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Abstract

We formulate homogeneous filtered value measures with Fatou, comonotone, and law invariant properties under multiperiod "Poisson-type" collective risk processes as a system of differential equations. We also derive a limit theorem for homogeneous filtered value measures under the same processes and calculate some numerical examples.

Keywords: coherent value measure, collective risk process, multiperiod, homogeneous filtered value measures.

1 Introduction

Artzner et al. [1997] introduced the concept of coherent risk-adjusted value measure (a.k.a. coherent risk measure), an axiomatic approach to determine an acceptance set of maps from L^∞ to $\mathbf{R}$ supposed to satisfy some coherence requirement for expressing the risk-adjusted value of the future (random) net worth. The definition is as follows.

Definition 1.1 *We say a map $\phi : L^\infty \to \mathbf{R}$ is a coherent value measure, if the following conditions are satisfied.*
(1) *If $X \geq 0$, then $\phi(X) \geq 0$.*
(2) $\phi(X_1 + X_2) \geq \phi(X_1) + \phi(X_2)$.
(3) *for $\lambda > 0$, we have $\phi(\lambda X) = \lambda\phi(X)$.*
(4) *for any constant c, we have $\phi(X + c) = \phi(X) + c$.*

Delbaen [2000] proved that, on general probability spaces, ϕ is a coherent risk measure with the Fatou property if and only if there is a set of probability measures $\mathcal{Q}$ such that any $Q \in \mathcal{Q}$ satisfies $Q \ll P$ and for any $X \in L^\infty$, $\phi(X) = \inf\{E^Q[X]; Q \in \mathcal{Q}\}$. Furthermore, Kusuoka [2001] specified the concept as follows. Let $\mathcal{L}$ be the set of probability measures on $\mathbf{R}$, $\mathcal{L}_p$, $p \in [1, \infty)$, be $\{\nu \in \mathcal{L}; \int_{\mathbf{R}} |x|^p \nu(dx) < \infty\}$, and $\mathcal{L}_\infty = \{\nu \in \mathcal{L}; \nu(\mathbf{R} \setminus [-M, M]) = 0$ for some $M > 0\}$. Also, let $\mathcal{M}_{[0,1]}$ be the set of probability measures on $[0, 1]$. Then he

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defined a value measure as a map from $\mathcal{L}_p$ to $\mathbf{R}$, $1 \leq p \leq \infty$, and proved that $\eta = \eta^m$ is a coherent value measure with law invariant, comonotone and the Fatou property if and only if there exists $m \in \mathcal{M}_{[0,1]}$ such that

$$\eta(\mu) = \int_0^1 \eta_\alpha(\mu) m(d\alpha), \quad \mu \in \mathcal{L}_p, p \in [1, \infty], \tag{1.1}$$

where η_α, usually called Conditional Value at Risk (CVaR), Tail-VaR (TVaR), or Conditional Tail Expectation (CTE), is defined as,

$$\eta_\alpha(\nu) = \frac{1}{\alpha} \int_0^\alpha Z(x; \nu) dx \quad \nu \in \mathcal{L}_1$$

with $Z(x; \nu) = \inf\{z; F_\nu(z) > x\}$. $Z(\cdot, \nu) : [0, 1] \to \mathbf{R}$ is non-decreasing and right continuous, and the probability law of $Z(\cdot, \nu)$ under Lebesgue measure on $[0, 1]$ is ν. We use η as a function from L^p to $\mathbf{R}$ as well; $\eta(X) = \eta(\mu_X)$ where μ_X denotes the probability measure of the random variable $X \in \mathcal{L}_p$.

Coherent value measure was originally designed for determining the risk-adjusted value (or risk) of financial instruments. On the other hand, actuarial mathematics have been using a similar approach for determining the insurance premium calculation. It is called the premium calculation principle; a functional H from the set of insurance risks (non-negative random variables in L^∞) to $[0, \infty]$. Gerber [1979] introduced five desireble properties and checked whether traditional risk measures were satisfied with the properties. Wang, et al. [1997] took an axiomatic approach and proved that H has a Choquet integral representation if and only if H satisfies some axioms they defined. Kusuoka [2001] showed that the equation (1.1) can be expressed as the same representation as H. Thus, in this paper, we use the representation as in the equation (1.1).

Recently, the concept of value measure is being applied in a dynamic framework. For example, Artzner et al. [2002] expanded the concept of coherent value measure to an analysis of multiperiod financial risk. The idea of expansion to dynamic setting is practically useful. In insurance business, for example, insurers have to keep reserves, the value necessary to cover the future claims less the future insurance premiums of the existing policies. As in usual actuarial method, reserves can be represented by recursive formula of the reserves at next term and cashflows during now and then. This representation is just same as the idea for multiperiod value measure. We will formulate the idea as follows.

Let $(\Omega, \mathcal{F}, P)$ be a standard probability space and $T \in (0, \infty)$. For any integrable random variable X and a sub-σ algebra $\mathcal{G}$, we define a $\mathcal{G}$-measurable random variable $\eta(X|\mathcal{G})$, which we call conditional value measure, by

$$\eta(X|\mathcal{G}) = \eta\left(P\left(X \in dx|\mathcal{G}\right)\right).$$

We partition T into N and let $h = \frac{T}{N}$ and $t_n = nh$, $n = 0, 1, 2, \ldots, N$. Let $\{\mathcal{F}_{t_n}\}_{n=0}^N$ be a filtration and X be an $\mathcal{F}_T$-measurable integrable random variable. Then we can define an adapted process $\{Z_n = Z(n; N, m)\}_{n=0}^N$ inductively using η defined above by

$$\begin{aligned} Z_N &= X, \\ Z_{n-1} &= \eta\left(Z_n|\mathcal{F}_{t_{n-1}}\right), \quad n = N-1, N-2, \ldots, 1. \end{aligned} \tag{1.2}$$

We denote an $\mathcal{F}_0$-measurable random variable Z_0 by $\eta(X|\{\mathcal{F}_{t_n}\}_{n=0}^N)$ and call the map a filtered value measure.

Next we specify an N-period collective risk process for expressing insurance portfolio behavior. Let $\ell \in \mathbf{N}$ be the number of policies at time zero and $L = \{1, 2, \ldots, \ell\}$. We assume that all contracts start at time zero and will end at time T. When a claim occurs, insurer pays the claim amount immediately (no time lag)[1]. Let τ_i, a stopping time, be the timing of the claim occurrence such that

$$P(\tau_i > t) = \exp(-\lambda_i t), \quad t > 0$$

for $\lambda_i > 0$, $i \in L$ and $D_i : \Omega \to \mathbf{R}$, a random variable with distribution function ν_i, be the net claim amount at time τ_i. Note that D_i is negative if net payment from the insurer to i-th policyholder is positive. We assume that D_i and τ_i, $i \in L$, are independent and that a contract would terminate just after the claim occurrence[2]; i.e., contract term for contract j is $[0, \tau_j \wedge T]$.

We also define insurance premiums. Insurance premium can be collected in various ways, but in this paper we assume that premium is collected continuously until the contract terminates. Let $q_i \in \mathbf{R}$ be the premium per unit time; i.e., i-th policyholder will pay $q_i(t \wedge \tau_i)$ to the insurer until time t. Then X_t^i, total cashflow for i-th policy in $[0, t]$ is

$$X_t^i = q_i(t \wedge \tau_i) + D_i 1_{\{\tau_i \leq t\}},$$

and total cashflow for the portfolio in $[0, t]$ is

$$X_t^I = \sum_{i \in I} X_t^i, \quad I \subset L.$$

We set $\mathcal{F}_t$ as $\sigma\{X_s^i : s \leq t, i \in L\}$. Then, the value measure for the insurance portfolio can be expressed as in the equation (1.2) by replacing X into X_T^L. Finally we define the reserve at time t_n as a difference between Z_n and $X_{t_n}^L$.

Let $\mathcal{I} = \{I : I \subset L\}$, $q^I = \sum_{i \in I} q_i$ and $\lambda^I = \sum_{i \in I} \lambda_i$. For any $\nu \in \mathcal{L}$ and $a \in \mathbf{R}$, $\nu + a$ denotes a probability measure on $\mathbf{R}$, given by

$$(\nu + a)(A) = \nu\left(\{x \in \mathbf{R}; x - a \in A\}\right) \quad \text{for any Borel set } A,$$

and we introduce a function F from $\mathcal{L}$ to $\mathbf{R}$ such that

$$F(\nu; a, b) = \inf_{\gamma} \left\{ \int_0^{\gamma} Z(x; \nu) dx; a \leq \gamma \leq b \right\}, \quad 0 \leq a \leq b \leq 1. \tag{1.3}$$

Also, let $\vec{x} = \{x_I : I \in \mathcal{I}\}$,

$$\nu^I = \nu^I(\vec{x}) = \frac{\sum_{j \in I} \lambda_j \left(\nu_j + x_{I \setminus \{j\}}\right)}{\lambda^I}, \tag{1.4}$$

and

$$\Phi_h^I(\vec{x}; m) = q^I + \lambda^I \int_0^1 \frac{1}{\alpha} F\left(\nu^I - x_I; 0 \vee \left(1 - \frac{1 - \alpha}{\lambda^I h}\right), \frac{\alpha}{\lambda^I h} \wedge 1\right) m(d\alpha). \tag{1.5}$$

Regarding Φ_h^I and m^N, $N = 1, 2, \ldots$, we assume the following.

[1]In real situation, there is a time lag between the claim occurrence and actual payment. Also, insurer may not be able to pay the amount because of the financial difficulty. But we do not take such situations into consideration in this paper.

[2]Usually, life insurance contract would terminate when insurance event occurs, while non-life insurance contract usually does not terminate in the middle of the contract term even though the insurance event occurs.

(A-1) $\Phi^I_{T/N}\left(\vec{x}; m^N\right)$ converges to a function $\Phi^I(\vec{x})$ for any $\vec{x}$ as $N \to \infty$.

(A-2) $\displaystyle\sup_N \int_0^1 \left(\lambda^I T N^{-1} \vee \alpha\right)^{-1} m^N(d\alpha) = C^* < \infty.$

(A-3) There exists $p \in [1, 2)$ such that $\nu_i \in \mathcal{L}_p$ for any $i \in L$ and that

$$\sup_N \int_0^1 \left(\alpha^{-1/p} \wedge \frac{(1-\alpha)^{1-1/p}}{\alpha}\right) m^N(d\alpha) < \infty.$$

We show later (Corollary 2.8) that under Assumptions (A-1) and (A-2), Φ^I satisfies Lipschitz condition. Thus a following system of equations for a value measure defined in equation (1.1) has a C^1 solution.

$$\frac{dx(t,I)}{dt} = \Phi^I(\vec{x}(t)), \quad I \in \mathcal{I}, \tag{1.6}$$

with initial conditions

$$\begin{aligned} x(0,I) &= 0, \quad I \in \mathcal{I}, \\ x(t,\emptyset) &= 0, \quad t \in (0,T), \end{aligned} \tag{1.7}$$

where

$$\vec{x}(t) = \{x(t,I) : I \in \mathcal{I}\}. \tag{1.8}$$

Then our main result is as follows.

