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Discriminant of certain $K3$ surfaces

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Dedicated to Professor Kyoji Saito on his sixtieth birthday
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Summary. In this article, we study the discriminant of those $K3$ surfaces with involution which were introduced and investigated by Matsumoto, Sasaki, and Yoshida. We extend several classical results on the discriminant of elliptic curves to the discriminant of Matsumoto-Sasaki-Yoshida's $K3$ surfaces.

Contents

1 Introduction – Discriminant of elliptic curves

Let $M(n,2n;\mathbf{C})$ be the vector space of all complex $n \times 2n$-matrices and consider the following subset

$$M^o(n,2n) = \{(\mathbf{a}_1,\ldots,\mathbf{a}_{2n}) \in M(n,2n;\mathbf{C});\ \mathbf{a}_{i_1} \wedge \cdots \wedge \mathbf{a}_{i_n} \neq \mathbf{0},\ \forall\, i_1 < \cdots < i_n\}\,.$$

On $M(n,2n;\mathbf{C})$, acts the group $GL_n(\mathbf{C}) \times (\mathbf{C}^*)^{2n}$ by

$$(g,\lambda_1,\ldots,\lambda_{2n}) \cdot A = g \cdot A \cdot \mathrm{diag}(\lambda_1,\ldots,\lambda_{2n}), \qquad A \in M(n,2n;\mathbf{C})$$

where $\mathrm{diag}(\lambda_1,\ldots,\lambda_{2n})$ denotes the diagonal matrix $(\lambda_i\,\delta_{ij})$. The configuration space of $2n$ points (or $2n$ hyperplanes) in gerenal position of $\mathbf{P}^{n-1}$ is defined as

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$$X^o(n,2n) = GL_n(\mathbf{C})\backslash M^o(n,2n)/(\mathbf{C}^*)^{2n}.$$

Let us consider the case $n = 2$. On $M^o(2,4)$, we have a family of curves $\pi\colon \mathcal{E} \to M^o(2,4)$ with fiber

$$\pi^{-1}(A) = E_A = \{((x_1 : x_2), y) \in \mathcal{O}_{\mathbf{P}^1}(2);\ y^2 = \prod_{i=1}^{4}(a_{i1}\,x_1 + a_{i2}\,x_2)\}, \tag{1.1}$$

where $(x_1 : x_2)$ denotes the homogeneous coordinates of $\mathbf{P}^1$. The natural projection $\mathrm{pr}_1\colon E_A \to \mathbf{P}^1$ is a double covering with 4 branch points, so that E_A is an elliptic curve. It is classical that $X^o(2,4)$ is a moduli space of elliptic curves with level 2 structure.

We define the discriminant of $A \in M^o(2,4)$ by

$$\varDelta_{(2,4)}(A) = \prod_{\{i,j\}\cup\{k,l\}=\{1,2,3,4\},\, i<j,\, k<l} \det(\mathbf{a}_i, \mathbf{a}_j)\,\det(\mathbf{a}_k, \mathbf{a}_l). \tag{1.2}$$

Set $dx = x_2\,dx_1 - x_1\,dx_2 = x_2^2 d(x_1/x_2)$, and define the norm of $\varDelta_{(2,4)}(A)$ by

$$\|\varDelta_{(2,4)}(A)\|^2 = \left(\frac{\mathrm{i}}{2\pi}\int_{E_A} \frac{dx}{y} \wedge \overline{\left(\frac{dx}{y}\right)}\right)^6 |\varDelta_{(2,4)}(A)|^2. \tag{1.3}$$

Since $\|\varDelta_{(2,4)}\|^2$ is invariant under the action of $GL_2(\mathbf{C}) \times (\mathbf{C}^*)^4$, it descends to a function on $X^o(2,4)$. There is an analytic expression of $\|\varDelta_{(2,4)}\|$.

Let $\det^* \square_A$ be the regularized determinant of the Laplacian of E_A with respect to the normalized flat Kähler metric of volume 1. Since the isomorphism class of E_A is constant along the $GL_2(\mathbf{C}) \times (\mathbf{C}^*)^4$-orbit, $\det^*\square_A$ is constant on each $GL_2(\mathbf{C}) \times (\mathbf{C}^*)^4$-orbit. For all $A \in M^o(2,4)$, we get by [11]

$$\det{}^* \square_A = \|\varDelta_{(2,4)}(A)\|^{-1/3}. \tag{1.4}$$

In fact, Eq. (1.4) follows from the classical Kronecker limit formula, which can be seen as follows. For $z \in \mathbf{C}$ and $\tau \in \mathbf{H} := \{\tau \in \mathbf{C};\ \mathrm{Im}\,\tau > 0\}$, let

$$\wp(z,\tau) = \frac{1}{z^2} + \sum_{(m,n)\in\mathbf{Z}^2\setminus\{(0,0)\}} \left\{\frac{1}{(z+m\tau+n)^2} - \frac{1}{(m\tau+n)^2}\right\}$$

be the Weierstrass $\wp$-function and set

$$A(\tau) = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & -\wp(\frac{1}{2},\tau) & -\wp(\frac{\tau}{2},\tau) & -\wp(\frac{1+\tau}{2},\tau) \end{pmatrix} \in M^o(2,4).$$

By setting $x_1 = u\,x_2$ and $y = v\,x_2^2/2$ in (1.1), $E_{A(\tau)}$ is isomorphic to the cubic curve in $\mathbf{P}^2$ defined by the inhomogeneous equation in the variables u, v:

$$\begin{aligned} v^2 &= 4\left\{u - \wp\left(\frac{1}{2},\tau\right)\right\}\left\{u - \wp\left(\frac{\tau}{2},\tau\right)\right\}\left\{u - \wp\left(\frac{1+\tau}{2},\tau\right)\right\} \\ &= 4u^3 - g_2(\tau)\,u - g_3(\tau) \end{aligned}$$

with $g_2(\tau) = 60\, E_4(\tau)$ and $g_3(\tau) = 140\, E_6(\tau)$, where $E_k(\tau)$ denotes the Eisenstein series of weight k (cf. [28, p.11]). Hence the complex torus $\mathbf{C}/\mathbf{Z} + \tau\,\mathbf{Z}$, $\tau \in \mathbf{H}$, is isomorphic to $E_{A(\tau)}$ via the map

$$(1.5) \qquad f\colon \mathbf{C}/\mathbf{Z} + \tau\,\mathbf{Z} \ni z \to ((\wp(z):1), \wp'(z)\, x_2^2/2) \in \mathcal{O}_{\mathbf{P}^2}(2).$$

Since $dx/y = 4\, f^*(dz)$ by (1.5) and since $\Delta_{(2,4)}(A(\tau))^2 = g_2(\tau)^3 - 27\, g_3(\tau)^2$, Eq. (1.4) is deduced from the Kronecker limit formula:

$$(1.6) \qquad \det{}^* \square_{A(\tau)} = C_1\, \|\Delta(\tau)\|^{-1/6},$$

where $C_1 \neq 0$ is an absolute constant, $\Delta(\tau) = (2\pi)^{-12}(g_2(\tau)^3 - 27\, g_3(\tau)^2)$ is the Jacobi Δ-function and $\|\Delta(\tau)\|^2 = (\mathrm{Im}\tau)^{12}|\Delta(\tau)|^2$ is its Petersson norm. Recall that one has the following expressions of the Jacobi Δ-function:

$$(1.7) \qquad \Delta(\tau) = \left(\prod_{\text{even}} \theta_{ab}(\tau) \right)^8 = q \prod_{n=1}^{\infty} (1-q^n)^{24}, \qquad q = e^{2\pi i \tau},$$

where $\theta_{ab}(\tau)$ denotes the theta constants.

In [39], we extended Eq. (1.6) to $K3$ surfaces with involution. Let us explain our results briefly. Let X be a $K3$ surface and let $\iota\colon X \to X$ be a holomorphic involution acting non-trivially on holomorphic 2-forms on X. Let $H^2_+(X, \mathbf{Z})$ be the invariant part of the ι-action on $H^2(X, \mathbf{Z})$. The free $\mathbf{Z}$-module $H^2(X, \mathbf{Z})$ of rank 22 endowed with the cup-product, is an even unimodular lattice of signature $(3, 19)$ isometric to the $K3$ lattice $\mathbb{L}_{K3}$. By Nikulin, the topological type of ι is determined by $H^2_+(X, \mathbf{Z})$, which is a primitive 2-elementary hyperbolic sublattice of $H^2(X, \mathbf{Z})$. Let $M \subset \mathbb{L}_{K3}$ be a primitive 2-elementary hyperbolic sublattice with rank $r(M)$. The pair (X, ι) is called a 2-elementary $K3$ surface of type M if $H^2_+(X, \mathbf{Z}) \cong M$. The period of a 2-elementary $K3$ surface of type M lies in $\Omega^o_M = \Omega_M \setminus \mathcal{D}_M$, where Ω_M is isomorphic to a symmetric bounded domain of type IV of dimension $20 - r(M)$ and $\mathcal{D}_M$ is a divisor of Ω_M, called the discriminant locus. The moduli space of 2-elementary $K3$ surfaces of type M is isomorphic to the quotient Ω^o_M/Γ_M, where Γ_M is an arithmetic subgroup of $\mathrm{Aut}(\Omega_M)$. We assume that $r(M) \leq 17$.

For a 2-elementary $K3$ surface (X, ι) of type M, we constructed an invariant $\tau_M(X, \iota)$ by using the equivariant analytic torsion of (X, ι) (cf. [5]). We regard τ_M as a function on the moduli space Ω^o_M/Γ_M. The main result of [39] is that τ_M is expressed as the norm of the "automorphic form" Φ_M on Ω_M characterizing the discriminant locus $\mathcal{D}_M$. Here Φ_M is an automorphic form with values in some Γ_M-equivariant coherent sheaf λ_M on Ω_M. If the fixed point set of ι consists of only rational curves, then $\lambda_M \cong \mathcal{O}_{\Omega_M}$ and hence Φ_M is an automorphic form in the classical sense. By Nikulin [33], there exist only seven isometry classes of lattices M with $r(M) \leq 17$ such that the fixed point set of a 2-elementary $K3$ surface of type M consists of only rational curves. Let $\mathbb{S}_k$, $1 \leq k \leq 7$ be those seven lattices, where $r(\mathbb{S}_k) = 10 + k$. In Sect. 6, we shall express $\Phi_{\mathbb{S}_k}$ as a Borcherds product [8]. Thus the infinite product expansion (1.7) shall be extended to 2-elementary $K3$ surfaces of type $\mathbb{S}_k$.

The case $k = 6$ is of particular interest. In [30], [31], [36], 2-elementary $K3$ surfaces of type $\mathbb{S}_6$ with level 2 structure were studied by Matsumoto, Sasaki, Yoshida; they proved that the moduli space of 2-elementary $K3$ surfaces of type $\mathbb{S}_6$ with level 2 structure is isomorphic to $X^o(3,6)$. In Sect. 7.3, we shall extend the definitions (1.2), (1.3) to 3×6-matrices and get a function $\|\Delta_{(3,6)}\|$ on the configuration space $X^o(3,6)$. By Freitag, there exist theta functions $\{\Theta\binom{a}{b}\}$ on the period domain $\Omega_{\mathbb{S}_6}$, ten of which are called even. We define the Matsumoto-Sasaki-Yoshida form $\Delta_{\mathrm{MSY}}(W)$ as the product of all even Freitag theta functions: $\Delta_{\mathrm{MSY}}(W) := \prod_{\mathrm{even}} \Theta\binom{a}{b}$. Let $\|\Delta_{\mathrm{MSY}}\|$ denote the Petersson norm of Δ_{MSY}, which descends to a function on $X^o(3,6)$. The main result of this article is the following identity, which can be regarded as an analogue of Eqs. (1.4), (1.6), (1.7) in dimension 2:

Theorem 1.1. *The following identity of functions on $X^o(3,6)$ holds*

$$\tau_{\mathbb{S}_6} = C_2 \,\|\Delta_{(3,6)}\|^{-1/4} = C_3 \,\|\Delta_{\mathrm{MSY}}\|^{-1/2}, \tag{1.8}$$

where C_2, C_3 are non-zero absolute constants.

This article is organized as follows. In Sect. 2, we recall $K3$ surfaces with involution and their moduli spaces. In Sect. 3, we recall automorphic forms on the moduli space. In Sect. 4, we recall the invariant τ_M. In Sect. 5, we recall Borcherds products. In Sect. 6, we give an expression of $\tau_{\mathbb{S}_k}$ as the Petersson norm of an interesting Borcherds product, whose proof shall be given in the forthcoming paper [41]. In Sect. 7, we prove Eq. (1.8). In Sect. 8, we prove that the discriminant of smooth quartic hypersurfaces of $\mathbf{P}^3$ is expressed as the norm of an interesting Borcherds product.

2 $K3$ surfaces with involution and their moduli spaces

In this section, we recall the definition of $K3$ surfaces with involution. We refer to [39] for more details about $K3$ surfaces with involution.

Let X be a compact, connected, smooth complex surface with canonical line bundle K_X. Then X is called a $K3$ surface if

$$H^1(X, \mathcal{O}_X) = 0, \qquad K_X \cong \mathcal{O}_X.$$

Every $K3$ surface is Kähler [2, Chap. 8 Th. 14.5]. By the second condition, there exists a nowhere vanishing holomorphic 2-form η_X on X. Notice that η_X is uniquely determined up to a nonzero constant. The cohomology group $H^2(X, \mathbf{Z})$ is a free $\mathbf{Z}$-module endowed with the cup-product pairing. There exists an isometry of lattices:

$$\alpha\colon H^2(X, \mathbf{Z}) \cong \mathbb{L}_{K3} := \mathbb{U} \oplus \mathbb{U} \oplus \mathbb{U} \oplus \mathbb{E}_8 \oplus \mathbb{E}_8.$$

Here $\mathbb{U} = \binom{0\,1}{1\,0}$ and $\mathbb{E}_8$ is the *negative-definite* Cartan matrix of type E_8 under the identification of a lattice with its Gram matrix. The isometry α as above is called a *marking*, and the pair (X, α) is called a *marked $K3$ surface*. For a marked $K3$ surface (X, α), the point

$$\pi(X, \alpha) := [\alpha(\eta_X)] \in \mathbf{P}(\mathbb{L}_{K3\,\mathbf{C}}), \qquad \eta_X \in H^0(X, K_X) \setminus \{0\}$$

is called the period of X, where $L_{\mathbf{K}} := L \otimes \mathbf{K}$ for a lattice L and a field $\mathbf{K}$.

For a lattice L with bilinear form $\langle \cdot, \cdot \rangle$, we denote by $L(k)$ the lattice with bilinear form $k\langle \cdot, \cdot \rangle$. The set of roots of L is defined by $\Delta_L := \{d \in L;\ \langle d, d\rangle = -2\}$. The isometry group of L is denoted by $O(L)$. Let $L^\vee = \mathrm{Hom}_{\mathbf{Z}}(L, \mathbf{Z})$ be the dual lattice of L, which is naturally embedded into $L_{\mathbf{Q}}$. The finite abelian group $A_L := L^\vee/L$ is called the *discriminant group* of L. For a primitive sublattice $L \subset \mathbb{L}_{K3}$, $L^\perp$ denotes the orthogonal complement of L in $\mathbb{L}_{K3}$.

Definition 2.1. *For a primitive hyperbolic sublattice $S \subset \mathbb{L}_{K3}$, define*

$$\Omega_S = \Omega_{S^\perp} := \{[x] \in \mathbf{P}(S^\perp_{\mathbf{C}});\ \langle x, x\rangle = 0,\ \langle x, \bar{x}\rangle > 0\}.$$

We set $r(S) := \mathrm{rank}_{\mathbf{Z}} S$. Then $\dim \Omega_S = 20 - r(S)$. There are two connected components of Ω_S, each of which is biholomorphic to a symmetric bounded domain of type IV of dimension $20 - r(S)$ (cf. [2, Chap. 8, Lemma 20.1]).

Definition 2.2. *An even lattice S is said to be 2-elementary if there is an integer $l \geq 0$ with $A_S \cong (\mathbf{Z}/2\mathbf{Z})^l$. For a 2-elementary lattice S, set $l(S) := \dim_{\mathbf{F}_2} A_S$.*

Let $M \subset \mathbb{L}_{K3}$ be a primitive 2-elementary hyperbolic sublattice. Let I_M be the involution on $M \oplus M^\perp$ defined by

$$I_M(x, y) = (x, -y).$$

Then I_M extends uniquely to an involution on $\mathbb{L}_{K3}$. For $l \in M^\perp_{\mathbf{R}}$, we set

$$\mathcal{H}_l := \{[x] \in \Omega_M;\ \langle x, l\rangle = 0\}.$$

Then $\mathcal{H}_l \neq \emptyset$ if and only if $\langle l, l\rangle < 0$. We define

$$\mathcal{D}_M := \bigcup_{d \in \Delta_{M^\perp}} \mathcal{H}_d, \qquad \Omega_M^o := \Omega_M \setminus \mathcal{D}_M.$$

We regard $\mathcal{D}_M$ as a reduced divisor of Ω_M.

Definition 2.3. *A $K3$ surface X equipped with a holomorphic involution $\iota\colon X \to X$ is called a 2-elementary $K3$ surface if*

$$\iota^*|_{H^0(X, K_X)} = -1.$$

The pair (X, ι) is called a 2-elementary $K3$ surface of type M if there exists a marking α of X with $\iota^* = \alpha^{-1} \circ I_M \circ \alpha$.

