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Abstract

In [1], the author showed the absolute continuity of a measure induced by infinite dimensional stochastic differential equations of the type $dX_t = dW_t + A(X_t)dW_t + b(X_t)dt$ under the condition that the modified Malliavin covariance is non-degenerate. We give a sufficient condition for the non-degeneracy of the modified Malliavin covariance.

1 Introduction

Let (B, H, μ) be an abstract Wiener space. We consider the following type of (infinite dimensional) stochastic differential equations on (B, H, μ):

$$dX_t = dW_t + A(X_t)dW_t + b(X_t)dt, \quad 0 \le t \le T \tag{1.1}$$

with $X_0 = 0$, where W_t is a B*-valued Wiener process* and $A : B \to H \otimes H$, $b : B \to H$ are measurable maps. In the previous work [1], we showed that the distribution of X_T is absolute continuous with respect to μ_T (the distribution of $x \in B \mapsto \sqrt{T}x \in B$ under μ) if

$$E\Big[|(tI_H + \sigma(t))^{-1}|^p_{L(H;H)}\Big] < \infty, \quad \text{for all } p \in (1, \infty) \text{ and } t \in (0, T], \tag{1.2}$$

where $\sigma(t)$ is the modified Malliavin covariance defined in Section 3 of [1] and $|\cdot|_{L(H;H)}$ is the operator norm on H. We also showed that (1.2) holds in the uniformly elliptic case in Section 6 of [1]. The main purpose in the present paper is to give more general sufficient condition for (1.2).

We briefly provide the notation used in the present paper. The reader refers to [1] for details.

Let $\mathbf{W} = \{w \in C([0, T] \to B); w_0 = 0\}$, $\mathbf{H} = \{\mathbf{h} \in \mathbf{W}; \int_0^T |\dot{\mathbf{h}}(t)|^2_H dt < \infty\}$ and P be a standard Wiener measure on $\mathbf{W}$. The triple $(\mathbf{W}, \mathbf{H}, P)$ is also an abstract Wiener space. Let $\mathcal{F}_t = \sigma\{w_s;\ 0 \le s \le t\}$. The modified Malliavin covariance $\sigma(t) \in H \otimes H$ is defined by

$$\langle (tI_H + \sigma(t))h, g\rangle_H = \Big(D\langle X_t, h\rangle_H, D\langle X_t, g\rangle_H\Big)_{\mathbf{H}}, \quad h, g \in H,$$

where D is the $\mathbf{H}$-derivative.

Let E and F be separable Hilbert spaces. $L^k_{(2)}(E;F)$ denotes the Hilbert space consisting of Hilbert-Schmidt multi-linear operators from $\underbrace{E\times\cdots\times E}_{k}$ to F. We denote $L^1_{(2)}(E;F)$ simply by $L_{(2)}(E;F)$ and often identify $L_{(2)}(E;F)$ with $E\otimes F$. $\mathcal{L}^p_2(H\otimes E)$, $p\in(1,\infty)$ denotes the collection of $(\mathcal{F}_t)$-adapted $H\otimes E$-valued processes Φ such that

$$E\Big[\Big\{\int_0^T |\Phi_t|^2_{H\otimes E}dt\Big\}^{p/2}\Big]<\infty.$$

For $\Phi\in\mathcal{L}^p_2(H\otimes E)$, we can define the stochastic integral $\int_0^t \Phi_s dW_s$ with respect to the B-valued Wiener process W_t as an element of $L^p(\mathbf{W};E)$. In the case where $E=\mathbf{R}$, we often denote $\int_0^t \Phi_s dW_s$ by $\int_0^t\langle \Phi_s, dW_s\rangle_H$. $\mathcal{L}^p_1(E)$, $p\in(1,\infty)$ denotes the collection of $(\mathcal{F}_t)$-adapted E-valued processes ϕ such that

$$\int_0^T E[|\phi_t|^p_E]^{1/p}dt<\infty.$$

Given a separable Hilbert space E, we say that a map $f:B\to E$ is *continuously H-Fréchet differentiable* if there exists a continuous map $f^{(1)}:B\to L_{(2)}(H;E)$ such that

$$\lim_{|h|_H\to 0}\frac{|f(x+h)-f(x)-f^{(1)}(x)h|_E}{|h|_H}=0$$

for each $x\in B$. We can define inductively n-times continuously H-Fréchet differentiability and n-times H-Fréchet derivative $f^{(n)}$ by $f^{(n)}=(f^{(1)})^{(n-1)}$, $n=2,3,\ldots$. We denote by $\mathcal{CH}^\infty_b(E)$ the collection of infinitely many times continuously H-Fréchet differentiable function $f:B\to E$ such that $\sup_{x\in B}|f^{(n)}(x)|_{L^n_{(2)}(H;E)}<\infty$ for all $n\in\mathbf{Z}_+$.

Let us restate the main theorem in [1]:

Theorem 1.1. *Assume that $A\in\mathcal{CH}^\infty_b(H\otimes H)$, $b\in\mathcal{CH}^\infty_b(H)$ and (1.2) holds. Then the distribution of X_T is absolutely continuous with respect to μ_T. Moreover, its Radon-Nikodým density $\rho_T(x)$ with respect to μ_T satisfies*

$$\int_B \rho_T(x)(\log\rho_T(x)\vee 1)^\alpha\mu_T(dx)<\infty \tag{1.3}$$

for any $\alpha\in[0,1/2)$.

