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Real $K3$ surfaces without real points, equivariant determinant of the Laplacian, and the Borcherds Φ-function

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REAL $K3$ SURFACES WITHOUT REAL POINTS, EQUIVARIANT DETERMINANT OF THE LAPLACIAN, AND THE BORCHERDS Φ-FUNCTION

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Abstract. We consider an equivariant analogue of a conjecture of Borcherds. Let (Y, σ) be a real $K3$ surface without real points. We shall prove that the equivariant determinant of the Laplacian of (Y, σ) with respect to a σ-invariant Ricci-flat Kähler metric is expressed as the norm of the Borcherds Φ-function at the "period point". Here the period of (Y, σ) is not the one in algebraic geometry.

1. Introduction

Let Y be an algebraic $K3$ surface defined over the real number field $\mathbb{R}$. Let $\sigma\colon Y \to Y$ be the anti-holomorphic involution on Y induced by the complex conjugation. Denote by $\mathbb{Z}_2 = \langle\sigma\rangle$ the group of order 2 of C^∞ diffeomorphisms of Y generated by σ. Recall that a point of Y is real if it is fixed by σ.

By [17], there exists a σ-invariant Ricci-flat Kähler metric g on Y with Kähler form ω_g. Since Y is defined over $\mathbb{R}$, there exists a nowhere vanishing holomorphic 2-form η_g on Y such that

$$\eta_g \wedge \overline{\eta}_g = 2\omega_g^2, \qquad \sigma^*\eta_g = \overline{\eta}_g.$$

Notice that the choice of η_g is unique up to a sign. We identify ω_g and η_g with their cohomology classes.

Let $\mathbb{L}_{K3}$ be the $K3$ lattice, which is an even unimodular lattice with signature $(3, 19)$. Then $H^2(Y, \mathbb{Z})$ equipped with the cup-product is isometric to $\mathbb{L}_{K3}$. By [13] or [6], there exists an isometry of lattices $\alpha\colon H^2(Y, \mathbb{Z}) \cong \mathbb{L}_{K3}$ such that the point $[\alpha(\omega_g + \sqrt{-1}\mathrm{Im}\,\eta_g)] \in \mathbb{P}(\mathbb{L}_{K3} \otimes \mathbb{C})$ lies in the period domain for Enriques surfaces.

Let $\Delta_{Y,g}$ be the Laplacian of (Y, g) acting on $C^\infty(Y)$. Following [2] and [11], one can define the equivariant determinant of the Laplacian $\Delta_{Y,g}$ with respect to the anti-holomorphic $\mathbb{Z}_2$-action on Y. Notice that σ acts on the vector space $C^\infty(Y)$ while it does not act on the vector space of C^∞ (p, q)-forms on Y unless $p = q$. Denote by $\det^*_{\mathbb{Z}_2}\Delta_{Y,g}(\sigma)$ the equivariant determinant of the Laplacian $\Delta_{Y,g}$ with respect to σ. (See Sect. 4.2.)

Recall that Borcherds [3] constructed a very interesting automorphic form on the period domain for Enriques surfaces, which is called the Borcherds Φ-function and is denoted by Φ. Let $\|\Phi\|$ denote the Petersson norm of Φ. Then $\|\Phi\|^2$ is a C^∞ function on the period domain for Enriques surfaces, which is invariant under the complex conjugation of the period domain. Our result is the following:

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Main Theorem 1.1. *There exists an absolute constant $C > 0$ such that for every real $K3$ surface without real points (Y, σ) and for every σ-invariant Ricci-flat Kähler metric g on Y with volume 1,*

$$\det{}^*_{\mathbb{Z}_2} \Delta_{Y,g}(\sigma) = C \, \|\Phi([\alpha(\omega_g + \sqrt{-1}\mathrm{Im}\,\eta_g)])\|^{\frac{1}{4}}.$$

Notice that the point $[\alpha(\omega_g + \sqrt{-1}\mathrm{Im}\,\eta_g)]$ is *not* the period of the marked $K3$ surface (Y, α), because $\omega_g + \sqrt{-1}\mathrm{Im}\,\eta_g$ is not a holomorphic 2-form on Y. Since ω_g is the Kähler form of (Y, g), the Main Theorem 1.1 may be regarded as a symplectic analogue of [18, Th. 8.3]. A typical example of a real $K3$ surface without real points is the quartic surface of $\mathbb{P}^3(\mathbb{C})$ defined by the equation $z_0^4 + z_1^4 + z_2^4 + z_3^4 = 0$.