Let (X, ι) be a 2-elementary $K3$ surface of type M and let α be a marking with $\iota^* = \alpha^{-1} \circ I_M \circ \alpha$. Let $\eta_X \in H^0(X, K_X) \setminus \{0\}$. Then $\pi(X, \alpha) \in \Omega^o_M$. The $O(M^\perp)$-orbit of $\pi(X, \alpha)$ is independent of the choice of a marking α with $\iota^* = \alpha^{-1} \circ I_M \circ \alpha$. The Griffiths period of (X, ι), which is denoted by $\varpi_M(X, \iota)$, is defined as the $O(M^\perp)$-orbit

$$\varpi_M(X, \iota) := O(M^\perp) \cdot \pi(X, \alpha) \in \Omega^o_M / O(M^\perp).$$

Theorem 2.4. *The coarse moduli space of* 2*-elementary* $K3$ *surfaces of type* M *is isomorphic to the analytic space* $\Omega^o_M / O(M^\perp)$*.*

Proof. See [39, Th. 1.8]. □

We set

$$\mathcal{M}_M := \Omega_M / O(M^\perp), \qquad \mathcal{M}^o_M := \Omega^o_M / O(M^\perp).$$

Let $\Omega^\pm_M$ be the connected components of Ω_M and set

$$O^+(M^\perp) := \{g \in O(M^\perp);\, g(\Omega^\pm_M) = \Omega^\pm_M\}.$$

Then $O^+(M^\perp) \subset O(M^\perp)$ is a subgroup of index 2 with $\mathcal{M}_M = \Omega^+_M / O(M^\perp)^+$ and $\mathcal{M}^o_M = (\Omega^+_M \setminus \mathcal{D}_M)/O^+(M^\perp)$. We consider Ω^+_M as the period domain for 2-elementary $K3$ surfaces of type M. By Baily-Borel-Satake, both of $\mathcal{M}_M$ and $\mathcal{M}^o_M$ are quasi-projective algebraic varieties.

The topological type of the set of fixed points of (X, ι) was determined by Nikulin. We need the following partial result. See [33] for the general cases.

Lemma 2.5. *Let* (X, ι) *be a* 2*-elementary* $K3$ *surface of type* M *and let*

$$X^\iota := \{x \in X;\, \iota(x) = x\}.$$

If $r(M) + l(M) = 22$*, then* X^ι *is the disjoint union of* $(r(M) - 10)$*-smooth rational curves.*

By the adjunction formula, a smooth irreducible curve of a $K3$ surface is rational if and only if its self-intersection number is equal to -2.

3 Automorphic forms on the moduli space

Throughout this section, we assume that $M \subset \mathbb{L}_{K3}$ *is a primitive* 2*-elementary hyperbolic sublattice*. In this section, we recall the definition of automorphic forms on the period domain Ω^+_M and give its differential geometric characterization.

Let us fix a vector $\mathbf{l}_M \in M^\perp_{\mathbf{R}}$ with $\langle \mathbf{l}_M, \mathbf{l}_M \rangle \geq 0$. We set

$$j_M(\gamma, [z]) := \frac{\langle \gamma \cdot z, \mathbf{l}_M \rangle}{\langle z, \mathbf{l}_M \rangle} \qquad [z] \in \Omega^+_M, \quad \gamma \in O^+(M^\perp).$$

Since $\mathcal{H}_{\mathbf{l}_M} = \emptyset$, $j_M(\gamma, \cdot)$ is a nowhere vanishing holomorphic function on Ω^+_M.

Definition 3.1. *Let $\Gamma \subset O^+(M^\perp)$ be a cofinite subgroup. A holomorphic function $f \in \mathcal{O}(\Omega_M^+)$ is called an automorphic form for Γ of weight p if*

$$f(\gamma \cdot [z]) = \chi(\gamma)\, j_M(\gamma, [z])^p f([z]), \qquad [z] \in \Omega_M^+, \quad \gamma \in \Gamma,$$

where $\chi \colon \Gamma \to \mathbf{C}^*$ is a character.

Let $K_M([z])$ be the Bergman kernel function of Ω_M^+:

$$K_M([z]) := \frac{\langle z, \bar{z} \rangle}{|\langle z, \mathbf{l}_M \rangle|^2}.$$

For an automorphic form of weight p, the Petersson norm of f is the function on Ω_M^+ defined as

$$\|f([z])\|^2 := K_M([z])^p\, |f([z])|^2.$$

If $r(M) \leq 17$ and if $\Gamma \subset O^+(M^\perp)$ is a cofinite subgroup, then $\|f\|^2$ is a Γ-invariant C^∞ function on Ω_M^+, because the group $\Gamma/[\Gamma, \Gamma]$ is finite and Abelian in this case.

Let ω_M be the Kähler form of the Bergman metric on Ω_M^+:

$$\omega_M := -dd^c \log K_M, \tag{3.1}$$

where $d^c = (\partial - \bar{\partial})/4\pi\mathrm{i}$ and hence $dd^c = \bar{\partial}\partial/2\pi\mathrm{i}$ for complex manifolds. For a divisor D on Ω_M^+, let δ_D be the Dirac δ-current on Ω_M^+ with support D.

Theorem 3.2. *Let $p \in \mathbf{N}$ and let D be a divisor on Ω_M^+. Let $\Gamma \subset O^+(M^\perp)$ be a cofinite subgroup. Let φ be a non-negative, Γ-invariant C^∞ function on $\Omega_M \setminus D$ satisfying $\log \varphi \in L^1_{\mathrm{loc}}(\Omega_M)$ and the equation of currents on Ω_M^+:*

$$dd^c \log \varphi = \delta_D - p\, \omega_M. \tag{3.2}$$

If $r(M) \leq 17$, then there exists an automorphic form F for Γ of weight p with zero divisor D such that $\varphi = \|F\|^2$.

Proof. Set $\psi = \varphi\, K_M^{-p}$. Then $\log \psi \in L^1_{\mathrm{loc}}(\Omega_M^+)$. We get the following equation of currents on Ω_M^+ by (3.1), (3.2):

$$dd^c \log \psi = \delta_D,$$

so that $\partial \log \psi$ is a meromorphic 1-form on Ω_M^+ with at most logarithmic poles along D. Fix a point $[\eta_0] \in \Omega_M^+ \setminus D$, and set

$$F([\eta]) := \exp\left(\int_{[\eta_0]}^{[\eta]} \partial \log \psi \right), \qquad [\eta] \in \Omega_M^+.$$

Since the residues of $\partial \log \psi$ are integers, we get $F \in \mathcal{O}(\Omega_M^+)$ and

$$d \log F = \partial \log \psi, \qquad \mathrm{div}(F) = D.$$

Let $\gamma \in \Gamma$. By the identity $K_M(\gamma[\eta]) = |j_M(\gamma, [\eta])|^{-2} K_M([\eta])$ and by the Γ-invariance of φ, we get $\psi(\gamma[\eta]) = |j_M(\gamma, [\eta])|^{2p} \psi([\eta])$, which yields that

$$\gamma^* \partial \log \psi = \partial \log \psi + p \cdot d \log j_M(\gamma, \cdot).$$

Namely, $d \log(\gamma^* F / j_M(\gamma, \cdot)^p F) = 0$ and hence

$$\chi(\gamma) := F(\gamma[\eta])\, j_M(\gamma, [\eta])^{-p} F([\eta])^{-1}$$

is a non-zero constant on Ω_S^+. Then χ is a character of Γ because for every $\gamma, \gamma' \in \Gamma$,

$$\begin{aligned}\chi(\gamma\gamma') &= \frac{F(\gamma\gamma'[\eta])}{j_M(\gamma\gamma', [\eta])^p F([\eta])} \\ &= \frac{F(\gamma\gamma'[\eta])}{j_M(\gamma,\, \gamma'[\eta])^p F(\gamma'[\eta])} \times \frac{j_M(\gamma,\, \gamma'[\eta])^p F(\gamma'[\eta])}{j_M(\gamma\gamma', [\eta])^p F([\eta])} = \chi(\gamma)\chi(\gamma').\end{aligned}$$

Hence F is an automorphic form on Ω_M^+ for Γ of weight p with character χ such that $\mathrm{div}(F) = D$. Since $r(M) \leq 17$, $\|F\|$ is Γ-invariant. By the Poincaré-Lelong formula, the following equation of currents on Ω_M^+ holds:

$$dd^c \log \|F\|^2 = \delta_D - p\, \omega_M. \tag{3.3}$$

By comparing (3.1) and (3.3), $\log(\varphi/\|F\|^2)$ is a Γ-invariant pluriharmonic function on Ω_M^+, so that $\log(\varphi/\|F\|^2)$ descends to a pluriharmonic function on $\mathcal{M}_M$. Since $\overline{\mathcal{M}}_M$, the Baily-Borel-Satake compactification of $\mathcal{M}_M$, is a normal projective variety with $\mathrm{codim}(\overline{\mathcal{M}}_M \setminus \mathcal{M}_M) \geq 2$ when $r(M) \leq 17$, $\log(\varphi/\|F\|^2)$ extends to a pluriharmonic function on $\overline{\mathcal{M}}_M$ by Grauert-Remmert. Since $\overline{\mathcal{M}}_M$ is compact, $\log(\varphi/\|F\|^2)$ is constant by the maximum principle for pluriharmonic functions. This proves the existence of a positive constant C with $\varphi = C\, \|F\|^2$. $\square$

4 Equivariant analytic torsion and 2-elementary $K3$ surfaces

4.1 Equivariant analytic torsion

Let (X, κ) be a compact Kähler manifold. Let G be a finite group acting holomorphically on X and preserving κ. Let $\square_q = (\bar\partial + \bar\partial^*)^2$ be the $\bar\partial$-Laplacian acting on C^∞ $(0, q)$-forms on X. Let $\sigma(\square_q)$ be the spectrum of $\square_q$. For $\lambda \in \sigma(\square_q)$, let $E_q(\lambda)$ be the eigenspace of $\square_q$ with respect to the eigenvalue λ. Since G preserves κ, $E_q(\lambda)$ is a finite-dimensional unitary representation of G. For $g \in G$ and $s \in \mathbf{C}$, set

$$\zeta_q(g)(s) := \sum_{\lambda \in \sigma(\square_q) \setminus \{0\}} \mathrm{Tr}\,(g|_{E_q(\lambda)})\, \lambda^{-s}.$$

Then $\zeta_q(g)(s)$ converges absolutely when $\mathrm{Re}\, s > \dim X$, admits a meromorphic continuation to the complex plane $\mathbf{C}$, and is holomorphic at $s = 0$.

Definition 4.1. *The equivariant analytic torsion of (X,κ) is the class function on G defined by*

$$\tau_G(X,\kappa)(g) := \exp[-\sum_{q\geq 0}(-1)^q q\, \zeta_q'(g)(0)], \qquad g \in G.$$

When $g=1$, $\tau_G(X,\kappa)(1)$ is denoted by $\tau(X,\kappa)$ and is called the analytic torsion of (X,κ).

4.2 An invariant for 2-elementary $K3$ surfaces

Let (X,ι) be a 2-elementary $K3$ surface of type M. Let $\mathbf{Z}_2$ be the subgroup of $\mathrm{Aut}(X)$ generated by ι. Let κ be a $\mathbf{Z}_2$-invariant Kähler form on X. Set $\mathrm{vol}(X,\kappa) := (2\pi)^{-2}\int_X \kappa^2/2!$. Let η_X be a nowhere vanishing holomorphic 2-form on X. The L^2-norm of η_X is defined as $\|\eta_X\|_{L^2}^2 := (2\pi)^{-2}\int_X \eta_X \wedge \bar{\eta}_X$.

Let $X^\iota = \sum_i C_i$ be the decomposition of the fixed point set of ι into the connected components. Set $\mathrm{vol}(C_i,\kappa|_{C_i}) := (2\pi)^{-1}\int_{C_i}\kappa|_{C_i}$. Let $c_1(C_i,\kappa|_{C_i})$ be the Chern form of $(TC_i,\kappa|_{C_i})$.

Definition 4.2. *Define*

$$\begin{aligned}\tau_M(X,\iota) :&= \mathrm{vol}(X,\kappa)^{\frac{14-r(M)}{4}}\tau_{\mathbf{Z}_2}(X,\kappa)(\iota)\prod_i \mathrm{vol}(C_i,\kappa|_{C_i})\tau(C_i,\kappa|_{C_i}) \\ &\times \exp\left[\frac{1}{8}\int_{C_i}\log\left(\frac{\eta_X\wedge\bar{\eta}_X}{\kappa^2/2!}\cdot\frac{\mathrm{vol}(X,\kappa)}{\|\eta_X\|_{L^2}^2}\right)\Bigg|_{C_i} c_1(C_i,\kappa|_{C_i})\right].\end{aligned}$$

Obviously, $\tau_M(X,\iota)$ is independent of the choice of η_X. It is worth remarking that if κ is *Ricci-flat*, then

$$\tau_M(X,\iota) = \mathrm{vol}(X,\kappa)^{\frac{14-r(M)}{4}}\tau_{\mathbf{Z}_2}(X,\kappa)(\iota)\prod_i \mathrm{vol}(C_i,\kappa|_{C_i})\tau(C_i,\kappa|_{C_i}).$$

Theorem 4.3. *Let $M \subset \mathbb{L}_{K3}$ be a primitive 2-elementary hyperbolic sublattice satisfying $11 \leq r(S) \leq 17$ and $r(M)+l(M)=22$. Then there exists an automorphic form Φ_M on Ω_M^+ for $O^+(M^\perp)$ of weight $(r(S)-6)$ with zero divisor $\mathcal{D}_S$ such that for every 2-elementary $K3$ surface (X,ι) of type M and for every $\mathbf{Z}_2$-invariant Kähler form κ on X,*

$$\tau_M(X,\iota) = \|\Phi_M(\varpi_M(X,\iota))\|^{-\frac{1}{2}}.$$

Proof. Theorem 4.3 follows from the following two claims:
- The number $\tau_M(X,\iota)$ is independent of the choice of a $\mathbf{Z}_2$-invariant Kähler form, and it gives rise to a function τ_M on $\mathcal{M}_M^o$.
- Regarded as a Γ_M-invariant function on Ω_M^o, $\log\tau_M$ lies in $L^1_{\mathrm{loc}}(\Omega_M)$ and satisfies the following equation of currents on Ω_M:

$$dd^c \log \tau_M = \frac{r(M)-6}{4}\,\omega_M - \frac{1}{4}\delta_{\mathcal{D}_M}. \tag{4.1}$$

The first claim follows immediately from the curvature formula for equivariant Quillen metrics [7], [29]. To prove the second claim, it suffices to determine the singularity of τ_M near the divisor $\mathcal{D}_M$. Let $\gamma\colon \Delta \to \Omega_M$ be a holomorphic curve intersecting $\mathcal{D}^o$ transversally at $t = 0$. Then Eq. (4.1) is deduced from the following estimate:

$$\log \tau_M(\gamma(t)) = -\frac{1}{4}\log|t|^2 + O(\log(-\log|t|)), \qquad t \to 0. \tag{4.2}$$

Under a certain technical assumption on the curve γ, Eq. (4.2) follows from the embedding formula of Bismut [5] for equivariant Quillen metrics. See [39] for more details about the proof. □

In the cases $M = \mathbb{U}(2) \oplus \mathbb{E}_8(2)$ and $M = \mathbb{U} \oplus \mathbb{E}_8(2)$, an explicit formula for Φ_M was given in [39, Sect. 8]; in the first case, Φ_M is given by the Borcherds Φ-function of dimension 10; in the second case, Φ_M is given by the restriction of the Borcherds Φ-function of dimension 26 to Ω_M.

5 The Borcherds products

In this section, we recall Borcherds products. For simplicity, we restrict our explanation for those lattices that splits into two hyperbolic lattices.

Let $Mp_2(\mathbf{Z})$ be the metaplectic group (cf. [8], [9]):

$$Mp_2(\mathbf{Z}) := \left\{ \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \sqrt{c\tau + d} \right);\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbf{Z}),\ \sqrt{c\tau+d} \in \mathcal{O}(\mathbf{H}) \right\},$$

which is generated by the following two elements

$$S := \left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \sqrt{\tau} \right), \qquad T := \left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, 1 \right).$$

Let K be an even hyperbolic lattice with signature $(1, b^- - 1)$. Let $N \in \mathbf{N}$. Let $\mathbf{f}$, $\mathbf{f}'$ be a basis of $\mathbb{U}(N)$ such that

$$\mathbf{f}\cdot\mathbf{f} = \mathbf{f}'\cdot\mathbf{f}' = 0, \qquad \mathbf{f}\cdot\mathbf{f}' = N.$$

Set

$$L := \mathbb{U}(N) \oplus K.$$

The signature of L is $(2, b^-)$. Let $l \in \mathbf{N}$ be the *level* of L; i.e., l is the smallest natural number such that $l\langle\gamma,\gamma\rangle/2 \in \mathbf{Z}$ and $l\langle\gamma,\delta\rangle \in \mathbf{Z}$ for all $\gamma, \delta \in A_L$.

Let $\mathbf{C}[A_L]$ be the group ring of the discriminant group A_L. Let $\{\mathbf{e}_\gamma\}_{\gamma\in A_L}$ be the standard basis of $\mathbf{C}[A_L]$. Let $\rho_L\colon Mp_2(\mathbf{Z}) \to \mathrm{GL}(\mathbf{C}[A_L])$ be the *Weil representation*. Namely, for the generators S and T of $Mp_2(\mathbf{Z})$, we define

$$(5.1) \qquad \rho_L(T)\,\mathbf{e}_\gamma = e^{\pi \mathrm{i} \gamma^2}\mathbf{e}_\gamma, \qquad \rho_L(S)\,\mathbf{e}_\gamma = \frac{\mathrm{i}^{\frac{b^- -2}{2}}}{\sqrt{|A_L|}} \sum_{\delta \in A_L} e^{-2\pi \mathrm{i} \gamma\cdot\delta}\mathbf{e}_\delta.$$

Here the bilinear form on L is denoted by $x \cdot y = \langle x, y\rangle$ for simplicity. Then ρ_L extends to a group homomorphism from $Mp_2(\mathbf{Z})$ to $\mathrm{GL}(\mathbf{C}[A_L])$.