Let E be a separable Hilbert space. We say that a bounded bilinear operator $T:H\times H\to E$ is in $\mathcal{T}(E)$ if

$$|T|_{\mathcal{T}(E)}=\sup\sum_{i=1}^\infty |T(e_i,f_i)|_E<\infty,$$

where the supremum is taken over all complete orthonormal systems $\{e_i\}$ and $\{f_i\}$ in H. It is easy to see that $\mathcal{T}(E)$ forms a Banach space with the norm $|\cdot|_{\mathcal{T}(E)}$. For $F\in$

$L^k_{(2)}(H,E)$, $k\geq 2$, define a map $F_{[2]}: H\times H\to L^{k-2}_{(2)}(H;E)$ by $F_{[2]}(h,g)=F[h,g,\cdots,\cdot]$. We denote by $\mathcal{TH}^\infty_b(E)$ the collection of $f\in\mathcal{CH}^\infty_b(E)$ such that, for every $x\in B$ and $k=2,3,\dots$, the map $f^{(k)}_{[2]}(x): H\times H\to L^{k-2}_{(2)}(H;E)$ is in $\mathcal{T}(L^{k-2}_{(2)}(H;E))$ and

$$\sup_{x\in B}|f^{(k)}_{[2]}(x)|_{\mathcal{T}(L^{k-2}_{(2)}(H;E))}<\infty.$$

For $U,V\in\mathcal{CH}^\infty_b(H)$, the *Lie bracket* $[U,V]\in\mathcal{CH}^\infty_b(H)$ is defined by

$$[U,V](x)=V^{(1)}(x)[U(x)]-U^{(1)}(x)[V(x)],\quad x\in B.$$

Fix a complete orthonormal system $\{e_i\}$ in H. Let $V_i(x)=e_i+A(x)e_i$. For $j\in\mathbf{N}$, define

$$\Sigma_j=\{[V_{i_1},[V_{i_2}\cdots[V_{i_{j-1}},V_{i_j}],\cdots]];i_1,i_2,\dots,i_j=1,2,\cdots\}$$

and $\tilde\Sigma_j=\bigcup_{i=1}^j\Sigma_i$.

Throughout this paper, we assume the following conditions.

(C1) $A\in\mathcal{TH}^\infty_b(H\otimes H)$ and $b\in\mathcal{CH}^\infty_b(H)$.

(C2) There exists $N\in\mathbf{N}$ such that the closure of the linear subspace of H spanned by $\{V(0);V\in\tilde\Sigma_N\}$ coincides to H.

Our main theorem is following:

Theorem 1.2. *Under the conditions* **(C1)** *and* **(C2)**,

$$E[|(tI_H+\sigma(t))^{-1}|^p_{L(H;H)}]<\infty$$

holds for all $p\in(1,\infty)$ and $t\in(0,T]$. In particular, the distribution of X_T is absolutely continuous with respect to μ_T and (1.3) *holds.*

2 Preliminary results

Let

$$\tilde\sigma(t)=(I_H+\tilde J_t)(tI_H+\sigma(t))(I_H+\tilde J_t^*)-tI_H,$$

where $\tilde J_t\in H\otimes H$ is determined by

$$\begin{aligned}\tilde J_t h=&-\int_0^t(I_H+\tilde J_s)A^{(1)}(X_s)[h]_H dW_s-\int_0^t(I_H+\tilde J_s)b^{(1)}(X_s)[h]_H ds\\&+\int_0^t\Big(\sum_{i=1}^\infty(I_H+\tilde J_s)A^{(1)}(X_s)\Big[A^{(1)}(X_s)[h]_He_i\Big]_He_i\Big)ds,\quad h\in H.\end{aligned}$$

Then $E[|\tilde J_t|^p_{H\otimes H}]<\infty$ for all $p\in(1,\infty)$ and

$$tI_H+\tilde\sigma(t)=\int_0^t(I_H+\tilde J_s)(I_H+A(X_s))(I_H+A(X_s)^*)(I_H+\tilde J_s{}^*)ds \tag{2.1}$$

(cf. Section 3 in [1]). The following is well known (cf. Kusuoka-Stroock [3]).

Lemma 2.1. *Let E be a separable Hilbert space. Let $\Phi \in \mathcal{L}_2^p(H\otimes E)$ and $I_t = \int_0^t \Phi_s dW_s$. If $K \equiv \sup_{0\le t<\infty} \sup_{w\in \mathbf{W}} |\Phi_t(w)|_{H\otimes E} < \infty$, then*

$$E\left[\exp\left\{\frac{\alpha}{2K^2t}\sup_{0\le s\le t}|I_s|_E^2\right\}\right] \le \frac{e}{(1-\alpha)^{1/2}}$$

for every $\alpha \in (0,1)$ and $t \in (0,\infty)$. In particular, there exist constants $C, C' > 0$ such that

$$P\Big(\sup_{0\le s\le t}|I_s|_E \ge r\Big) < Ce^{-C'r^2/t}$$

for any $t > 0$ and $r > 0$

The following is easily derived from Lemma 2.1 (cf. Section 6 in [1]).

Lemma 2.2. *There exist constants $C, C' > 0$ such that*

$$P\Big(\sup_{0\le s\le t}|X_s|_B \ge r\Big) + P\Big(\sup_{0\le s\le t}|\tilde{J}_t|_{H\otimes H} \ge r\Big) \le Ce^{-C'r^2/t}$$

for any $t > 0$ and $r \in (0,1)$.

The following is also well known (see Section 6 in Shigekawa [5] for the proof).