Theorem 1.2 *Assume that* (A-1), (A-2) *and* (A-3) *hold. Let* $x(t,I)$ *be the solution of the system of equations* (1.6) *with initial conditions* (1.7)*. Then*

$$\eta^{m^N}(X^I_T | \{\mathcal{F}_{t_n}\}_{n=0}^N) \to x(T,I) \quad N \uparrow \infty, \quad I \in \mathcal{I}.$$

2 Preparations

In order to prove Theorem 1.2, we will prepare some estimates. Most of them are cited from Kusuoka and Morimoto [2004]. We will show proofs since they are simple. For $m \in \mathcal{M}_{[0,1]}$ let

$$\Delta_p(m) = \int_0^1 \left(\alpha^{-1/p} \wedge \frac{(1-\alpha)^{1-1/p}}{\alpha}\right) m(d\alpha).$$

Then we have the following.

Proposition 2.1 (1) *For any* $0 < \alpha \leq 1$ *and* $\nu \in \mathcal{L}_p$, $p \in (1, \infty]$,

$$|\eta_\alpha(\nu)| \leq \alpha^{-1/p} \left(\int_0^\alpha |Z(x;\nu)|^p dx\right)^{1/p}.$$

(2) *For any* $0 < \alpha \leq 1$ *and* $\nu \in \mathcal{L}_p$, $p \in (1, \infty]$, *with* $\int_{\mathbf{R}} x\nu(dx) = 0$,

$$|\eta_\alpha(\nu)| \leq \frac{(1-\alpha)^{1-1/p}}{\alpha} \left(\int_0^1 |Z(x;\nu)|^p dx\right)^{1/p}.$$

Proof. For $0 < \alpha \le 1$, we have

$$|\eta_\alpha(\nu)| = \frac{1}{\alpha}\left|\int_0^\alpha Z(x;\nu)dx\right| \le \frac{1}{\alpha}\alpha^{1-1/p}\left(\int_0^\alpha |Z(x;\nu)|^p dx\right)^{1/p}.$$

Also, if $\int_{\mathbf{R}} x\nu(dx) = 0$,

$$|\eta_\alpha(\nu)| = \frac{1}{\alpha}\left|\int_\alpha^1 Z(x;\nu)dx\right| \le \frac{(1-\alpha)^{1-1/p}}{\alpha}\left(\int_0^1 |Z(x;\nu)|^p dx\right)^{1/p}.$$

Hence we get the result. ∎

Corollary 2.2 *For any $\nu \in \mathcal{L}_p$, $p \in (1,\infty]$, with $\int_{\mathbf{R}} x\nu(dx) = 0$,*

$$|\eta^m(\nu)| \le \Delta_p(m)\left(\int_{\mathbf{R}} |x|^p \nu(dx)\right)^{1/p}.$$

Corollary 2.3 *If $\nu \in \mathcal{L}_p$, $p \in (1,\infty]$, then $\alpha^{1/p}\eta_\alpha(\nu)$ is continuous on $\alpha \in [0,1]$, and*

$$\alpha^{1/p}\eta_\alpha(\nu) \to 0, \quad \alpha \to 0.$$

Proposition 2.4 *For any $X_1, X_2 \in L^p$, $p \ge 1$,*

$$|\eta^m(X_1) - \eta^m(X_2)| \le (1 + 2\Delta_p(m))E\left[|X_1 - X_2|^p\right]^{1/p}.$$

Proof. From (3) of Definition 1.1 and Corollary 2.2,

$$\begin{aligned}\eta^m(X_1) - \eta^m(X_2) &\le |-\eta^m(X_2 - X_1) + E[X_2 - X_1]| + |E[|X_2 - X_1|]| \\ &\le \Delta_p(m)E[|X_2 - X_1 - E[X_2 - X_1]|^p]^{1/p} + E[|X_2 - X_1|^p]^{1/p} \\ &\le (1 + 2\Delta_p(m))E[|X_1 - X_2|^p]^{1/p}.\end{aligned}$$

From the symmetry, we have our assertion. ∎

Proposition 2.5 *Let $\nu, \mu \in \mathcal{L}$. Then for any λ, $\alpha \in [0,1]$, we have*

$$\begin{aligned}&\int_0^\alpha Z(x;\lambda\nu + (1-\lambda)\mu)dx \\ &= \inf\left\{\lambda\int_0^\beta Z(x;\nu)dx + (1-\lambda)\int_0^\gamma Z(x;\mu)dx; \beta,\gamma \in [0,1], \lambda\beta + (1-\lambda)\gamma = \alpha\right\}.\end{aligned}$$

Proof. Let X be a random variable defined on $[0,1)$ such that for $\lambda \in (0,1)$,

$$X = Z\left(\lambda^{-1}x;\nu\right)1_{[0,\lambda)}(x) + Z\left((1-\lambda)^{-1}(x-\lambda);\mu\right)1_{[\lambda,1)}(x), \quad x \in [0,1).$$

Then the distribution law of X under Lebesgue measure is $\lambda\nu + (1-\lambda)\mu$. Note that

$$\int_0^\alpha Z(x;\lambda\nu+(1-\lambda)\mu)dx = \inf\left\{\int_A X(x)dx; A \in \mathcal{B}([0,1)), \int_A dx = \alpha\right\}$$
$$= \inf\left\{\int_0^a Z\left(\frac{x}{\lambda};\nu\right)dx + \int_\lambda^{\lambda+b} Z\left(\frac{x-\lambda}{1-\lambda};\mu\right)dx; a \in [0,\lambda), b \in [0,1-\lambda), a+b=\alpha\right\}.$$

This implies our assertion. ∎

Corollary 2.6 $\nu, \mu \in \mathcal{L}$. *Then for any* $\lambda \in [0,1]$ *and* $0 \le a \le b \le 1$, *we have*

$$F(\lambda\nu+(1-\lambda)\mu;a,b)$$
$$= \inf\left\{\lambda\int_0^\beta Z(x;\nu)dx + (1-\lambda)\int_0^\gamma Z(x;\mu)dx; \beta,\gamma \in [0,1], \lambda\beta+(1-\lambda)\gamma \in [a,b]\right\}.$$

We also show the Lipschitz property of Φ_h^I defined on (1.5).

Proposition 2.7 *Let* $I \in \mathcal{I}$. *For any* $\vec{x} = \{x_J; J \in \mathcal{I}\}$ *and* $\vec{y} = \{y_J; J \in \mathcal{I}\}$,

$$|\Phi_h^I(\vec{x};m) - \Phi_h^I(\vec{y};m)| \le 2\lambda^I \left(\int_0^1 (\lambda^I h \vee \alpha)^{-1} m(d\alpha)\right) \max_{J \subset I} |x_J - y_J|.$$

Proof. Since

$$Z(x;\nu+k) = Z(x;\nu)+k, \quad k \in \mathbf{R},$$

we have

$$F(\nu+k_1;a,b) = \inf_\gamma\left\{\int_0^\gamma (Z(x;\nu+k_2)+k_1-k_2)\,dx; a \le \gamma \le b\right\}$$
$$\le F(\nu+k_2;a,b) + b|k_1-k_2|, \quad k_1,k_2 \in \mathbf{R}.$$

Then, we get

$$|F(\nu+k_1;a,b) - F(\nu+k_2;a,b)| \le b|k_1-k_2|, \quad k_1,k_2 \in \mathbf{R}.$$

Thus

$$|\Phi_h^I(\vec{x};m) - \Phi_h^I(\vec{y};m)|$$
$$\le \lambda^I \int_0^1 \frac{1}{\alpha}\left(\frac{\alpha}{\lambda^I h} \wedge 1\right)\left|\left(\sum_{j\in I}\frac{\lambda_j x_{I\setminus\{j\}}}{\lambda^I} - x_I\right) - \left(\sum_{j\in I}\frac{\lambda_j y_{I\setminus\{j\}}}{\lambda^I} - y_I\right)\right| m(d\alpha)$$
$$\le \lambda^I \int_0^1 \left(\lambda^I h \vee \alpha\right)^{-1}\left(\max_{i\in I}\left|x_{I\setminus\{i\}} - y_{I\setminus\{i\}}\right| + |x_I - y_I|\right).$$

This completes the proof. ∎

Corollary 2.8 *If Assumptions* (A-1) *and* (A-2) *hold, then*

(1) $|\Phi^I(\vec{x};m) - \Phi^I(\vec{y};m)| \le 2\lambda^I C^* \max_{J\subset I}|x_J - y_J|$

(2) $\Phi_{T/N}^I(\vec{x};m^N)$ *converges to* $\Phi^I(\vec{x})$ *uniformly on compact sets as* $N \to \infty$.

3 Proof of Theorem 1.2

Next, we specify the transition of reserves in N-period model. We define $\xi_n = \xi(n; N, m)$, the reserve at time $t_n = t_n^N = tn/N$ in N-period model, as

$$\xi_n = \xi(n; N, m) = Z(n; N, m) - X_{t_n}^L, \quad n = 0, 1, \ldots, N-1$$

where $Z(n; N, m)$ is defined as

$$\begin{aligned} Z(N; N, m) &= X_T^L, \\ Z(n-1; N, m) &= \eta^m \left(Z(n; N, m) | \mathcal{F}_{t_{n-1}} \right), \quad n = N-1, N-2, \ldots, 1. \end{aligned} \tag{3.1}$$

Let $J_n = \{i; \tau_i > t_n\}$. Recall that $h = \frac{T}{N}$.

Proposition 3.1 (1) *Let*

$$\mu_k(A) = P\left(q_k(\tau_k - t_n) \in A | t_n < \tau_k \leq t_{n+1} \right).$$

Then

$$\mu_k(A) = \int_A 1_{(0, q_k h]} \frac{\lambda_k e^{-\frac{\lambda_k x}{q_k}}}{1 - e^{-\lambda_k h}} dx, \quad \textit{for any Borel set } A.$$

(2) *Let*

$$P(I, J) = P(J_{n+1} = J | J_n = I), \quad J \subset I.$$

Then

$$P(I, J) = \prod_{k \in I \setminus J} \left(1 - e^{-\lambda_k h} \right) \prod_{j \in J} e^{-\lambda_j h}.$$

(3) *Furthermore, the following equation holds.*

$$\begin{aligned} P(\{D_i \in A_i\} \cap \{q_i(\tau_i - t_n) \in B_i\}, i \in I \setminus J, J_{n+1} = J, J_n = I | \mathcal{F}_{t_n}) \\ = 1_{\{J_n = I\}} \prod_{i \in I \setminus J} \nu_i(A_i) \mu_i(B_i) P(I, J). \end{aligned}$$

Proof. (1) Since τ_k is exponentially distributed with a parameter λ_k, we can rewrite μ_k using $W_k = q_k(\tau_k - t_n)$,

$$\mu_k(A) = \frac{P\left(W_k \in A \cap (0, hq_k] \right)}{P(W_k \in (0, hq_k])} = \int_A 1_{(0, q_k h]} \frac{\lambda_k e^{-\frac{\lambda_k x}{q_k}}}{1 - e^{-\lambda_k h}} dx.$$

This implies our assertion.