Definition 5.1. *A $\mathbf{C}[A_L]$-valued holomorphic function $F(\tau)$ on the complex upper-half plane $\mathbf{H}$ is a modular form of weight $1 - \frac{b^-}{2}$ of type ρ_L if the following conditions are satisfied:*
(1) *For all $(\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right), \sqrt{c\tau+d}) \in Mp_2(\mathbf{Z})$ and $\tau \in \mathbf{H}$,*

$$F\left(\frac{a\tau+b}{c\tau+d}\right) = \sqrt{c\tau+d}^{\,2-b^-}\rho_L\left(\begin{pmatrix} a & b \\ c & d\end{pmatrix}, \sqrt{c\tau+d}\right)\cdot F(\tau).$$

(2) *$F(\tau)$ is meromorphic at $+\mathrm{i}\infty$ and admits the integral Fourier expansion:*

$$F(\tau) = \sum_{\gamma \in A_L} \mathbf{e}_\gamma \sum_{k \in \frac{1}{l}\mathbf{Z}} c_\gamma(k)\, e^{2\pi \mathrm{i} k\tau},$$

where $c_\gamma(k) \in \mathbf{Z}$ for all $k \in \frac{1}{l}\mathbf{Z}$ and $c_\gamma(k) = 0$ for $k \ll 0$.

By [8, p.512 Th. 5.3], $F(\tau)$ induces an elliptic modular form $F_K(\tau)$ of the same weight $1 - \frac{b^-}{2}$ of type ρ_K.

As before, define

$$\Omega_L := \{[x] \in \mathbf{P}(L_\mathbf{C});\ \langle x, x\rangle = 0,\ \langle x, \bar{x}\rangle > 0\}.$$

For $\lambda \in L_\mathbf{R}$ with $\langle \lambda, \lambda\rangle < 0$, we define $\mathcal{H}_\lambda$ as before in Sect. 2. Let

$$C_K = \{v \in K_\mathbf{R};\ \langle v, v\rangle > 0\}$$

be the *light cone* of K. Then the tube domain $K_\mathbf{R} + \mathrm{i}\,C_K$ is identified with Ω_L by the map

$$(5.2) \qquad K_\mathbf{R} + \mathrm{i}\,C_K \ni z \to \left[\mathbf{f} - \frac{\langle z, z\rangle}{2}\,\mathbf{f}' + z\right] \in \mathbf{P}(L_\mathbf{C}).$$

Since K is hyperbolic, C_K consists of two connected components. Let C_K^+ be one of them. Let Ω_L^+ be the component of Ω_L corresponding to $K_\mathbf{R} + \mathrm{i}\,C_K^+$ via the isomorphism (5.2).

By [8, p.517], $F_K(\tau)$ induces a chamber structure of the cone C_K^+. Each chamber of C_K^+ is called a *Weyl chamber*. Let W be a Weyl chamber of C_K^+. The dual cone of W is defined by

$$W^\vee = \{v \in K_\mathbf{R};\ \langle v, w\rangle > 0,\ \forall\, w \in W\}.$$

Theorem 5.2. *(Borcherds) There exists an automorphic form $\Psi_L(z,F)$ on Ω_L^+ with the following properties:*
(1) $\Psi_L(z,F)$ is an automorphic form of weight $c_0(0)/2$ for a cofinite subgroup of $O^+(L)$.
(2) The divisor of $\Psi_L(\cdot,F)$ is given by

$$\mathrm{div}(\Psi_L(\cdot,F)) = \sum_{\lambda\in L^\vee,\,\lambda^2<0} c_\lambda\left(\frac{\lambda^2}{2}\right)\mathcal{H}_\lambda.$$

(3) There exists a vector $\rho=\rho(K,F_K(\tau),W)\in K_{\mathbf{Q}}$ determined by K, $F_K(\tau)$ and W such that $\Psi_L(z,F)$ admits the following infinite product expansion for $z\in K_{\mathbf{R}}+iW$ with $\langle z,z\rangle\gg 0$:

$$\Psi_L(z,F) = e^{2\pi i\rho\cdot z}\prod_{\lambda\in K^\vee\cap W^\vee}\prod_{n\in\mathbf{Z}/N\mathbf{Z}}\left(1-e^{2\pi i(\lambda\cdot z+\frac{n}{N})}\right)^{c_{\lambda+\frac{n}{N}\mathbf{f}'}(\lambda^2/2)}.$$

The automorphic form $\Psi_L(z,F)$ is called the Borcherds product associated with L and $F(\tau)$. The vector ρ is called the Weyl vector of $\Psi_L(z,F)$.

Proof. See [8] and [12].

6 Borcherds products for odd unimodular lattices

Define the symmetric unimodular matrix $\mathbb{I}_{1,m}$ of rank $m+1$ and with signature $(1,m)$ by

$$\mathbb{I}_{1,m} := \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & -1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

We identify $\mathbb{I}_{1,m}$ with the corresponding unimodular hyperbolic lattice. Define 2-elementary lattices $\mathbb{S}_k$, $\mathbb{T}_k$ $(1\le k\le 9)$ by

$$\mathbb{T}_k := \mathbb{U}(2)\oplus\mathbb{I}_{1,9-k}(2), \qquad \mathbb{S}_k := \mathbb{T}_k^\perp.$$

Then $\mathbb{S}_k$ verifies the conditions in Theorem 4.3:
(6.1)

$$11\le r(\mathbb{S}_k) = 22-r(\mathbb{T}_k)\le 17, \qquad r(\mathbb{S}_k)+l(\mathbb{S}_k) = 22-r(\mathbb{T}_k)+l(\mathbb{T}_k) = 22.$$

By Nikulin [33, Th. 4.2.2, P.1434 table 1], $\mathbb{S}_k$ are the only 2-elementary hyperbolic lattices satisfying (6.1), up to an automorphism of $\mathbb{L}_{K3}$. In this section, we give an explicit expression of the automorphic form $\Phi_{\mathbb{S}_k}$ in Theorem 4.3 as a Borcherds product.

We define the Weyl vector of $\mathbb{I}_{1,9-k}(2)$ by

$$\rho_k := \frac{1}{2}(3,-1,\dots,-1) \in \mathbb{I}_{1,9-k}(2)^\vee.$$

We set

$$V := S^{-1}T^2S = \left(\begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}, \sqrt{-2\tau+1}\right) \in Mp_2(\mathbf{Z})$$

and we define $\mathbf{e}_0, \mathbf{e}_1, \mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \in \mathbf{C}[A_{\mathbb{T}_k}]$ by

$$\mathbf{e}_0 := \mathbf{e}_{(0,0,0)}, \qquad \mathbf{e}_1 := \mathbf{e}_{(0,0,\rho_k)}, \qquad \mathbf{v}_i := \sum_{\delta\in A_{\mathbb{T}_k},\, 2\langle\delta,\delta\rangle\equiv i \bmod 4} \mathbf{e}_\delta.$$

where vectors in $\mathbb{T}_k$ are denoted by (m,n,λ), $m,n\in\mathbf{Z}$, $\lambda\in\mathbb{I}_{1,9-k}(2)$.

Set $q = e^{2\pi i\tau}$. For $\tau\in\mathbf{H}$, let $\eta(\tau) = q^{1/24}\prod_{n=1}^\infty(1-q^n)$ be the Dedekind η-function and let

$$\theta_2(\tau) = \sum_{m\in\mathbb{Z}} q^{(m+\frac{1}{2})^2/2}, \qquad \theta_3(\tau) = \sum_{m\in\mathbb{Z}} q^{m^2/2}, \qquad \theta_4(\tau) = \sum_{m\in\mathbb{Z}} (-1)^m q^{m^2/2}$$

be Jacobi's theta functions. Notice that we use the notation $q = e^{2\pi i\tau}$ while $q = e^{\pi i\tau}$ in [13, Chap. 4]. For $\delta\in\{0,1/2\}$, let $\theta_{\mathbb{A}_1+\delta/2}(\tau)$ be the theta function of the A_1-lattice

$$\theta_{\mathbb{A}_1}(\tau) := \theta_3(2\tau), \qquad \theta_{\mathbb{A}_1+1/2}(\tau) := \theta_2(2\tau).$$

Define holomorphic functions $f_k^{(0)}(\tau)$, $f_k^{(1)}(\tau)$ and the series $\{c_k^{(0)}(l)\}_{l\in\mathbf{Z}}$, $\{c_k^{(1)}(l)\}_{l\in\mathbf{Z}+1/4}$ by

$$f_k^{(0)}(\tau) := \frac{\eta(2\tau)^8\,\theta_{\mathbb{A}_1}(\tau)^k}{\eta(\tau)^8\eta(4\tau)^8} = \sum_{l\in\mathbf{Z}} c_k^{(0)}(l)\,q^l = q^{-1}+8+2k+O(q),$$

$$f_k^{(1)}(\tau) := -16\frac{\eta(4\tau)^8\,\theta_{\mathbb{A}_1+1/2}(\tau)^k}{\eta(2\tau)^{16}} = \sum_{l\in 1/4+\mathbf{Z}} 2c_k^{(1)}(l)\,q^l.$$

We define holomorphic functions $g_k^{(i)}(\tau)$, $i\in\mathbf{Z}/4\mathbf{Z}$ by

$$g_k^{(i)}(\tau) = \sum_{l\equiv i \bmod 4} c_k^{(0)}(l)\,q^{l/4}.$$

By definition,

$$\sum_{i\in\mathbf{Z}/4\mathbf{Z}} g_k^{(i)}(\tau) = \frac{\eta(\tau/2)^8\,\theta_{\mathbb{A}_1}(\tau/4)^k}{\eta(\tau)^8\eta(\tau/4)^8} = f_k^{(0)}(\tau/4).$$

Define a $\mathbf{C}[A_{\mathbb{T}_k}]$-valued holomorphic function on $\mathbf{H}$ by

$$F_k(\tau) := f_k^{(0)}(\tau)\,\mathbf{e}_0 + f_k^{(1)}(\tau)\,\mathbf{e}_1 + \sum_{i\in\mathbf{Z}/4\mathbf{Z}} g_k^{(i)}(\tau)\,\mathbf{v}_i.$$

Theorem 6.1. *For $1 \leq k \leq 9$, the following hold:*
(1) *$F_k(\tau)$ is a modular form for $Mp_2(\mathbf{Z})$ of type $\rho_{\mathbb{T}_k}$ and of weight $(k-8)/2$;*
(2) *the Weyl vector of $\Psi_{\mathbb{T}_k}(z, F_k)$ is given by $2\rho_k$;*
(3) *there exists a generalized Kac-Moody superalgebra with denominator function $\Phi_{\mathbb{S}_k}$;*
(4) *if $k < 8$, then there exists a constant $C_k \neq 0$ such that*

$$\Phi_{\mathbb{S}_k}(z)^2 = C_k \, \Psi_{\mathbb{T}_k}(z, F_k).$$

The modular form $F_k(\tau)$ for $Mp_2(\mathbf{Z})$ is induced from the modular form $f_k^{(0)}(\tau)$ for $\Gamma_0(4)$. The modular form $f_k^{(0)}(\tau)$ is *reflective* for $\mathbb{T}_k$ in the sense of Borcherds [9, Sect. 11, pp.350-351].

Remark 6.2. Theorem 6.1 (2) is closely related to an example of Borcherds [8, Example 15.3]. Theorem 6.1 (3), (4) seem to be closely related to a problem of Borcherds [8, Problem 16.2] and conjectures of Harvey-Moore [20, Sect. 7 Conjecture] and Gritsenko-Nikulin [17]. See [40, Sect. 7] for more explanations. The automorphic form $\Phi_{\mathbb{S}_7}$ was already found by Gritsenko-Nikulin [19].

We shall give a detailed proof of Theorem 6.1 in the forthcoming paper [41]. In fact, the norm of $\Phi_{\mathbb{S}_k}$ is regarded as an invariant of certain Calabi-Yau threefolds, which was introduced by Bershadsky-Cecotti-Ooguri-Vafa [4] and by Fang-Lu-Yoshikawa [15] using analytic torsion.

7 $K3$ surfaces of Matsumoto-Sasaki-Yoshida

In Sections 4 and 6, we extended Eqs. (1.6) and (1.7) to 2-elementary $K3$ surfaces of type $\mathbb{S}_k$. In this section, we consider an analogue of Eq. (1.4) in dimension 2. We focus on 2-elementary $K3$ surfaces of type $\mathbb{S}_6$. Those $K3$ surfaces were studied in detail by Matsumoto-Sasaki-Yoshida [30], [31], [36].

7.1 The construction of Matsumoto-Sasaki-Yoshida

Recall that

$$M^o(3,6) := \{A = (\mathbf{a}_1, \ldots, \mathbf{a}_6) \in M(3,6);\ \mathbf{a}_i \wedge \mathbf{a}_j \wedge \mathbf{a}_k \neq 0 \text{ for } i < j < k\}.$$

For $A \in M^o(3,6)$, we define

$$S_A := \{((x_1 : x_2 : x_3), y) \in \mathcal{O}_{\mathbf{P}^2}(3);\ y^2 = \prod_{i=1}^{6}(a_{1i}\, x_1 + a_{2i}\, x_2 + a_{3i}\, x_3)\}.$$

The natural projection $p = \mathrm{pr}_2 \colon S_A \to \mathbf{P}^2$ is a double covering with branch divisor

$$L_{A,1} \cup \cdots \cup L_{A,6},$$

where

$$L_{A,i} := \{(x_1 : x_2 : x_3) \in \mathbf{P}^2;\ a_{1i}\, x_1 + a_{2i}\, x_2 + a_{3i}\, x_3 = 0\} \cong \mathbf{P}^1.$$

Set $E_{A,ij} := L_{A,i} \cap L_{A,j}$. Corresponding to the 15 points $\{E_{A,ij}\, i \neq j\} \subset \mathbf{P}^2$, S_A has 15 ordinary double points. By [31], [36, Sect. 9.1], the minimal resolution of S_A, i.e., the blowing-up of these 15 singular points, is a $K3$ surface. In fact, the following 2-form η_A on X_A is nowhere vanishing:

$$\eta_A := \frac{dx}{y} = \frac{dx}{\prod_{i=1}^{6}(a_{1i}\, x_1 + a_{2i}\, x_2 + a_{3i}\, x_3)^{1/2}}, \tag{7.1}$$

where

$$dx = x_1\, dx_2 \wedge dx_3 - x_2\, dx_1 \wedge dx_3 + x_3\, dx_1 \wedge dx_2.$$

Let $\theta_A \colon S_A \to S_A$ be the involution defined as the non-trivial covering transformation of the double covering $p \colon S_A \to \mathbf{P}^2$.

Definition 7.1. *Let X_A be the minimal resolution of S_A, and let $\iota_A \colon X_A \to X_A$ be the involution on X_A induced from θ_A.*
(1) *The pair (X_A, ι_A) is called a Matsumoto-Sasaki-Yoshida (MSY) $K3$ surface associated with A.*
(2) *Let $L_A := (L_{A,1}, \cdots, L_{A,6})$ be the* ordered *set of lines of $\mathbf{P}^2$ associated with A. The triple (X_A, ι_A, L_A) is called a MSY-$K3$ surface with level 2 structure associated with A.*

Two MSY-$K3$ surfaces with level 2 structure (X_A, ι_A, L_A) and (X_B, ι_B, L_B) are isomorphic if there exists an isomorphism $\varphi \colon X_A \to X_B$ such that

$$\varphi \circ \iota_A = \iota_B \circ \varphi, \qquad \varphi(L_A) = L_B.$$

Let $\widetilde{E}_{A,ij} \subset X_A$ be the proper transform of $E_{A,ij}$ by the blowing-up $X_A \to S_A$. Let $H_A \subset \mathbf{P}^2$ be a line which does not pass any points $E_{A,ij}$, and let $\widetilde{H}_A \subset X_A$ be the proper transform of $p^{-1}(H_A)$ by the blowing-up $X_A \to S_A$. Let $\widetilde{L}_{A,i}$ be the proper transform of $p^{-1}(L_{A,i})$ by the blowing-up $X_A \to S_A$. By [31, Prop. 2.1.5], there exists a system of generators E_{ij} $(1 \leq i < j \leq 6)$, H, L_i $(1 \leq i \leq 6)$ of $\mathbb{S}_6$ such that for every MSY-$K3$ surfaces with level 2 structure (X_A, ι_A, L_A), there exists a marking α with

$$\alpha^{-1}(E_{ij}) = c_1([\widetilde{E}_{A,ij}]), \qquad \alpha^{-1}(H) = c_1([\widetilde{H}_A]), \qquad \alpha^{-1}(L_i) = c_1([\widetilde{L}_{A,i}]).$$

Here $[D]$ denotes the line bundle on X_A associated with the divisor D. The triple (X_A, ι_A, L_A) defines a $\mathbb{S}_6$-polarized $K3$ surface in the sense of Dolgachev [14]. A marking of (X_A, ι_A) satisfying these conditions is called a marking of MSY-$K3$ surfaces with level 2 structure (X_A, ι_A, L_A).

Define

$$O(\mathbb{T}_6)(2) := \ker\{O(\mathbb{T}_6) \to O(A_{\mathbb{T}_6})\}.$$

If α, β are markings of (X_A, ι_A, L_A), then

$$\beta \circ \alpha^{-1}|_{\mathbb{S}_6} = \mathrm{id}_{\mathbb{S}_6}, \qquad \beta \circ \alpha^{-1}|_{\mathbb{T}_6} \in O(\mathbb{T}_6)(2).$$

Since $\beta \circ \alpha^{-1}|_{\mathbb{T}_6} \in O(\mathbb{T}_6)(2)$, the $O(\mathbb{T}_6)(2)$-orbit of the period $\pi(X_A, \iota_A, \alpha)$ is independent of the choice of a marking of MSY-$K3$ surface with level 2 structure. The $O(\mathbb{T}_6)(2)$-orbit

$$O(\mathbb{T}_6)(2) \cdot \pi(X_A, \iota_A, \alpha) \in \Omega^o_{\mathbb{S}_6}/O(\mathbb{T}_6)(2)$$

is called the *Griffiths period* of a MSY-$K3$ surface (X_A, ι_A, L_A).