Lemma 2.3. *Let M_t be an $\mathbf{R}$-valued local martingale with $M_0 = 0$ and $\langle M\rangle_t$ be its quadratic variation. Then*

$$P\Big[\sup_{0\le s\le t}|M_s| \ge \delta,\ \langle M\rangle_t \le \varepsilon\Big] \le 2e^{-\delta^2/2\varepsilon}$$

for any $t \ge 0$

Using the idea in Norris [4], we have the following.

Lemma 2.4. *Let $Y(t)$ be an $\mathbf{R}$-valued semimartingale expressed as*

$$Y(t) = y + \int_0^t \langle \psi(s), dW_s\rangle_H + \int_0^t a(s)ds$$

for some $\psi \in \mathcal{L}_2^p(H)$ and $a \in \mathcal{L}_1^p(\mathbf{R})$, $p \in (1,\infty)$. Then, for every $\alpha,\ \beta > 0$, there exists a constant $C = C(\alpha,\beta) > 0$ such that

$$P\Big[\int_0^\varepsilon Y(t)^2dt \le \alpha\varepsilon^{11n},\ \int_0^\varepsilon |\psi(t)|_H^2dt \ge \beta\varepsilon^n,\ \sup_{0\le t\le\varepsilon}(|\psi(t)|_H \vee |a(t)|) \le \varepsilon^{-n}\Big] \le Ce^{-1/2\varepsilon}$$

for any $\varepsilon > 0$ and $n \in \mathbf{N}$.

Proof. Let

$$\varepsilon_0 = \frac{1}{\alpha + 4\sqrt{\alpha}}\Big(1 \wedge \frac{\beta^2}{4}\Big).$$

It suffices to show the claim for $\varepsilon \in (0, \varepsilon_0)$. Using Itô formula, we have

$$\begin{aligned}
&\int_0^t |\psi(s)|^2 ds \\
&= Y(t)^2 - y^2 - 2\int_0^t Y(s) dY(s) \\
&= Y(t)^2 - y^2 - 2\int_0^t Y(s)a(s)ds - 2\int_0^t \langle Y(s)\psi(s), dW_s\rangle_H \\
&\le Y(t)^2 + 2\Big\{\int_0^t |Y(s)|^2 ds\Big\}^{1/2}\Big\{\int_0^t |a(s)|^2 ds\Big\}^{1/2} + 2\Big|\int_0^t \langle Y(s)\psi(s), dW_s\rangle_H\Big|.
\end{aligned} \tag{2.2}$$

If $\sup_{0\le t\le\varepsilon}\Big|\int_0^t \langle Y(s)\psi(s), dW_s\rangle_H\Big| < \sqrt{\alpha}\varepsilon^{4n}$, $\int_0^\varepsilon Y(t)^2 dt \le \alpha\varepsilon^{11n}$ and $\sup_{0\le t\le\varepsilon}(|\psi(t)|_H \vee |a(t)|) \le \varepsilon^{-n}$, then we have, by (2.2),

$$\begin{aligned}
&\int_0^\varepsilon \Big(\int_0^t |\psi(s)|_H^2 ds\Big) dt \\
&\le \int_0^\varepsilon Y(t)^2 dt + 2\varepsilon^{3/2-n}\Big\{\int_0^\varepsilon |Y(t)|^2 dt\Big\}^{1/2} + 2\varepsilon \sup_{0\le t\le\varepsilon}\Big|\int_0^t \langle Y(s)\psi(s), dW_s\rangle_H\Big| \\
&\le \alpha\varepsilon^{11n} + 2\sqrt{\alpha}\varepsilon^{(9n+3)/2} + 2\sqrt{\alpha}\varepsilon^{4n+1} \\
&\le (\alpha + 4\sqrt{\alpha})\varepsilon^{4n+1}.
\end{aligned}$$

Hence, letting $\gamma = (\alpha + 4\sqrt{\alpha})^{1/2}\varepsilon^{3n+1/2}$, we have

$$\begin{aligned}
\int_0^\varepsilon |\psi(t)|_H^2 dt &= \int_0^{\varepsilon-\gamma} |\psi(t)|_H^2 dt + \int_{\varepsilon-\gamma}^\varepsilon |\psi(t)|_H^2 dt \\
&\le \gamma^{-1}\int_{\varepsilon-\gamma}^\varepsilon \Big(\int_0^t |\psi(s)|_H^2 ds\Big) dt + \int_{\varepsilon-\gamma}^\varepsilon |\psi(t)|_H^2 dt \\
&\le \gamma^{-1}(\alpha + 4\sqrt{\alpha})\varepsilon^{4n+1} + \gamma\varepsilon^{-2n} \\
&= 2(\alpha + 4\sqrt{\alpha})^{1/2}\varepsilon^{n+1/2} < \beta\varepsilon^n.
\end{aligned}$$

Hence, by Lemma 2.2, we have

$$\begin{aligned}
&P\Big[\int_0^\varepsilon Y(t)^2 dt \le \alpha\varepsilon^{11n},\ \int_0^\varepsilon |\psi(t)|^2 dt \ge \beta\varepsilon^n,\ \sup_{0\le t\le\varepsilon}(|\psi(t)|_H \vee |a(t)|) \le \varepsilon^{-n}\Big] \\
&\le P\Big[\int_0^\varepsilon |Y(t)\psi(t)|_H^2 dt \le \alpha\varepsilon^{9n},\ \sup_{0\le t\le\varepsilon}\Big|\int_0^t \langle Y(s)\psi(s), dW_s\rangle_H\Big| \ge \sqrt{\alpha}\varepsilon^{4n}\Big] \\
&\le 2e^{-1/2\varepsilon^n}
\end{aligned}$$

$$\leq \quad 2e^{-1/2\varepsilon}.$$

□

Lemma 2.5. (1) *For any* $V \in \mathcal{CH}_b^\infty(H)$,

$$\sup_{x\in B}\sum_{i=1}^\infty |[V_i, V](x)|_H^2 < \infty.$$

(2) *If* $U, V \in \mathcal{TH}_b^\infty(H)$, *then* $[U,V] \in \mathcal{TH}_b^\infty(H)$. *In particular,* $\bigcup_{j=1}^\infty \Sigma_j \subset \mathcal{TH}_b^\infty(H)$.

