. Take $Q \in \mathcal{Q}^{(n)}$ and fix $\omega \in \Omega$. We denote $q_{\omega,k,j} = Q[Z_k = z_j|\mathcal{F}_{k-1}](\omega),\ \ k = 1, 2, \ldots, n,\ j = 1, 2, \ldots, N$. We see that

$$\begin{aligned}
1 = E^Q[Y_{i,k}^{(n)}|\mathcal{F}_{k-1}](\omega) &= \sum_{j=1}^N q_{\omega,k,j}(a_{il}^{(n)}) \\
&= 1 + E^Q[Z_{i,k}|\mathcal{F}_{k-1}](\omega)\sqrt{\frac{T}{n}} + b_i\frac{T}{n} \\
&\quad + \sum_{j=1}^N q_{\omega,k,j}\{\exp(z_{ij}\sqrt{\frac{T}{n}} + b_i\frac{T}{n}) - (1 + z_{ij}\sqrt{\frac{T}{n}} + b_i\frac{T}{n})\}.
\end{aligned}$$

Then we have

$$E^Q[|Y_{i,k}^{(n)}|^{2m}|\mathcal{F}_{k-1}](\omega)$$

$$
\begin{aligned}
&= 1 + 2mE^Q[Z_{i,k}|\mathcal{F}_{k-1}](\omega)\sqrt{\frac{T}{n}} + 2mb_i\frac{T}{n} \\
&\quad + \sum_{j=1}^{N} q_{\omega,k,j}\{\exp\big(2mz_{ij}\sqrt{\frac{T}{n}} + 2mb_i\frac{T}{n}\big) - (1 + 2mz_{ij}\sqrt{\frac{T}{n}} + 2mb_i\frac{T}{n})\},
\end{aligned}
$$

and

$$
\begin{aligned}
&E^Q[|Y_{i,k}^{(n)}|^{2m}|\mathcal{F}_{k-1}](\omega) \\
&\le 1 + \sum_{j=1}^{N}\{2m|\exp\big(z_{ij}\sqrt{\frac{T}{n}} + b_i\frac{T}{n}\big) - (1 + z_{ij}\sqrt{\frac{T}{n}} + b_i\frac{T}{n})| \\
&\qquad + |\exp\big(2mz_{ij}\sqrt{\frac{T}{n}} + 2mb_i\frac{T}{n}\big) - (1 + 2mz_{ij}\sqrt{\frac{T}{n}} + 2mb_i\frac{T}{n})|\}.
\end{aligned}
$$

for $k = 1, 2, \ldots, n$. We denote by $b^{(n)}$ the right term of this inequality. Note that $b^{(n)}$ is independent of k, ω and Q. We see that $\lim_{n\to\infty} n(b^{(n)} - 1)$ exists and $|n(b^{(n)} - 1)| \le b,\ n \in \mathbf{N}$, for some $b > 0$. Then we have

$$
\begin{aligned}
&E^Q[|X_{i,n}^{(n)}|^{2m}|] \le E^Q[|X_{i,n-1}^{(n)}|^{2m} E^Q[|Y_{i,n}^{(n)}|^{2m}|\mathcal{F}_{n-1}]] \\
&\le (1 + \frac{b}{n})E^Q[|X_{i,n-1}^{(n)}|^{2m}] \le \cdots \le (1 + \frac{b}{n})^n, \quad Q \in \mathcal{Q}^{(n)}.
\end{aligned}
$$

This implies that $\limsup_{n\to\infty} \sup_{Q\in\mathcal{Q}^{(n)}} E^Q[|X_{i,n}^{(n)}|^{2m}] \le e^b$. Hence we have the assertion.

We show that assertion (3) holds. We see that there exists some $c > 0$ such that $E^Q[|Y_{i,k+1}^{(n)} - 1|^4|\mathcal{F}_k](\omega) \le c(\frac{T}{n})^2$, for each $i \le M,\ k \le n-1,\ Q$ and ω by an argument similar to that of (2). Then using the result of (2), we have $E^Q[|X_{i,k+1}^{(n)} - X_{i,k}^{(n)}|^4|] \le c(T/n)^2 E^Q[|X_{i,k}^{(n)}|^4] \le c'(T/n)^2$ for some $c' > 0$. Hence we have assertion (3) by Burkholder's Inequality. $\square$

4.2 Proposition on Limit of Value

Let $\bar{\mathcal{P}}$ be the set of probability measures $\bar{P}$ on $(C[0,\infty)^M, \mathcal{B}[C[0,\infty)^M])$ such that the following is satisfied:

> The coordinate function $w(\cdot) = (w_1(\cdot), \ldots, w_M(\cdot))$ is a positive martingale with respect to $\{\mathcal{B}_t\}_{t\in[0,T]}$ under $\bar{P}$, where we define $\{\mathcal{B}_t\}_{t\in[0,T]}$ by $\mathcal{B}_t = \sigma(w(u); u \le t), \quad 0 \le t \le T$.

$$\bar{P}(w_i(0) = 1,\ i = 1, 2, \ldots, M) = 1.$$

$$\begin{aligned}
&\langle w_i, w_{i'}\rangle_{\cdot},\ i, i' \in \{1, 2, \ldots, M\} \text{ are absolutely continuous} \\
&\text{with respect to Lebesgue measure, and} \\
&\Big(\frac{1}{w_i(u)w_{i'}(u)}\frac{d\langle w_i, w_{i'}\rangle}{dt}(u)\Big)_{i,i'=1,2,\ldots,M} \in \Gamma,\ u \in [0, T],\ \bar{P}\ a.s.
\end{aligned}$$

Our purpose in this subsection is to prove the following.

Proposition 4.2. *Take $g \in \hat{C}([0,\infty)^M : \mathbf{R})$ and an arbitrary subsequence $(\bar{n})$ of $(n)_{n\in\mathbf{N}}$.*
(1) There exist a subsequence $(\bar{n}(k))_{k\in\mathbf{N}}$ of $(\bar{n})$ and a continuous mapping $W_g : [0,T] \times [0,\infty)^M \to \mathbf{R}$ such that

$$W_g(t, x) = \lim_{k\to\infty}(H^{(\bar{n}(k))})^{\bar{n}(k)-[\bar{n}(k)t/T]}g(x),$$

for any $(t, x) \in [0, T] \times [0,\infty)^M$, where $[x]$ represents the greatest integer that is not greater than x. Also this convergence is uniform on any compact subsets on $[0, T] \times [0,\infty)^M$.
(2) $W_g(t, \cdot)$ belongs to $\hat{C}([0,\infty)^M : \mathbf{R})$ for each $t \in [0, T]$.
(3) There exists $\bar{P}_{t,x} \in \bar{\mathcal{P}}$ such that $W_g(t, x) = E^{\bar{P}_{t,x}}[g(xw(T-t))]$ holds for any $(t, x) \in [0, T] \times [0,\infty)^M$,
(4) $W_g(t, x) = \lim_{k\to\infty}(H^{(\bar{n}(k))})^{[\frac{\bar{n}(k)t'}{T}]}W_g(t + t', \cdot)(x)$, for $t, t' \in [0, T]$ with $t + t' \in [0, T]$ and $x \in [0,\infty)^M$.
(5) W_g is a viscosity subsolution of Hamilton-Jacobi-Bellman equation (13).