Lemma 7.2. *A MSY-$K3$ surface is a 2-elementary $K3$ surface of type $\mathbb{S}_6$.*

Proof. Let (X_A, ι_A) be a MSY-$K3$ surface. Since X_A/ι_A is the blowing-up of $\mathbf{P}^2$ at the 15 points $\{E_{A,ij}\}_{i<j}$ and is a rational surface, ι_A acts non-trivially on $H^0(X_A, K_{X_A})$. The type of (X_A, ι_A) is $\mathbb{S}_6$ by [31, Prop. 2.1.5]. $\square$

We have a family of $K3$ surfaces with involution $\pi\colon (\mathcal{X}, \iota) \to M^o(3,6)$ such that $\pi^{-1}(A) = (X_A, \iota_A)$. On $M^o(3,6)$, acts the group $GL_3(\mathbf{C}) \times (\mathbf{C}^*)^6$ by

$$(g, \lambda_1, \cdots, \lambda_6) \cdot A := g \cdot A \cdot \mathrm{diag}(\lambda_1, \cdots, \lambda_6).$$

Definition 7.3. *Define the configuration space of six lines in gerenal position on $\mathbf{P}^2$ by*

$$X^o(3,6) := GL_3(\mathbf{C}) \backslash M^o(3,6)/(\mathbf{C}^*)^6.$$

The configuration space $X^o(3,6)$ is a Zariski open subset of $\mathbf{C}^4$. In fact, every element of $X^o(3,6)$ has a unique representative of the form (cf. [36, Chap. 7 Sect. 2]):

$$\begin{pmatrix} a_1 & a_2 & 1 & 1 & 0 & 0 \\ a_3 & a_4 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{pmatrix}, \qquad a_1, \dots, a_4 \in \mathbf{C}.$$

Hence there exists an embedding $j\colon X^o(3,6) \hookrightarrow M^o(3,6)$ with

$$j(X^o(3,6)) = \left\{ \begin{pmatrix} a_1 & a_2 & 1 & 1 & 0 & 0 \\ a_3 & a_4 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{pmatrix} \in M^o(3,6);\, a_1, a_2, a_3, a_4 \in \mathbf{C} \right\}. \tag{7.2}$$

By the expression (7.2), there exist 15 hyperplanes $H_1, \dots, H_{15} \subset \mathbf{C}^4$ and a hyperquadric $Q \in \mathbf{C}^4$ such that $X^o(3,6) = \mathbf{C}^4 \setminus H_1 \cup \dots \cup H_{15} \cup Q$.

The permutation group on 6 letters $\mathfrak{S}_6$ acts on $M(3,6;\mathbf{C})$ by

$$\sigma \cdot (\mathbf{a}_1, \dots, \mathbf{a}_6) := (\mathbf{a}_{\sigma(1)}, \dots, \mathbf{a}_{\sigma(6)}), \qquad (\mathbf{a}_1, \dots, \mathbf{a}_6) \in M(3,6;\mathbf{C}), \quad \sigma \in \mathfrak{S}_6.$$

Following [36, Chap. 7 Sect. 3], we define an automorphism of $M^o(3,6)$ by

$$T(U,V) := (\det(U){}^tU^{-1}, \det(V){}^tV^{-1}), \qquad U, V \in GL_3(\mathbf{C}).$$

Notice that the (i,j)-entry of $\det(U){}^tU^{-1}$ is the (i,j)-minor of U for $U \in GL_3(\mathbf{C})$. For all $A \in M^o(3,6)$, $g \in GL_3(\mathbf{C})$, $\lambda_1, \ldots, \lambda_6 \in \mathbf{C}^*$, one has

$$\sigma(gA) = g\sigma(A), \qquad \sigma(A \cdot \mathrm{diag}(\lambda_1, \cdots, \lambda_6)) = \sigma(A) \cdot \mathrm{diag}(\lambda_{\sigma(1)}, \cdots, \lambda_{\sigma(6)})$$

$$T(gA) = {}^tg^{-1}T(A), \quad T(A \cdot \mathrm{diag}(\lambda_1, \cdots, \lambda_6)) = T(A) \cdot \mathrm{diag}(\mu_{\sigma(1)}, \cdots, \mu_{\sigma(6)})$$

where $\mu_i = \lambda_1\lambda_2\lambda_3/\lambda_i$ for $i = 1,2,3$ and $\mu_j = \lambda_4\lambda_5\lambda_6/\lambda_j$ for $j = 4,5,6$. Hence the actions of $\mathfrak{S}_6$ and T on $M^o(3,6)$ descend to the ones on $X^o(3,6)$.

Let $\langle T \rangle \cong \mathbf{Z}_2$ be the subgroup of $\mathrm{Aut}(X^o(3,6))$ generated by T. Let G be the finite automorphism group of $X^o(3,6)$ generated by $\mathfrak{S}_6$ and T. Since $\mathfrak{S}_6$ commutes with $\langle T \rangle$ by [36, Chap. 7 Prop. 3.3], one has $G \cong \mathfrak{S}_6 \times \mathbf{Z}_2$.

Set

$$\mathcal{M}^o_{\mathbb{S}_6}(2) := \Omega^o_{\mathbb{S}_6}/O(\mathbb{T}_6)(2).$$

Theorem 7.4. *(Matsumoto-Sasaki-Yoshida) The period map for the family of MSY-$K3$ surfaces with level 2 structure $\pi\colon (\mathcal{X}, \iota) \to M^o(3,6)$ with fiber $\pi^{-1}(A) = (X_A, \iota_A, L_A)$, induces an isomorphisms of analytic spaces*

$$X^o(3,6) \cong \mathcal{M}^o_{\mathbb{S}_6}(2), \qquad X^o(3,6)/G \cong \mathcal{M}^o_{\mathbb{S}_6}.$$

In particular, $X^o(3,6)/G$ (resp. $X^o(3,6)$) is a coarse moduli space of MSY-$K3$ *surfaces (resp. with level 2 structure).*

Proof. By [31, Prop. 2.10.1] [36, Sect. 9.5], the period map for the family $\pi\colon (\mathcal{X}, \iota) \to M^o(3,6)$ induces an isomorphism of analytic spaces $\varphi\colon X^o(3,6) \cong \mathcal{M}^o_{\mathbb{S}_6}(2)$ such that the following diagram is commutative:

$$\begin{array}{ccc} M^o(3,6) & \xrightarrow{\mathrm{id}} & M^o(3,6) \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle \phi} \\ X^o(3,6) & \xrightarrow{\varphi} & \mathcal{M}^o_{\mathbb{S}_6}(2), \end{array} \tag{7.3}$$

where $q\colon M^o(3,6) \to X^o(3,6)$ is the natural projection and $\phi\colon M^o(3,6) \to \mathcal{M}^o_{\mathbb{S}_6}(2)$ is the period map for the family $\pi\colon (\mathcal{X}, \iota) \to M^o(3,6)$. This proves the first isomorphism. Since the isomorphism φ induces an isomorphism of groups $G \cong O(\mathbb{T}_6)/O(\mathbb{T}_6)(2)$ by [36, Prop. 9.4], we have $X^o(3,6)/G \cong \Omega^o_{\mathbb{S}_6}/O(\mathbb{T}_6) = \mathcal{M}^o_{\mathbb{S}_6}$. This proves the second assertion. See [30], [31, Prop. 2.10.1, p.22, l.7], [36] for more details. □

7.2 The Freitag theta functions

Let $M(2,\mathbf{C})$ denote the vector space of 2×2 complex matrices. Then $\Omega_{\mathbb{S}_6}$ is biholomorphic to a tube domain $\mathbf{H}_2 \subset M(2,\mathbf{C})$ defined by

$$\mathbf{H}_2 := \Lambda + \mathrm{i}\, C_\Lambda = \left\{ W \in M(2,\mathbf{C});\ \frac{W - W^*}{2\mathrm{i}} > 0 \right\}, \qquad W^* := {}^t\overline{W}.$$

The isomorphism between $\Omega_{\mathbb{S}_6}$ and $\mathbf{H}_2$ is given as follows. Let Λ be the real vector space of 2×2 Hermitian matrices:

$$\Lambda := \left\{ \begin{pmatrix} u & w \\ \bar{w} & v \end{pmatrix} \in M(2,\mathbf{C});\ u,v\in\mathbf{R},\ w\in\mathbf{C} \right\}.$$

Let $C_\Lambda := \{H\in\Lambda:\ H>0\}$ be the light cone of Λ, where $H>0$ if and only if H is positive-definite. Let $\{\mathbf{h},\mathbf{d}_1,\mathbf{d}_2,\mathbf{d}_3\}$ be the basis of Λ defined as

$$\mathbf{h}=\begin{pmatrix} 1 & \frac{1}{1+\mathrm{i}} \\ \frac{1}{1-\mathrm{i}} & 1 \end{pmatrix},\quad \mathbf{d}_1=\begin{pmatrix} 1 & \frac{1}{1+\mathrm{i}} \\ \frac{1}{1-\mathrm{i}} & 0 \end{pmatrix},\quad \mathbf{d}_2=\begin{pmatrix} 0 & \frac{1}{1+\mathrm{i}} \\ \frac{1}{1-\mathrm{i}} & 1 \end{pmatrix},\quad \mathbf{d}_3=\begin{pmatrix} 0 & \frac{\mathrm{i}}{1+\mathrm{i}} \\ \frac{-\mathrm{i}}{1-\mathrm{i}} & 0 \end{pmatrix}$$

We consider the following coordinates $\mathbf{y}=(y_0,y_1,y_2,y_3)$ on $\mathbf{H}_2=\Lambda+\mathrm{i}\,C_\Lambda$:

$$\begin{aligned}\mathbf{y} &= y_0\,\mathbf{h}+y_1\,\mathbf{d}_1+y_2\,\mathbf{d}_2+y_3\,\mathbf{d}_3\\ &= \begin{pmatrix} y_0+y_1 & (y_0+y_1+y_2+\mathrm{i}y_3)/(1+\mathrm{i}) \\ (y_0+y_1+y_2-iy_3)/(1-\mathrm{i}) & y_0+y_2 \end{pmatrix}\in\mathbf{H}_2.\end{aligned}$$

The period domain $\Omega_{\mathbb{S}_6}$ is isomorphic to the tube domain $\mathbf{H}_2$ by the map:

$$\mu\colon \mathbf{H}_2\ni\mathbf{y}\to(1:-\det(\mathbf{y}):y_0:y_1:y_2:y_3)\in\Omega_{\mathbb{S}_6}. \tag{7.4}$$

Definition 7.5. (1) *For $a,b\in\{0,\frac{1+\mathrm{i}}{2}\}^2$ and $W\in\mathbf{H}_2$, define*

$$\Theta\begin{pmatrix} a\\ b\end{pmatrix}(W)=\sum_{m\in\mathbf{Z}[\mathrm{i}]^2}\exp\pi\mathrm{i}\left\{\left(m+\frac{a}{1+\mathrm{i}}\right)^*W\left(m+\frac{a}{1+\mathrm{i}}\right)+2\mathrm{Re}\left(\frac{b}{1+\mathrm{i}}\right)^*m\right\}.$$

The Freitag theta function $\Theta\binom{a}{b}(W)$ is said to be even if $a^*b\in\mathbf{Z}$.
(2) *Define the Matsumoto-Sasaki-Yoshida form Δ_{MSY} by*

$$\Delta_{\mathrm{MSY}}(W):=\prod_{\binom{a}{b}\ \mathrm{even}}\Theta\begin{pmatrix} a\\ b\end{pmatrix}(W).$$

Let $\mathcal{P}$ be the set of all partitions $\binom{ijk}{lmn}$ of the set $\{1,\dots,6\}$, where

$$\begin{pmatrix} ijk\\ lmn\end{pmatrix}:=\{i,j,k\}\cup\{l,m,n\}=\{1,\dots,6\},\qquad i<j<k,\quad l<m<n.$$

There exists a one to one correspondence between $\mathcal{P}$ and the set of even Freitag theta functions. Since $\#\mathcal{P}=10$, there exists ten even Freitag theta functions. The Freitag theta function corresponding to the partition $\binom{ijk}{lmn}$ is denoted by $\Theta\binom{ijk}{lmn}(W)$. Hence

$$\Delta_{\mathrm{MSY}}(W)=\prod_{\binom{ijk}{lmn}\in\mathcal{P}}\Theta\begin{pmatrix} ijk\\ lmn\end{pmatrix}(W).$$

See [30, Sect. 2.3], [36, Sect. 9.12.5] for the explicit correspondence between the even characteristics $\{\binom{a}{b}\}$ and the partitions $\{\binom{ijk}{lmn}\}$.

Proposition 7.6. *Under the identification* $\mu\colon \mathbf{H}_2 \cong \Omega^+_{\mathbb{S}_6}$, *the Matsumoto-Sasaki-Yoshida form* $\Delta_{\mathrm{MSY}}(W)$ *is an automorphic form on* $\mathbf{H}_2$ *for* $O(\mathbb{T}_6)^+$ *of weight* 10 *with*

$$\operatorname{div}(\Delta_{\mathrm{MSY}}) = \mathcal{D}_{\mathbb{S}_6} = \sum_{\delta \in \Delta_{\mathbb{T}_6}} \mathcal{H}_\delta.$$

Proof. See [30, Lemma 2.3.1 and Prop. 3.1.1]. □

7.3 The discriminant of MSY $K3$ surfaces

We introduce an analogue of the function $\Delta_{(2,4)}$ in the case $M^o(3,6)$.

Definition 7.7. (1) *For* $A = (\mathbf{a}_1, \dots, \mathbf{a}_6) \in M^o(3,6)$ *and a partition* $\binom{ijk}{lmn} \in \mathcal{P}$, *define*

$$D\binom{ijk}{lmn}(A) := \det(\mathbf{a}_i, \mathbf{a}_j, \mathbf{a}_k)\, \det(\mathbf{a}_l, \mathbf{a}_m, \mathbf{a}_n).$$

(2) *Define a holomorphic function* $\Delta_{(3,6)}$ *on* $M^o(3,6)$ *by*

$$\Delta_{(3,6)}(A) := \prod_{\binom{ijk}{lmn} \in \mathcal{P}} D\binom{ijk}{lmn}(A) = \prod_{\binom{ijk}{lmn} \in \mathcal{P}} \det(\mathbf{a}_i, \mathbf{a}_j, \mathbf{a}_k) \cdot \det(\mathbf{a}_l, \mathbf{a}_m, \mathbf{a}_n).$$

(3) *Define a real-valued function* $\|\Delta_{(3,6)}\|$ *on* $M^o(3,6)$ *by*

$$\|\Delta_{(3,6)}(A)\| := \left(\frac{1}{(2\pi)^2} \int_{X_A} \eta_A \wedge \overline{\eta_A} \right)^{10} |\Delta_{(3,6)}(A)|.$$

Lemma 7.8. (1) $M^o(3,6) = M(3,6;\mathbf{C}) \setminus \operatorname{div}(\Delta_{(3,6)})$.
(2) $\|\Delta_{(3,6)}\|$ *is* $GL_3(\mathbf{C}) \times (\mathbf{C}^*)^6$*-invariant.*

Proof. **(1)** The first assertion follows from the definition of $M^o(3,6)$.
(2) Let $A = (a_{ij}) \in M^o(3,6)$ and $g \in GL_3(\mathbf{C})$. We write $gA = (a^{(g)}_{ij})$. We identify g with the corresponding projective transformation. Then the projective transformation $\mathbf{P}^2 \ni [x] \to [{}^t g^{-1} x] \in \mathbf{P}^2$ lifts to an isomorphism $f_g\colon X_A \to X_{gA}$ such that

$$\begin{aligned} f_g^*(\eta_{gA}) &= f_g^* \left(\frac{dx}{\prod_{i=1}^6 (a^{(g)}_{1i}\, x_1 + a^{(g)}_{2i}\, x_2 + a^{(g)}_{3i}\, x_3)^{1/2}} \right) \\ &= \frac{d({}^t g^{-1} x)}{\prod_{i=1}^6 (a_{1i}\, x_1 + a_{2i}\, x_2 + a_{3i}\, x_3)^{1/2}} = \det(g)^{-1}\, \eta_A. \end{aligned}$$

This, together with $\Delta_{(3,6)}(gA) = \det(g)^{20}\, \Delta_{(3,6)}(A)$, implies the $GL_3(\mathbf{C})$-invariance of $\|\Delta_{(3,6)}\|$. Let us see the $(\mathbf{C}^*)^6$-invariance of $\|\Delta_{(3,6)}\|$. Identify $\lambda = (\lambda_i)_{i=1}^6 \in (\mathbf{C}^*)^6$ with the invertible diagonal matrix $\lambda = (\delta_{ij}\lambda_i)_{1 \le i,j \le 6} \in GL_6(\mathbf{C})$. Since $\eta_{A\lambda} = (\det \lambda)^{-1/2} \eta_A$ and $\Delta_{(3,6)}(A\lambda) = \det(\lambda)^{10}\, \Delta_{(3,6)}(A)$, we get the $(\mathbf{C}^*)^6$-invariance of $\|\Delta_{(3,6)}\|$. □

By Lemma 7.8, $\|\varDelta_{(3,6)}\|$ descends to a function on $X^o(3,6)$. We identify $\|\varDelta_{(3,6)}\|$ with the corresponding function on $X^o(3,6)$.