Proof. (1) By definition, we have

$$[V_i, V](x) = V^{(1)}(x)[e_i] + V^{(1)}(x)[A(x)e_i] - A^{(1)}(x)[V(x)]e_i.$$

Note that $\sum_{i=1}^\infty |V^{(1)}(x)[e_i]|_H^2 \leq |V^{(1)}(x)|^2_{L_{(2)}(H;H)}$,

$$\sum_{i=1}^\infty |V^{(1)}(x)[A(x)e_i]|_H^2 \leq |V^{(1)}(x)|^2_{L_{(2)}(H;H)}|A(x)|^2_{H\otimes H}$$

and

$$\sum_{i=1}^\infty |A^{(1)}(x)[V(x)]e_i|_H^2 \leq |V(x)|_H^2|A^{(1)}(x)|^2_{L_{(2)}(H;H\otimes H)}.$$

These imply our assertion.

(2) Let $F(x) = U^{(1)}(x)[V(x)]$, $U, V \in \mathcal{TH}_b^\infty(H)$. It suffices to show that $F \in \mathcal{TH}_b^\infty(H)$. Let $\{e_i\}$ and $\{f_i\}$ be complete orthonormal systems in H. By Leibniz' rule, we have

$$\begin{aligned}
&\left|F_{[2]}^{(k)}(x)[e_i,f_i]\right|_{L_{(2)}^{k-2}(H;H)} \\
&\leq \sum_{l=0}^k \binom{k-2}{l}\left|U_{[2]}^{(k-l+1)}(x)[e_i,f_i]\right|_{L_{(2)}^{k-l-1}(H;H)}\left|V^{(l)}(x)\right|_{L_{(2)}^{l}(H;H)} \\
&\quad + \sum_{l=1}^k \binom{k-2}{l-1}\left|U_{[2]}^{(k-l+1)}(x)[e_i,\,\cdot\,]\right|_{L_{(2)}^{k-l}(H;H)}\left|V_{[2]}^{(l)}(x)[f_i,\,\cdot\,]\right|_{L_{(2)}^{l-1}(H;H)} \\
&\quad + \sum_{l=1}^k \binom{k-2}{l-1}\left|U_{[2]}^{(k-l+1)}(x)[f_i,\,\cdot\,]\right|_{L_{(2)}^{k-l}(H;H)}\left|V_{[2]}^{(l)}(x)[e_i,\,\cdot\,]\right|_{L_{(2)}^{l-1}(H;H)} \\
&\quad + \sum_{l=2}^k \binom{k-2}{l-2}\left|U^{(k-l+1)}(x)\right|_{L_{(2)}^{k-l+1}(H;H)}\left|V_{[2]}^{(l)}(x)[e_i,f_i]\right|_{L_{(2)}^{l-2}(H;H)}
\end{aligned}$$

for $k = 2, 3, \ldots$. Hence

$$\sum_{i=1}^\infty \left|F_{[2]}^{(k)}(x)[e_i,f_i]\right|_{L_{(2)}^{k-2}(H;H)}$$

$$\begin{aligned}
\leq\ & \sum_{l=0}^{k}\binom{k-2}{l}\Big|U_{[2]}^{(k-l+1)}(x)\Big|_{\mathcal{T}(L_{(2)}^{k-l-1}(H;H))}\Big|V^{(l)}(x)\Big|_{L_{(2)}^{l}(H;H)} \\
& +2\sum_{l=1}^{k}\binom{k-2}{l-1}\Big|U^{(k-l+1)}(x)\Big|_{L_{(2)}^{k-l+1}(H;H)}\Big|V^{(l)}(x)\Big|_{L_{(2)}^{l}(H;H)} \\
& +\sum_{l=2}^{k}\binom{k-2}{l-2}\Big|U^{(k-l+1)}(x)\Big|_{L_{(2)}^{k-l+1}(H;H)}\Big|V_{[2]}^{(l)}(x)\Big|_{\mathcal{T}(L_{(2)}^{l-2}(H;H))}.
\end{aligned}$$

This implies our assertion. □

3 Proof of Theorem

Let $S=\{h\in H;|h|_H=1\}$, $H_0=\{h\in H;(I_H+A(0)^*)h=0\}$ and $S_0=S\cap H_0$. For $m,j\in\mathbf{N}$, let

$$\Sigma_j^m=\{[V_{i_1},[V_{i_2}\cdots[V_{i_{j-1}},V_{i_j}],\cdots]];i_1,i_2,\ldots,i_j=1,2,\cdots,m\}$$

and $\tilde{\Sigma}_j^m=\bigcup_{i=1}^{j}\Sigma_i^m$. Define

$$I_j^m(t;h)=\int_0^t\sum_{V\in\tilde{\Sigma}_j^m}\langle(I+\tilde{J}_s)V(X_s),h\rangle_H^2ds,\quad h\in H.$$

Note that

$$I_1^m(t;h)=\int_0^t\sum_{i=1}^{m}\langle(I+\tilde{J}_s)V_i(X_s),h\rangle_H^2ds\leq\langle(tI_H+\tilde{\sigma}(t))h,h\rangle_H$$

for any $m\in\mathbf{N}$.