(2) Since

$$P(I, J) = \frac{P(\tau_k \leq t_n, t_n < \tau_i \leq t_{n+1}, \tau_j > t_{n+1}; k \in I^c, i \in I \setminus J, j \in J)}{P(\tau_k \leq t_n, t_n < \tau_i; k \in I^c, i \in I)}$$

where $I^c = L \setminus I$. Since τ_i, $i \in L$, are independently exponentially distributed, we have our assertion.

(3) Since D_i and τ_i, $i \in L$, are mutually independent and τ_i are exponentially distributed, we have

$$
\begin{aligned}
&P\left(\{D_i \in A_i\} \cap \{q_i(\tau_i - t_n) \in B_i\}, i \in I \setminus J, J_{n+1} = J, J_n = I | \mathcal{F}_{t_n}\right) \\
&= P\Bigg(\bigcap_{i \in I \setminus J} \left(\{D_i \in A_i\} \cap \{W_i \in B_i\} \cap \{t_n < \tau_i \leq t_{n+1}\} \right) \\
&\qquad\qquad \cap \left(\bigcap_{k \in I^c} \tau_k \leq t_n \right) \cap \left(\bigcap_{j \in J} \tau_j > t_{n+1} \right) \Bigg| \mathcal{F}_{t_n} \Bigg) \\
&= 1_{\{J_n = I\}} \prod_{i \in I \setminus J} \left(P(D_i \in A_i | \mathcal{F}_{t_n}) P(W_i \in B_i, 0 < W_i \leq h q_i | \mathcal{F}_{t_n}) \right) \prod_{j \in J} P(\tau_j > t_{n+1} | \mathcal{F}_{t_n}) \\
&= 1_{\{J_n = I\}} \prod_{i \in I \setminus J} \nu_i(A_i) \int_{B_i} 1_{(0, q_k h]} \lambda_i e^{-\frac{\lambda_i x}{q_k}} dx \prod_{j \in J} e^{-\lambda_j h} \\
&= 1_{\{J_n = I\}} \prod_{i \in I \setminus J} \nu_i(A_i) \mu_i(B_i) P(I, J).
\end{aligned}
$$

This completes the proof. ∎

Then the following lemma holds.

Lemma 3.2 *Let $\zeta_n(I) = \zeta_n(I; N, m) : \mathcal{I} \to \mathbf{R}$ be determined by the following system of the recursive formula.*

$$
\begin{aligned}
\zeta_0(I) &= 0, \quad I \in \mathcal{I} \\
\zeta_n(\emptyset) &= 0, \quad n = 1, 2, \ldots, N-1 \\
\zeta_{n+1}(I) &= \eta^m \left(\sum_{J \subset I} P(I, J) \left(\prod_{k \in I \setminus J}^{*} (\nu_k * \mu_k) + \sum_{j \in J} q_j h + \zeta_n(J) \right) \right)
\end{aligned}
\tag{3.2}
$$

where $\prod^*$ and $*$ stand for the convolution. Then we have

$$
\xi_n = \sum_{I \in \mathcal{I}} 1_{\{J_n = I\}} \zeta_{N-n}(I). \tag{3.3}
$$

Proof. Note that

$$
\xi_n 1_{\{J_n = I\}} = \eta^m \left((X^I_{t_{n+1}} - X^I_{t_n} + \xi_{n+1}) 1_{\{J_n = I\}} | \mathcal{F}_{t_n} \right).
$$

Since $\xi_N = 0$, the equation (3.3) holds when $n = N$. Assume that the equation holds when $n = k$. Then, in the case of $n = k - 1$,

$$
\begin{aligned}
\xi_{k-1} 1_{\{J_{k-1} = I\}} &= \eta^m ((X^I_{t_k} - X^I_{t_{k-1}} + \xi_k) 1_{\{J_{k-1} = I\}} | \mathcal{F}_{t_{k-1}}) \\
&= \eta^m \left(\left(X^I_{t_k} - X^I_{t_{k-1}} + \sum_{J \subset I} \zeta_{N-k}(J) 1_{\{J_k = J\}} \right) 1_{\{J_{k-1} = I\}} \middle| \mathcal{F}_{t_{k-1}} \right) \\
&= \eta^m \left(\sum_{J \subset I} (X^I_{t_k} - X^I_{t_{k-1}} + \zeta_{N-k}(J)) 1_{\{J_{k-1} = I, J_k = J\}} \middle| \mathcal{F}_{t_{k-1}} \right).
\end{aligned}
$$

However, from Proposition 3.1, for any Borel set A with $A \subset \mathbf{R} \setminus \{0\}$,

$$
\begin{aligned}
&P\left(\sum_{J\subset I}(X^I_{t_k} - X^I_{t_{k-1}} + \zeta_{N-k}(J))1_{\{J_k=J, J_{k-1}=I\}} \in A \,\middle|\, \mathcal{F}_{t_{k-1}}\right) \\
&= \sum_{J\subset I} P\left((X^I_{t_k} - X^I_{t_{k-1}} + \zeta_{N-k}(J)1_{\{J_k=J,J_{k-1}=I\}}) \in A|\mathcal{F}_{t_{k-1}}\right) \\
&= \sum_{J\subset I}\left(\prod_{k\in I\setminus J}{}^{*}(\nu_k * \mu_k) + h\sum_{j\in J} q_j + \zeta_{N-k}(J)\right)(A)P(I,J).
\end{aligned}
$$

Thus, it is also true for any Borel set A. Then we have

$$
\begin{aligned}
\xi_{k-1}1_{\{J_{k-1}=I\}} &= \eta^m\left(\sum_{J\subset I} P(I,J)\left(\prod_{k\in I\setminus J}{}^{*}(\nu_k * \mu_k) + h\sum_{j\in J} p_j + \zeta_{N-k}(J)\right)\right) \\
&= \zeta_{N-k+1}(I).
\end{aligned}
$$

Hence we have our assertion. ∎

Now fix an $I \in \mathcal{I}$ until the end of this section. Using $\zeta_n(I)$ defined above, we will prove the following lemma.

Lemma 3.3 *Assume $\nu_i \in \mathcal{L}_i$, $i \in I$ and let $\vec{\zeta_n} = \{\zeta_n(J); J \in \mathcal{I}\}$. Then,*

$$
\left|\zeta_{n+1}(I) - \zeta_n(I) - h\Phi^I_h(\vec{\zeta_n}; m)\right| \le \left(C_1 + C_2 \max_{J\subset I}|\zeta_n(J)|\right) h^{2/p},
$$

where

$$
C_1 = (1 + 2\Delta_p(m))\left((\lambda^L)^{2/p}\ell(1+2^{\ell+2})\left(\sum_{k\in L}\left(\int |x|^p \nu_k(dx)\right)^{1/p} + q^L h\right) + (q^L)^2\right)
$$

and

$$
C_2 = 2\,(1 + 2\Delta_p(m))\left((\lambda^L)^{2/p}\ell(1+2^{\ell+2})\right).
$$

In order to prove the lemma, we prepare for the following. Let $(\Omega_0, \mathcal{F}_0, P_0)$ and $(\Omega_1, \mathcal{F}_1, P_1)$ be non-atomic, standard probability spaces and let $\Omega = \Omega_0 \times \Omega_1$, $\mathcal{F} = \mathcal{F}_0 \otimes \mathcal{F}_1$, and $P = P_0 \otimes P_1$. Note that

$$
P(I,I) = e^{-\lambda^I h} \ge 1 - \lambda^I h,
$$

and

$$
P(I, I\setminus\{i\}) = \left(1 - e^{-\lambda_i h}\right)\prod_{j\in I\setminus\{I\}} e^{-\lambda_j h} \le \left(1 - e^{-\lambda_i h}\right) \le \lambda_i h.
$$

On Ω_0, we take disjoint subsets A_J, $J \subset I$, where $P_0(A_J) = P(I,J)$. Also, because of the note above, we can define disjoint subsets B_i, $i \in \{0\} \cup I$, such that

$$
\begin{aligned}
P_0(B_0) &= 1 - \lambda^I h, \quad B_0 \subset A_I \\
P_0(B_i) &= \lambda_i h, \quad B_i \supset A_{I\setminus\{i\}}, i \in I.
\end{aligned}
$$

On Ω_1, we take independent random variables $\tilde{U}_i$ and $\tilde{V}_i$, $i \in I$, such that the probability distribution of $\tilde{U}_i$ and $\tilde{V}_i$ are $\nu_i + \zeta_n(I \setminus \{i\}) - \zeta_n(I)$ and $\mu_i - q_i h$, respectively. Let $U_{I\setminus\{i\}} = \tilde{U}_i + \tilde{V}_i$. Also we take random variables U_J for $J \subset I$ with $\sharp(I \setminus J) \geq 2$ such that the distribution law is

$$\prod_{k\in I\setminus J}{}^{*}(\nu_k * \mu_k - q_k h) + \zeta_n(J) - \zeta_n(I).$$

Also let $U_I = 0$. Then we see that

$$\max_{J\subset I} E\left[|U_J|^p\right]^{1/p} \vee \max_{i\in I} E\left[|\tilde{U}_i|^p\right]^{1/p} \leq \sum_{k\in L}\left(\int |x|^p \nu_k(dx)\right)^{1/p} + q^L h + 2\max_{J\subset I}|\zeta_n(J)|.$$

Using these notations, we define two random variables W and $\tilde{W}$ on $\Omega_0 \times \Omega_1$ as

$$\begin{aligned} W(\omega_0,\omega_1) &= \sum_{J\subset I} 1_{A_J}(\omega_0) U_J(\omega_1), \\ \tilde{W}(\omega_0,\omega_1) &= \sum_{i\in I} 1_{B_i}(\omega_0)\tilde{U}_i(\omega_1) + 1_{B_0}(\omega_0)\cdot 0. \end{aligned}$$

Then the next proposition holds.