To prove the Main Theorem 1.1, we consider an equivariant analogue of the conjecture of Borcherds: Let X be the differentiable manifold underlying a $K3$ surface. In [4, Example 15.1], Borcherds conjectured that the regularized determinant of the Laplacian, regarded as a function on the moduli space of Ricci-flat metrics on X with volume 1, coincides with the automorphic form on the Grassmann $G(\mathbb{L}_{K3})$ associated to the elliptic modular form $E_4(\tau)/\Delta(\tau)$; it is worth remarking that the regularized determinant of the Laplacian of a Ricci-flat $K3$ surface can be regarded as an analytic torsion of certain elliptic complex [12].

As an equivariant analogue of the Borcherds conjecture, we shall compare the following two functions on the space of σ-invariant Ricci-flat metrics on X; one is the equivariant determinant of the Laplacian, and the other is the pull-back of the norm of the Borcherds Φ-function via the "period map". (See Sect. 3.4 for the definition of the period map.) It is a trick of Donaldson [6], [8] that relates these two objects: Let (I, J, K) be a hyper-Kähler structure on (X, g) with $Y = (X, J)$. Then σ is holomorphic with respect to another complex structure I, while σ is anti-holomorphic with respect to the initial complex structure J. We shall show that the equivariant determinant of the Laplacain of (Y, σ) coincides with the equivariant analytic torsion of (X, I, σ). (See Sect. 3.3 and Sect. 4.) After this observation, the Main Theorem is a consequence of our result [18, Main Theorem and Th. 8.2].

This note is organized as follows. In Sect. 2, we recall the notion of hyper-Kähler structure on a $K3$ surface. In Sect. 3, we recall the trick of Donaldson. In Sect. 4, we study equivariant determinant of the Laplacian as a function on the space of σ-invariant Ricci-flat metrics on a $K3$ surface, and we prove the Main Theorem.

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2. $K3$ surfaces and hyper-Kähler structures

2.1. $K3$ surfaces

A compact, connected, smooth complex surface is a $K3$ *surface* if it is simply connected and has trivial canonical line bundle. Every $K3$ surface is diffeomorphic to a smooth quartic surface in $\mathbb{P}^3(\mathbb{C})$ (cf. [1, Chap. 8 Cor. 8.6]). Throughout this note, X denotes the C^∞ *differentiable manifold underlying a $K3$ surface*, and X is equipped with the orientation as a complex submanifold of $\mathbb{P}^3(\mathbb{C})$. For a complex structure I on X, X_I denotes the $K3$ surface (X, I).

Let $\mathbb{U}$ be the lattice of rank 2 associated with the symmetric matrix $\binom{0\,1}{1\,0}$, and let $\mathbb{E}_8$ be the root lattice of the simple Lie algebra of type E_8. We assume that $\mathbb{E}_8$ is *negative-definite*. The even unimodular lattice with signature $(3, 19)$

$$\mathbb{L}_{K3} := \mathbb{U} \oplus \mathbb{U} \oplus \mathbb{U} \oplus \mathbb{E}_8 \oplus \mathbb{E}_8$$

is called *the* $K3$ *lattice*. Then $H^2(X,\mathbb{Z})$ equipped with the cup-product $\langle\cdot,\cdot\rangle$, is isometric to $\mathbb{L}_{K3}$ (cf. [1, Chap. 8, Prop. 3.2]).

2.2. Hyper-Kähler structures on X

In this subsection, we recall Hitchin's result [9]. Let $\mathcal{E}$ be the set of all Ricci-flat metrics on X with volume 1. For every complex structure I on X, there exists a Kähler metric on X_I by [1, Chap. 8, Th. 14.5]. For every Kähler class κ on X_I, there exists by [17] a unique Ricci-flat Kähler form on X_I representing κ. Hence $\mathcal{E}\neq\emptyset$. For $g\in\mathcal{E}$, let dV_g denote the volume element of (X,g). Then $\int_X dV_g=1$ by our assumption.