Lemma 3.1. *Let*

$$\tau_1=\inf\{t\geq 0\ ;|\tilde{J}_t|_{H\otimes H}\geq 1\}.$$

For $m,j\in\mathbf{N}$ and $\alpha>0$, there exists a constant $C=C(\alpha,m,j)>0$ such that

$$\sup_{h\in S}P\Big[\tau_1\geq\varepsilon,\quad I_{j+1}^m(\varepsilon;h)\geq\alpha\varepsilon^n,\quad I_j^m(\varepsilon;h)<\alpha\varepsilon^{11n}\Big]\leq Ce^{-1/2\varepsilon}$$

for any $\varepsilon>0$ and $n\in\mathbf{N}$.

Proof. Let $h\in S$. For $U\in\bigcup_{j=1}^{\infty}\Sigma_j$, define

$$Y_U(t)=\langle(I+\tilde{J}_t)U(X_t),h\rangle_H.$$

Using Itô formula, we have

$$Y_U(t)=\langle U(0),h\rangle_H+\int_0^t\langle\psi_U(s),dW_s\rangle_H+\int_0^t a_U(s)ds,$$

where

$$\psi_U(s) = \sum_{i=1}^{\infty} \langle (I + \tilde{J}_s)[V_i, U](X_s), h\rangle_H e_i$$

and

$$\begin{aligned} a_U(s) &= \langle (I_H + \tilde{J}_s)[b, U](X_s), h\rangle_H \\ &+ \frac{1}{2}\sum_{i=1}^{\infty} \langle (I_H + \tilde{J}_s) U^{(2)}(X_s)[e_i, (I_H + A(X_s)(I_H + A(X_s)^*)e_i], h\rangle_H \\ &+ \sum_{i=1}^{\infty} \langle (I_H + \tilde{J}_s) V_i^{(1)}(X_s)[[U, V_i](X_s)], h\rangle_H . \end{aligned}$$

By Lemma 2.5, there is a constant $K = K(m, j) > 0$ such that

$$\sup_{0 \le t \le \tau_1} (|\psi_U(t)|_H \vee |a_U(t)|) < K$$

for all $U \in \tilde{\Sigma}_j^m$. We may assume $\varepsilon < 1 \wedge K^{-1}$. Since $I_{j+1}^m(\varepsilon; h) \le \int_0^\varepsilon \sum_{U \in \tilde{\Sigma}_j^m} |\psi_U(s)|_H^2 ds$, we have

$$\begin{aligned} &\{I_{j+1}^m(\varepsilon; h) \ge \alpha\varepsilon^n, \quad I_j^m(\varepsilon; h) < \alpha\varepsilon^{11n}\} \\ &\quad \subset \bigcup_{U \in \tilde{\Sigma}_j^m} \left\{ \int_0^\varepsilon |\psi_U(s)|_H^2 ds \ge \alpha'\varepsilon^n, \quad \int_0^\varepsilon Y_U(t)^2 dt < \alpha\varepsilon^{11n} \right\} \end{aligned} \tag{3.1}$$

where $\alpha' = \alpha / \sharp(\tilde{\Sigma}_j^m)$. Here $\sharp(A)$ denotes the cardinal of a set A. But by Lemma 3.1, we have

$$\begin{aligned} &P\Big[\tau_1 \ge \varepsilon, \quad \int_0^\varepsilon |\psi_U(s)|_H^2 ds \ge \alpha'\varepsilon^n, \quad \int_0^\varepsilon Y_U(t)^2 dt < \alpha\varepsilon^{11n}\Big] \\ &\quad \le \quad P\Big[\int_0^\varepsilon |\psi_U(s)|_H^2 ds \ge \alpha'\varepsilon^n, \quad \int_0^\varepsilon Y_U(t)^2 dt < \alpha\varepsilon^{11n}, \\ &\qquad\qquad\qquad \sup_{0 \le t \le \varepsilon} (|\psi_U(t)|_H \vee |a(t)|) < \varepsilon^{-n}\Big] \\ &\quad \le \quad C(\alpha, m, j) e^{-1/2\varepsilon}. \end{aligned}$$

Combining this with (3.1), we have our assertion. □

Lemma 3.2. *There exists $M \in \mathbf{N}$ such that*

$$\min_{h \in S_0} \sum_{V \in \tilde{\Sigma}_N^M} \langle V(0), h\rangle_H^2 > 0.$$

Proof. Let $U_m = \Big\{h \in H; \sum_{V\in\tilde{\Sigma}_N^m} \langle V(0), h\rangle_H^2 > 0\Big\}$, $m \in \mathbf{N}$. Note that for each $m \in \mathbf{N}$ the map $h \mapsto \sum_{V\in\tilde{\Sigma}_N^m} \langle V(0), h\rangle_H^2$ is continuous. Then each U_m is a open set and $S_0 \subset \bigcup_{m=1}^\infty U_m$ by virtue of **(C2)**. Since S_0 is compact, we can find M such that $S_0 \subset U_M$. Namely, $\sum_{V\in\tilde{\Sigma}_N^M} \langle V(0), h\rangle_H^2 > 0$ for all $h \in S_0$. Hence $\min_{h\in S_0} \sum_{V\in\tilde{\Sigma}_N^M} \langle V(0), h\rangle_H^2 > 0$. □