Proposition 3.4 *Assume that there exists $p > 1$ such that $\nu_k \in \mathcal{L}_p$ for $k \in I$. Let W and $\tilde{W}$ as above. Then*

$$\left|\eta^m(W) - \eta^m(\tilde{W})\right| \leq \left(C_1 + C_2 \max_{J\subset I}|\zeta_n(J)|\right) h^{2/p} \tag{3.4}$$

Proof. Since

$$\left|W - \tilde{W}\right| \leq \sum_{i\in I}\left(1_{B_i\setminus A_{I\setminus\{i\}}}\cdot 2\max_{J\subset I}|U_J| + 1_{A_{I\setminus\{i\}}} q_i h\right) + 1_{A_I\setminus B_0}\cdot \max_{i\in I}|\tilde{U}_i|,$$

we get

$$\begin{aligned} \left\|W - \tilde{W}\right\|_{L^p} \leq \sum_{i\in I}\Bigg(2P(B_i \setminus A_{I\setminus\{i\}})^{1/p}\sum_{J\subset I} E\left[|U_J|^p\right]^{1/p} + P(A_{I\setminus\{i\}})^{1/p} q_i h\Bigg) \\ + P(A_I \setminus B_0)^{1/p}\sum_{i\in I} E\left[|\tilde{U}_i|^p\right]^{1/p}. \end{aligned}$$

Also we have

$$P(A_I \setminus B_0) = e^{-\lambda^I h} - 1 + \lambda_I h \leq (\lambda^I h)^2$$

and

$$\begin{aligned} P(B_i \setminus A_{I\setminus\{i\}}) &= \lambda_i h - (1 - e^{-\lambda_i h})\prod_{j\in I\setminus\{i\}} e^{-\lambda_j h} \leq \lambda_i h + e^{-\lambda^I h}\sum_{m=1}^{\infty}\frac{(-\lambda_i h)^m}{m!} \\ &\leq \lambda_i h(1 - e^{-\lambda^I h}) + (\lambda_i h)^2 e^{-\lambda_i h} e^{-\lambda^I h} \leq 2(\lambda^I h)^2. \end{aligned}$$

Thus,

$$\left\|W - \tilde{W}\right\|_{L^p} \leq \left(4\ell(\lambda^L)^{2/p} 2^\ell E\left[|U_J|^p\right]^{1/p} + (q^L)^2 + (\lambda^L)^{2/p}\ell E\left[|\tilde{U}_i|^p\right]^{1/p}\right) h^{2/p}$$

and using Proposition 2.4, we get the result. ∎

Proposition 3.5 *For W and $\tilde{W}$ above, the following hold.*

(1) $\eta^m(W) = \zeta_{n+1}(I) - \zeta_n(I) - hq^I$

(2) $\eta^m(\tilde{W}) = h\Phi_h^I(\vec{\zeta}_n; m) - hq^I$

Proof. (1) We get by definition

$$\begin{aligned}
\eta^m(W) &= \eta^m\left(\sum_{J\subset I} P(I,J)\left(\prod_{k\in I\setminus J}^{*}(\nu_k * \mu_k - q_k h) + \zeta_n(J) - \zeta_n(I)\right)\right)\\
&= \eta^m\left(\sum_{J\subset I} P(I,J)\left(\prod_{k\in I\setminus J}^{*}(\nu_k * \mu_k - q_k h) + hq^I + \zeta_n(J)\right)\right) - \zeta_n(I) - hq^I\\
&= \zeta_{n+1}(I) - \zeta_n(I) - hq^I.
\end{aligned}$$

(2) Using the equation (1.4), $\mu_{\tilde{W}}$, the distribution law of $\tilde{W}$, can be written as $\lambda^I h(\nu^I(\vec{\zeta}_n) - \zeta_n(I)) + (1-\lambda^I h)\delta_0$ where δ_0 is a probability measure such that $\delta_0(\{0\}) = 1$. Then, using the proposition 2.5, we get

$$\begin{aligned}
&\int_0^\alpha Z(x;\mu_{\tilde{W}})dx\\
&= \inf\left\{\lambda^I h\int_0^\beta Z(x;\nu^I(\vec{\zeta}_n) - \zeta_n(I))dx; \beta,\gamma\in[0,1], \lambda^I h\beta + (1-\lambda^I h)\gamma = \alpha\right\}\\
&= \inf\left\{\lambda^I h\int_0^\beta Z(x;\nu^I(\vec{\zeta}_n) - \zeta_n(I))dx; \beta\in[0,1], 1-\frac{1-\alpha}{\lambda^I h}\le\beta\le\frac{\alpha}{\lambda^I h}\right\}.\\
&= \lambda^I hF\left(\nu^I(\vec{\zeta}_n) - \zeta_n(I); 0\vee\left(1-\frac{1-\alpha}{\lambda^I h}\right), \frac{\alpha}{\lambda^I h}\wedge 1\right).
\end{aligned}$$

Thus,

$$\eta^m(\tilde{W}) = \lambda^I h\int_0^1 \frac{1}{\alpha}F\left(\nu^I(\vec{\zeta}_n) - \zeta_n(I); 0\vee\left(1-\frac{1-\alpha}{\lambda^I h}\right), \frac{\alpha}{\lambda^I h}\wedge 1\right) m(d\alpha).$$

Using the definition of Φ_h^I on the equation (1.5), we complete the proof. ∎

Then simply by substituting the equations in Proposition 3.5 for the equation (3.4), we complete the proof of Lemma 3.3.

Finally we need the following proposition in order to complete the proof of Lemma 3.3.

Proposition 3.6 *Assume that Assumptions* (A-1), (A-2) *and* (A-3) *hold. Let $x(t,I)$ be the solution of the system of equations* (1.6) *with initial conditions* (1.7) *and*

$$\epsilon_n^I = \max_{J\subset I}\left|\zeta_n(J;N,m^N) - x\left(t_n^N, J\right)\right|,$$

for $I\subset L$, where $t_n^N = nT/N$. Then,

$$\max_{1\le n\le N}\epsilon_n^I \to 0, \quad N\to\infty.$$

Proof. By examining the difference between $\zeta(I)$ and $x(\cdot, I)$, we get,

$$\begin{aligned}
&\left|\zeta_{n+1}\left(I;N,m^N\right) - x\left(t_{n+1}^N, I\right)\right| \\
&\leq \left|\zeta_{n+1}(I) - \zeta_n(I) - \frac{T}{N}\Phi^I_{T/N}(\vec{\zeta}_n)\right| + \left|x\left(t_{n+1}^N, I\right) - x\left(t_n^N, I\right) - \frac{T}{N}\Phi^I\left(\vec{x}\left(t_n^N\right)\right)\right| \\
&\quad + \left|\zeta_n(I) - x\left(t_n^N, I\right)\right| + \frac{T}{N}\left(\left|\Phi^I_{T/N}(\vec{\zeta}_n) - \Phi^I_{T/N}(\vec{x}\left(t_n^N\right))\right| + \left|\Phi^I_{T/N}(\vec{x}\left(t_n^N\right)) - \Phi^I(\vec{x}\left(t_n^N\right))\right|\right) \\
&\leq \left(C_1 + C_2\left(\max_{J\subset I}\left|\zeta_n(J) - x\left(t_n^N, J\right)\right| + \max_{J\subset I}\left|x\left(t_n^N, J\right)\right|\right)\right)\left(\frac{T}{N}\right)^{2/p} \\
&\quad + \frac{T}{N}\max_{0\leq s\leq T/N}\left|\Phi^I\left(\vec{x}\left(t_n^N + s\right)\right) - \Phi^I\left(\vec{x}\left(t_n^N\right)\right)\right| \\
&\quad + \left|\zeta_n(I) - x\left(t_n^N, I\right)\right| + \frac{T}{N}\left|\Phi^I_{T/N}(\vec{x}\left(t_n^N\right)) - \Phi^I(\vec{x}\left(t_n^N\right))\right| \\
&\quad + \frac{2T\lambda^L}{N}\left(\int_0^1 (\lambda^I T/N \vee \alpha)^{-1} m^N(d\alpha)\right)\max_{J\subset I}\left|\zeta_n(J) - x\left(t_n^N, J\right)\right|.
\end{aligned}$$

Let

$$\begin{aligned}
r_N = &\left(C_1 + C_2\max_{\substack{J\subset I\\ 0\leq n\leq N}}\left|x\left(t_n^N, J\right)\right|\right)\left(\frac{T}{N}\right)^{2/p} \\
&+ \left(\max_{\substack{0\leq s\leq T/N\\ 0\leq n\leq N}}\left|\Phi^I\left(\vec{x}\left(t_n^N + s\right)\right) - \Phi^I\left(\vec{x}\left(t_n^N\right)\right)\right| + \left|\Phi^I_{T/N}(\vec{x}(t_n^N)) - \Phi^I(\vec{x}(t_n^N))\right|\right)\frac{T}{N}
\end{aligned}$$

and

$$C = C_2\left(\frac{T}{N}\right)^{2/p} + 2\lambda^L\int_0^1 (\lambda^I h \vee \alpha)^{-1} m^N(d\alpha).$$

Then we get

$$\begin{aligned}
\epsilon_{n+1}^I &\leq \left(1 + \frac{CT}{N}\right)\epsilon_n^I + r_N \\
\left(1 + \frac{CT}{N}\right)^{-(n+1)}\epsilon_{n+1}^I &\leq \left(1 + \frac{CT}{N}\right)^{-n}\epsilon_n^I + r_N
\end{aligned}$$

Since $\epsilon_0 = 0$ and

$$\lim_{N\to\infty} N r_N = 0,$$

we get by (2) of Corollary 2.8

$$e^{-CT}\max_{1\leq n\leq N}\epsilon_n^I \leq \sum_{n=0}^{N-1} r_N \to 0.$$

This completes the proof. ∎

Then we are ready to prove Theorem 1.2. We will prove more generalized one.

Theorem 3.7 *Assume that* (A-1), (A-2) *and* (A-3) *hold. Let $x(t,I)$ be the solution of the system of equations* (1.6) *with initial conditions* (1.7) *and*

$$n(t,N) = \left[\frac{tN}{T}\right], \quad 0 \le t \le T.$$

Then

$$\xi_{n(t,N)} \to \sum_{I \in \mathcal{I}} 1_{\{J_t = I\}} x(T-t, I), \quad N \uparrow \infty.$$

In particular,

$$\xi_0 = \eta^{m^N}(X_T | \{\mathcal{F}_{t_n}\}_{n=0}^N) \to x(T, L) \quad N \uparrow \infty.$$

Proof. From Proposition 3.6, the difference between $\zeta(I)$ and $x(\cdot, I)$ goes to zero as $N \to \infty$ and

$$\xi_{n(t,N)} = \sum_{I \in \mathcal{I}} 1_{\{J_n(t,N) = I\}} \zeta_{N-n(t,N)}(I) \to \sum_{I \in \mathcal{I}} 1_{\{J_t = I\}} x(T-t, I).$$

This proves our assertion. ∎

4 Some properties of $x(t,I)$

In this section, we introduce some properties of $x(t,I)$.