We need two lemmas to prove Lemma 4.2. Let us define $X(t, \omega; n) = (X_1(t, \omega; n), X_2(t, \omega; n), \ldots, X_M(t, \omega; n)),\ t \in [0, T],\ \omega \in \Omega,\ n \in \mathbf{N}$ by

$$\begin{aligned}
&X(t, \omega; n) \equiv \frac{nt - T[nt/T]}{T}X^{(n)}_{[nt/T]+1} + \frac{T([nt/T] + 1) - nt}{T}X^{(n)}_{[nt/T]},\ t \neq T, \\
&X(T, \omega; n) \equiv X^{(n)}_n, \qquad (32)
\end{aligned}$$

i.e., $X(t; n)$ is the linear interpolation of $X^{(n)}_{\cdot}$.

Lemma 4.3. *Let $g \in \hat{C}([0,\infty)^M : \mathbf{R})$. Then we have*

$$|(H^{(n)})^{k'}g(x) - (H^{(n)})^k g(x)| \leq \bar{K}|x|\sqrt{\frac{T}{n}|k' - k|},\quad k, k' \in \{0, 1, \ldots, n\},$$

for some $\bar{K} > 0$ which does not depend on $n \in \mathbf{N}$.

Proof. We may assume that $k' > k$. Let $K > 0$ be a constant such that $|g(x)| \le K(1+|x|^{2m}), |g(x)-g(x')| \le K|x-x'|,\ x, x' \in [0,\infty)^M$ holds. By virtue of Lemma 4.1(3), we have

$$|E^Q[g(xX_{k'}^{(n)})] - E^Q[g(xX_k^{(n)})]| \le E^Q[|g(xX_{k'}^{(n)}) - g(xX_k^{(n)})|]$$
$$\le K|x|E^Q[|X_{k'}^{(n)} - X_k^{(n)}|^4]^{\frac{1}{4}} \le \bar{K}|x|\sqrt{\frac{T}{n}|k'-k|}$$

for $Q \in \mathcal{Q}^{(n)}$. Hence we have the assertion. $\square$

Lemma 4.4. *Let $(Q^{(n)})_{n\in\mathbf{N}},\quad Q^{(n)} \in \mathcal{Q}^{(n)}$ be an arbitrary sequence. We define a probability measure $P^{(n)}$ on $(C[0,\infty)^M, \mathcal{B}[C[0,\infty)^M])$ by $P^{(n)} = Q^{(n)} \circ X(\cdot;n)^{-1}$ for each $n \in \mathbf{N}$. Then the sequence $(P^{(n)})_{n\in\mathbf{N}}$ is tight. Moreover any cluster point of $(P^{(n)})_{n\in\mathbf{N}}$ belongs to $\bar{\mathcal{P}}$.*

Proof. We can easily see that

$$E^{P^{(n)}}[|w(t)-w(t')|^4] = E^{Q^{(n)}}[|X(t';n) - X(t;n)|^4]$$
$$\le K|t'-t|^2,\ t.t' \in [0,T],$$

for some $K > 0$ by Lemma 4.1. Hence $(P^{(n)})_{n\in\mathbf{N}}$ is tight.

Let $\bar{P}$ be a cluster point of $(P^{(n)})_{n\in\mathbf{N}}$. Obviously we see that the coordinate function $w(\cdot) = (w_1(\cdot), w_2(\cdot), \dots, w_M(\cdot))$ is a positive martingale with respect to $\{\mathcal{B}_t\}_{t\in[0,T]}$ under $\bar{P}$, and $\bar{P}(w_i(0) = 1,\ i = 1,2,\dots,M) = 1$.

For each $n \in \mathbf{N}$ and $\tilde{\omega} \in \Omega$, we define a probability measure $\hat{Q}_{\tilde{\omega},k}^{(n)} \in \hat{\mathcal{Q}},\ k = 0,1,\dots,n-1$ by $\hat{Q}_{\tilde{\omega},k}^{(n)}[\hat{A}] = E^{\hat{P}}[\rho_k^{(n)}(\tilde{\omega},\hat{Z})1_{\hat{A}}]$, where $E[dQ^{(n)}/dP|\mathcal{F}_n] = \prod_{l=1}^n \rho_l^{(n)}(\cdot, Z_i)$. Let $(g_{ii'})_{i,i'=1,2,\dots,M} \in C([0,T]\times C[0,\infty)^M)^{M\times M}$ be a matrix valued function such that each $g_{ii'},\ i,i' = 1,2,\dots,M$ is bounded $\{\mathcal{B}_t\}$-adapted function, and $(g_{ii'}(u,w)) \in \mathbf{R}^{M\times M}$ is nonnegative definite for all $(u,w) \in [0,T]\times C[0,\infty)^M$. For $n \in \mathbf{N}, k = 1,2,\dots,n$ we have

$$E^{P^{(n)}}[\sum_{i,i'=1}^M g_{ii'}(\frac{kT}{n})(w_i(\frac{kT}{n}+\frac{T}{n}) - w_i(\frac{kT}{n}))(w_{i'}(\frac{kT}{n}+\frac{T}{n}) - w_{i'}(\frac{kT}{n}))]$$

$$= E^{Q^{(n)}}[\sum_{i,i'=1}^M g_{ii'}(\frac{kT}{n}, X(\cdot;n))E^{Q^{(n)}}[(X_{i,k+1}^{(n)} - X_{i,k}^{(n)})(X_{i',k+1}^{(n)} - X_{i',k}^{(n)})|\mathcal{F}_k]]$$

$$= E^{Q^{(n)}}[\sum_{i,i'=1}^M g_{ii'}(\frac{kT}{n}, X(\cdot;n))X_{i,k}X_{i',k}\frac{E^{\hat{Q}_{\omega,k}^{(n)}}[(\hat{Y}_i^{(n)}-1)(\hat{Y}_{i'}^{(n)}-1)]}{T/n}\frac{T}{n}]$$