Throughout the sequel, we fix $M \in \mathbf{N}$ such as in Lemma 3.2. Let

$$\alpha_N = \frac{1}{4} \min_{h\in S_0} \sum_{V\in\tilde{\Sigma}_N^M} \langle V(0), h\rangle_H^2 > 0.$$

We can find $\eta \in (0,1]$ such that

$$\sum_{V\in\tilde{\Sigma}_N^M} |(I_H + J)V(x) - V(0)|_H^2 < \alpha_N$$

for any $x \in B$ and $J \in H \otimes H$ with $|x|_B < \eta$ and $|J|_{H\otimes H} < \eta$. Define

$$\tau = \inf\{t \in [0,T] \ ; \ |X_t| \geq \eta \text{ or } |\tilde{J}_t| \geq \eta\}.$$

Lemma 3.3. *There exist constants $C, C' > 0$ such that*

$$\sup_{h\in S_0} P\Big[I_1^M(\varepsilon; h) < \alpha_N \varepsilon^{11^{N-1}}\Big] \leq Ce^{-C'/\varepsilon}$$

for any $\varepsilon > 0$.

Proof. Let $h \in S_0$. If $\tau \geq \varepsilon$, then

$$\begin{aligned} I_N^M(\varepsilon; h) &= \int_0^\varepsilon \sum_{V\in\tilde{\Sigma}_N^M} \langle (I + \tilde{J}_t)V(X_t), h\rangle_H^2 dt \\ &\geq \frac{1}{2}\varepsilon \sum_{V\in\tilde{\Sigma}_N^M} \langle V(0), h\rangle_H^2 - \int_0^\varepsilon \sum_{V\in\tilde{\Sigma}_N^M} |(I_H + \tilde{J}_t)V(X_t) - V(0)|_H^2 dt \\ &\geq \alpha_N \varepsilon. \end{aligned}$$

Hence we see that

$$\{\tau \geq \varepsilon\} \subset \{I_N^M(\varepsilon; h) \geq \alpha_N \varepsilon\}. \tag{3.2}$$

Let

$$\mathcal{W}(\varepsilon; h) = \bigcup_{j=1}^{N-1} \{\tau \geq \varepsilon, \ \ I_{j+1}^M(\varepsilon; h) \geq \alpha_N \varepsilon^{11^{N-j-1}}, \ \ I_j^M(\varepsilon; h) < \alpha_N \varepsilon^{11^{N-j}}\}.$$

Then, by Lemma 3.1 we see that there exists a constant $C > 0$ such that

$$\sup_{h\in S_0} P(\mathcal{W}(\varepsilon; h)) \leq Ce^{-1/2\varepsilon}. \tag{3.3}$$

If $w \notin \mathcal{W}(\varepsilon; h)$ and $I_1^M(\varepsilon; h) < \alpha_N \varepsilon^{11^{N-1}}$, then $I_j^M(\varepsilon; h) < \alpha_N \varepsilon$ for $j = 1, 2, \ldots, N$. Therefore we see that

$$\{I_1^M(\varepsilon, h) < \alpha_N \varepsilon^{11^{N-1}}\} \cap \mathcal{W}(\varepsilon; h)^c \quad \subset \quad \{I_N^M(\varepsilon, h) < \alpha_N \varepsilon\}.$$

Hence, by (3.2), we have

$$P(I_1^M(\varepsilon, h) < \alpha_N \varepsilon^{11^{N-1}}) \quad \leq \quad P(\tau < \varepsilon) + P(\mathcal{W}(\varepsilon; h)).$$

So we have our assertion by (3.3) and Lemma 2.3. □

Let $c_N = \dfrac{\alpha_N}{16(1+\sqrt{2})(1+A_\infty)^2}$, where $A_\infty = \sup_{x \in B} |A(x)|_{H \otimes H}$. Define

$$\tilde{S}(\varepsilon) = \{h \in S;\ |(I_H - P_0)h|_H < c_N \varepsilon^{11^{N-1}-1}\}$$

where P_0 is the orthogonal projection from H onto H_0.

Proposition 3.4. *There exist constants $C_1, C_2 > 0$ such that*

$$P\Big(\inf_{h \in \tilde{S}(\varepsilon)} I_1^M(\varepsilon; h) < \frac{1}{2}\alpha_N \varepsilon^{11^{N-1}}\Big) \leq C_1 \varepsilon^{-C_1} \exp\left(-C_2/\varepsilon\right)$$

for any $\varepsilon > 0$.

Proof. Let $n_0 = \dim H_0$. Since S_0 is contained in an n_0-dimensional hypercube with side-length 2, for every $\delta > 0$ there exist $h_1, h_2, \ldots, h_d \in S_0$ such that

$$S_0 \subset \bigcup_{k=1}^d B(h_k; \delta)$$

and $d \leq (4\sqrt{n_0})^{n_0} \delta^{-n_0}$, where $B(h_k; \delta) = \{h \in H; |h - h_k|_H \leq \delta\}$. Applying this fact for $\delta = c_N \varepsilon^{11^{N-1}-1}$, we can find $h_1, h_2, \ldots, h_d \in S_0$ such that

$$\tilde{S}(\varepsilon) \subset \bigcup_{k=1}^d B\Big(h_k; (1+\sqrt{2}) c_N \varepsilon^{11^{N-1}-1}\Big) \tag{3.4}$$

and $d \leq C\varepsilon^{-C'}$, where $C = (4\sqrt{n_0})^{n_0} c_N$ and $C' = (11^{N-1} - 1)n_0$. On the other hand, if $\tau \geq \varepsilon$, then