Theorem 7.9. (1) *There exist non-zero constants C_1, C_2 such that the following identity holds under the identification* (7.4)*:*

$$\varPhi_{\mathbb{S}_6} = C_1\, \varDelta_{\mathrm{MSY}} = C_2\, \varPsi_{\mathbb{T}_6}(\cdot, F_6)^{1/2}.$$

(2) *There exists an absolute constant $C_3 \neq 0$ such that for all $A \in M^o(3,6)$,*

$$\tau_{\mathbb{S}_6}(X_A, \iota_A) = C_3\, \|\varDelta_{(3,6)}(A)\|^{-1/4}.$$

By Theorems 6.1 and 7.9 (1), we get an infinite product expansion of the Igusa cusp form, i.e., the restriction of $\varDelta_{\mathrm{MSY}}$ to the Siegel upper-half space $\mathfrak{S}_2 = \{W \in \mathbf{H}_2;\, {}^tW = W\}$. The infinite product expansion of the Igusa cusp form was first obtained by Gritsenko-Nikulin [18].

For the proof of Theorem 7.9, we recall the results of Matsumoto-Saasaki-Yoshida in more details.

7.4 A compactification of $X^o(3,6)$

For $1 \leq i < j < k \leq 6$, we define

$$M_{ijk}(3,6) := \left\{ A \in M(3,6;\mathbf{C});\ \begin{matrix} \mathbf{a}_i \wedge \mathbf{a}_j \wedge \mathbf{a}_k = \mathbf{0} & \text{if } (k,l,m) = (i,j,k) \\ \mathbf{a}_l \wedge \mathbf{a}_m \wedge \mathbf{a}_n \neq \mathbf{0} & \text{if } (k,l,m) \neq (i,j,k) \end{matrix} \right\},$$

$$X_{ijk}(3,6) := GL_3(\mathbf{C})\backslash M_{ijk}(3,6)/(\mathbf{C}^*)^6,$$

and we set

$$M^*(3,6) := M^o(3,6) \cup \amalg_{i<j<k} M_{ijk}(3,6)$$
$$X^*(3,6) := X^o(3,6) \cup \amalg_{i<j<k} X_{ijk}(3,6).$$

Notice that if $i < j < k$, $l < m < n$ and $(i,j,k) \neq (l,m,n)$, then

$$M_{ijk}(3,6) \cap M_{lmn}(3,6) = \emptyset, \qquad X_{ijk}(3,6) \cap X_{lmn}(3,6) = \emptyset.$$

The subset $M^*(3,6)$ is open in $M(3,6;\mathbf{C})$.

For $A \in \amalg_{i<j<k} M_{ijk}(3,6)$, we define S_A and $L_{A,i}$, $i = 1, \ldots, 6$ as in Sect. 7.1. Then $\mathrm{Sing}\, S_A$ consists of only rational double points, i.e., 12 ordinary double points and one A_3-singularity. For $A \in M^*(3,6)$, we define η_A as in (7.1). Since η_A is nowhere vanishing on the regular part of S_A, the minimal resolution of S_A, denoted again by X_A, is a $K3$ surface. We have a flat family of surfaces $\pi\colon \mathcal{S} \to M^*(3,6)$ with fiber $\pi^{-1}(A) = S_A$.

With respect to the trivial $GL_3(\mathbf{C}) \times (\mathbf{C}^*)^6$-action on $\mathbf{P}^{29}$, there exists by [31], [36] a $GL_3(\mathbf{C}) \times (\mathbf{C}^*)^6$-equivariant holomorphic map $F\colon M^*(3,6) \to \mathbf{P}^{29}$ that induces an injection $f\colon X^*(3,6) \hookrightarrow \mathbf{P}^{29}$. We consider the topology on $X^*(3,6)$ induced from the one on $f(X^*(3,6))$ via f; we identify $X^*(3,6)$ with $f(X^*(3,6))$ as a topological space. Let $\overline{X}(3,6)$ be the closure of $f(X^*(3,6))$ in $\mathbf{P}^{29}$ and let $\overline{X}_{ijk}(3,6)$ be the closure of $f(X_{ijk}(3,6))$ in $\mathbf{P}^{29}$.

Set $\mathcal{M}_{\mathbb{S}_6}(2) := \varOmega^+_{\mathbb{S}_6}/O^+(\mathbb{T}_6)(2)$. Since $O^+(\mathbb{T}_6)(2)$ is generated by reflections by [31, Prop. 2.5.2], $\mathcal{M}_{\mathbb{S}_6}(2)$ is smooth.

Theorem 7.10. (1) $\overline{X}(3,6)$ *is a projective variety of dimension* 4*. The isomorphism* φ *in* (7.3) *extends to an isomorphism* $\overline{\varphi}$ *between* $\overline{X}(3,6)$ *and the Baily-Borel-Satake compactification of* $\overline{\mathcal{M}_{\mathbb{S}_6}(2)}$*.*
(2) $X^*(3,6) \subset \overline{X}(3,6)_{\mathrm{reg}} := \overline{X}(3,6) \setminus \mathrm{Sing}\,\overline{X}(3,6)$*.*
(3) $X^*(3,6)$ *is a Zariski open subset of* $\overline{X}(3,6)$ *with* $\dim \overline{X}(3,6) \setminus X^*(3,6) \leq 2$*.*
(4) $\overline{X}_{ijk}(3,6) \cap X^*(3,6)$ *is a smooth hypersurface of* $X^*(3,6)$*.*

Proof. See [31, Th. A6.2] for the first part of (1) and [30, Th. 3.2.4, Cor. 4.4.2] for the second part of (1). Since X_A is a $K3$ surface with at most rational double points for $A \in \amalg_{i<j<k} M_{ijk}(3,6)$, $\overline{X}_{ijk}(3,6)$ is identified with a divisor of $\mathcal{M}_{\mathbb{S}_6}(2)$ via $\overline{\varphi}$. Hence $X^*(3,6)$ is regarded as a subset of $\overline{\mathcal{M}_{\mathbb{S}_6}(2)}$ via $\overline{\varphi}$. Since $\mathcal{M}_{\mathbb{S}_6}(2)$ is smooth, $X^*(3,6)$ consists of smooth points of $\overline{X}(3,6)$. This proves (2). See also [36, p.244] for the proof of (2). See [31, Prop. A5.3, Cor. A5.4, Th. A6.2] for the proof of (3). Consider the following subset of $M^*(3,6)$:

$$\mathcal{U} := \left\{ \begin{pmatrix} 1 & 0 & a & 0 & 1 & c \\ 0 & 1 & b & 0 & 1 & d \\ 0 & 0 & z & 1 & 1 & 1 \end{pmatrix} \in M^*(3,6);\ a,b,c,d,z \in \mathbf{C} \right\}.$$

Let $U_{123} \subset X^*(3,6)$ be the image of $\mathcal{U}$ by the natural projection $M^*(3,6) \to X^*(3,6)$. By [31, Lemmas A6.8 and A6.9 and their proofs], U_{123} is an open subset of $X^*(3,6)$ containing $X^o(3,6) \cup X_{123}(3,6)$. Since U_{123} is isomorphic to an open subset of $\mathbf{P}^2 \times \mathbf{C}^2$ and since $X_{123}(3,6) \cap U_{123}$ is defined by the equation $z = 0$, $X_{123}(3,6)$ is a smooth hypersurface of $X^*(3,6)$. This proves (4). By [36, p.244], $X_{ijk}(3,6)$ is identified with a certain smooth hypersurface of $\mathcal{M}_{\mathbb{S}_6}(2)$, which also proves (4). $\square$

See [36, Chap. 7 Sect. 5] for the interpretation of the boundary locus $\overline{X}(3,6) \setminus X^o(3,6)$ in terms of degenerate matrices in $M(3,6;\mathbf{C})$.

Define a function K on $M^*(3,6)$ by

$$K(A) := \int_{X_A} \eta_A \wedge \bar{\eta}_A, \qquad A \in M^*(3,6).$$

Lemma 7.11. *K is a nowhere vanishing continuous function on $M^*(3,6)$.*

Proof. Let $\mathfrak{U} \cong \Delta^{18}$ be a small neighborhood of A in $M^*(3,6)$ such that $\mathfrak{U} \cap \amalg_{i<j<k} M_{ijk}(3,6) \cong \Delta^{17}$. By [25, Th. 4.28], there exists a finite holomorphic map $h\colon \mathfrak{V} \to \mathfrak{U}$ with branch divisor $\mathfrak{U} \cap \amalg_{i<j<k} M_{ijk}(3,6)$ such that the family $\mathrm{pr}_2\colon \mathcal{S} \times_{\mathfrak{U}} \mathfrak{V} \to \mathfrak{V}$ induced from $\pi\colon \mathcal{S} \to M^*(3,6)$ by $h\colon \mathfrak{V} \to \mathfrak{U}$ admits a simultaneous resolution. Namely, there exist a complex manifold $\mathcal{X}$, holomorphic maps $p\colon \mathcal{X} \to \mathcal{S} \times_{\mathfrak{U}} \mathfrak{V}$ and $\widetilde{\pi}\colon \mathcal{X} \to \mathfrak{V}$ such that the diagram

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{p} & \mathcal{S} \times_{\mathfrak{U}} \mathfrak{V} \\ {\scriptstyle\widetilde{\pi}}\downarrow & & \downarrow{\scriptstyle\mathrm{pr}_2} \\ \mathfrak{V} & \xrightarrow{\mathrm{id}} & \mathfrak{V} \end{array}$$

is commutative and such that $p\colon X_B := \widetilde{\pi}^{-1}(B) \to S_B$ is the minimal resolution for all $B \in \mathfrak{V}$. Hence $\{p^*\eta_B\}_{B\in\mathfrak{U}}$ is a nowhere vanishing relative holomorphic 2-form on $\mathcal{X}$. Since every fiber of $\widetilde{\pi}$ is smooth and since $h^*K(B) = \int_{X_B} p^*\eta_B \wedge p^*\bar{\eta}_B$ for all $B \in \mathfrak{V}$, h^*K is a continuous function on $\mathfrak{V}$. Since $p^*\eta_B \neq 0$ for all $B \in \mathfrak{V}$, h^*K is nowhere vanishing on $\mathfrak{B}$. This proves the assertion on $\mathfrak{U}$. Since $A \in \amalg_{i<j<k} M_{ijk}(3,6)$ is an arbitrary point, K is a nowhere vanishing continuous function on $M^*(3,6)$. $\square$

7.5 An intermediate modular variety

Let $z = (z_{\binom{ijk}{lmn}})_{\binom{ijk}{lmn}\in\mathcal{P}}$ be the homogeneous coordinates of $\mathbf{P}^9$ and define

$$Z := \{z \in \mathbf{P}^9;\ \mathrm{Plk}_{ij}(z) = 0\ \text{ for all } i<j\},$$

where $\mathrm{Plk}_{ij}(z) := z_{\binom{ijk}{lmn}} - z_{\binom{ijl}{mnk}} + z_{\binom{ijm}{nkl}} - z_{\binom{ijn}{klm}}$ are the Plucker relations. Then $Z \subset \mathbf{P}^9$ is a linear subspace of dimension 4.

After Matsumoto, we define

$$\mathrm{Pr}\colon X(3,6) \ni [A] \to (\cdots : D\binom{ijk}{lmn}(A) : \cdots)_{\binom{ijk}{lmn}\in\mathcal{P}} \in \mathbf{P}^9$$

and

$$\varTheta\colon \mathbf{H}_2 \ni W \to (\cdots : \varTheta\binom{ijk}{lmn}^2(W) : \cdots)_{\binom{ijk}{lmn}\in\mathcal{P}} \in \mathbf{P}^9.$$

Recall that the period map $\phi\colon M^o(3,6) \to \mathcal{M}^o_{\mathbb{S}_6}(2)$ induces the isomorphism $\varphi\colon X^o(3,6) \to \mathcal{M}^o_{\mathbb{S}_6}(2)$ in (7.3). Let $\varGamma_M(1+\mathrm{i}) \subset \mathrm{Aut}(\mathbf{H}_2)$ be the subgroup corresponding to $O^+(\mathbb{T}_6)(2) \subset \mathrm{Aut}(\varOmega_{\mathbb{S}_6})$ via the isomorphism $\mu\colon \mathbf{H}_2 \cong \varOmega_{\mathbb{S}_6}$. Let $\overline{\mathbf{H}_2/\varGamma_M(1+\mathrm{i})}$ be the Baily-Borel-Satake compactification of $\mathbf{H}_2/\varGamma_M(1+\mathrm{i})$.

Theorem 7.12. (1) *The images of* Pr *and* $\varTheta$ *are contained in* Z*;*
(2) Pr *extends to a double covering* $\overline{\mathrm{Pr}}\colon \overline{X}(3,6) \to Z$*;*
(3) $\varTheta$ *induces a double covering* $\overline{\varTheta}\colon \overline{\mathbf{H}_2/\varGamma_M(1+\mathrm{i})} \to Z$*;*
(4) *The period map for the family* $\pi\colon (\mathcal{X},\iota) \to M^o(3,6)$ *induces an isomorphism* $\psi\colon \overline{X}(3,6) \to \overline{\mathbf{H}_2/\varGamma_M(1+\mathrm{i})}$ *such that the following diagram is commutative:*

$$\begin{array}{ccc} \overline{X}(3,6) & \xrightarrow{\ \psi\ } & \overline{\mathbf{H}_2/\varGamma_M(1+\mathrm{i})} \\ {\scriptstyle\overline{\mathrm{Pr}}}\downarrow & & \downarrow{\scriptstyle\overline{\varTheta}} \\ Z & \xrightarrow{\ \mathrm{id}\ } & Z \end{array} \tag{7.5}$$

Proof. See [30, Th. 4.4.1, Cor. 4.4.2] and [36, Chap. 7 Prop. 6.2, Chap. 9 Th. 12.7] $\square$

We regard the monomial $\prod_{\mathcal{P}} z_{\binom{ijk}{lmn}}$ as an element of $H^0(\mathbf{P}^9, \mathcal{O}_{\mathbf{P}^9}(10))$. Let $\|\cdot\|_{\mathcal{O}_{\mathbf{P}^9}(10)}$ be the standard Hermitian metric on $\mathcal{O}_{\mathbf{P}^9}(10)$ whose Chern form is proportional to the Fubini-Study form on $\mathbf{P}^9$. Then

$$\frac{\|\Delta_{(3,6)}(A)\|^2}{\overline{\mathrm{Pr}}^* \|\prod_{\mathcal{P}} z_{\binom{ijk}{lmn}}\|^2_{\mathcal{O}_{\mathbf{P}^9}(10)}(A)} = \frac{K(A)^{20}}{(\sum_{\mathcal{P}} |D\binom{ijk}{lmn}(A)|^2)^{10}}, \qquad A \in M^*(3,6).$$

Since $K(A)^{20}/(\sum_{\mathcal{P}} |D\binom{ijk}{lmn}(A)|^2)^{10}$ descends to a nowhere vanishing continuous function on $X^*(3,6)$ by Lemmas 7.8 and 7.11, there exists a continuous Hermitian metric $\|\cdot\|'$ on $\overline{\mathrm{Pr}}^* \mathcal{O}_{\mathbf{P}^9}(10)$ such that

$$\|\overline{\mathrm{Pr}}^* \prod_{\mathcal{P}} z_{\binom{ijk}{lmn}}\|' = \|\Delta_{(3,6)}\|.$$

Lemma 7.13. *Let $\gamma\colon \Delta \to X^*(3,6)$ be a holomorphic curve that intersects $\amalg_{i<j<k} X_{ijk}(3,6)$ transversally at $\gamma(0)$. Then as $t \to 0$,*

$$\log \|\Delta_{(3,6)}(\gamma(t))\|^2 = \log |t|^2 + O(1).$$

Proof. Let $\gamma(0) \in X_{ijk}(3,6)$. Let f be a local holomorphic function defining the divisor $X_{ijk}(3,6)$ near $\gamma(0)$. Since $\|\cdot\|'$ is a continuous metric on $\overline{\mathrm{Pr}}^* \mathcal{O}_{\mathbf{P}^9}(10)$ and since $\prod_{\mathcal{P}} z_{\binom{ijk}{lmn}} \in H^0(\overline{X}(3,6), \overline{\mathrm{Pr}}^* \mathcal{O}_{\mathbf{P}^9}(10))$, we get

$$\log \|\Delta_{(3,6)}(\gamma(t))\|^2 = \log(\|\overline{\mathrm{Pr}}^* \prod_{\mathcal{P}} z_{\binom{ijk}{lmn}}(\gamma(t))\|')^2 = (\mathrm{mult}_{t=0}\gamma^* f)\, \log |t|^2 + O(1).$$

Since γ intersects $X_{ijk}(3,6)$ transversally at $\gamma(0)$, we get $\mathrm{mult}_{t=0}\gamma^* f = 1$. □

Proof of Theorem 7.9

We keep the notation in (7.3) and (7.5).
(1) The identity $\Phi_{\mathbb{S}_6} = C_2 \Psi_{\mathbb{T}_6}(\cdot, F_6)$ follows from Theorem 6.1. We compare the weights and the zeros of $\Phi_{\mathbb{S}_6}$ and Δ_{MSY}. By Theorem 4.3 and Proposition 7.6, both of $\Phi_{\mathbb{S}_6}$ and Δ_{MSY} have the same weight 10 and the same zero divisor $\mathcal{D}_{\mathbb{S}_5}$. From the Köcher principle, the assertion follows.
(2) By Theorems 4.3 and 7.9 (1), it suffices to prove that

$$\|\Delta_{(3,6)}\|^2 = \mathrm{Const.}\, \varphi^* \|\Delta_{\mathrm{MSY}}\|^2. \tag{7.6}$$