$$\le E^{P^{(n)}}[\max_{\hat{Q}}\sum_{i,i'=1}^M g_{ii'}(\frac{kT}{n})w_i(\frac{kT}{n})w_{i'}(\frac{kT}{n})\frac{E^{\hat{Q}}[(Y_i^{(n)}-1)(Y_{i'}^{(n)}-1)]}{T/n}\frac{T}{n}],$$

where $\hat{Q}$ runs $\hat{\mathcal{Q}} \cap \bigcap_{i=1}^{M} \{\hat{Q} \in \hat{\mathcal{P}} \mid E^{\hat{Q}}[\hat{Y}_i^{(n)}] = 1\}$. Then we see that

$$\int_s^t \sum_{i,i'=1}^{M} g_{ii'}(u,w) d\langle w_i, w_{i'} \rangle_u \leq \int_s^t (\max_{\gamma \in \Gamma} \sum_{i,i'=1}^{M} \gamma_{ii'} g_{ii'}(u,w) w_i(u) w_{i'}(u)) du,$$

for $t, s \in [0,T]$. Also we have

$$\int_s^t \sum_{i,i'=1}^{M} g_{ii'}(u,w) d\langle w_i, w_{i'} \rangle_u \geq \int_s^t (\min_{\gamma \in \Gamma} \sum_{i,i'=1}^{M} \gamma_{ii'} g_{ii'}(u,w) w_i(u) w_{i'}(u)) du,$$

for $t, s \in [0,T]$, by the same way. Hence we deduce that $\bar{P} \in \bar{\mathcal{P}}$. This completes the proof. $\square$

Now we show Proposition 4.2. Let $K > 0$ be a positive number such that $|g(x) - g(x')| \leq K|x - x'|$, $x, x' \in [0,\infty)^M$ and $|g(x)| \leq K(1 + |x|^{2m})$, $x \in [0,\infty)^M$ hold. We denote $W_g^{(n)}(t,x) = (H^{(n)})^{n-[nt/T]} g(x)$. Then using Lemma 4.3, we see that

$$\begin{aligned} &|W_g^{(n)}(t,x) - W_g^{(n)}(t,x')| \leq K|x - x'|, \\ &|W_g^{(n)}(t,x)| \leq K(1 + |x|^{2m}), \\ &|W_g^{(n)}(t,x) - W_g^{(n)}(t',x)| \leq K|x|\sqrt{\frac{T}{n}|[\frac{nt}{T}] - [\frac{nt'}{T}]|}, \end{aligned} \tag{33}$$

for some $K > 0$ which is independent of $n \in \mathbf{N}$.

(1): We see that the family $\{W_g^{(\bar{n})}(t,\cdot)\}_{n \in \mathbf{N},\ t \in [0,T]} \subset \hat{C}([0,\infty)^M : \mathbf{R})$ is uniformly bounded and equicontinuous on any compact set of $[0,\infty)^M$. Then using Ascoli-Arzela's theorem we see that there exists a continuous function $W_{1,g,t}$ on $[0,1]^M$ for each $t \in [0,T]$ and a subsequence $(\bar{n}_1)$ of $(\bar{n})$, which does not depend on t, such that

$$\sup_{x \in [0,1]^M} |W_g^{(\bar{n}_1)}(t,x) - W_{1,g,t}(x)| \to 0,\ \bar{n}^1 \to \infty. \tag{34}$$

Also we see by the same way that there exists a continuous function $W_{2,g,t}$ on $[0,2]^M$ for each $t \in [0,T]$ and a subsequence $(\bar{n}_2)$ of $(\bar{n}_1)$, which does not depend on t, such that

$$\sup_{x \in [0,2]^M} |W_g^{(\bar{n}_2)}(t,x) - W_{2,g,t}(x)| \to 0,\ \bar{n}^2 \to \infty. \tag{35}$$

Obviously $W_{1,g,t} = W_{2,g,t}$ on $[0,1]^M$, for each $t \in [0,T]$.

Then for each $l \in \mathbf{N}$, we can inductively define continuous functions $W_{l+1,g,t}$ on $[0, l+1]^M$ for $t \in [0,T]$ and subsequences $(\bar{n}_{l+1})$ of $(\bar{n}_l)$, which does not depend on t, such that $W_{1,g,t} = W_{2,g,t} = \cdots = W_{l+1,g,t}$ on $[0,l]^M$. We also define continuous functions $W_{g,t}$ on $[0,\infty)^M$ for $t \in [0,T]$ by $W_{g,t}(x) = W_{l,g,t}(x)$ on $[0,l]^M$. Obviously $W_{g,t}$ is well-defined, and we see that the sequence $(\bar{n}(k))$, $\bar{n}(k) = \bar{n}_k(k)$, $k \in \mathbf{N}$ satisfies $\lim_{k\to\infty} W_g^{(\bar{n}(k))}(t,x) = W_{g,t}(x)$ for $(t,x) \in [0,T] \times [0,\infty)^M$. Using (33), we have

$$|W_{g,t}(x) - W_{g,t'}(x')| \leq K|x - x'| + K|x|\sqrt{|t-t'|} \tag{36}$$

for $(t,x),(t',x') \in [0,T] \times [0,\infty)^M$. This implies that the mapping $(t,x) \to W_{g,t}(x)$ is continuous. Then we can define a continuous mapping $W_g : [0,T] \times [0,\infty)^M$ by $W_g(t,x) = W_{g,t}(x)$. Obviously, W_g and $(\bar{n}(k))_{k\in\mathbf{N}}$ satisfy the assertion.

(2): We can easily show the assertion by letting $k \to \infty$ on both sides of the inequalities (33).

(3): Fix $(t,x) \in [0,T] \times [0,\infty)^M$. Hereafter we simply write $(n)_{n\in\mathbf{N}}$ instead of $(\bar{n}(k))_{k\in\mathbf{N}}$ for convention. We see by Lemma 4.1 that there exists $Q_l^{(n)} \in \mathcal{Q}^{(n)}$ for each n, l such that $W_g^{(n)}(t,x) \geq E^{Q_l^{(n)}}[g(xX^{(n)}_{n-[nt/T]})] \geq W_g^{(n)}(t,x) - 1/l$. Let $P_l^{(n)} = Q_l^{(n)} \circ X(\cdot;n)^{-1}$. Since $(P_l^{n)})_{n,l\in\mathbf{N}}$ is tight by Lemma 4.4, then there exist a cluster point $P_{t,x} \in \bar{\mathcal{P}}$. Obviously $P_{t,x}$ satisfies the condition.