Definition 2.1. A complex structure I on X is *compatible* with $g\in\mathcal{E}$ if g is a Kähler metric on X_I, i.e., I is parallel with respect to the Levi-Civita connection of (X,g). For $g\in\mathcal{E}$, let $\mathcal{C}_g$ denote the set of all complex structures on X compatible with g.

Let $g\in\mathcal{E}$. By Hitchin [9, Sect. 2, (i) $\Leftrightarrow$ (iii)], we get $\mathcal{C}_g\neq\emptyset$. For $I\in\mathcal{C}_g$, we define a real closed 2-form γ_I on X by

$$\gamma_I(u,v):=g(Iu,v),\qquad u,v\in TX. \tag{2.1}$$

Then γ_I is a Ricci-flat Kähler form on X_I such that

$$\gamma_I^2=2dV_g.$$

Definition 2.2. Let $I,J,K\in\mathcal{C}_g$. The ordered triplet (I,J,K) is called a *hyper-Kähler structure* on (X,g) if

$$IJ=-JI=K. \tag{2.2}$$

Let $*_g\colon \bigwedge^p T^*X\to\bigwedge^{4-p}T^*X$ be the Hodge star-operator on (X,g). Since $\dim_{\mathbb{R}}X=4$, we have $*_g^2=1$ on $\bigwedge^2 T^*X$. Recall that a 2-form f on X is *seld-dual* with respect to g if $*_g f=f$. Let $\mathcal{H}^2_+(g)$ be the real vector space of self-dual, real harmonic 2-forms on (X,g). Every vector of $\mathcal{H}^2_+(g)$ is parallel with respect to the Levi-Civita connection by [9].

Theorem 2.3. *Let $I\in\mathcal{C}_g$, and let η be a nowhere vanishing holomorphic 2-form on X_I such that $\eta\wedge\bar\eta=2\gamma_I^2$. Then there exist complex structures $J,K\in\mathcal{C}_g$ satisfying*
(1) (I,J,K) is a hyper-Kähler structure on (X,g) with $\eta=\gamma_J+\sqrt{-1}\gamma_K$;
(2) $\mathcal{H}^2_+(g)$ is a 3-dimensional real vector space spanned by $\{\gamma_I,\gamma_J,\gamma_K\}$;
(3) $\mathcal{C}_g=\{aI+bJ+cK;\ (a,b,c)\in\mathbb{R}^3,\ a^2+b^2+c^2=1\}$.

Proof. See [9, Sect. 2, (i) $\Leftrightarrow$ (iii)] for (1) and (2). Let $I'\in\mathcal{C}_g$. Since $\gamma_{I'}\in\mathcal{H}^2_+(g)$ by [9, Sect. 2, (i) $\Leftrightarrow$ (iii)], we can write $\gamma_{I'}=a\gamma_I+b\gamma_J+c\gamma_K$ for some $a,b,c\in\mathbb{R}$. We get $a^2+b^2+c^2=1$ by the relations $\gamma_{I'}^2=\gamma_I^2=2dV_g$, $\gamma\wedge\eta=0$, and $\eta\wedge\bar\eta=2\gamma_I^2$. □

Lemma 2.4. *Let (I,J,K) be a hyper-Kähler structure on (X,g). The map from $SO(3)$ to the set of all hyper-Kähler structures on (X,g) defined by*

$$A=(a_{ij})\mapsto(a_{11}I+a_{12}J+a_{13}K,\ a_{21}I+a_{22}J+a_{23}K,\ a_{31}I+a_{32}J+a_{33}K)$$

is a bijection.