Let $\Pi\colon \Omega_{\mathbb{S}_6} \to \Omega_{\mathbb{S}_6}/O^+(\mathbb{T}_6)(2)$ be the natural projection, and set

$$f := \Pi^*(\varphi^{-1})^* \log \|\Delta_{(3,6)}\|^2.$$

We compute the $(1,1)$-current $dd^c f$ on $\Omega_{\mathbb{S}_6}$. By Definition 7.7 (3) and the definition of the Bergman metric, we get on $M^o(3,6)$

$$q^* dd^c \log \|\Delta_{(3,6)}\|^2 = -20\, \phi^* \omega_{\mathbb{S}_6}.$$

Since $X^o(3,6)$ is regarded as a subvariety of $M^o(3,6)$ via the embedding (7.2), we get on $X^o(3,6)$

$$dd^c \log \|\Delta_{(3,6)}\|^2 = -20\, \varphi^* \omega_{\mathbb{S}_6} \tag{7.7}$$

because

$$\text{L.H.S.} = j^* q^* dd^c \log \|\Delta_{(3,6)}\|^2 = -20\, j^* \phi^* \omega_{\mathbb{S}_6} = -20\, j^* q^* \varphi^* \omega_{\mathbb{S}_6} = \text{R.H.S.}$$

Since $\Pi^{-1} \circ \varphi(X^o(3,6)) = \Omega^o_{\mathbb{S}_6}$, we deduce from (7.7) the following equation on $\Omega^o_{\mathbb{S}_6}$

$$dd^c f = -20\, \omega_{\mathbb{S}_6}. \tag{7.8}$$

By the commutativity of (7.5), we get the equation of sets on $\mathbf{H}_2/\Gamma_M(1+\mathrm{i})$:

$$(\psi^{-1})^* \mathrm{div} \Delta_{(3,6)} = \mathrm{div} \Delta_{\mathrm{MSY}}. \tag{7.9}$$

Let $\overline{\mu}\colon \mathbf{H}_2/\Gamma_M(1+\mathrm{i}) \to \mathcal{M}_{\mathbb{S}_6}(2) = \Omega^+_{\mathbb{S}_6}/O^+(\mathbb{T}_6)(2)$ be the isomorphism induced from the isomorphism $\mu\colon \mathbf{H}_2 \cong \Omega_{\mathbb{S}_6}$. Set $\overline{\mathcal{D}}_{\mathbb{S}_6} := \mathcal{D}_{\mathbb{S}_6}/O^+(\mathbb{T}_6)(2)$. Since $\varphi = \overline{\mu} \circ \psi$, we get by Proposition 7.6 and (7.9)

$$(\varphi^{-1})^* \mathrm{div} \Delta_{(3,6)} = (\overline{\mu}^{-1})^* (\psi^{-1})^* \mathrm{div} \Delta_{(3,6)} = (\overline{\mu}^{-1})^* \mathrm{div} \Delta_{\mathrm{MSY}} = \overline{\mathcal{D}}_{\mathbb{S}_6},$$

which yields the equation of sets

$$\Pi^* (\varphi^{-1})^* \mathrm{div} \Delta_{(3,6)} = \mathcal{D}_{\mathbb{S}_6}. \tag{7.10}$$

Set $\mathcal{D}^o_{\mathbb{S}_6} := \amalg_{i<j<k} \Pi^* (\varphi^{-1})^* (X_{ijk}(3,6))$. By Theorem 7.10 (3), $\mathcal{D}^o_{\mathbb{S}_6}$ is smooth and it is a dense Zariski open subset of $\mathcal{D}_{\mathbb{S}_6}$. Let $x \in \mathcal{D}^o_{\mathbb{S}_6}$ be an arbitrary point. Let $\gamma\colon \Delta \to X^*(3,6)$ be a holomorphic curve intersecting $\amalg_{i<j<k} X_{ijk}(3,6)$ transversally at $\gamma(0)$. By [39, (2.3)], there exists a holomorphic curve $c\colon \Delta \to \Omega_{\mathbb{S}_6}$ intersecting $\mathcal{D}^o_{\mathbb{S}_6}$ transversally at $c(0)$ such that

$$\Pi \circ c(t) = \varphi \circ \gamma(t^2), \qquad t \in \Delta.$$

By Lemma 7.13, we get

$$f(c(t)) = 2\, \log |t|^2 + O(1) \tag{7.11}$$

because

$$\log \|\Delta_{(3,6)}(\varphi^{-1} \circ \Pi \circ c(t))\|^2 = \log \|\Delta_{(3,6)}(\gamma(t^2))\|^2 = 2 \log |t|^2 + O(1).$$

Since $c(0)$ is an arbitrary point of $\mathcal{D}^o_{\mathbb{S}_6}$ and since $c(t)$ intersects $\mathcal{D}^o_{\mathbb{S}_6}$ transversally at $c(0)$, we deduce from (7.8), (7.11) the following equation of currents on $\Omega_{\mathbb{S}_6}$:

$$dd^c f = -20\,\omega_{\mathbb{S}_6} + 2\,\delta_{\mathcal{D}_{\mathbb{S}_6}}. \tag{7.12}$$

Since f is $O^+(\mathbb{T}_6)(2)$-invariant, it follows from Theorem 3.2 and (7.12) the existence of an automorphic form F for $O^+(\mathbb{T}_6)(2)$ of weight 20 with zero divisor $2\mathcal{D}_{\mathbb{S}_6}$ such that $f = \log \|F\|^2$. Comparing the weights and the zeros of F and $\varDelta_{\mathrm{MSY}}^2$, we get $F = \mathrm{Const.}\,\varDelta_{\mathrm{MSY}}^2$. This proves (7.6). □

Question 7.14. Recall that the constant C_3 was defined as the ratio of $\tau_{\mathbb{S}_6}$ and $\|\varDelta_{(3,6)}\|^{-1/4}$ in Theorem 7.9. Is it possible to compute $\log C_3$ in $\mathbf{R}/\mathbf{Q}\log 2$ by using the arithmetic Lefschetz-Riemann-Roch theorem [5], [23]? The corresponding question for the family of elliptic curves over the configuration space $\pi\colon \mathcal{E} \to M^o(2,4;\mathbf{C})$ was considered by Bost [11], who obtained Eq. (1.4) from the arithmetic Riemann-Roch theorem [6], [16].

Question 7.15. Let L be an even lattice of signature $(2, b^-)$. In [8, Th. 14.3], Borcherds constructed a correspondence from modular forms for $Mp_2(\mathbf{Z})$ of type ρ_L with weight $1 + m^+ - b^-/2$ to automorphic forms on $\varOmega_L$ for some cofinite subgroup of $O^+(L)$ of weight m^+. We call this correspondence the *Borcherds additive lifting*, while we call the correspondence in Theorem 5.2 the *multiplicative Borcherds product*. Is it true that the even Freitag theta functions $\{\varTheta\binom{ijk}{lmn}\}$ are the Borcherds additive lifting of some modular forms for $Mp_2(\mathbf{Z})$ of type $\rho_{\mathbb{T}_6}$? If it is the case, Theorem 7.9 (1) may be expressed as follows:

$$\prod_{\text{finite}} (\text{additive Borcherds lifting}) = (\text{multiplicative Borcherds product})^{\text{integer}}. \tag{7.13}$$

There are some examples of Eq. (7.13) given by Allcock-Freitag [1] and Kondo [26]; Allcock-Freitag gave an example where the multiplicative Borcherds product is the one given by Borcherds [10] characterizing the discriminant locus on the moduli space of cubic surfaces; Kondo gave an example where the multiplicative Borcherds product is the Borcherds $\varPhi$-function of dimension 10 characterizing the discriminant locus on the moduli space of Enriques surfaces. It may be worth asking the existence of additive Borcherds liftings such that Eq. (7.13) holds for the automorphic forms $\varPhi_{\mathbb{S}_k}$ in Theorem 6.1. Are there many examples of Eq. (7.13)?

Question 7.16. In [27], Krieg studied automorphic forms on the period domain $\varOmega_{\mathbb{S}_4}$. There exist analogues of the Freitag theta functions on the period domain $\varOmega_{\mathbb{S}_4}$. Is it true that the automorphic form $\varPhi_{\mathbb{S}_4}$ has an expression in terms of those theta functions similar to the Matsumoto-Sasaki-Yoshida form $\varDelta_{\mathrm{MSY}}$?

Question 7.17. In [24], the moduli space and the period map for 2-elementary $K3$ surfaces of type $\mathbb{S}_4$ were studied by Koike, Shiga, Takayama, and Tsutsui. They proved that a general 2-elementary $K3$ surface of type $\mathbb{S}_4$ is obtained as the minimal resolution of the following double covering of $\mathbf{P}^1 \times \mathbf{P}^1$:

$$S(x) := \{((s,t),w) \in \mathcal{O}_{\mathbf{P}^1}(4) \boxtimes \mathcal{O}_{\mathbf{P}^1}(4);\, w^2 = \prod_{k=1}^{4} (x_1^{(k)}\, st + x_2^{(k)}\, s + x_3^{(k)}\, t + x_4^{(k)})\},$$

where s, t denote the inhomogeneous coordinates of the first $\mathbf{P}^1$ and the second $\mathbf{P}^1$, respectively. Following Koike-Shiga-Takayama-Tsutsui, we set

$$x_k := \begin{pmatrix} x_1^{(k)} & x_2^{(k)} \\ x_3^{(k)} & x_4^{(k)} \end{pmatrix} \in M(2, \mathbf{C}), \qquad 1 \leq k \leq 4$$

and define for $x = (x_1, x_2, x_3, x_4) \in M(2, \mathbf{C})^4$

$$\eta_x := \prod_{k=1}^{4} \frac{ds \wedge dt}{(x_1^{(k)}\, st + x_2^{(k)}\, s + x_3^{(k)}\, t + x_4^{(k)})^{1/2}} \in H^0(S(x), K_{S(x)}) \setminus \{0\}.$$

Then the following function seems to be an analogue of $\Delta_{(3,6)}$ in the case of 2-elementary $K3$ surfaces of type $\mathbb{S}_4$:

$$\|\Delta_{\mathrm{KSTT}}(x)\|^2 := \left| \prod_{k=1}^{4} \det(x_k) \right|^2 \left(\frac{1}{(2\pi)^2} \int_{S(x)} \eta_x \wedge \bar{\eta}_x \right)^4 .$$

Let $\widetilde{S}(x)$ be the minimal resolution of $S(x)$. It may be worth asking if an analogue of Theorem 7.9 (2) holds in this case, i.e., the existences of a rational number ν and a non-zero real number C with

$$\tau_{\mathbb{S}_4}(\widetilde{S}(x)) = C\, \|\Delta_{\mathrm{KSTT}}(x)\|^{\nu}.$$

Question 7.18. Let $A \in M^o(3,6)$. Assume that there exists a smooth conic Q_A such that all the six lines $L_{1,A}, \ldots, L_{6,A}$ are tangent to Q_A. Then X_A is a *Kummer surface.* Let C_A be the double covering of Q_A with 6 branch points $L_{1,A} \cap Q_A, \cdots, L_{6,A} \cap Q_A$. Then C_A is a curve of genus 2 and X_A is the Kummer surface associated with the Jacobian variety of C_A, i.e., $X_A = \mathrm{Km}(\mathrm{Jac}(C_A))$. Let $\tau(C_A)$ be the analytic torsion of C_A with respect to the metric induced from the flat Kähler metric on $\mathrm{Jac}(C_A)$. By e.g. [38], $\tau(C_A)$ is expressed as the Petersson norm of the Igusa cusp form. Explain the coincidence of $\tau(C_A)$ and $\tau_{\mathbb{S}_6}(X_A, \iota_A)$.

8 Discriminant of quartic surfaces

8.1 Discriminant of quartic hypersurfaces of $\mathbf{P}^3$

Let $(Z_0 : Z_1 : Z_2 : Z_3)$ be the homogeneous coordinates of $\mathbf{P}^3$. Let $H = \mathcal{O}_{\mathbf{P}^3}(1)$ be the hyperplane bundle over $\mathbf{P}^3$. We identify $Z_0, \ldots, Z_3$ as a basis of $H^0(\mathbf{P}^3, H)$. For an index $I = (i_0, i_1, i_2, i_3)$, we set $|I| = i_0 + \cdots + i_3$ and define $Z^I := Z_0^{i_0} Z_1^{i_1} Z_2^{i_2} Z_3^{i_3}$. Then $\{Z^I\}_{|I|=4}$ is a basis of $H^0(\mathbf{P}^3, 4H)$. Let $\{\xi_I\}_{|I|=4}$

be the coordinates of $H^0(\mathbf{P}^3, 4H)$ with respect to the basis $\{Z^I\}_{|I|=4}$. Then $\{\xi_I\}_{|I|=4}$ is regarded as a basis of the dual vector space $H^0(\mathbf{P}^3, 4H)^\vee$.

Let $\Phi_{|4H|} \colon \mathbf{P}^3 \ni Z \hookrightarrow (Z^I) \in \mathbf{P}(H^0(\mathbf{P}^3, 4H))^\vee$ be the projective embedding associated with the very ample line bundle $4H$. Let $\Phi_{|4H|}(\mathbf{P}^3)^\vee \subset \mathbf{P}(H^0(\mathbf{P}^3, 4H))$ be the projective dual variety of $\Phi_{|4H|}(\mathbf{P}^3)$ (cf. [22]). Then $\Phi_{|4H|}(\mathbf{P}^3)^\vee$ is a hypersurface of $\mathbf{P}(H^0(\mathbf{P}^3, 4H))$. The discriminant of quartic hypersurfaces of $\mathbf{P}^3$ is the reduced homogeneous polynomial $\Delta_{(\mathbf{P}^3,4H)}(\xi) \in \mathbf{Z}[\xi]$ such that

$$\Phi_{|4H|}(\mathbf{P}^3)^\vee = \mathrm{div}\Delta_{(\mathbf{P}^3,4H)}(\xi). \tag{8.1}$$

The choice of $\Delta_{(\mathbf{P}^3,4H)}(\xi)$ is unique, up to a constant. We fix one polynomial $\Delta_{(\mathbf{P}^3,4H)}(\xi)$ satisfying (8.1).

We define

$$F(Z, \xi) := \sum_{|I|=4} \xi_I Z^I \in H^0(\mathbf{P}^3, 4H)^\vee \otimes H^0(\mathbf{P}^3, 4H).$$

Set

$$\mathbb{P} := \mathbf{P}(H^0(\mathbf{P}^3, 4H)),$$
$$X_\xi := \{[Z] \in \mathbf{P}^3;\, F(Z, \xi) = 0\}, \qquad \xi \in \mathbb{P}.$$

and

$$\mathcal{X} := \{([Z], \xi) \in \mathbf{P}^3 \times \mathbb{P};\, F(Z, \xi) = 0\}, \qquad \pi := \mathrm{pr}_2.$$

Then $\pi \colon \mathcal{X} \to \mathbb{P}$ is a universal family of quartic hypersurfaces of $\mathbf{P}^3$ with fiber $\pi^{-1}(\xi) = X_\xi$. Let $\mathfrak{D}$ be the discriminant locus of the family $\pi \colon \mathcal{X} \to \mathbb{P}$:

$$\mathfrak{D} := \{\xi \in \mathbb{P};\, \mathrm{Sing}\, X_\xi \neq \emptyset\},$$

which is an irreducible divisor of $\mathbb{P}$ such that

$$\mathfrak{D} = \mathrm{div}\Delta_{(\mathbf{P}^3,4H)}(\xi) = \Phi_{|4H|}(\mathbf{P}^3)^\vee.$$

By a formula of Katz [22, Cor. 5.6], we have

$$\deg \mathfrak{D} = \deg \Delta_{(\mathbf{P}^3,4H)}(\xi) = (-1)^3 \int_{\mathbf{P}^3} \frac{c(T\mathbf{P}^3)}{(1 + 4\, c_1(H))^2} = 108, \tag{8.2}$$

where $c(T\mathbf{P}^3) = (1 + c_1(H))^4$ denotes the total Chern class of $T\mathbf{P}^3$.

For $\xi \in \mathbb{P} \setminus \mathfrak{D}$, $(X_\xi, H|_{X_\xi})$ is a polarized $K3$ surface of degree 4, i.e., a $K3$ surface equipped with an ample line bundle of degree 4. For $\xi \in \mathbb{P} \setminus \mathfrak{D}$, set

$$\eta_\xi := \mathrm{Res}_{X_\xi} \left(\frac{\sum_{\sigma \in \mathfrak{S}_4} \mathrm{sgn}\, \sigma\, Z_{\sigma(1)}\, dZ_{\sigma(2)} \wedge dZ_{\sigma(3)} \wedge dZ_{\sigma(4)}}{F(Z, \xi)} \right).$$

Then η_ξ is a non-zero holomorphic 2-form on X_ξ.

Definition 8.1. *The norm of* $\varDelta_{(\mathbf{P}^3,4H)}(\xi)$ *is defined by*

$$\|\varDelta_{(\mathbf{P}^3,4H)}(\xi)\|^2 := \left(\frac{1}{(2\pi)^2}\int_{X_\xi} \eta_\xi \wedge \bar{\eta}_\xi\right)^{108} |\varDelta_{(\mathbf{P}^3,4H)}(\xi)|^2.$$

By (8.2), $\|\varDelta_{(\mathbf{P}^3,4H)}(\xi)\|$ is a C^∞ function on $\mathbb{P} \setminus \mathfrak{D}$. In this section, we prove that $\|\varDelta_{(\mathbf{P}^3,4H)}(\xi)\|$ is expressed as the norm of a Borcherds product on the period domain for polarized $K3$ surfaces of degree 4 (cf. Theorem 8.11).