Proposition 4.1 *For any I, I_1, and $I_2 \subset L$ such that $I = I_1 \cup I_2$ and that $I_1 \cap I_2 = \emptyset$ and for any t, $x(t,I) \ge x(t,I_1) + x(t,I_2)$.*

Proof. First we will prove $\zeta_n(I) \ge \zeta_n(I_1) + \zeta_n(I_2)$. It is true when $n = 0$ from the definition. Assume that the inequation holds when $n = k$ for any $I \subset L$. Then,

$$\begin{aligned}
&\sum_{J \subset I} \left(X^I_{t_{N-k}} - X^I_{t_{N-k-1}} + \zeta_k(J) \right) 1_{\{J_{N-k-1} = I, J_{N-k} = J\}} \\
&\ge \sum_{J \subset I} \left(X^{I_1}_{t_{N-k}} - X^{I_1}_{t_{N-k-1}} + X^{I_2}_{t_{N-k}} - X^{I_2}_{t_{N-k-1}} + \zeta_k(J \cap I_1) + \zeta_k(J \cap I_2) \right) 1_{\{J_{N-k-1} = I, J_{N-k} = J\}} \\
&= \sum_{J \subset I} \left(X^{I_1}_{t_{N-k}} - X^{I_1}_{t_{N-k-1}} + \zeta_k(J \cap I_1) \right) 1_{\{J_{N-k-1} = I, J_{N-k} = J\}} \\
&\quad + \sum_{J \subset I} \left(X^{I_2}_{t_{N-k}} - X^{I_2}_{t_{N-k-1}} + \zeta_k(J \cap I_2) \right) 1_{\{J_{N-k-1} = I, J_{N-k} = J\}}.
\end{aligned}$$

Note that for any $J_1 \subset I_1$,

$$E\left[\sum_{J \in I, J \cap I_1 = J_1} 1_{\{J_{N-k-1} = I, J_{N-k} = J\}} \middle| \mathcal{F}_{t_{N-k-1}} \right] = E\left[1_{\{J_{N-k-1} = I_1, 1_{N-k} = J_1\}} \middle| \mathcal{F}_{t_{N-k-1}} \right].$$

Thus,

$$\begin{aligned}
&\eta^m \left(\sum_{J \subset I} \left(X^{I_1}_{t_{N-k}} - X^{I_1}_{t_{N-k-1}} + \zeta_k(J \cap I_1) \right) 1_{\{J_{N-k-1} = I, J_{N-k} = J\}} \middle| \mathcal{F}_{t_{N-k-1}} \right) \\
&= \eta^m \left(\sum_{J_1 \subset I_1} \left(X^{I_1}_{t_{N-k}} - X^{I_1}_{t_{N-k-1}} + \zeta_k(J_1) \right) 1_{\{J_{N-k-1} = I_1, J_{N-k} = J_1\}} \middle| \mathcal{F}_{t_{N-k-1}} \right) = \zeta_{k+1}(I_1).
\end{aligned}$$

Using the equations shown on Lemma 3.2, we get

$$\zeta_{k+1}(I) \geq \zeta_{k+1}(I_1) + \zeta_{k+1}(I_2).$$

Thus from Proposition 3.6, we have our assertion. ∎

Proposition 4.2 *If $\nu_i((-\infty, 0]) = 1$ and $q_i \geq 0$ for any $i \in L$, then following properties hold.*

(1) *$x(t, I) \leq q^I t$ for any $I \subset L$ and t.*

(2) *$x(t, I_1) \leq x(t, I_2) + q^{I_1 \setminus I_2} t$ for any $I_2 \subset I_1 \subset L$ and t.*

Proof. (1) We will prove that $\zeta_n(I) \leq q^I nh$. When $n = 0$, it is obvious. Assume that when $n = k$, the inequation holds; i.e., $\zeta_k(I) \leq q^I kh$. Then since $\nu_i((-\infty, 0]) = 1$, we have

$$X^I_{t_{N-k}} - X^I_{t_{N-k-1}} + \zeta_k(J) \leq q^I h + q^I kh = q^I (k+1)h$$

for any $J \subset I$. Thus,

$$\zeta_{k+1}(I) \leq \eta^m(q^I(k+1)h) = q^I(k+1)h.$$

It completes the proof.

(2) Again, we will prove that $\zeta_n(I_1) \leq \zeta_n(I_2) + q^{I_1 \setminus I_2} nh$. When $n = 0$, it is obvious, Assume that when $n = k$, the inequation holds. Then,

$$\begin{aligned} X^{I_1}_{t_{N-k}} - X^{I_1}_{t_{N-k-1}} + \zeta_k(J) &\leq X^{I_2}_{t_{N-k}} - X^{I_2}_{t_{N-k-1}} + q^{I_1 \setminus I_2} h + \zeta_k(J \cap I_2) + q^{J \setminus (J \cap I_2)} kh \\ &\leq X^{I_2}_{t_{N-k}} - X^{I_2}_{t_{N-k-1}} + q^{I_1 \setminus I_2}(k+1)h \end{aligned}$$

for any $J \subset I_1$. It implies our assertion as in (1). ∎

5 Example

In this section we specify m^N in order to examine the model in detail.

5.1 Example of the measure m^N

We only focus on the N-period model with $N > 2\lambda^L T$ in this section. We express measure m^N by functions $g^N : [0, \infty) \to \mathbf{R}^+$, $h^N : [0, 1] \to \mathbf{R}^+$, and $f^N : [0, \infty) \to \mathbf{R}^+$ depending on the partition size N as follows for scrutinizing the effect of the tail;

$$m^N(d\alpha) = \left(g^N(N\alpha) + h^N(\alpha) + f^N(N(1-\alpha))\right) d\alpha. \tag{5.1}$$

We impose the following assumptions on the functions g^N, h^N and f^N.

(B-1) $g^N(u) = f^N(u) = 0$ for $u \in [N/2, \infty)$.

(B-2) $\displaystyle \frac{1}{N}\int_0^\infty g^N(u)du + \int_0^1 h^N(u)du + \frac{1}{N}\int_0^\infty f^N(u)du = 1.$

(B-3) $\lim_{\epsilon \to 0} \sup_N \int_0^\epsilon \frac{h^N(\alpha)}{\alpha} d\alpha = 0, \quad \lim_{\epsilon \to 0} \sup_N \int_{1-\epsilon}^1 \frac{h^N(\alpha)}{\alpha} d\alpha = 0.$

(B-4) $\sup_N \int_0^1 \frac{h^N(\alpha)}{\alpha} d\alpha < \infty$ and $\int_0^1 \frac{h^N(\alpha)}{\alpha} d\alpha \to C_h \in \mathbf{R}^+$ as $N \to \infty$.

(B-5) $(N-u)^{-1} f^N(u) du$ converges weakly to $m_f(du)$ as $N \to \infty$, i.e., for any bounded function $k(u) : [0, \infty) \to \mathbf{R}$, $k(u)(N-u)^{-1} f^N(u) du \to k(u) m_f(du)$.

(B-6) $\sup_N \int_0^\infty g^N(u) du < \infty$.

(B-7) There exists $p \in (1, 2)$ satisfying that $\nu_i \in \mathcal{L}_p$, $i \in L$, and that $u^{-1/p} g^N(u) du$ converges weakly to $m_g(du)$ as $N \to \infty$.

Note that Assumption (B-2) guarantees for m^N to be probability distribution.

Then, under the assumptions above, we will prove the following propositions.

Proposition 5.1 *If m^N above satisfies Assumptions* (B-1) - (B-7)*, then,*

$$\sup_N \int_0^1 (\lambda^I T N^{-1} \vee \alpha)^{-1} m^N(d\alpha) < \infty.$$

Proof. Note that $\lambda^I T/N < 1/2$. We get

$$\begin{aligned} \int_0^1 \left(\frac{N}{\lambda^I T} \wedge \frac{1}{\alpha} \right) m^N(d\alpha) &\leq \int_0^1 \frac{N g^N(N\alpha)}{\lambda^I T} d\alpha + \int_0^1 \frac{h^N(\alpha)}{\alpha} d\alpha + \int_{1/2}^1 \frac{f^N\ (N(1-\alpha))}{\alpha} d\alpha \\ &\leq \int_0^\infty \frac{g^N(u)}{\lambda^I T} du + \int_0^1 \frac{h^N(\alpha)}{\alpha} d\alpha + \frac{2}{N} \int_0^\infty f^N(u) du. \end{aligned}$$

Then, by (B-6), (B-4) and (B-2), we complete the proof. ∎

Proposition 5.2 *If m^N above satisfies Assumptions* (B-1) - (B-7). *Then, for $p \in (1, 2)$ on Assumption* (B-7)*, we have*

$$\sup_N \Delta_p(m^N) < \infty.$$

Proof. Since

$$\begin{aligned} \Delta_p(m^N) &\leq \int_0^{1/2} \frac{g^N(N\alpha)}{\alpha^{1/p}} d\alpha + \int_0^1 \frac{h^N(\alpha)}{\alpha^{1/p}} d\alpha + \int_{1/2}^1 \frac{(1-\alpha)^{1-1/p}}{\alpha} f^N\left(N(1-\alpha)\right) d\alpha \\ &\leq N^{1/p-1} \int_0^\infty \frac{g^N(u)}{u^{1/p}} du + \int_0^1 \frac{h^N(\alpha)}{\alpha^{1/p}} d\alpha + N^{1/p-1} \int_0^{N/2} \frac{u^{1-1/p}}{N-u} f^N(u) du \\ &\leq N^{1/p-1} \int_0^\infty \frac{g^N(u)}{u^{1/p}} du + \int_0^1 \frac{h^N(\alpha)}{\alpha} d\alpha + \int_0^\infty \frac{f^N(u)}{N-u} du, \end{aligned}$$

by (B-7), (B-4) and (B-5), we complete the proof. ∎

Proposition 5.3 *Under Assumption* (B-4), *we get*

$$\left|\int_0^1 \left(F(\nu^I - x_I; 0, 1) - F\left(\nu^I(\vec{x}) - x_I; 0 \vee \left(1 - \frac{N(1-\alpha)}{\lambda^I T}\right), 1 \wedge \frac{N\alpha}{\lambda^I T}\right)\right) \frac{h^N(\alpha)}{\alpha} d\alpha\right| \to 0,$$

as $N \to \infty$.

Proof. We get

$$\begin{aligned}
&\left|\int_0^1 \left(F(\nu^I - x_I; 0, 1) - F\left(\nu^I - x_I; 0 \vee \left(1 - \frac{N(1-\alpha)}{\lambda^I T}\right), 1 \wedge \frac{N\alpha}{\lambda^I T}\right)\right) \frac{h^N(\alpha)}{\alpha} d\alpha\right| \\
&\le \int_0^{\lambda^I T/N} \left|F(\nu^I - x_I; 0, 1) - F\left(\nu^I - x_I; 0, 1 \wedge \frac{N\alpha}{\lambda^I T}\right)\right| \frac{h^N(\alpha)}{\alpha} d\alpha \\
&\quad + \int_{1-\lambda^I T/N}^1 \left|F(\nu^I - x_I; 0, 1) - F\left(\nu^I - x_I; 0 \vee \left(1 - \frac{N(1-\alpha)}{\lambda^I T}\right), 1\right)\right| \frac{h^N(\alpha)}{\alpha} d\alpha \\
&\le \left(\left|F(\nu^I - x_I; 0, 1)\right| + \left|F(x^I - \nu^I; 0, 1)\right|\right) \left(\int_0^{\lambda^I T/N} \frac{h^N(\alpha)}{\alpha} d\alpha + \int_{1-\lambda^I T/N}^1 \frac{h^N(\alpha)}{\alpha} d\alpha\right).
\end{aligned}$$

This proves our assertion. ∎

Then using the proposition, we get the following.

Proposition 5.4 *Assume that* (B-1) - (B-7) *hold. Also let*

$$\begin{aligned}
\Phi^I(\vec{x}) = \lambda^I \Bigg(&\int_0^\infty F\left(\nu^I(\vec{x}) - x_I; 0, 1 \wedge \frac{u}{\lambda^I T}\right) \tilde{m}_g(du) + F(\nu^I(\vec{x} - x_I; 0, 1) C_h \\
&+ \int_0^\infty F\left(\nu^I(\vec{x}) - x_I; 0 \vee \left(1 - \frac{u}{\lambda^I T}\right), 1\right) m_f(du)\Bigg) + q^I,
\end{aligned} \tag{5.2}$$

where $\tilde{m}_g(du) = u^{1/p-1} m_g(du)$. *Then* $\Phi^I_{T/N}(\vec{x}; m^N)$ *converges to* $\Phi^I(\vec{x})$ *uniformly on compact as* $N \to \infty$.