Proof. It is obvious that the map defined as above is injective. Let (I',J',K') be an arbitrary hyper-Kähler structure on (X,g). By Theorem 2.3 (3), there is a real 3×3 matrix $B=(b_{ij})$ with

$$I'=b_{11}I+b_{12}J+b_{13}K,\quad J'=b_{21}I+b_{22}J+b_{23}K,\quad K'=b_{31}I+b_{32}J+b_{33}K.$$

We get $B \in SO(3)$ by the relations $(I')^2 = (J')^2 = (K')^2 = -1_{TX}$ and $I'J' = -J'I' = K'$. This proves the surjectivity. □

By Lemma 2.4, the element $\gamma_I \wedge \gamma_J \wedge \gamma_K \in \det \mathcal{H}^2_+(g)$ is independent of the choice of a hyper-Kähler structure (I, J, K) on (X, g), and it defines an orientation on $\mathcal{H}^2_+(g)$. In this note, $\mathcal{H}^2_+(g)$ is equipped with this orientation.

Let $A^p(X)$ denote the real vector space of real C^∞ p-forms on X. For a complex structure I on X, $A^{p,q}(X_I)$ denotes the complex vector space of C^∞ (p,q)-forms on X_I, and $\Omega^p_{X_I}$ denotes the sheaf of holomorphic p-forms on X_I.

Recall that the L^2-inner product on $A^p(X)$ with respect to g is defined by

$$(f, f')_{L^2} := \int_X f \wedge *_g f' = \int_X \langle f, f' \rangle_x \, dV_g(x), \qquad f, f' \in A^p(X).$$

Equipped with the restriction of $(\cdot,\cdot)_{L^2}$, $\mathcal{H}^2_+(g)$ is a metrized vector space. Then $\{\gamma_I/\sqrt{2}, \gamma_J/\sqrt{2}, \gamma_K/\sqrt{2}\}$ is an oriented orthonormal basis of $\mathcal{H}^2_+(g)$ for every hyper-Kähler structure (I, J, K) on (X, g), because $\gamma = \gamma_I \in A^{1,1}(X_I)$ and $\eta = \gamma_J + \sqrt{-1}\gamma_K \in H^0(X_I, \Omega^2_{X_I})$ satisfy the equations $\gamma \wedge \eta = \eta^2 = 0$.

Lemma 2.5. *The map from the set of hyper-Kähler structures on (X, g) to the set of oriented orthonormal basis of $\mathcal{H}^2_+(g)$ defined by $(I, J, K) \mapsto \{\gamma_I/\sqrt{2}, \gamma_J/\sqrt{2}, \gamma_K/\sqrt{2}\}$, is a bijection.*

Proof. The result is an immediate consequence of Lemma 2.4. □

3. Hyperbolic involutions on $K3$ surfaces and Ricci-flat metrics

In this section, we recall a trick of Donaldson that relates real $K3$ surfaces and $K3$ surfaces with anti-symplectic holomorphic involution. We follow [6, Chap. 6, Sect. 15] and [8, Sect. 2 pp.21-22].

3.1. Hyperbolic Involution

For a C^∞ involution ι on X, we set

$$H^2_\pm(X, \mathbb{Z}) := \{l \in H^2(X, \mathbb{Z});\, \iota^*(l) = \pm l\}, \qquad r(\iota) := \mathrm{rank}_\mathbb{Z}\, H^2_+(X, \mathbb{Z}).$$

By [13, Cor. 1.5.2], $H^2_+(X, \mathbb{Z}) \subset H^2(X, \mathbb{Z})$ is primitive and 2-elementary.

Definition 3.1. A C^∞ involution $\iota\colon X \to X$ is *hyperbolic* if the following two conditions are satisfied:
(1) $H^2_+(X, \mathbb{Z})$ has signature $(1, r(\iota) - 1)$;
(2) ι is holomorphic with respect to a complex structure on X.

Remark 3.2. The second condition of Definition 3.1 does not seem very natural. We do no know if it is deduced from the first condition. Are there any C^∞ involution on X which is never holomorphic with respect to any complex structure on X, such that the invariant lattice $H^2_+(X, \mathbb{Z})$ is hyperbolic?