(4): Fix $t, t' \in [0,T]$, $x \in [0,\infty)^M$. First we claim that

$$|(H^{(n)})^{[nt'/T]} W_g^{(n)}(t+t',\cdot)(x) - (H^{(n)})^{[nt'/T]} W_g(t+t',\cdot)(x)| \to 0, \tag{37}$$

as $n \to \infty$. Fix $\varepsilon > 0$. Since the convergence $W_g^{(n)}(t+t',\cdot) \to W_g(t+t',\cdot)$, $n \to \infty$ is uniform on $\prod_{i=1}^{M}[0, x_i R]$, for each $R > 0$, there exists $n(R) \in \mathbf{N}$ such that

$$\sup_{y \in \prod_{i=1}^M [0,x_i R]} |W_g^{(n)}(t+t',y) - W_g(t+t',y)| < \varepsilon/2,$$

for $n > n(R)$. We denote

$$F^{(n)} = W_g^{(n)}(t+t',\cdot)(xX^{(n)}_{[nt'/T]}) - W_g(t+t',\cdot)(xX^{(n)}_{[nt'/T]}), \tag{38}$$

for $n \in \mathbf{N}$. Then for each $Q^{(n)} \in \mathcal{Q}^{(n)}$, $n > n(R)$ we have

$$|E^{Q^{(n)}}[F^{(n)}]|$$

$$
\begin{aligned}
&\le E^{Q^{(n)}}[|F^{(n)}|1_{\{|X^{(n)}_{[nt'/T]}|\le R\}}] + E^{Q^{(n)}}[|F^{(n)}|1_{\{|X^{(n)}_{[nt'/T]}|>R\}}] \\
&\le \frac{\varepsilon}{2} + E^{Q^{(n)}}[|F^{(n)}|^2]^{\frac{1}{2}} Q^{(n)}[|X^{(n)}_{[nt'/T]}|>R]^{\frac{1}{2}} \\
&\le \frac{\varepsilon}{2} + \frac{K}{R} E^{Q^{(n)}}[1+|X^{(n)}_{[nt'/T]}|^{2m}]^{\frac{1}{2}} E^{Q^{(n)}}[|X^{(n)}_{[nt'/T]}|^2]^{\frac{1}{2}} \\
&\le \frac{\varepsilon}{2} + \frac{K'}{R},
\end{aligned}
$$

where K, K' are positive numbers which do not depend on R, n, and $Q^{(n)}$. Let $R = \varepsilon/2K'$ and $n_0 = n(\varepsilon/2K')$. Then we have $|E^{Q^{(n)}}[F^{(n)}]| < \varepsilon$ and

$$
|(H^{(n)})^{[nt'/T]} W^{(n)}_g(t+t', \cdot)(x) - (H^{(n)})^{[nt'/T]} W_g(t+t', \cdot)(x)| \le \varepsilon,
$$

for $n > n_0$. Hence we have the claim.

Also we see that $|W^{(n)}_g(t,x) - W_g(t,x)| \to 0$, as $n \to \infty$ and,

$$
\begin{aligned}
&|(H^{(n)})^{[\frac{nt'}{T}]} W^{(n)}_g(t+t', \cdot)(x) - W^{(n)}_g(t,x)| \\
&= |(H^{(n)})^{n-[n(t+t')/T]+[nt'/T]} g(x) - (H^n)^{n-[nt/T]} g(x)| \\
&\le K|x| \sqrt{\frac{T}{n} |[\frac{nt'}{T}] + [\frac{nt}{T}] - [\frac{n(t+t')}{T}]|} \to 0,
\end{aligned}
$$

as $n \to \infty$. We have the assertion from these results.

(5): Take $(\bar{t}, \bar{x}) \in [0,T) \times [0,\infty)^M$. Let $\hat{U} \in C^\infty([0,T] \times [0,\infty)^M)$ be a function such that $\hat{U}(\bar{t}, \bar{x}) = W_g(\bar{t}, \bar{x})$ and $\hat{U} \ge W_g$ on some neighbourhood V of $(\bar{t}, \bar{x})$ hold. We can easily see by (2) that $W_g(t,x) \le K(1+|x|^{2m})$ for some $K > 0$, $m \in \mathcal{M}$. Then we may assume that $\hat{U}(t,x) \ge W_g(t,x)$, and $|\hat{U}(t,x)| + |(\partial \hat{U}/\partial t)(t,x)| + \sum_{i,i'} |x_i x_{i'} (\partial^2 \hat{U}/\partial x_i \partial x_{i'})(t,x)| \le K(1+|x|^{2m})$, for any $(t,x) \in [0,T] \times [0,\infty)^M$. Let $\bar{\gamma} = (\bar{\gamma}_{i,i'}) \in \Gamma$ be the element such that

$$
\begin{aligned}
&\sum_{i,i'=1}^M \frac{1}{2} \bar{\gamma}_{i,i'} \bar{x}_i \bar{x}_{i'} \frac{\partial^2 \hat{U}}{\partial x_i x_{i'}}(\bar{t}, \bar{x}) = \min_{\gamma \in \Gamma} \sum_{i,i'=1}^M \frac{1}{2} \gamma_{i,i'} \bar{x}_i \bar{x}_{i'} \frac{\partial^2 \hat{U}}{\partial x_i x_{i'}}(\bar{t}, \bar{x}) \\
&= \min_{\hat{Q}} \sum_{i,i'=1}^M \frac{1}{2} E^{\hat{Q}}[\hat{Z}_i \hat{Z}_{i'}] \bar{x}_i \bar{x}_{i'} \frac{\partial^2 \hat{U}}{\partial x_i x_{i'}}(\bar{t}, \bar{x}),
\end{aligned}
$$

where $\hat{Q}$ runs $\hat{\mathcal{Q}} \cap \bigcap_{i=1}^M \{\hat{Q} \in \hat{\mathcal{P}} \mid E^{\hat{Q}}[\hat{Z}_i] = 0\}$.

Also, let $\hat{Q}^{(n)} \in \hat{\mathcal{Q}} \cap \bigcap_{i=1}^M \{\hat{Q} \in \hat{\mathcal{P}} \mid E^{\hat{Q}}[\hat{Y}^{(n)}_i] = 1\}$, $n \in \mathbf{N}$ be measures that attain the minimal of

$$
\min_{\hat{Q}} \sum_{i,i'=1}^M \frac{1}{2} \frac{E^{\hat{Q}}[(\hat{Y}^{(n)}_i - 1)(\hat{Y}^{(n)}_{i'} - 1)]}{T/n} \bar{x}_i \bar{x}_{i'} \frac{\partial^2 \hat{U}}{\partial x_i x_{i'}}(\bar{t}, \bar{x}), \quad n \in \mathbf{N},
$$