Proof. From the definition of $\Phi^I_{T/N}$, we get

$$\begin{aligned}
\frac{\Phi^I_{T/N}(\vec{x}) - q^I}{\lambda^I} &= \int_0^1 \frac{1}{\alpha} F\left(\nu^I - x_I; 0, 1 \wedge \frac{N\alpha}{\lambda^I T}\right) g^N(N\alpha) d\alpha \\
&\quad + \int_0^1 \frac{1}{\alpha} F\left(\nu^I - x_I; 0 \vee \left(1 - \frac{N(1-\alpha)}{\lambda^I T}\right), 1 \wedge \frac{N\alpha}{\lambda^I T}\right) h^N(\alpha) d\alpha \\
&\quad + \int_0^1 \frac{1}{\alpha} F\left(\nu^I - x_I; 0 \vee \left(1 - \frac{N(1-\alpha)}{\lambda^I T}\right), 1\right) f^N(N(1-\alpha)) d\alpha \\
&= \int_0^\infty F\left(\nu^I - x_I; 0, \frac{u}{\lambda^I T} \wedge 1\right) \frac{g^N(u)}{u} du \\
&\quad + \int_0^1 F\left(\nu^I - x_I; 0 \vee \left(1 - \frac{N(1-\alpha)}{\lambda^I T}\right), 1 \wedge \frac{N\alpha}{\lambda^I T}\right) \frac{h^N(\alpha)}{\alpha} d\alpha \\
&\quad + \int_0^\infty F\left(\nu^I - x_I; 0 \vee \left(1 - \frac{u}{\lambda^I T}\right), 1\right) \frac{f^N(u)}{N - u} du.
\end{aligned}$$

For $u \le \lambda^I T$,

$$\frac{u^{1/p}}{u} F\left(\nu^I - x_I; 0, 1 \wedge \frac{u}{\lambda^I T}\right) = \frac{1}{(\lambda^I T)^{1-1/p}} \left(\frac{u}{\lambda^I T}\right)^{1/p} \eta_{u/\lambda^I T}(\nu^I - x_I),$$

and for $u > \lambda^I T$, the left hand side of the equation above is equal to $u^{1/p-1} E[(D^I - x_I) \wedge 0]$, which is bounded. Thus, from Assumption (B-7) and Corollary 2.3, we get

$$F\left(\nu^I - x_I; 0, 1 \wedge \frac{u}{\lambda^I T}\right) \frac{g^N(u)}{u} du \to u^{1/p-1} F\left(\nu^I - x_I; 0, 1 \wedge \frac{u}{\lambda^I T}\right) m_g(du), \quad N \to \infty$$

Also, using the boundedness of $F(\nu^I - x_I; 0 \vee (1 - u/\lambda^I T), 1)$ and the proposition above, we complete the proof. ∎

From Propositions 5.1, 5.2, and 5.4, we see that Assumptions (A-2), (A-3) and (A-1) hold under Assumptions (B-1) - (B-7).

5.2 Homogeneous Collective Risk

In this subsection, we focus on the homogeneous collective risk process, i.e., $\lambda_i = \lambda$, $\nu_i = \nu$ and $q_i = q$. In this case it is obvious that $x_{J_1} = x_{J_2}$ and $\Phi^{J_1}(\vec{x}) = \Phi^{J_2}(\vec{x})$ if $\sharp(J_1) = \sharp(J_2)$. Thus we use the following notations in this subsection.

$$\begin{aligned}
x^{(n)}(t) &= x(t, I), \quad \sharp(I) = n, n = 0, 1, 2, \ldots \\
\tilde{x}^{(n)}(t) &= \frac{x^{(n)}(t)}{n} \\
\Delta x^{(n)}(t) &= x^{(n)}(t) - x^{(n-1)}(t) \\
\Phi^{(n)}(y) &= n\lambda \Bigg(\int_0^\infty F\left(\nu + y; 0, 1 \wedge \frac{u}{\lambda n T}\right) \tilde{m}_g(du) + F(\nu + y; 0, 1) C_h \\
&\qquad + \int_0^\infty F\left(\nu + y; 0 \vee \left(1 - \frac{u}{\lambda n T}\right), 1\right) m_f(du) \Bigg) + nq \\
\tilde{\Phi}^{(n)}(y) &= \frac{\Phi^{(n)}(y)}{n}.
\end{aligned}$$

Then, the equation (1.6) can be rewritten as

$$\frac{d\tilde{x}^{(n)}(t)}{dt} = \tilde{\Phi}^{(n)}\left(-\Delta x^{(n)}(t)\right).$$

Also, we assume $\nu((-\infty, 0]) = 1$.

What we want to show in this subsection is where $\tilde{x}^{(n)}(t)$ converges as $n \to \infty$. First, we show some properties of $x^{(n)}(t)$.

Proposition 5.5

(1) $x^{(1)}(t) \le x^{(n+1)}(t) - x^{(n)}(t) \le qt$.

(2) $\left|\tilde{x}^{(n)}(t)\right| \le Ct$ *for a* $C > 0$.

(3) $\tilde{x}^{(n)}(t)$ *converges as* $n \to \infty$.

Proof. (1) This is a simple outcome from Propositions 4.1 and 4.2.
(2) From (1), we get

$$nx^{(1)}(t) \le x^{(n)}(t) \le nqt$$

and from the shape of the differential equation of $x^{(1)}(t)$, we can get

$$\left(\Phi^{(1)}(0) - q\right) t \le x^{(1)}(t).$$

Thus, by setting $C = \max\{\left|\Phi^{(1)}(0) - q\right|, q\}$, we get the result.
(3) We will prove that $\sup_n \tilde{x}^{(n)}(t) \le \liminf_n \tilde{x}^{(n)}(t)$. We can find a subsequence $\{N_k\} \subset \mathbf{N}$, $k = 1, 2, ...$ such that $\tilde{x}^{(N_k)}(t) \to \liminf_n \tilde{x}^{(n)}(t)$, $k \to \infty$. For any $n \in \mathbf{N}$, let $a_n(k) = [N_k/n]$ and $b_n(k) = N_k - n[N_k/n]$. Then, we get

$$x^{(N_k)}(t) = x^{(a_n(k)n + b_n(k))}(t) \ge a_n(k)x^{(n)}(t) + x^{(b_n(k))}(t),$$
$$\tilde{x}^{(N_k)}(t) \ge \frac{a_n(k)}{N_k} x^{(n)}(t) + \frac{x^{(b_n(k))}(t)}{N_k}.$$

Since

$$\frac{a_n(k)}{N_k} \to \frac{1}{n},$$
$$\left|\frac{x^{(b_n(k))}(t)}{N_k}\right| \le \frac{\max_{0 \le m \le n} \left|x^{(m)}(t)\right|}{N_k} \to 0$$

when $k \to \infty$, we get

$$\liminf_{k \to \infty} \tilde{x}^{(N_k)}(t) \ge \tilde{x}^{(n)}(t).$$

It completes the proof. ∎

Proposition 5.6 *For any m, $n \in \mathbf{N}$ such that $m \ge n$,*

$$\tilde{\Phi}^{(m)}(y) \ge \tilde{\Phi}^{(n)}(y).$$

Proof. Since for any $u \in [0, \infty)$,

$$F\left(\nu + y; 0, 1 \wedge \frac{u}{\lambda m T}\right) \ge F\left(\nu + y; 0, 1 \wedge \frac{u}{\lambda n T}\right)$$

and

$$F\left(\nu + y; 0 \vee \left(1 - \frac{u}{\lambda m T}\right), 1\right) \ge F\left(\nu + y; 0 \vee \left(1 - \frac{u}{\lambda n T}\right), 1\right),$$

we have our assertion. ∎

Proposition 5.7 *There exists a $C > 0$ such that for any m, $n \in \mathbf{N}$,*

$$\left|\tilde{\Phi}^{(m)}(y) - \tilde{\Phi}^{(n)}(y)\right| \le C \left|\frac{1}{n} - \frac{1}{m}\right|.$$

Proof. Assume $m > n$. Since

$$\begin{aligned}
&\left|\int_0^\infty \left(F\left(\nu+y;0,1\wedge\frac{u}{\lambda Tn}\right)-F\left(\nu+y;0,1\wedge\frac{u}{\lambda Tm}\right)\right)\tilde{m}_g(du)\right| \\
&=\left|\int_0^{\lambda Tm} \left(F\left(\nu+y;0,1\wedge\frac{u}{\lambda Tn}\right)-F\left(\nu+y;0,1\wedge\frac{u}{\lambda Tm}\right)\right)\tilde{m}_g(du)\right| \\
&\le \int_0^{\lambda Tm} \left|F\left(\nu+y;0,1\wedge\frac{u}{\lambda Tn}\right)-F\left(\nu+y;0,1\wedge\frac{u}{\lambda Tm}\right)\right|\tilde{m}_g(du) \\
&\le \int_0^{\lambda Tm}\left(\int_{\frac{u}{\lambda Tm}}^{\frac{u}{\lambda Tn}\wedge 1}(|y|-Z(x;\nu))dx\right)\tilde{m}_g(du) \\
&\le \left(\frac{1}{n}-\frac{1}{m}\right)\frac{1}{\lambda T}\int_0^{\lambda Tm} u\left(|y|-Z\left(\frac{u}{\lambda Tm};\nu\right)\right)\tilde{m}_g(du) \\
&\le \left(\frac{1}{n}-\frac{1}{m}\right)\frac{1}{\lambda T}\int_0^{\infty} u\left(|y|-F\left(\nu;0,1\wedge\frac{u}{\lambda Tm}\right)\right)\tilde{m}_g(du) \\
&\le \left(\frac{1}{n}-\frac{1}{m}\right)\frac{1}{\lambda T}\left(|y|-\eta_1(\nu)\right)\int_0^\infty u^{1/p}m_g(du)
\end{aligned}$$

and since

$$\begin{aligned}
&\left|\int_0^\infty \left(F\left(\nu+y;0\vee\left(1-\frac{u}{\lambda Tn}\right),1\right)-F\left(\nu+y;0\vee\left(1-\frac{u}{\lambda Tm}\right),1\right)\right)m_f(du)\right| \\
&\le \int_0^{\lambda Tn}\left(\int_{\left(1-\frac{u}{\lambda Tn}\right)\vee 0}^{1-\frac{u}{\lambda Tm}}\left(|y|-|Z(x;\nu)|\right)dx\right)m_f(du) \\
&\le \left(\frac{1}{n}-\frac{1}{m}\right)\frac{|y|}{\lambda T}\int_0^{\lambda Tn} u m_f(du) \le \left(\frac{1}{n}-\frac{1}{m}\right)\frac{|y|}{\lambda T}\int_0^{\infty} u m_f(du),
\end{aligned}$$

by setting C as

$$C=\frac{1}{\lambda T}\left(\left(|y|-\eta_1(\nu)\right)\int_0^\infty u^{1/p}m_g(du)+|y|\int_0^\infty u m_f(du)\right),$$

we complete the proof. ∎

Since,

$$\int_0^\infty F\left(\nu+y;0,1\wedge\frac{u}{\lambda nT}\right)\tilde{m}_g(du)\to 0$$

and

$$F\left(\nu+y;0,1\vee\left(1-\frac{u}{\lambda nT}\right)\right)\to\eta_1(\nu)+y$$

when n goes to infinity, we get

$$\lim_{n\to\infty}\tilde{\Phi}^{(n)}(y)=\lambda\left(F(\nu+y;0,1)C_h+(\eta_1(\nu)+y)\int_0^\infty m_f(du)\right)+q\equiv\tilde{\Phi}^{(\infty)}(y).$$