Definition 3.3. For a hyperbolic involution $\iota\colon X \to X$, set

$$\mathcal{E}^\iota := \{g \in \mathcal{E};\, \iota^* g = g\}.$$

Proposition 3.4. *For every hyperbolic involution $\iota\colon X \to X$, one has $\mathcal{E}^\iota \neq \emptyset$.*

Proof. There exists a complex structure I on X such that ι is holomorphic with respect to I. Since X_I is Kähler, there exists an ι-invariant Kähler class κ on X_I. Let γ be the unique Ricci-flat Kähler form representing κ. Then $\iota^*\gamma = \gamma$ by the uniqueness of γ. Let g be the Kähler metric on X whose Kähler form is γ. Then g is Ricci-flat and ι-invariant. □

Let $\iota\colon X \to X$ be a hyperbolic involution, and let $g \in \mathcal{E}^{\iota}$. Then ι preserves $\mathcal{H}^2_+(g)$. By identifying a real harmonic 2-form on (X, g) with its cohomology class in $H^2(X, \mathbb{R})$, we regard $\mathcal{H}^2_+(g)$ as an oriented subspace of $H^2(X, \mathbb{R})$. Since $*_g = 1$ on $\mathcal{H}^2_+(g)$, the cup-product $\langle\cdot,\cdot\rangle$ is positive-definite on $\mathcal{H}^2_+(g) \subset H^2(X, \mathbb{R})$.

Proposition 3.5. *The orientation on $\mathcal{H}^2_+(g)$ is preserved by ι.*

Proof. Since ι is a diffeomorphism of X, the result follows from [7, Prop. 6.2]. □

Proposition 3.6. (1) *There exists a hyper-Kähler structure (I, J, K) on (X, g) with*

$$\iota_* I = I\iota_*, \qquad \iota_* J = -J\iota_*, \qquad \iota_* K = -K\iota_*. \tag{3.1}$$

(2) *If (I', J', K') is another hyper-Kähler structure satisfying* (3.1)*, then there exists $\psi \in \mathbb{R}$ satisfying one of the following two equations:*

$$(I', J', K') = \begin{cases} (I,\ \cos\psi J - \sin\psi K,\ \sin\psi J + \cos\psi K), \\ (-I,\ \cos\psi J + \sin\psi K,\ \sin\psi J - \cos\psi K). \end{cases} \tag{3.2}$$

Proof. Set $\Pi(g)_\pm := \{\gamma \in \mathcal{H}^2_+(g);\ \iota^*\gamma = \pm\gamma\}$. Since the cup-product is positive definite on $\mathcal{H}^2_+(g)$, the hyperbolicity of ι implies that $\dim \Pi(g)_+ \leq 1$. Since $\det \iota^*|_{\mathcal{H}^2_+(g)} = 1$ by Proposition 3.5, we get $\dim \Pi(g)_+ = 1$ and $\dim \Pi(g)_- = 2$. Since ι is an involution preserving the L^2-inner product $(\cdot,\cdot)_{L^2}$, ι^* is symmetric with respect to $(\cdot,\cdot)_{L^2}$. Hence there exists an oriented orthonormal basis $\{\gamma_1, \gamma_2, \gamma_3\} \subset \mathcal{H}^2_+(g)$ with

$$\iota^*\gamma_1 = \gamma_1, \qquad \iota^*\gamma_2 = -\gamma_2, \qquad \iota^*\gamma_3 = -\gamma_3. \tag{3.3}$$

By Lemma 2.5, there exists a hyper-Kähler structure (I, J, K) on (X, g) satisfying $\gamma_1 = \gamma_I/\sqrt{2}$, $\gamma_2 = \gamma_J/\sqrt{2}$, $\gamma_3 = \gamma_K/\sqrt{2}$. These equations, together with (2.1), (3.3) and $\iota^* g = g$, yields (3.1). This proves (1).

Since $\dim \Pi(g)_+ = 1$, there exists $l \in \mathbb{R}$ such that $\gamma_{I'} = l\gamma_I$. This, together with $\gamma_I^2 = \gamma_{I'}^2 = 2dV_g$, implies that $I' = \pm I$. Since $\{\omega_J/\sqrt{2}, \omega_K/\sqrt{2}\}$ and $\{\omega_{J'}/\sqrt{2}, \omega_{K'}/\sqrt{2}\}$ are orthonormal bases of $\Pi(g)_-$, there exists $\psi \in \mathbb{R}$ with

$$(J', K') = (\cos\psi J \mp \sin\psi K, \sin\psi J \pm \cos\psi K).$$

Since $J'K' = I$ when $I' = I$ and since $J'K' = -I$ when $I' = -I$, we get (3.2). □

Definition 3.7. A hyper-Kähler structure (I, J, K) on (X, g) is *compatible* with ι if Eq. (3.1) holds.