8.2 The polarized period for quartic surfaces

Fix a primitive vector $\mathbf{h} \in \mathbb{L}_{K3}$ of norm 4. The choice of $\mathbf{h}$ is unique up to an automorphism of $\mathbb{L}_{K3}$. Set

$$\mathbb{T} := \mathbf{h}^\perp \cong \mathbb{U} \oplus \mathbb{U} \oplus \mathbb{E}_8 \oplus \mathbb{E}_8 \oplus \langle -4 \rangle.$$

A marking of (X_ξ, H) is an isometry $\alpha\colon H^2(X_\xi, \mathbf{Z}) \cong \mathbb{L}_{K3}$ with $\alpha(c_1(H)) = \mathbf{h}$. There exists a marking of (X_ξ, H). The triple (X_ξ, H, α) is called a marked polarized $K3$ surface of degree 4. The polarized period of (X_ξ, H, α) is the point of $\varOmega_{\mathbb{T}}$ defined by

$$\pi(X_\xi, H, \alpha) := [\alpha(\eta_\xi)].$$

We define

$$\mathcal{M}_4 := \varOmega_{\mathbb{T}}/O(\mathbb{T}).$$

The Griffiths period of (X_ξ, H) is then defined as the orbit

$$\varpi(X_\xi, H) := O(\mathbb{T}) \cdot [\alpha(\eta_\xi)] \in \mathcal{M}_4.$$

Let $\varpi^o\colon \mathbb{P} \setminus \mathfrak{D} \to \mathcal{M}_4$ be the period map for the universal family of quartic surfaces $\pi\colon (\mathcal{X}, (\mathrm{pr}_1)^*H)|_{\mathbb{P}\setminus\mathfrak{D}} \to \mathbb{P} \setminus \mathfrak{D}$

$$\varpi^o(\xi) := \varpi(X_\xi, H), \qquad \xi \in \mathbb{P} \setminus \mathfrak{D}.$$

As in Section 2, we define the discriminant locus of $\varOmega_{\mathbb{T}}$ by

$$\mathcal{D}_{\mathbb{T}} := \bigcup_{d \in \Delta_{\mathbb{T}}} \mathcal{H}_d$$

and set $\overline{\mathcal{D}}_{\mathbb{T}} := \mathcal{D}_{\mathbb{T}}/O(\mathbb{T}) \subset \mathcal{M}_4$. We regard $\mathcal{D}_{\mathbb{T}}$ as a reduced divisor of $\varOmega_{\mathbb{T}}$.

Lemma 8.2. *One has* $\varpi^o(\mathbb{P} \setminus \mathfrak{D}) \subset \mathcal{M}_4 \setminus \overline{\mathcal{D}}_{\mathbb{T}}$.

Proof. Let $\xi \in \mathbb{P} \setminus \mathfrak{D}$ and assume that $\varpi(\xi) \in \overline{\mathcal{D}}_{\mathbb{T}}$. There is a marking α of X_ξ and a root $\delta \in \Delta_{\mathbb{T}}$ such that $\pi(X_\xi, H, \alpha) \in \mathcal{H}_\delta$. By the Riemann-Roch theorem, there exists an effective divisor E of X_ξ with $\alpha(c_1([E])) = \pm\delta$. Since $\langle \mathbf{h}, \delta \rangle = 0$, we get $\deg H|_E = 0$, which contradicts the ampleness of H. Hence $\varpi(\xi) \in \mathcal{M}_4 \setminus \overline{\mathcal{D}}_{\mathbb{T}}$. $\square$

Define

$$\mathfrak{D}^o := \{\xi \in \mathbb{P};\ \mathrm{Sing}\, X_\xi \text{ consists of a unique ordinary double point}\}.$$

Let $\mathfrak{D}^o_{\mathrm{reg}} := \mathfrak{D}^o \setminus \mathrm{Sing}\, \mathfrak{D}^o$ be the regular part of $\mathfrak{D}^o$. Since $\mathfrak{D}^o$ is a dense Zariski open subset of $\mathfrak{D}$ by [22, Prop. 3.2], so is $\mathfrak{D}^o_{\mathrm{reg}}$. By the Borel-Kobayashi-Ochiai extension theorem, the period map ϖ^o extends to a holomorphic map from $(\mathbb{P} \setminus \mathfrak{D}) \cup \mathfrak{D}^o_{\mathrm{reg}}$ to $\overline{\mathcal{M}}_4$, the Bail-Borel-Satake compactification of $\mathcal{M}_4$. This extension of ϖ^o is denoted by ϖ.

Let us make a geometric construction of the Borel-Kobayashi-Ochiai extension ϖ. Let $\mathcal{Z} \subset \mathbf{P}^3 \times \varDelta$ be a smooth complex threefold such that $p := \mathrm{pr}_2 \colon \mathcal{Z} \to \varDelta$ is a proper, surjective holomorphic function without critical points on $\mathcal{Z} \setminus p^{-1}(0)$. Set $Z_t = p^{-1}(t)$ for $t \in \Delta$. Then $p \colon \mathcal{Z} \to \varDelta$ is called an *ordinary singular family* of quartic surfaces if p has a unique, non-degenerate critical point on Z_0 and if Z_t is a quartic surface for all $t \in \Delta$.

We define

$$\mathcal{H}^o_\delta := \mathcal{H}_\delta \setminus \bigcup_{d \in \Delta_{\mathbb{T}} \setminus \{\pm\delta\}} \mathcal{H}_d, \qquad \mathcal{D}^o_{\mathbb{T}} := \sum_{d \in \Delta_{\mathbb{T}}} \mathcal{H}^o_d$$

and set $\overline{\mathcal{D}}^o_{\mathbb{T}} := \mathcal{D}^o_{\mathbb{T}} / O(\mathbb{T})$.

Lemma 8.3. *Let $p \colon \mathcal{Z} \to \varDelta$ be an ordinary singular family of quartic surfaces. Let $\overline{c} : \varDelta^* \to \mathcal{M}_4 \setminus \overline{\mathcal{D}}_{\mathbb{T}}$ be the Griffiths period map for $p \colon \mathcal{Z} \to \Delta$. Let $\varPi : \varOmega_{\mathbb{T}} \to \mathcal{M}_4$ be the natural projection. Then there exist a holomorphic curve $c \colon \varDelta \to \varOmega_{\mathbb{T}}$ and a root $\delta \in \Delta_{\mathbb{T}}$ satisfying*
(1) *$\varPi \circ c(t) = \overline{c}(t^2)$ for all $t \in \varDelta$ and $c(0) \in \mathcal{H}^o_\delta$;*
(2) *c intersects $\mathcal{H}^o_\delta$ transversally at $c(0)$.*

Proof. **(1)** Let $\widetilde{\varDelta}$ be another disc and let $\mathcal{Z} \times_\Delta \widetilde{\varDelta}$ be the induced family over $\widetilde{\varDelta}$ by the map $\widetilde{\varDelta} \ni t \to t^2 \in \Delta$. By e.g. [25, Th. 4.28], there exists a simultaneous resolution $\pi : \widetilde{\mathcal{Z}} \to \mathcal{Z} \times_\Delta \widetilde{\varDelta}$, i.e., a resolution satisfying the commutative diagram

$$\begin{array}{ccc} \widetilde{\mathcal{Z}} & \xrightarrow{\ \pi\ } & \mathcal{Z} \times_\Delta \widetilde{\varDelta} \\ {\scriptstyle \widetilde{p}}\downarrow & & \downarrow{\scriptstyle \mathrm{pr}_2} \\ \widetilde{\varDelta} & \xrightarrow{\ \mathrm{id}\ } & \widetilde{\varDelta} \end{array}$$

such that $\pi|_{\widetilde{p}^{-1}(t)} \colon \widetilde{p}^{-1}(t) \to \mathrm{pr}_2^{-1}(t)$ is an isomorphism for $t \neq 0$ and is the minimal resolution for $t = 0$. In particular, $\widetilde{p}$ is a smooth morphism. Set $\widetilde{\pi} := \mathrm{pr}_1 \circ \pi \colon \widetilde{\mathcal{Z}} \to \mathcal{Z}$. For $t \in \widetilde{\varDelta}$, we set $\widetilde{Z}_t := \widetilde{p}^{-1}(t)$ and $\widetilde{\pi}_t := \widetilde{\pi}|_{\widetilde{Z}_t} : \widetilde{Z}_t \to Z_{t^2}$. Then $\widetilde{\pi}_t$ is an isomorphism for $t \in \widetilde{\varDelta}^*$ and is the minimal resolution for $t = 0$.

Since the family $\widetilde{p} : \widetilde{\mathcal{Z}} \to \widetilde{\varDelta}$ is differentiably trivial, it admits a marking α such that $(\widetilde{Z}_t, \widetilde{\pi}^* H, \alpha_t := \alpha|_{\widetilde{Z}_t})$ is a marked polarized $K3$ surface of degree 4 for $t \in \widetilde{\varDelta}^*$. Let $c \colon \widetilde{\varDelta} \to \varOmega_{\mathbb{T}}$ be the period map for the marked family $(\widetilde{p} : \widetilde{\mathcal{Z}} \to \widetilde{\varDelta}, \alpha)$. Since $(\widetilde{Z}_t, \widetilde{\pi}^* H) \cong (Z_{t^2}, H)$ for $t \neq 0$, we have $\varPi \circ c(t) = \overline{c}(t^2)$.

Let $E_0 \subset \widetilde{Z}_0$ be the exceptional curve of $\widetilde{p}_0$. Since Z_0 has a unique ordinary double point, the self-intersection number of E_0 is equal to -2. Set

$$\delta := \alpha_0(c_1([E_0])) \in \Delta_{\mathbb{L}_{K3}}.$$

Since E_0 is an algebraic cycle, we have $c(0) \in \mathcal{H}_\delta$. By the same argument as in [39, p.70 Claim 2], we get $c(0) \in \mathcal{H}_\delta^o$. This proves (1).
(2) Let $K_{\mathcal{Z}}$ be the canonical line bundle of $\mathcal{Z}$. Since $K_{\mathcal{Z}}$ is trivial by e.g. [39, Lemma 2.3], there exists a nowhere vanishing 3-form ξ on $\mathcal{Z}$. For $t \in \Delta$, set

$$\eta_t := \mathrm{Res}_{Z_t} \frac{\xi}{p(z) - t} \in H^0(Z_t, K_{Z_t}) \setminus \{0\}. \tag{8.3}$$

Then $\widetilde{\eta}_t := \eta_{t^2}$ is regarded as a holomorphic 2-form on $\widetilde{Z}_t$ for $t \neq 0$. There exists a system of coordinates (z_1, z_2, z_3) near the critical point of p with

$$p(z) = z_1^2 + z_2^2 + z_3^2. \tag{8.4}$$

In the local expression (8.4), the vanishing cycle $\alpha_t^{-1}(\delta) \in H^2(\widetilde{Z}_t, \mathbf{Z})$ is realized as the following embedded 2-sphere $E_t \subset \widetilde{Z}_t$ under the identification $\widetilde{Z}_t = Z_{t^2}$:
(8.5)

$$E_t := \left\{ (z_1, z_2, z_3) \in \mathbf{C}^3;\ \left(\frac{z_1}{t}\right)^2 + \left(\frac{z_2}{t}\right)^2 + \left(\frac{z_3}{t}\right)^2 = 1,\ \frac{z_1}{t}, \frac{z_2}{t}, \frac{z_3}{t} \in \mathbf{R} \right\}.$$

By (8.3), (8.4), (8.5), there exists a germ $\epsilon(t) \in \mathbf{C}\{t\}$ with

$$\langle \alpha_t(\widetilde{\eta}_t), \delta \rangle = \int_{E_t} \eta_{t^2} = t\,\epsilon(t), \qquad \epsilon(0) \neq 0.$$

Fix $l \in \mathbb{T}_{\mathbf{R}}$ with $\langle l, l \rangle \geq 0$. Since $\langle \cdot, \delta \rangle / \langle \cdot, l \rangle$ is an equation defining $\mathcal{H}_\delta^o$, $c(t)$ intersects $\mathcal{H}_\delta^0$ transversally at $c(0)$. This proves (2). □

Lemma 8.4. *The following hold:*
(1) $\varpi(\mathfrak{D}_{\mathrm{reg}}^o) \subset \overline{\mathcal{D}}_{\mathbb{T}}^o$;
(2) $\overline{\mathcal{D}}_{\mathbb{T}}^o \subset \mathcal{M}_4 \setminus \mathrm{Sing}\,\mathcal{M}_4$ *and* $\overline{\mathcal{D}}_{\mathbb{T}}^o \subset \overline{\mathcal{D}}_{\mathbb{T}} \setminus \mathrm{Sing}\,\overline{\mathcal{D}}_{\mathbb{T}}$;
(3) $\mathfrak{D}_{\mathrm{reg}}^o$ *is an irreducible divisor of* $(\mathbb{P} \setminus \mathfrak{D}) \cup \mathfrak{D}_{\mathrm{reg}}^o$.

Proof. **(1)** The result follows from Lemma 8.3 (1).
(2) One can prove the result by the same argument as in [39, Prop. 1.9].
(3) The result follows from the irreducibility of the divisor $\mathfrak{D}$ of $\mathbb{P}$. □

Let $L \subset \mathbb{P}$ be a line, i.e., a smooth rational curve of degree 1. Then L is *general* if the induced family $\pi|_L \colon \mathcal{X}|_L \to L$ is a Lefschetz pencil, i.e.,
(i) $\mathcal{X}|_L$ is a smooth threefold;
(ii) all the critical points of the projection $\pi|_L$ are non-degenerate;
(iii) any singular fiber of $\pi|_L$ has only one critical point of $\pi|_L$.

By [22, Cor. 3.2.1], the set of general lines of $\mathbb{P}$ is a dense Zariski open subset of the set of all lines of $\mathbb{P}$.

Lemma 8.5. *Let $L \subset \mathbb{P}$ be a general line. Let $\varpi|_L \colon L \to (\mathcal{M}_4 \setminus \overline{\mathcal{D}}_{\mathbb{T}}) \cup \overline{\mathcal{D}}^o_{\mathbb{T}}$ be the period map for $\pi|_L \colon \mathcal{X}|_L \to L$. Then $\varpi|_L$ intersects $\overline{\mathcal{D}}^o_{\mathbb{T}}$ transversally at $\varpi(L \cap \mathfrak{D})$.*

Proof. The result follows from Lemma 8.3 (2). □

Let $\omega_{\mathbb{T}}$ be the Kähler form of the Bergman metric on $\Omega_{\mathbb{T}}$:

$$\omega_{\mathbb{T}}([\eta]) = -dd^c \log \frac{\langle \eta, \bar{\eta} \rangle}{|\langle \eta, \mathbf{l} \rangle|^2}, \qquad [\eta] \in \Omega_{\mathbb{T}}, \tag{8.6}$$

where $\mathbf{l} \in \mathbb{T}_{\mathbf{R}}$ is a fixed vector with $\langle \mathbf{l}, \mathbf{l} \rangle \geq 0$. Since $\omega_{\mathbb{T}}$ is invariant under the action of $\mathrm{Aut}(\Omega_{\mathbb{T}})$, it descends to a Kähler form $\omega_{\mathcal{M}_4}$ on $\mathcal{M}_4$ in the sense of orbifolds. By (8.6) and the definition of the period map ϖ, we get the following equation of $(1,1)$-forms on $\mathbb{P} \setminus \mathfrak{D}$:

$$dd^c \log \|\Delta_{(\mathbf{P}^3, 4H)}(\xi)\|^2 = -108\, (\varpi^o)^* \omega_{\mathcal{M}_4}. \tag{8.7}$$

Lemma 8.6. *The semi-positive $(1,1)$-form $(\varpi^o)^* \omega_{\mathcal{M}_4}$ on $\mathbb{P} \setminus \mathfrak{D}$ has Poincaré growth along $\mathfrak{D}_{\mathrm{reg}}$. In particular, $(\varpi^o)^* \omega_{\mathcal{M}_4}$ extends trivially to a closed positive $(1,1)$-current on $\mathbb{P}$.*

Proof. By the same argument as in [39, Prop. 3.8 and Th. 3.9] using the Schwarz lemma for Bergman metrics on symmetric bounded domains, the semi-positive $(1,1)$-form $(\varpi^o)^* \omega_{\mathcal{M}_4}$ has Poincaré growth along $\mathfrak{D}_{\mathrm{reg}}$. It extends trivially to a closed positive $(1,1)$-current on $(\mathbb{P} \setminus \mathfrak{D}) \cup \mathfrak{D}_{\mathrm{reg}}$ by an extension theorem of Skoda. Since $\mathrm{Sing}\, \mathfrak{D}$ is a subvariety of $\mathbb{P}$ with codimension ≥ 2 and with $(\mathbb{P} \setminus \mathfrak{D}) \cup \mathfrak{D}_{\mathrm{reg}} = \mathbb{P} \setminus \mathrm{Sing}\, \mathfrak{D}$, the result follows from Siu's extension theorem [34, p.53 Th. 1]. □

The trivial extension of $(\varpi^o)^* \omega_{\mathcal{M}_4}$ from $\mathbb{P} \setminus \mathfrak{D}$ to $\mathbb{P}$ is denoted by $\varpi^* \omega_{\mathcal{M}_4}$.