where $\hat{Q}$ runs $\hat{\mathcal{Q}} \cap \bigcap_{i=1}^{M} \{\hat{Q} \in \hat{\mathcal{P}} \mid E^{\hat{Q}}[\hat{Y}_i^{(n)}] = 1\}$ for each $n \in \mathbf{N}$. Note that we can naturally regard $\hat{Q}^{(n)}$ as a probability measure on $(\Omega, \mathcal{F}_1)$ for each $n \in \mathbf{N}$. Take $\rho^{(n)} \in \mathcal{L}_1$ such that $d\hat{Q}^{(n)}/dP = \rho^{(n)}(\cdot, Z_1)$ holds. We define $\bar{Q}^{(n)} \in \mathcal{Q}^{(n)}$ by $E[\frac{d\bar{Q}^{(n)}}{dP}|\mathcal{F}_n] = \prod_{k=1}^{n} \bar{\rho}_k^{(n)}(\cdot, Z_k)$, where $\bar{\rho}_k^{(n)} = \rho^{(n)}, k = 1, 2, \dots, n,$, and $\bar{P}^{(n)} = \bar{Q}^{(n)} \circ X(\cdot; n)^{-1}$ for $n \in \mathbf{N}$. We see that there exists a cluster point $\bar{P} \in \mathcal{P}$ of $\{\bar{P}^{(n)}\}_{n \in \mathbf{N}}$ and $d\langle w_i, w_{i'} \rangle_u = \bar{\gamma}_{i,i'} w_i(u) w_{i'}(u) du$, $\bar{P}$-a.s., by an argument similar to that of the proof of Lemma 4.4. Then we have

$$\begin{aligned}
&\hat{U}(\bar{t}, \bar{x}) = W_g(\bar{t}, \bar{x}) = \lim_{k \to \infty} (H^{(n)})^{[nh/T]} W_g(\bar{t} + h, \cdot)(\bar{x}) \\
&\leq \limsup_{k \to \infty} E^{\bar{P}^{(n)}}[W_g(\bar{t} + h, \bar{x} w(\frac{T}{n}[\frac{nh}{T}]))] \\
&= E^{\bar{P}}[W_g(\bar{t} + h, \bar{x} w(h))] \leq E^{\bar{P}}[\hat{U}(\bar{t} + h, \bar{x} w(h))].
\end{aligned}$$

Then we see by Ito's formula that

$$0 \leq E^{\bar{P}}[\int_0^h (\frac{\partial \hat{U}}{\partial t} + \sum_{i,i'=1}^{M} \frac{1}{2} \bar{\gamma}_{i,i'} x_i x_{i'} \frac{\partial^2 \hat{U}}{\partial x_i x_{i'}})(\bar{t} + u, \bar{x} w(u)) du]. \tag{39}$$

Dividing both sides by $h > 0$ and letting $h \to \infty$, we have

$$\begin{aligned}
0 &\leq (\frac{\partial \hat{U}}{\partial t} + \sum_{i,i'=1}^{M} \frac{1}{2} \bar{\gamma}_{i,i'} x_i x_{i'} \frac{\partial^2 \hat{U}}{\partial x_i x_{i'}})(\bar{t}, \bar{x}) \\
&= (\frac{\partial \hat{U}}{\partial t} + \min_{\gamma \in \Gamma} \sum_{i,i'=1}^{M} \frac{1}{2} \gamma_{i,i'} \bar{x}_i \bar{x}_{i'} \frac{\partial^2 \hat{U}}{\partial x_i \partial x_{i'}})(\bar{t}, \bar{x}).
\end{aligned}$$

This completes the proof.

4.3 Conclusion

Now we prove Theorem 1.5, which is our main result in this paper. Take an arbitrary subsequence $(\bar{n})$ of $(n)_{n \in \mathbf{N}}$ and define W_f as in Proposition 4.2. Using Proposition 4.2 (3), We have $\inf_{\bar{P} \in \bar{\mathcal{P}}} E^{\bar{P}}[f(xw(T-t))] \leq W_f(t,x)$ for $(t,x) \in [0,T] \times [0,\infty)^M$. On the other hand, we have the inverse inequality by Theorem A.2, Proposition A.4, and Proposition 4.2 (3). Then we have

$$W_f(t,x) = \inf_{\bar{P} \in \bar{\mathcal{P}}} E^{\bar{P}}[f(xw(T-t))].$$

Since the subsequence $(\bar{n})$ is arbitrary, $U(t,x) = \lim_{n \to \infty} (H^{(n)})^{n-[nt/T]T/n} f(x)$ exists for any $t \in [0,T],\ x \in [0,\infty)^M$, and equals $\inf_{\bar{P} \in \bar{\mathcal{P}}} E^{\bar{P}}[f(xw(T-t))]$. Then we see that U is a viscosity solution of (13) because $U(t,x) = W_f(t,x) = E^{\bar{P}}[f(xw(T-t))]$ is both a supersolution and subsolution. Also the uniqueness holds by Corollary A.2. Hence we have the assertion from Theorem 3.2. This completes the proof.

A Some Remarks on a Bellman Equation and viscosity solution

We recall the definition and some property of viscosity solution in this appendix. The reader also refer to [6] for detail.

Definition A.1. *We say that a continuous function* $U : [0,T]\times[0,\infty)^M \to \mathbf{R}$ *is a viscosity supersolution* (*resp. subsolution*) *of Hamilton-Jacobi-Bellman equation* (13), *if*

$$(\frac{\partial \hat{U}}{\partial t} + \inf_{\gamma\in\Gamma} \sum_{i,i'=1}^{M} \frac{1}{2}\gamma_{i,i'} x_i x_{i'} \frac{\partial^2 \hat{U}}{\partial x_i x_{i'}})(\bar{t},\bar{x}) \leq 0 \ (resp.\ \geq 0)$$

holds for any $(\bar{t},\bar{x}) \in [0,T]\times[0,\infty)^M$ *and* $\hat{U} \in C^\infty([0,T]\times[0,\infty)^M)$ *such that* $\hat{U}(\bar{t},\bar{x}) = U(\bar{t},\bar{x})$ *and* $\hat{U} - U$ *takes its local maximum* (*resp. local minimum*) *value* 0 *at* $(\bar{t},\bar{x})$. *Also we say that a function* $U : [0,T]\times[0,\infty)^M \to \mathbf{R}$ *is a viscosity solution if it is both a viscosity supersolution and subsolution.*

We will need the following comparison theorem for a viscosity supersolution and a subsolution due to [14].