$$\begin{aligned}
&|I_1^M(\varepsilon; h) - I_1^M(\varepsilon; g)| \\
&= \Big| \int_0^\varepsilon \sum_{i=1}^M \Big\{ \langle (I + \tilde{J}_s)V_i(X_s), h\rangle_H^2 - \langle (I + \tilde{J}_s)V_i(X_s), g\rangle_H^2 \Big\} ds \Big| \\
&\leq \int_0^\varepsilon \Big\{ \sum_{i=1}^M \langle (I + \tilde{J}_s)V_i(X_s), h+g\rangle_H^2 \Big\}^{1/2} \Big\{ \sum_{i=1}^M \langle (I + \tilde{J}_s)V_i(X_s), h-g\rangle_H^2 \Big\}^{1/2} ds \qquad (3.5) \\
&\leq \int_0^\varepsilon \Big| (I_H + A(X_s)^*)(I + \tilde{J}_s^*)(h+g) \Big| \Big| (I_H + A(X_s)^*)(I + \tilde{J}_s^*)(h-g) \Big| ds \\
&\leq (1 + A_\infty)^2 (1+\eta)^2 |h+g|_H |h-g|_H \varepsilon \\
&\leq 8(1 + A_\infty)^2 |h-g|_H \varepsilon
\end{aligned}$$

for $h, g \in S$. By (3.4) and (3.5), for every $h \in \tilde{S}(\varepsilon)$, there exists $h_k \in \{h_1, h_2, \dots, h_d\}$ such that if $\tau \geq \varepsilon$, then

$$|I_1^M(\varepsilon; h) - I_1^M(\varepsilon; h_k)| \leq \frac{1}{2}\alpha_N \varepsilon^{11^{N-1}}.$$

Therefore, we have

$$\begin{aligned}
&P\Big(\inf_{h\in\tilde{S}(\varepsilon)} I_1^M(\varepsilon; h) < \frac{1}{2}\alpha_N \varepsilon^{11^{N-1}}\Big) \\
&\quad\leq\ P(\tau < \varepsilon) + P\Big(\tau \geq \varepsilon, \quad \inf_{h\in\tilde{S}(\varepsilon)} I_1^M(\varepsilon; h) < \frac{1}{2}\alpha_N \varepsilon^{11^{N-1}}\Big) \\
&\quad\leq\ P(\tau < \varepsilon) + \sum_{k=1}^{d} P\Big(\tau \geq \varepsilon, \quad I_1^M(\varepsilon; h_k) < \alpha_N \varepsilon^{11^{N-1}}\Big) \\
&\quad\leq\ P(\tau < \varepsilon) + C\varepsilon^{-C'} \sup_{h\in S_0} P\Big(I_1^M(\varepsilon; h) < \alpha_N \varepsilon^{11^{N-1}}\Big).
\end{aligned}$$

So we have our assertion by Lemma 2.3 and Lemma 3.3. □

Proposition 3.5. *For each $t \in (0, T]$, there exist constants $C_3, C_4 > 0$ such that*

$$P\Big(\inf_{h\in S\setminus\tilde{S}(\varepsilon)} \langle (tI_H + \tilde{\sigma}(t))h, h\rangle_H < \varepsilon^{4\cdot 11^{N-1}-1}\Big) \leq C_3 \exp(-C_4/\varepsilon)$$

for any $\varepsilon > 0$.

Proof. Let $\alpha = 2\cdot 11^{N-1}$. Since $A(0)$ is a compact operator, we can find a constant $\lambda_0 > 0$ such that $|(I_H + A(0)^*)h|_H \geq \lambda_0 |h|_H$ holds for all $h \in H_0^\perp$. Then, for all $h \in S\setminus\tilde{S}(\varepsilon)$, we have

$$\begin{aligned}
|(I_H + A(0)^*)h|_H &= |(I_H + A(0)^*)(I_H - P_0)h|_H \\
&\geq \lambda_0 |(I_H - P_0)h|_H \qquad (3.6)\\
&\geq \lambda_0 c_N \varepsilon^{(\alpha-2)/2}.
\end{aligned}$$

Moreover, using Itô formula, we have

$$A(X_t) - A(0) = I_t + \int_0^t a(s)ds \qquad (3.7)$$

where $I_t = \int_0^t A^{(1)}(X_s)[I_H + A(X_s)]dW_s$ and

$$a(s) = A^{(1)}(X_s)[b(X_s)] + \frac{1}{2}\sum_{i=1}^{\infty} A^{(2)}(X_s)[e_i, (I_H + A(X_s)(I_H + A(X_s)^*)e_i].$$

Since $A \in \mathcal{TH}_b^\infty(H \otimes H)$, there exists a constant $K > 0$ such that $\sup_{0\leq t\leq T} |a(t)|_{H\otimes H} \leq K$. Let

$$\varepsilon_0 = t \wedge \frac{3\lambda_0^2 c_N^2}{4(K^2 + 3)}.$$

We may assume that $0 < \varepsilon < \varepsilon_0$. Then $\varepsilon^\alpha \le t$ and

$$\frac{1}{2}\lambda_0^2 c^2 \varepsilon^{2\alpha-2} - \frac{2}{3}K^2\varepsilon^{3\alpha} \ge 2\varepsilon^{2\alpha-1}. \tag{3.8}$$