Lemma 8.7. *The function $\log \|\Delta_{(\mathbf{P}^3, 4H)}(\xi)\|^2$ is locally integrable on $\mathbb{P}$ and satisfies the following equation of $(1,1)$-currents on $\mathbb{P}$:*

$$dd^c \log \|\Delta_{(\mathbf{P}^3, 4H)}(\xi)\|^2 = \delta_{\mathfrak{D}} - 108\, \varpi^* \omega_{\mathcal{M}_4}. \tag{8.8}$$

Proof. By (8.7) and Siu's extension theorem, it suffices to prove the assertion on $(\mathbb{P} \setminus \mathfrak{D}) \cup \mathfrak{D}^o_{\mathrm{reg}}$. By the same argument as in [39, Prop. 3.11], it suffices to prove the following: let $\gamma \colon \Delta \to \mathbb{P}$ be a holomorphic curve intersecting $\mathfrak{D}^o_{\mathrm{reg}}$ transversally at $\gamma(0)$. Then

$$\log \|\Delta_{(\mathbf{P}^3, 4H)}(\gamma(t))\|^2 = \log |t|^2 + O(1). \tag{8.9}$$

Since $X_{\gamma(0)}$ has only one ordinary double point as its singular set, the function $\log(\int_{X_{\gamma(t)}} \eta_{\gamma(t)} \wedge \bar{\eta}_{\gamma(t)})$ is bounded as $t \to 0$ by [37, Proof of Theorem 8.1]. By Definition 8.1, we get (8.9). □

8.3 A Borcherds product

Let $\mathbb{D}_7$ be the root lattice of type D_7, which is assumed to be *negative-definite*. Then $\mathbb{D}_7$ is a primitive sublattice of $\mathbb{E}_8$ with $\mathbb{D}_7^{\perp} = \langle -4 \rangle$. Hence $\mathbb{T}$ is regarded as the orthogonal complement of $\mathbb{D}_7$ in $\mathbb{U} \oplus \mathbb{U} \oplus \mathbb{E}_8 \oplus \mathbb{E}_8 \oplus \mathbb{E}_8$:

$$\mathbb{T} = \{(x, y, a, b, c) \in \mathbb{U} \oplus \mathbb{U} \oplus \mathbb{E}_8 \oplus \mathbb{E}_8 \oplus \mathbb{E}_8;\ \langle c, \mathbb{D}_7 \rangle = 0\}.$$

Since $\mathbb{U} \oplus \mathbb{U} \oplus \mathbb{E}_8 \oplus \mathbb{E}_8 \oplus \mathbb{E}_8$ is unimodular, we get

$$A_{\mathbb{T}} = A_{\mathbb{D}_7} = A_{\langle -4 \rangle} = \frac{1}{4}\mathbf{Z}/\mathbf{Z} = \left\{0, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}\right\}. \tag{8.10}$$

In what follows, $0, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}$ often denote the corresponding elements of the discriminant group $A_{\mathbb{T}} = A_{\mathbb{D}_7} = A_{\langle -4 \rangle}$.

Let $\mathbf{e}_0$, $\mathbf{e}_{1/4}$, $\mathbf{e}_{2/4}$, $\mathbf{e}_{3/4}$ be the standard basis of $\mathbf{C}[A_{\mathbb{D}_7}]$. Let $\Theta_{\mathbb{D}_7}(\tau)$ be the theta series of the lattice $\mathbb{D}_7$:

$$\Theta_{\mathbb{D}_7}(\tau) := \theta_{\mathbb{D}_7}(\tau)\, \mathbf{e}_0 + \theta_{\mathbb{D}_7+1/4}(\tau)\, \mathbf{e}_{1/4} + \theta_{\mathbb{D}_7+2/4}(\tau)\, \mathbf{e}_{2/4} + \theta_{\mathbb{D}_7+3/4}(\tau)\, \mathbf{e}_{3/4},$$

where

$$\theta_{\mathbb{D}_7+\delta/4}(\tau) := \sum_{l \in \mathbb{D}_7+\delta/4} q^{-\langle l, l \rangle}, \qquad q = e^{2\pi i \tau}.$$

Notice that $\mathbb{D}_7$ is negative-definite.

Lemma 8.8. *$\Theta_{\mathbb{D}_7}(\tau)/\Delta(\tau)$ is a modular form for $Mp_2(\mathbf{Z})$ of type $\rho_{\mathbb{T}}$ of weight $-17/2$.*

Proof. Since $\Delta(\tau)$ is a modular form for $SL_2(\mathbf{Z})$ of weight 12 and since $\rho_{\mathbb{T}} = \rho_{\mathbb{D}_7}$ by (8.10), it suffices to prove that $\Theta_{\mathbb{D}_7}(\tau)$ is a modular form for $Mp_2(\mathbf{Z})$ of weight 7/2 and of type $\rho_{\mathbb{D}_7}$. This follows from [8, Th. 4.1]. $\square$

Lemma 8.9. *The following identity holds:*

$$\Theta_{\mathbb{D}_7}(\tau)/\Delta(\tau) \equiv (q^{-1} + 108)\, \mathbf{e}_0 + 2^6\, q^{-1/8}\, \mathbf{e}_{1/4} + 14\, q^{-1/2}\, \mathbf{e}_{2/4} + 2^6\, \mathbf{e}_3 \mod q.$$

Proof. Recall that the Jacobi theta functions $\theta_2(\tau)$, $\theta_3(\tau)$, $\theta_4(\tau)$ were defined in Sect. 6. By [13, Chap. 4, p.118, Eqs. (8.7), (8.8), (8.9)], we get

$$\Theta_{\mathbb{D}_7}(\tau) = \frac{\theta_3(\tau)^7 + \theta_4(\tau)^7}{2}\, \mathbf{e}_0 + \frac{\theta_2(\tau)^7}{2}\, \mathbf{e}_{1/4} + \frac{\theta_3(\tau)^7 - \theta_4(\tau)^7}{2}\, \mathbf{e}_{2/4} + \frac{\theta_2(\tau)^7}{2}\, \mathbf{e}_{3/4}. \tag{8.11}$$

By the definitions of the Jacobi theta functions, we get

$$\theta_2(\tau)^7 = 2^7\, q^{7/8} + 7 \cdot 2^7\, q^{15/8} + O(q^2), \quad \theta_3(\tau)^7 = 1 + 14\, q^{1/2} + 84\, q + O(q^{3/2}),$$
$$\theta_4(\tau)^7 = 1 - 14\, q^{1/2} + 84\, q + O(q^{3/2}),$$

which, together with (8.11), yield that

$$\Theta_{\mathbb{D}_7}(\tau) \equiv (1 + 84\, q)\, \mathbf{e}_0 + 2^6 q^{7/8}\, \mathbf{e}_{1/4} + 14\, q^{1/2}\, \mathbf{e}_{2/4} + 2^6\, q^{7/8}\, \mathbf{e}_3 \mod q^{3/2}. \tag{8.12}$$

The result follows from (8.12) and the identity $1/\Delta(\tau) = q^{-1} + 24 + O(q)$. $\square$

By Lemma 8.8, we can apply Theorem 5.2 to the lattice $\mathbb{T}$ and the modular form $\Theta_{\mathbb{D}_7}(\tau)/\Delta(\tau)$.

Lemma 8.10. *Let $\Psi_{\mathbb{T}}(\cdot, \Theta_{\mathbb{D}_7}/\Delta)$ be the Borcherds product associated with $\mathbb{T}$ and $\Theta_{\mathbb{D}_7}(\tau)/\Delta(\tau)$. Then $\Psi_{\mathbb{T}}(\cdot, \Theta_{\mathbb{D}_7}/\Delta)$ has weight 54 and the zero divisor*

$$\operatorname{div} \Psi_{\mathbb{T}}(\cdot, \Theta_{\mathbb{D}_7}/\Delta) = \mathcal{D}_{\mathbb{T}} + 2^7 \sum_{d\in\mathbb{T}+1/4,\, d^2=-1/4} \mathcal{H}_d + 14 \sum_{d\in\mathbb{T}+2/4,\, d^2=-1} \mathcal{H}_d.$$

Proof. By Theorem 5.2 (1) and Lemma 8.9, the weight of $\Psi_{\mathbb{T}}(\cdot, \Theta_{\mathbb{D}_7}/\Delta)$ is given by $c_0(0)/2 = 108/2 = 54$. By Theorem 5.2 (2) and Lemma 8.9, we get

$$\begin{aligned}\operatorname{div} \Psi_{\mathbb{T}}(\cdot, \Theta_{\mathbb{D}_7}/\Delta) &= \sum_{d\in\Delta_{\mathbb{T}}} \mathcal{H}_d + 2^6 \sum_{d\in\mathbb{T}+1/4,\, d^2=-1/4} \mathcal{H}_d + 14 \sum_{d\in\mathbb{T}+2/4,\, d^2=-1} \mathcal{H}_d \\ &\quad +2^6 \sum_{d\in\mathbb{T}+3/4,\, d^2=-1/4} \mathcal{H}_d \\ &= \mathcal{D}_{\mathbb{T}} + 2^7 \sum_{d\in\mathbb{T}+1/4,\, d^2=-1/4} \mathcal{H}_d + 14 \sum_{d\in\mathbb{T}+2/4,\, d^2=-1} \mathcal{H}_d,\end{aligned}$$

where we used $\mathcal{H}_d = \mathcal{H}_{-d}$ to get the second equality. □

Define the effective divisor $\mathcal{D}'$ on $\Omega_{\mathbb{T}}$ by

$$\mathcal{D}' := 2^7 \sum_{d\in\mathbb{T}+1/4,\, d^2=-1/4} \mathcal{H}_d + 14 \sum_{d\in\mathbb{T}+2/4,\, d^2=-1} \mathcal{H}_d$$

and set $\overline{\mathcal{D}}' := \mathcal{D}'/O(\mathbb{T})$. Then $\overline{\mathcal{D}}'$ is an effective divisor of $\mathcal{M}_4$.

The discriminant $\Delta_{(\mathbf{P}^3,4H)}(\xi)$ is expressed as the Borcherds product:

Theorem 8.11. *There exists a non-zero constant C such that the following identity of C^∞ functions on $\mathbb{P} \setminus \mathfrak{D}$ holds:*

$$\|\Delta_{(\mathbf{P}^3,4H)}\|^2 = C\, \varpi^* \|\Psi_{\mathbb{T}}(\cdot, \Theta_{\mathbb{D}_7}/\Delta)\|^4.$$

Proof. By the Poincaré-Lelong formula and Lemma 8.10, we get the following equation of currents on $\Omega_{\mathbb{T}}$:

$$dd^c \log \|\Psi_{\mathbb{T}}(\cdot, \Theta_{\mathbb{D}_7}/\Delta)\|^2 = \delta_{\mathcal{D}_{\mathbb{T}}} + \delta_{\mathcal{D}'} - 54\, \omega_{\mathbb{T}},$$

which descends to the following equation of currents on $\mathcal{M}_4$:

$$dd^c \log \|\Psi_{\mathbb{T}}(\cdot, \Theta_{\mathbb{D}_7}/\Delta)\|^2 = \frac{1}{2}\delta_{\overline{\mathcal{D}}_{\mathbb{T}}} + \delta_{\overline{\mathcal{D}}'} - 54\, \omega_{\mathcal{M}_4}. \tag{8.13}$$

In (8.13), the coefficient 1/2 of $\delta_{\overline{\mathcal{D}}_{\mathbb{T}}}$ is necessary because the natural projection $\Omega_{\mathbb{T}} \to \mathcal{M}_4$ doubly ramifies along $\mathcal{D}_{\mathbb{T}}$ (cf. [39, Prop. 1.9 (4)]).

Since $\varpi^*\overline{\mathcal{D}}^o_{\mathbb{T}} \subset \mathfrak{D}^o_{\mathrm{reg}}$ by Lemma 8.4 (1) and since $\mathfrak{D}^o_{\mathrm{reg}}$ is an *irreducible* divisor of $(\mathbb{P} \setminus \mathfrak{D}) \cup \mathfrak{D}^o_{\mathrm{reg}}$ by Lemma 8.4 (3), there exists an integer $\nu \geq 1$ with

$$\varpi^*\overline{\mathcal{D}}^o_{\mathbb{T}} = \nu\,\mathfrak{D}^o_{\mathrm{reg}}. \tag{8.14}$$

Let $L \subset \mathbb{P}$ be a general line. We compute the intersection number of L and the divisor $\varpi^*\overline{\mathcal{D}}^o_{\mathbb{T}}$. Since the period map $\varpi|_L \colon L \to (\mathcal{M}_4 \setminus \overline{\mathcal{D}}_{\mathbb{T}}) \cup \overline{\mathcal{D}}^o_{\mathbb{T}}$ intersects $\overline{\mathcal{D}}^o_{\mathbb{T}}$ transversally at $\varpi(L \cap \mathfrak{D}^o_{\mathrm{reg}})$ by Lemma 8.5, we get by (8.14)

$$\nu\,\#(L \cap \mathfrak{D}^o_{\mathrm{reg}}) = \#(L \cap \varpi^*\overline{\mathcal{D}}^o_{\mathbb{T}}) = \#(\varpi(L) \cap \overline{\mathcal{D}}^o_{\mathbb{T}}) = \#(L \cap \mathfrak{D}^o_{\mathrm{reg}}),$$

which yields that $\nu = 1$.

Let x be an arbitrary point of $\mathcal{M}_4$. Let $f = 0$ be a local equation near x defining the divisor $\overline{\mathcal{D}}_{\mathbb{T}} + 2\,\overline{\mathcal{D}}'$. (When $x \notin \overline{\mathcal{D}}_{\mathbb{T}} + 2\,\overline{\mathcal{D}}'$, we can choose f to be a non-zero constant function.) By (8.13), $\log(\|\Psi_{\mathbb{T}}(\cdot, \Theta_{\mathbb{D}_7}/\Delta)\|^4/|f|^2)$ is a local potential function for $-108\,\omega_{\mathcal{M}_4}$:

$$dd^c \log(\|\Psi_{\mathbb{T}}(\cdot, \Theta_{\mathbb{D}_7}/\Delta)\|^4/|f|^2) = -108\,\omega_{\mathcal{M}_4} \tag{8.15}$$

as currents on an open subset of $\mathcal{M}_4$. Let $\xi \in \mathbb{P}$ be a point with $\varpi(\xi) = x$. Since $\log(\|\Psi_{\mathbb{T}}(\cdot, \Theta_{\mathbb{D}_7}/\Delta)\|^4/|f|^2)$ is locally bounded near x, we deduce from [39, Prop. 3.11] the following equation of currents near ξ:

$$dd^c \varpi^* \log(\|\Psi_{\mathbb{T}}(\cdot, \Theta_{\mathbb{D}_7}/\Delta)\|^4/|f|^2) = -108\,\varpi^*\omega_{\mathcal{M}_4}. \tag{8.16}$$

Since x is an arbitrary point of $\mathcal{M}_4$ and hence ξ is an arbitrary point of $(\mathbb{P} \setminus \mathfrak{D}) \cup \mathfrak{D}^o_{\mathrm{reg}}$, we deduce from (8.16) and $\nu = 1$ the following equation of currents on $(\mathbb{P} \setminus \mathfrak{D}) \cup \mathfrak{D}^o_{\mathrm{reg}}$:

$$dd^c \varpi^* \log \|\Psi_{\mathbb{T}}(\cdot, \Theta_{\mathbb{D}_7}/\Delta)\|^2 = \frac{1}{2}\,\delta_{\mathfrak{D}^o_{\mathrm{reg}}} + \delta_{\varpi^*\overline{\mathcal{D}}'} - 54\,\varpi^*\omega_{\mathcal{M}_4}. \tag{8.17}$$

Comparing (8.8) and (8.17), we get the equation of currents on $(\mathbb{P} \setminus \mathfrak{D}) \cup \mathfrak{D}^o_{\mathrm{reg}}$:

$$dd^c \log \frac{\|\Delta_{(\mathbf{P}^3, 4H)}\|^2}{\|\Psi_{\mathbb{T}}(\cdot, \Theta_{\mathbb{D}_7}/\Delta)\|^4} = -2\,\delta_{\varpi^*\overline{\mathcal{D}}'}. \tag{8.18}$$

Set $F := \log(\|\Delta_{(\mathbf{P}^3, 4H)}\|^2/\|\Psi_{\mathbb{T}}(\cdot, \Theta_{\mathbb{D}_7}/\Delta)\|^4)$. Since $\mathfrak{D} \setminus \mathfrak{D}^o_{\mathrm{reg}}$ is a subvariety of $\mathbb{P}$ whose codimension is strictly greater than 1, we deduce from Siu's extension theorem [34, p.53 Th. 1] that $F \in L^1(\mathbb{P})$ and that Eq. (8.18) holds on $\mathbb{P}$. Assume that $\varpi^*\overline{\mathcal{D}}' \neq \emptyset$. Let $L \subset \mathbb{P}$ be a general line. By (8.18), $\partial F|_L$ is a logarithmic 1-form on L with $\mathrm{div}(\partial F|_L) = (\varpi^*\overline{\mathcal{D}}') \cap L$. Since $\varpi^*\overline{\mathcal{D}}'$ is an effective divisor and hence so is $(\varpi^*\overline{\mathcal{D}}') \cap L$, the sum of the residues of $\partial F|_L$ does not vanish, which contradicts the residue theorem. Hence $\varpi^*\overline{\mathcal{D}}' = \emptyset$ and F is a constant function on $\mathbb{P}$. This proves the theorem. □

We do not know if $\|\Delta_{(\mathbf{P}^3, 4H)}\|$ admits an analytic expression using (equivariant) analytic torsion. After Beauville [3, Sect. 6], Voisin [35], Huybrechts [21, Example 2.7], it is possible to associate to X an irreducible compact holomorphic symplectic 4-fold with anti-symplectic involution as follows.

For a smooth quartic surface $X \subset \mathbf{P}^3$, let $\mathrm{Hilb}^{(2)}(X)$ denote the Hilbert scheme of zero-cycles of degree 2 of X, which is a symplectic resolution of the second symmetric product of X. Since X is a quartic surface, $\mathrm{Hilb}^{(2)}(X)$ has a natural involution defined as follows. Let $P_1 + P_2$, $P_1 \neq P_2$, be a point of $\Sigma^{(2)}X$, the second symmetric product of X. Let L be the line of $\mathbf{P}^3$ connecting P_1 and P_2. Then there exist $P_3, P_4 \in X$ such that $X \cap L = \{P_1, P_2, P_3, P_4\}$. Let Δ be the diagonal locus of $\Sigma^{(2)}X$. We define the involution $\theta\colon \Sigma^{(2)}X \setminus \Delta \to \Sigma^{(2)}X \setminus \Delta$ by $\theta(P_1 + P_2) := P_3 + P_4$. By [3, Sect. 6 Prop. 11], θ extends to an anti-symplectic holomorphic involution on $\mathrm{Hilb}^{(2)}(X)$. As an analogue of Theorem 4.3, it may be worth asking the following:

Question 8.12. Is it possible to express $\|\Delta_{(\mathbf{P}^3,4H)}\|^2$ as a combination of the equivariant analytic torsions of the bundles $\Omega^p_{\mathrm{Hilb}^{(2)}(X)}$, $p \geq 0$?

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