Theorem A.2. *Let* $\overline{U}$ *and* $\underline{U}$ *be a viscosity supersolution and a subsolution of Hamilton-Jacobi-Bellman equation* (13). *If the following inequalities*:

$$\begin{aligned}
&\sup_{[0,T)\times[0,\infty)^M} \underline{U}(t,x)/(|x|^2+1)^m < \infty, \\
&\inf_{[0,T)\times[0,\infty)^M} \overline{U}(t,x)/(|x|^2+1)^m > -\infty, \quad m > 0, \\
&\underline{U}(T,x) \leq \overline{U}(T,x), \quad x \in [0,\infty)^M,
\end{aligned}$$

hold, then we have $\underline{U}(t,x) \leq \overline{U}(t,x)$, $(t,x) \in [0,T)\times[0,\infty)^M$. *In particular, the viscosity solution* U *of Hamilton-Jacobi-Bellman equation* (13) *satisfying* $U \in \hat{C}([0,\infty)^M : \mathbf{R})$ *is unique.*

Before we state a proposition on Hamilton-Jacobi-Bellman equation (13), we prove a lemma.

Lemma A.3. *There exists $C_m > 0$ for each $m \in \mathbf{N}$ such that*

$$E^{\bar{P}}[\max_{u\in[0,t]} |w(u) - \mathbf{1}|^{2m}] \le C_m t^m,$$
$$E^{\bar{P}}[\max_{u\in[0,T]} |w(u)|^{2m}] \le C_m,\ t \in [0,T],\ \bar{P} \in \bar{\mathcal{P}}.$$

Proof. We see by Burkholder's inequality that

$$\begin{aligned}
&E^{\bar{P}}[\max_{u\in[0,t]} |w(u) - \mathbf{1}|^{2m}] \le cE^{\bar{P}}[\sum_{i=1}^{M} \langle w_i \rangle_u^m] \\
&\le cE^{\bar{P}}[(\int_0^t \max_{\gamma\in\Gamma} \sum_{i=1}^{M} \gamma_{ii} |w_i(u)|^2 du)^m] \le c\overline{\gamma}^m E^{\bar{P}}[(\int_0^t |w(u)|^2 du)^m] \quad (40) \\
&\le ct^m E^{\bar{P}}[\int_0^t |w(u)|^{2m} du] \le c + ct^m E^{\bar{P}}[\int_0^t \max_{s\in[0,u]} |w(u) - \mathbf{1}|^{2m} du]
\end{aligned}$$

where all c stand for positive numbers (not necessarily equal) which do not depend on $\bar{P} \in \bar{\mathcal{P}}$ and t. Then we have the assertion by Gronwall's inequality. □

Let $\bar{U}(t,x) = \inf_{\bar{P}\in\bar{\mathcal{P}}} E^{\bar{P}}[\tilde{f}(xw(T-t))]$. Using Lemma A.3, We can easily see that $\bar{U} \in \hat{C}([0,\infty)^M : \mathbf{R})$. Now we show the following.

Proposition A.4. *$\bar{U}(t,x)$ is a viscosity supersolution of* (13).

Proof. First we denote by Λ a set of control which is composed of pairs $\{(\Omega, \mathcal{F}, P; \{\mathcal{F}_t\}_{t\in[0,T]}), X\}$ such that the following satisfied:

$$\begin{aligned}
&(\Omega, \mathcal{F}, P; \{\mathcal{F}_t\}_{t\in[0,T]}) \text{ is a filtered probability space,} \\
&X = (X_1, X_2, \ldots, X_M) \text{ is a continuous positive martingale} \\
&\text{with respect to } \{\mathcal{F}_t\}_{t\in[0,T]} \text{ under } P, \\
&P(X_i(0) = 1,\ i = 1,2,\ldots,M) = 1. \\
&\langle X_i, X_{i'} \rangle_\cdot,\ i,i' \in \{1,2,\ldots,M\} \text{ are absolutely continuous} \\
&\text{with respect to Lebesgue measure, and} \\
&(\frac{1}{X_i(u)X_{i'}(u)} \frac{d\langle X_i, X_{i'}\rangle}{dt}(u))_{i,i'=1,2,\ldots,M} \in \Gamma,\ u \in [0,T],\ P \text{ a.s.}
\end{aligned} \quad (41)$$

We define $Q_t g(x), \ \ t \in [0,T), \ x \in [0,\infty)^M$, and $g \in \bar{C}([0,\infty)^M : \mathbf{R})$ by

$$Q_t g(x) = \inf_{\bar{P}\in\bar{\mathcal{P}}} E^{\bar{P}}[g(xw(t))]. \tag{42}$$

Then we can easily see that $Q_t g(x) = \inf_{\lambda\in\Lambda} E^P[g(xX_t)]$ since $PX^{-1} \in \bar{\mathcal{P}}$.

step1 : We show that $Q_{t+t'}g(x) \geq Q_t Q_{t'} g(x), \ t, t' \in [0,T], \ x \in [0,\infty)^M$. We define a filtration $\{\hat{\mathcal{B}}_u\}_{u\in[0,T]}$ on $(C[0,\infty)^M, \mathcal{B}[C[0,\infty)^M])$ and a M-dimensional $\{\hat{\mathcal{B}}_u\}$-adapted process $\hat{X}_u = (\hat{X}_{1,u}, \hat{X}_{2,u}, \ldots, \hat{X}_{M,u}), \ \ u \in [0,T]$, by

$$\hat{\mathcal{B}}_u = \begin{cases} \mathcal{B}_{t+u}, & 0 \leq u \leq T-t, \\ \mathcal{B}_T, & T-t < u \leq T. \end{cases}$$

$$\hat{X}_{i,u}(w) = \begin{cases} w_i(t+u)/w_i(t), & 0 \leq u \leq T-t, \\ w_i(T)/w_i(t), & T-t < u \leq T. \end{cases}$$

Since $(C[0,\infty)^M, \mathcal{B}[C[0,\infty)^M], \bar{P})$ is standard probability space for each $\bar{P} \in \bar{\mathcal{P}}$, there exists a regular conditional measure $\bar{P}_t(w,B) \ : \ C[0,\infty)^M \times \mathcal{B}[C[0,\infty)^M] \to [0,1], \ \ t \in [0,t]$. Then we can easily see that

$$((C[0,\infty)^M, \mathcal{B}[C[0,\infty)^M], \bar{P}_t(w,\cdot) \ ; \ \{\hat{\mathcal{B}}_u\}_{u\in[0,T]}), \ \hat{X}) \in \Lambda,$$

and

$$\begin{aligned} &E^{\bar{P}}[g(xw(t+t'))] = E^{\bar{P}}[E^{\bar{P}}[g(xw(t+t'))|\mathcal{B}_t]] \\ &= E^{\bar{P}}[\int g(xw(t')\hat{X}_{t'})\hat{P}_t(w,dw')] \geq E^{\hat{P}}[Q_{t'}g(xw(t))] \geq Q_t Q_{t'} g(x), \end{aligned}$$

for $\bar{P} \in \bar{\mathcal{P}}$. Hence we have the assertion.