By (3.6), (3.7) and (3.8), we have

$$\begin{aligned}
&\inf_{h\in S\setminus\tilde{S}(\varepsilon)} \langle (tI_H + \tilde{\sigma}(t))h, h\rangle_H \\
&\ge \inf_{h\in S\setminus\tilde{S}(\varepsilon)} \int_0^{\varepsilon^\alpha} |(I_H + A(X_s)^*)(I_H + \tilde{J}_s^*)h|_H^2 ds \\
&\ge \frac{1}{2}\int_0^{\varepsilon^\alpha} |(I_H + A(0)^*)h|_H^2 dt - \int_0^{\varepsilon^\alpha} \Big\{|A(0) - A(X_s)|_{H\otimes H} + (1 + A_\infty)|\tilde{J}_s|_{H\otimes H}\Big\}^2 ds \\
&\ge \frac{1}{2}\lambda_0^2 c_N^2 \varepsilon^{2\alpha-2} - 2\int_0^{\varepsilon^\alpha} K^2 s^2 ds - 2\int_0^{\varepsilon^\alpha} \{|I_s|_{H\otimes H} + (1+A_\infty)|\tilde{J}_s|_{H\otimes H}\}^2 ds \\
&\ge 2\varepsilon^{2\alpha-1} - 2\sup_{0\le s\le \varepsilon^\alpha} \{|I_s|_{H\otimes H} + (1+A_\infty)|\tilde{J}_s|_{H\otimes H}\}^2 \varepsilon^\alpha,
\end{aligned}$$

where $A_\infty = \sup_{x\in B}|A(x)|_{H\otimes H}$. Hence, by Lemma 2.1 and Lemma 2.2,

$$\begin{aligned}
&P\Big(\inf_{h\in S\setminus\tilde{S}(\varepsilon)} \langle (tI_H + \tilde{\sigma}(t))h, h\rangle_H < \varepsilon^{2\alpha-1}\Big) \\
&\le P\Big(\sqrt{2}\sup_{0\le s\le\varepsilon^\alpha} \{|I_s|_{H\otimes H} + (1+A_\infty)|\tilde{J}_s|_{H\otimes H}\} > \varepsilon^{(\alpha-1)/2}\Big) \\
&\le C_3 e^{-C_4/\varepsilon}
\end{aligned}$$

for some constants $C_3, C_4 > 0$. □

Now, let us prove Theorem 1.2.

Since $E[|\tilde{J}_t|_{H\otimes H}^p] < \infty$ for all $p \in (1,\infty)$, it suffices to show the claim for $\tilde{\sigma}(t)$ instead of $\sigma(t)$. By Proposition 3.4 and Proposition 3.5, if $\varepsilon \le t$ and $\varepsilon^{3\cdot 11^{N-1}-1} \le \frac{1}{2}\alpha_N$, then

$$\begin{aligned}
&P\Big(\inf_{h\in S}\langle (tI_H + \tilde{\sigma}(t))h, h\rangle_H < \varepsilon^{4\cdot 11^{N-1}-1}\Big) \\
&\quad\le P\Big(\inf_{h\in\tilde{S}(\varepsilon)}\langle (tI_H + \tilde{\sigma}(t))h, h\rangle_H < \varepsilon^{4\cdot 11^{N-1}-1}\Big) \\
&\qquad\qquad + P\Big(\inf_{h\in S\setminus\tilde{S}(\varepsilon)}\langle (tI_H + \tilde{\sigma}(t))h, h\rangle_H < \varepsilon^{4\cdot 11^{N-1}-1}\Big) \\
&\quad\le P\Big(\inf_{h\in\tilde{S}(\varepsilon)} I_1^M(\varepsilon; h) < \frac{1}{2}\alpha_N \varepsilon^{11^{N-1}}\Big) \\
&\qquad\qquad + P\Big(\inf_{h\in S\setminus\tilde{S}(\varepsilon)}\langle (tI_H + \tilde{\sigma}(t))h, h\rangle_H < \varepsilon^{4\cdot 11^{N-1}-1}\Big) \\
&\quad\le C\varepsilon^{-C} e^{-C'/\varepsilon}
\end{aligned}$$

for some constants $C, C' > 0$. Hence, for given $p \in (1, \infty)$, there exists $n_0 \in \mathbf{N}$ such that if $n \geq n_0$, then

$$P\Big(\inf_{h \in S} \langle (tI_H + \tilde{\sigma}(t))h, h \rangle_H < 2^{-n} \Big) \leq 2^{-n(p+1)}.$$

Hence we have

$$\begin{aligned}
&E[|(tI_H + \tilde{\sigma}(t))^{-1}|^p_{L(H;H)}] \\
&\quad \leq 1 + \sum_{n=0}^{\infty} E[|(tI_H + \tilde{\sigma}(t))^{-1}|^p, 2^{-(n+1)} \leq \inf_{h \in S} \langle (tI_H + \tilde{\sigma}(t))h, h \rangle_H < 2^{-n} \;] \\
&\quad \leq 1 + \sum_{n=0}^{\infty} 2^{(n+1)p} P\Big(\inf_{h \in S} \langle (tI_H + \tilde{\sigma}(t))h, h \rangle_H < 2^{-n} \Big) \\
&\quad \leq 1 + \sum_{n=0}^{n_0 - 1} 2^{(n+1)p} + \sum_{n=n_0}^{\infty} 2^{(n+1)p} 2^{-n(p+1)} \\
&\quad = 1 + \sum_{n=0}^{n_0 - 1} 2^{(n+1)p} + \sum_{n=n_0}^{\infty} 2^{p-n} < \infty.
\end{aligned}$$

This completes the proof of Theorem 1.2.

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