Proposition 5.8

$$\int_s^t \tilde{\Phi}^{(\infty)}\left(-\tilde{x}^{(\infty)}(u)\right)du \ge \tilde{x}^{(\infty)}(t)-\tilde{x}^{(\infty)}(s)$$

Proof. Since for $m < n$, we have

$$\begin{aligned}\tilde{\Phi}^{(m)}\left(-\frac{x^{(n)}(t)}{n}\right) &= \tilde{\Phi}^{(m)}\left(-\frac{1}{n}\sum_{k=1}^{n}\left(x^{(k)}(t)-x^{(k-1)}(t)\right)\right)\\ &\geq \frac{1}{n}\sum_{k=1}^{n}\tilde{\Phi}^{(m)}\left(-\left(x^{(k)}(t)-x^{(k-1)}(t)\right)\right)\\ &\geq \frac{1}{n}\sum_{k=1}^{n}\tilde{\Phi}^{(k)}\left(-\left(x^{(k)}(t)-x^{(k-1)}(t)\right)\right).\end{aligned}$$

By integrating both sides, we have

$$\int_s^t \tilde{\Phi}^{(m)}\left(-\frac{x^{(n)}(u)}{n}\right)du \geq \frac{1}{n}\sum_{k=1}^{n}\left(-\left(\tilde{x}^{(k)}(t)-\tilde{x}^{(k)}(s)\right)\right).$$

Thus, by increasing m and n to infinity, we get the result. ∎

Proposition 5.9 *Let $y(t)$ be the solution of the following differential equations.*

$$\begin{aligned}\frac{dy(t)}{dt} &= \tilde{\Phi}^{(\infty)}\left(-y(t)\right)\\ y(0) &= 0.\end{aligned} \tag{5.3}$$

Then, for any $n \in \mathbf{N}$ and $t \geq 0$,

$$y(t) \geq \tilde{x}^{(n)}(t).$$

Proof. Let $y^{(\epsilon)}(t)$ be the solution of the following differential equations.

$$\begin{aligned}\frac{dy^{(\epsilon)}(t)}{dt} &= \tilde{\Phi}^{(\infty)}\left(-y^{(\epsilon)}(t)\right)+\epsilon\\ y^{(\epsilon)}(0) &= \epsilon.\end{aligned}$$

Suppose there exists $t > 0$ such that $y^{(\epsilon)}(t) < \tilde{x}^{(\infty)}(t)$. Then, from Proposition 5.5, we can choose $\tau = \inf\{t > 0; y^{(\epsilon)}(t) < \tilde{x}^{(\infty)}(t)\}$, then for any $s \in [0,\tau)$, $y^{(\epsilon)}(s) > \tilde{x}^{(\infty)}(s)$ and $y^{(\epsilon)}(\tau) \leq \tilde{x}^{(\infty)}(\tau)$. Then, since

$$\int_s^\tau \left(\tilde{\Phi}^{(\infty)}\left(-y^{(\epsilon)}(t)\right)+\epsilon\right)dt = y^{(\epsilon)}(\tau)-y^{(\epsilon)}(s) < \tilde{x}^{(\infty)}(\tau)-\tilde{x}^{(\infty)}(s) \leq \int_s^\tau \tilde{\Phi}^{(\infty)}\left(-\tilde{x}^{(\infty)}(u)\right)du,$$

we get

$$\frac{1}{\tau-s}\int_s^\tau \left(\tilde{\Phi}^{(\infty)}\left(-y^{(\epsilon)}(t)\right)+\epsilon\right)dt < \frac{1}{\tau-s}\int_s^\tau \tilde{\Phi}^{(\infty)}\left(-\tilde{x}^{(\infty)}(t)\right)dt.$$

But this is contradiction to the assumption when $s \uparrow \tau$. Thus for any $t > 0$, $y^{(\epsilon)}(t) > \tilde{x}^{(n)}(t)$. Then taking ϵ to zero completes the proof. ∎

Proposition 5.10 *For a $\delta \in (0,1]$, let $z(t) = z(t;\delta)$ be the solution of the following differential equations.*

$$\begin{aligned}\frac{dz(t)}{dt} &= \tilde{\Phi}^{(\infty)}(-z(t)) - \delta \\ z(0) &= 0\end{aligned}$$

Then, for any $n \in \mathbf{N}$ and $t \geq 0$,

$$z(t,0) \leq \tilde{x}^{(\infty)}(t).$$

Proof. Let's take a $k \in \mathbf{N}$ large enough to satisfy $C \leq k$, where C satisfies

$$\begin{aligned}\tilde{\Phi}^{(\infty)}(x) - \frac{C}{n} \leq & \tilde{\Phi}^{(n)}(x) \leq \tilde{\Phi}^{(\infty)}(x) \\ \left|\tilde{x}^{(k)}(t)\right| &\leq Ct \\ |z(t)| &\leq Ct\end{aligned}$$

and let $\delta = C/k$. We can choose such C because of Propositions 5.5 and 5.7. Also, let

$$w_n(t) = nz(t) - 2Ckt.$$

Then we have

$$\begin{aligned}\frac{dw_{n+1}(t)}{dt} &= (n+1)\frac{dz(t)}{dt} - 2Ck \\ &= (n+t)\left(\tilde{\Phi}^{(\infty)}(-z(t)) - \delta\right) - 2Ck \\ &= (n+t)\left(\tilde{\Phi}^{(\infty)}(w_n(t) - w_{n+1}(t)) - \delta\right) - 2Ck\end{aligned}$$

and

$$w_k(t) \leq -Ckt \leq x^{(k)}(t).$$

What we want to prove here is for $n \geq k$, $w_n(t) \leq x^{(n)}(t)$, $0 \leq t \leq T$. When $n = k$, it is true from the inequation above. Assume it is true for n. Then, let $v(t)$ be as

$$\begin{aligned}\frac{dv(t)}{dt} &= (n+1)\left(\tilde{\Phi}^{(\infty)}(w_n(t) - v(t)) - \delta\right) - 2Ck \\ v(0) &= -\epsilon\end{aligned}$$

for a $\epsilon > 0$, and we want to show a claim $v(t) \leq x^{(n+1)}(t)$. If this claim is not true, their exists a posivive value $\tau = \inf\{t : v(t) > x^{(n+1)}(t)\} < T$. Then,

$$\begin{aligned}\left.\frac{d}{dt}v(t)\right|_{t=\tau} &= (n+1)\left(\tilde{\Phi}^{(\infty)}\left(w_n(\tau) - v(\tau)\right) - \delta\right) - 2Ck \\ &\leq (n+1)\left(\tilde{\Phi}^{(\infty)}\left(x^{(n)}(\tau) - x^{(n+1)}(\tau)\right) - \frac{C}{k}\right) - 2Ck \\ &\leq (n+1)\left(\tilde{\Phi}^{(\infty)}\left(x^{(n)}(\tau) - x^{(n+1)}(\tau)\right) - \frac{C}{n+1}\right) - 2Ck \\ &\leq (n+1)\left(\tilde{\Phi}^{(n+1)}\left(x^{(n)}(\tau) - x^{(n+1)}(\tau)\right)\right) - 2Ck \\ &< \left.\frac{d}{dt}x^{(n+1)}(t)\right|_{t=\tau}.\end{aligned}$$

Thus, there exists a $\delta' > 0$ such that for any $t \in [\tau, \tau + \delta']$,

$$\frac{dv(t)}{dt} < \frac{dx^{(n+1)}(t)}{dt},$$

so,

$$v(t) = v(\tau) + \int_\tau^t \frac{dv(s)}{ds} ds \leq x^{(n+1)}(\tau) \int_\tau^t \frac{dx^{(n+1)}(s)}{ds} ds = x^{(n+1)}(t)$$

but it is contradiction. Thus, $v(t) \leq x^{(n+1)}(t)$ for any $t \in [0, T]$ and by $\epsilon \to 0$, we have $v(t) \to w_{n+1}(t)$. Hence we get $w_{n+1}(t) \leq x^{(n+1)}(t)$ and by induction, it is true for any n. Thus,

$$nz(t) - 2Ckt \leq x^{(n)}(t)$$
$$z(t) - \frac{2Ckt}{n} \leq \tilde{x}^{(n)}(t)$$

and by $n \to \infty$, we have $z(t; \delta) \leq \tilde{x}^{(\infty)}(t)$. Finally, by $k \to \infty$, we get $z(t; 0) \leq \tilde{x}^{(\infty)}(t)$. It completes the proof. ∎

From Propositions above, $\tilde{x}^{(n)}(t) \to y(t)$ defined on Equation (5.3) as $n \to \infty$.

5.3 Numerical Examples

In this subsection, we will show some numerical results with concrete assumptions in the case of homogeneous collective risk process. Assume that D has a following distribution with a parameter $\beta \in (0, 1]$;

$$P(D < u) \equiv \nu_\beta((-\infty, u)) = \left(1 - \frac{\beta u}{1 - \beta}\right)^{-1/\beta}, \quad u \in [-\infty, 0]. \tag{5.4}$$

This is a version of generalized Pareto distribution with $E[D] = -1$ for any β.

Proposition 5.11 *Some properties of ν_β are as follows.*

(1) $\nu_\beta \in \mathcal{L}_{1/\beta}$

(2) $Z(u; \nu_\beta) = \frac{1 - \beta}{\beta} \left(1 - u^{-\beta}\right)$

(3) $\eta_\alpha(\nu_\beta) = \frac{1 - \beta - \alpha^{-\beta}}{\beta}$

We also specify the g^N, h^N and f^N in equation (5.1) as follows.

$$g^N(y) = w_g e^{-y} y^{-a_g} 1_{[0,N/2]}, \quad w_g \geq 0, a_g \in (0, 1)$$
$$h^N(y) = w_h^N \left(-(2y - 1)^{2a_h} + 1\right), \quad w_h > 0, a_h \in \mathbf{N}$$
$$f^N(y) = w_f e^{-a_f y} (N - y) 1_{[0,N/2]}, \quad w_f \geq 0, a_f \geq 0.$$

In order to satisfy the assumption (B-2), w_h^N should satisty the following equation (we assume that w_g and w_f are constant for any N).

$$\frac{w_g}{N} \Gamma(1 - a_g; 0, N/2) + \frac{2a_h w_h^N}{2a_h + 1} + \frac{w_f}{a_f} \left(1 - \frac{1 - e^{-a_h N/2}}{N a_f} - \frac{e^{-a_h N/2}}{2}\right) = 1,$$

where

$$\Gamma(z;a,b) = \int_a^b t^{z-1}e^{-t}dt.$$

Thus, we get

$$w_h = \lim_{N\to\infty} w_h^N = \frac{2a_h+1}{2a_h}\left(1-\frac{w_f}{a_f}\right),$$

Note that $w_f < a_f$ should be held. In this specification, w_f determines the "severeness" of the risk measure. When w_f is larger, then m^N has more weight on the neighborhood of 1, meaning that the expected value (η_1) is more weighted. On the other hand, when w_f is smaller, then CTEs with low α are more highly taken into calculation.