step2 : Take $(\bar{t},\bar{x}) \in [0,T) \times [0,\infty)^M$ and fix it. Let $\hat{U} \in C^\infty([0,T] \times [0,\infty)^M)$ be a function such that $\hat{U}(\bar{t},\bar{x}) = \overline{U}(\bar{t},\bar{x})$ and $\hat{U} \leq \overline{U}$on some neighbourhood V of $(\bar{t},\bar{x})$ hold. We may assume that $\hat{U}(t,x) \leq \overline{U}(t,x)$ and,

$$|\hat{U}(t,x)| + |\frac{\partial \hat{U}}{\partial t}(t,x)| + \sum_{i,i'} |x_i x_{i'} \frac{\partial^2 \hat{U}}{\partial x_i x_{i'}}(t,x)| \leq K(1+|x|^{2m}), \tag{43}$$

for any $(t,s) \in [0,T] \times [0,\infty)$, because $\overline{U} \in \hat{C}([0,\infty)^M : \mathbf{R})$.

From here to the end of this proof, $c > 0$ will stand for positive numbers (not necessarily equal) which do not depend on $\bar{P} \in \bar{\mathcal{P}}, \ t \in [0,T], \ \gamma \in \Gamma$, and $R>0$. First we claim that there exists $M(R)>0$ for each $R>0$ such that $M(R)$ does not depend on $\bar{P}\in\bar{\mathcal{P}},\ t\in[0,T],\ \gamma\in\Gamma$, and

$$
\begin{aligned}
&|E^{\bar{P}}[\int_0^{t-\bar{t}}\frac{\partial\hat{U}}{\partial t}(\bar{t}+u,\bar{x}w(u))du \\
&\quad+\int_0^{t-\bar{t}}\sum_{i,i'=1}^{M}\frac{1}{2}\bar{x}_i\bar{x}_{i'}\frac{\partial^2\hat{U}}{\partial x_i\partial x_{i'}}(\bar{t}+u,\bar{x}w(u))d\langle w_i,w_{i'}\rangle_u] \\
&-\{\frac{\partial\hat{U}}{\partial t}(\bar{t},\bar{x})(t-\bar{t})+E^{\bar{P}}[\sum_{i,i'=1}^{M}\frac{1}{2}\bar{x}_i\bar{x}_{i'}\frac{\partial^2\hat{U}}{\partial x_i\partial x_{i'}}(\bar{t},\bar{x})\langle w_i,w_{i'}\rangle_{t-\bar{t}}]\}| \\
&\leq\frac{c(t-\bar{t})}{R}+cM(R)(t-\bar{t})^{3/2},
\end{aligned}
$$

for $t>\bar{t}$. To show this claim, we estimate

$$
\begin{aligned}
&I_1+I_2 \\
&=E^{\bar{P}}[|\int_0^{t-\bar{t}}\sum_{i,i'=1}^{M}\frac{1}{2}\bar{x}_i\bar{x}_{i'}F_{ii'}(u,w)d\langle w_i,w_{i'}\rangle_u|1_{\{\max\limits_{u\in[0,t-\bar{t}]}|w(u)|>R\}}] \\
&\quad+E^{\bar{P}}[|\int_0^{t-\bar{t}}\sum_{i,i'=1}^{M}\frac{1}{2}\bar{x}_i\bar{x}_{i'}F_{ii'}(u,w)d\langle w_i,w_{i'}\rangle_u|1_{\{\max\limits_{u\in[0,t-\bar{t}]}|w(u)|\leq R\}}],
\end{aligned}
\tag{44}
$$

where $F_{i,i'}(u,w)=\frac{\partial^2\hat{U}}{\partial x_i\partial x_{i'}}(\bar{t}+u,\bar{x}w(u))-\frac{\partial^2\hat{U}}{\partial x_i\partial x_{i'}}(\bar{t},\bar{x}),\ i,i'=1,2,\ldots,M$, $R>0,\ \bar{P}\in\bar{\mathcal{P}}$, and

$$
\begin{aligned}
J_1+J_2=E^{\bar{P}}[&\int_0^{t-\bar{t}}|G(u,w)|du1_{\{\max\limits_{u\in[0,t-\bar{t}]}|w(u)|>R\}}] \\
&+E^{\bar{P}}[\int_0^{t-\bar{t}}|G(u,w)|du1_{\{\max\limits_{u\in[0,t-\bar{t}]}|w(u)|\leq R\}}],
\end{aligned}
\tag{45}
$$

where $G(u,w)=\frac{\partial\hat{U}}{\partial t}(\bar{t}+u,\bar{x}w(u))-\frac{\partial\hat{U}}{\partial t}(\bar{t},\bar{x}),\ R>0,\ \bar{P}\in\bar{\mathcal{P}}$. We have the claim by Lemma A.3, Hölder's inequality, Tchebychev's inequality, Burkholder's inequality, and Lipschitz continuity of all derivatives of $\hat{U}$ on a bounded interval. Also we have

$$
\begin{aligned}
&|E^{\bar{P}}[\int_0^{t-\bar{t}}(\min_{\gamma\in\Gamma}\sum_{i,i'=1}^{M}\frac{1}{2}\gamma_{ii'}\bar{x}_i\bar{x}_{i'}\frac{\partial^2\hat{U}}{\partial x_i\partial x_{i'}}(\bar{t},\bar{x})w_i(u)w_{i'}(u))du] \\
&\quad-(t-\bar{t})(\min_{\gamma\in\Gamma}\sum_{i,i'=1}^{M}\frac{1}{2}\gamma_{ii'}\bar{x}_i\bar{x}_{i'}\frac{\partial^2\hat{U}}{\partial x_i\partial x_{i'}}(\bar{t},\bar{x}))|\leq c|t-\bar{t}|^{3/2}.
\end{aligned}
$$

by Lemma A.3 and Hölder's inequality.

step3 : We see by Ito's formula that

$$
\begin{aligned}
&\hat{U}(\bar{t},\bar{x}) = \overline{U}(\bar{t},\bar{x}) = Q(T-\bar{t})\tilde{f}(\bar{t},\bar{x}) \\
&\geq Q(t-\bar{t})Q(T-\bar{t})\tilde{f}(x) = Q(t-\bar{t})\overline{U}(t,\cdot)(x) \\
&= \inf_{\bar{P}\in\bar{\mathcal{P}}} E^{\bar{P}}[\overline{U}(t,\bar{x}w(t-\bar{t}))] \geq \inf_{\bar{P}\in\bar{\mathcal{P}}} E^{\bar{P}}[\hat{U}(t,\bar{x}w(t-\bar{t}))] \\
&= \hat{U}(\bar{t},\bar{x}) + \inf_{\bar{P}\in\bar{\mathcal{P}}} E^{\bar{P}}[\int_0^{t-\bar{t}} \frac{\partial \hat{U}}{\partial t}(\bar{t}+u,\bar{x}w(u))du \\
&\qquad\qquad + \int_0^{t-\bar{t}} \sum_{i,i'=1}^{M} \frac{1}{2}\bar{x}_i\bar{x}_{i'}\frac{\partial^2 \hat{U}}{\partial x_i \partial x_{i'}}(\bar{t}+u,\bar{x}w(u))d\langle w_i,w_{i'}\rangle_u].
\end{aligned}
$$

Then we have

$$
\begin{aligned}
0 &\geq -c(M(R)+1)(t-\bar{t})^{3/2} - \frac{c(t-\bar{t})}{R} + \frac{\partial \hat{U}}{\partial t}(\bar{t},\bar{x})(t-\bar{t}) \\
&\quad + (t-\bar{t})(\min_{\gamma\in\Gamma} \sum_{i,i'=1}^{M} \frac{1}{2}\gamma_{ii'}\bar{x}_i\bar{x}_{i'}\frac{\partial^2 \hat{U}}{\partial x_i\partial x_{i'}}(\bar{t},\bar{x})), \quad \bar{t} < t \leq T,\ R > 0,
\end{aligned}
$$

by the consequence of step 2. Dividing both sides of the above inequality by $t-\bar{t}>0$ and letting $t\to\bar{t}$, we have

$$
0 \geq -\frac{c}{R} + \frac{\partial \hat{U}}{\partial t}(\bar{t},\bar{x}) + \min_{\gamma\in\Gamma}\sum_{i,i'=1}^{M}\frac{1}{2}\gamma_{ii'}\bar{x}_i\bar{x}_{i'}\frac{\partial^2 \hat{U}}{\partial x_i\partial x_{i'}}(\bar{t},\bar{x}),
$$

for all $R>0$. Hence we have the assertion. This completes the proof. □

Acknowledgement: The author would like to thank Professor Shigeo Kusuoka for his useful advices.

References

[1] Artzner, P., Delbaen, F., Eber, J-M., Heath, D. (1999) Coherent measures of risk. Math. Finance 9(3), 203-228.

[2] Artzner, P., Delbaen, F., Eber, J-M., Heath, D., Ku, H. (2002) Multiperiod risk and coherent multiperiod risk measurements of risk. Preprint, ETH Zürich.

[3] Acerbi, C., Tasche, D. (2001) On the Coherence of Expected Shortfall. Working Paper.

[4] P. Barrieu, N. El Karoui (2005) Inf-Convolution of risk measures and optimal risk transfer. Finance Stoch., 9, 269-298.

[5] Cheridito, P., Delbaen, F., Kupper, M. (2004) Coherent and convex risk measures for bounded cadlag processes. Stoc. Appl., 112, 1-22.

[6] Crandall, M. G., Ishii, H., Lions, P. L. (1993) User's guide to viscosity solutions of second order partial differential equations, Bull. A. M. S., 27, 1-67.

[7] Delbaen, F. (2001) Coherent risk measures. Lecture notes, Pisa.

[8] Delbaen, F. (2001) The structure of m-stable sets and in particular of the set of risk neutral measures, Preprint, ETH Zürich.

[9] Detlefsen, K., Scandolo, G. (2005) Conditional and dynamic convex risk measures. Finance Stoch., 9, 539-561.

[10] Föllmer, H., Leukert, P. (1999) Quantile Hedging. Finance Stoch., 3, 251-273.

[11] Föllmer, H., Leukert, P. (2000) Efficient hedges: cost versus shortfall risk. Finance Stoch., 4, 117-146.

[12] Föllmer, H., Schied, A. (2002) Convex measures of risk and trading constraints. Finance Stoch., 6, 429-447.

[13] Kusuoka, S., Morimoto, Y. (2004) Homogeneous law invariant coherent multiperiod value measures and their limits. Preprint, The University of Tokyo.

[14] Nagai, H. (1999) Stochastic Differential Equations, Kyoritsu (In Japanese).

[15] Roorda, B. (2002) Martingale characterizations of coherent acceptability measures. Working Paper, Twente University.

[16] Takahashi, W., (1976) Nonlinear variational inequalities and fixed point theorems. J. Math. Soc. Japan 28, No 1, 168-181.

UTMS

2006–3 Mourad Choulli, Oleg Yu. Imanuvilov and Masahiro Yamamoto: *Inverse source problem for the Navier-Stokes equations.*

2006–4 Takashi Taniguchi: *Distributions of discriminants of cubic algebras.*

2006–5 Jin Cheng, Yanbo Wang and Masahiro Yamamoto: *Regularity of the Tikhonov regularized solutions and the exact solution.*

2006–6 Keiichi Gunji: *The dimension of the space of siegel Eisenstein series of weight one.*

2006–7 Takehiro Fujiwara: *Sixth order methods of Kusuoka approximation.*

2006–8 Takeshi Ohtsuka: *Motion of interfaces by the Allen-Cahn type equation with multiple-well potentials.*

2006–9 Yoshihiro Sawano and Tsuyoshi Yoneda: *On the Young theorem for amalgams and Besov spaces.*

2006–10 Takahiko Yoshida: *Twisted toric structures.*

2006–11 Yoshihiro Sawano, Takuya Sobukawa and Hitoshi Tanaka: *Limiting case of the boundedness of fractional integral operators on non-homogeneous space.*

2006–12 Yoshihiro Sawano and Hitoshi Tanaka: *Equivalent norms for the (vector-valued) Morrey spaces with non-doubling measures.*

2006–13 Shigeo Kusuoka and Song Liang: *A mechanical model of diffusion process for multi-particles.*

2006–14 Yuji Umezawa: *A Limit Theorem on Maximum Value of Hedging with a Homogeneous Filtered Value Measure.*

The Graduate School of Mathematical Sciences was established in the University of Tokyo in April, 1992. Formerly there were two departments of mathematics in the University of Tokyo: one in the Faculty of Science and the other in the College of Arts and Sciences. All faculty members of these two departments have moved to the new graduate school, as well as several members of the Department of Pure and Applied Sciences in the College of Arts and Sciences. In January, 1993, the preprint series of the former two departments of mathematics were unified as the Preprint Series of the Graduate School of Mathematical Sciences, The University of Tokyo. For the information about the preprint series, please write to the preprint series office.

ADDRESS:
Graduate School of Mathematical Sciences, The University of Tokyo
3–8–1 Komaba Meguro-ku, Tokyo 153-8914, JAPAN
TEL +81-3-5465-7001 FAX +81-3-5465-7012