Using the specification, we can calculate the value measure. First, we can get C_h as follows.

$$C_h = \lim_{N\to\infty}\int_0^1 \frac{h^N(\alpha)}{\alpha}d\alpha = \int_0^1 \frac{w_h\left(-(2y-1)^{2a_h}+1\right)}{y}dy = -w_h\sum_{i=1}^{2a_h}\frac{{}_{2a_h}C_i(-2)^i}{i}.$$

Then,

$$\begin{aligned}\Phi^{(n)}(y) =& n\lambda\Bigg\{\frac{w_g}{\lambda nT}\left(\frac{1-\beta}{\beta}+y\right)\Gamma\left(1-a_g;0,k_g\right) - \frac{w_g}{(\lambda nT)^{1-\beta}\beta}\Gamma\left(1-\beta-a_g;0,k_g\right)\\ &-\left(1+\frac{\beta y}{1-\beta}\right)^{1-1/\beta}\left(\frac{w_g}{a_g}\left(k_g^{-a_g}e^{-k_g}-\Gamma(1-a_g:k_g,\infty)\right)+C_h+\frac{w_f e^{-a_f k_f}}{a_f}\right)\\ &+\frac{w_f}{a_f}\left(\frac{1-\beta}{\beta}+y\right)\left(1-e^{-a_f k_f}+\frac{k_f e^{-a_f k_f}}{\lambda nT}-\frac{1-e^{-a_f k_f}}{\lambda nT a_f}\right)\\ &-\frac{w_f}{\beta}\int_0^{k_f}\left(1-\frac{u}{\lambda nT}\right)^{1-\beta}e^{-a_f u}du\Bigg\}+nq\end{aligned}$$

where,

$$\begin{aligned}k_g =& n\lambda T\left(1+\frac{\beta y}{1-\beta}\right)^{-1/\beta},\\ k_f =& n\lambda T\left(1-\left(1+\frac{\beta y}{1-\beta}\right)^{-1/\beta}\right),\end{aligned}$$

and in the case of $y \le 0$,

$$\begin{aligned}\Phi^{(n)}(y) =& n\lambda\Bigg\{\frac{w_g}{\lambda nT}\left(\frac{1-\beta}{\beta}+y\right)\Gamma\left(1-a_g;0,\lambda nT\right) - \frac{w_g}{(\lambda nT)^{1-\beta}\beta}\Gamma\left(1-\beta-a_g;0,\lambda nT\right)\\ &-(1-y)\left(\frac{w_g\left((\lambda nT)^{-a_g}e^{-\lambda nT}-\Gamma(1-a_g:\lambda nT,\infty)\right)}{a_g}+C_h+\frac{w_f}{a_f}\right)\Bigg\}+nq.\end{aligned}$$

We also get

$$\tilde{\Phi}^{(\infty)}(y) = \begin{cases}\lambda\left(-\left(1+\frac{\beta y}{1-\beta}\right)^{1-1/\beta}C_h+(-1+y)\frac{w_f}{a_f}\right)+q & y>0,\\ \lambda(-1+y)\left(C_h+\frac{w_f}{a_f}\right)+q & y\le 0.\end{cases}$$

Now we can calculate the transition of the value measure by substituting real numbers for each parameter. For example, let $\beta = 0.6$, $\lambda = 0.02$ (i.e., the expected loss from an insurance contract is 0.02), $a_f = 1$, $a_g = 0.1$, $a_h = 20$, $w_f = 0.4$ and $w_g = 0.05$. We show $\tilde{x}^{(n)}(t)$ with different level of n in Figure 1 in the Appendix. In this case, even in the case of $q = 0.05$, i.e., insurance premium is 2.5 times higher than the expected loss, the value measure is negative. This is because of the shape of m^N showing in Figure 2. Figure 3 and 4 is the same graphs with $w_f = 0.8$. Since m has more weight on the neighborhood of 1, the value measure becomes positive.

Appendix: Figures

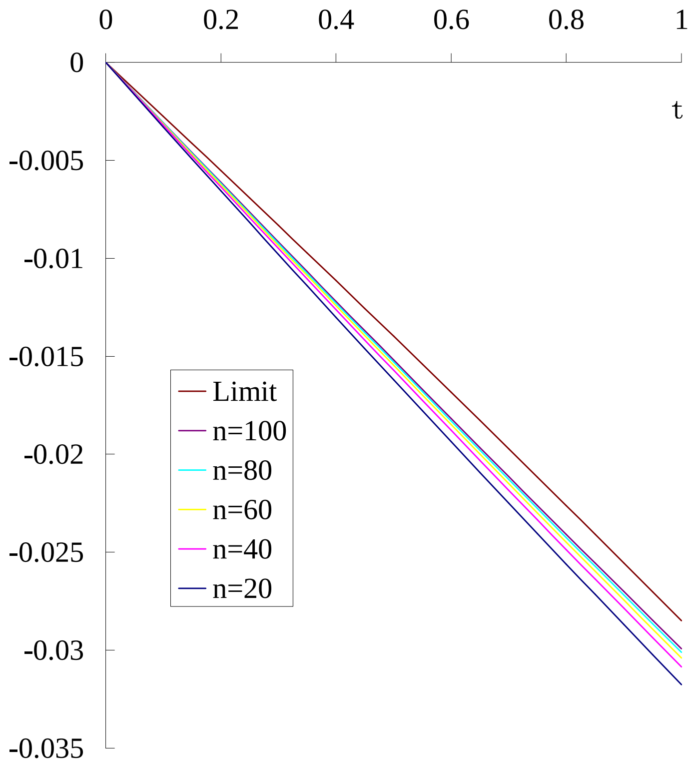

Figure 1: $\tilde{x}^{(n)}(t)$ with $\beta = 0.6$, $\lambda = 0.02$, $a_f = 1$, $a_g = 0.1$, $a_h = 20$, $w_f = 0.4$ and $w_g = 0.05$

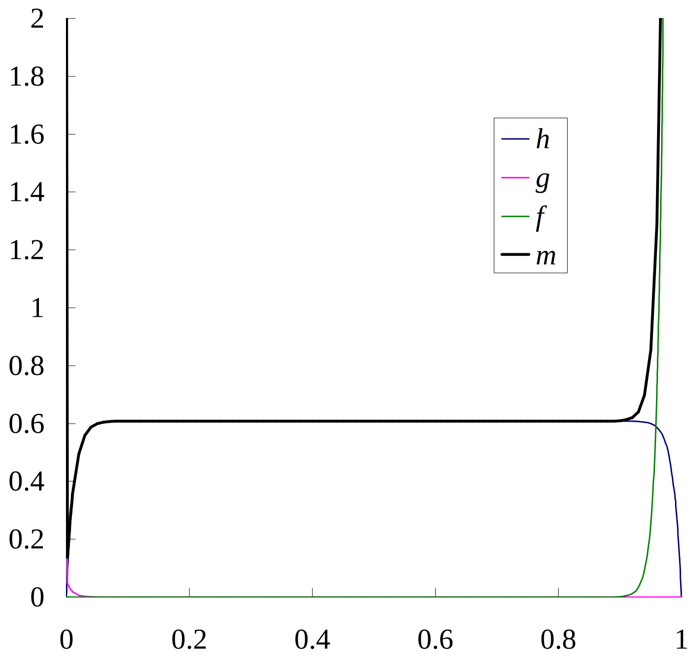

Figure 2: m^N with the same parameters as in Figure 1 and $N = 100$.

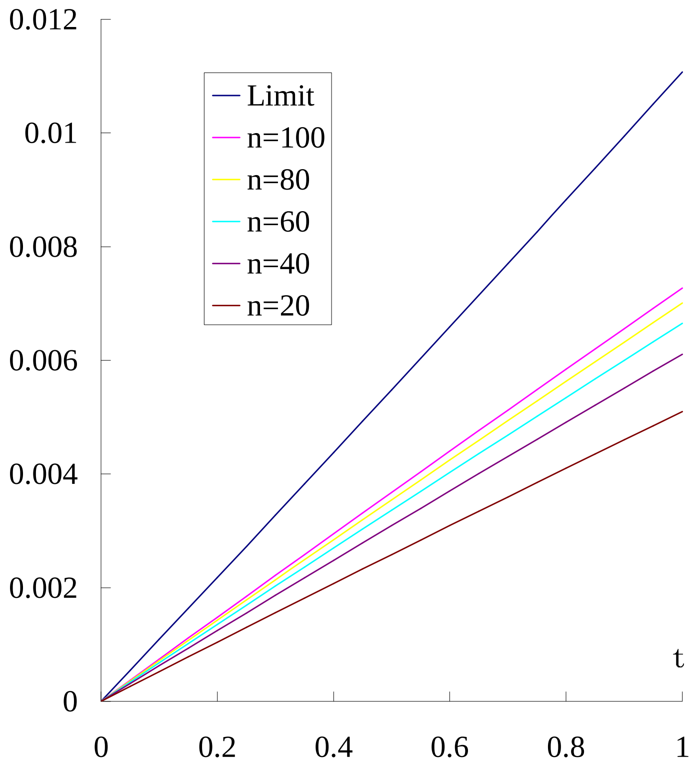

Figure 3: $\tilde{x}^{(n)}(t)$ with the same parameters as in Figure 1 except $w_f = 0.8$.

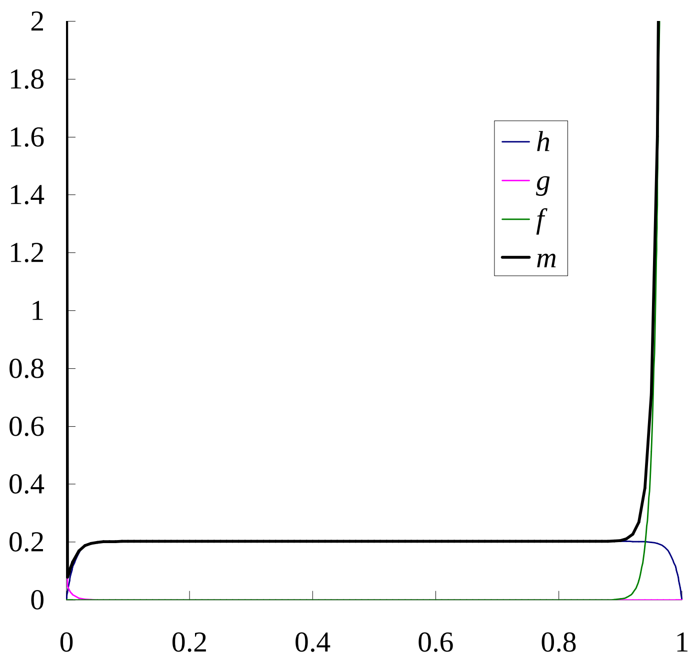

Figure 4: m^N with the same parameters as in Figure 3 and $N = 100$

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