3.2. **2-elementary $K3$ surfaces.** Let Y be a $K3$ surface, and let $\theta\colon Y \to Y$ be a holomorphic involution. Then θ is *anti-symplectic* if

$$\theta^*\eta = -\eta, \qquad \forall\, \eta \in H^0(Y, \Omega^2_Y). \tag{3.4}$$

Definition 3.8. A $K3$ surface equipped with an anti-symplectic holomorphic involution is called a 2-*elementary $K3$ surface.*

Proposition 3.9. *Let (Y, θ) be a 2-elementary $K3$ surface equipped with a θ-invariant Ricci-flat Kähler metric g. Let I be the complex structure on X such that $Y = X_I$, let η be a holomorphic 2-form on Y such that $\eta \wedge \bar{\eta} = 2\gamma_I^2$, and let $J, K \in \mathcal{C}_g$ be the complex structures such that $\gamma_J = \mathrm{Re}(\eta)$ and $\gamma_K = \mathrm{Im}(\eta)$. Then*
(1) θ is a hyperbolic involution and $g \in \mathcal{E}^\theta$;
(2) the hyper-Kähler structure (I, J, K) on (X, g) is compatible with θ.

Proof. By (3.4) and the θ-invariance of γ_I, we get (3.1). The hyperbolicity of θ follows from e.g. [6], [13], [18, Lemma 1.3 (1)]. □

We refer to [6], [14], [18] for more details about 2-elementary $K3$ surfaces.

3.3. **Real $K3$ surfaces**

After [6], [10], [15, Sect. 2 and Sect. 3], we make the following:

Definition 3.10. A $K3$ surface equipped with an *anti-holomorphic* involution is called a *real $K3$ surface.* A point of a real $K3$ surface is *real* if it is fixed by the anti-holomorphic involution.

Example 3.11. Let Y be an algebraic $K3$ surface defined over $\mathbb{R}$. Then there exists a projective embedding $j: Y \hookrightarrow \mathbb{P}^N(\mathbb{C})$ defined over $\mathbb{R}$. The complex conjugation $\mathbb{P}^N(\mathbb{C}) \ni (z_1 : \cdots : z_N) \to (\bar{z}_1 : \cdots : \bar{z}_N) \in \mathbb{P}^N(\mathbb{C})$ acts on Y as an anti-holomorphic involution. Let $\sigma: Y \to Y$ be the involution induced by the complex conjugation on $\mathbb{P}^N(\mathbb{C})$. Then the pair (Y, σ) is a real $K3$ surface. We refer to [6], [10], [13], [15, Sect. 2] for more details about this example.

Let (Y, σ) be a real $K3$ surface. Let g be a Kähler metric on Y with Kähler form γ. Then $\sigma^* g$ is a Kähler metric with Kähler form $-\sigma^*\gamma$. Indeed, if $Y = X_J$, we get

(3.5)
$$(\sigma^* g)(J(u), v) = g(\sigma_* J(u), \sigma_*(v)) = -g(J\sigma_*(u), \sigma_*(v)) = -(\sigma^*\gamma)(u, v)$$

for all $u, v \in TX$. Hence Y admits a σ-invariant Kähler metric e.g. $g + \sigma^* g$. By (3.5), the Kähler form and the Kähler class of a σ-invariant Kähler metric are anti-invariant with respect to the σ-action. In particular, there exists a Kähler class κ on Y with $\sigma^*\kappa = -\kappa$.

Lemma 3.12. (1) *There exists $\eta \in H^0(Y, \Omega_Y^2) \setminus \{0\}$ with*

$$\sigma^*\eta = \bar{\eta}. \tag{3.6}$$

(2) *Let κ be a Kähler class on Y with $\sigma^*\kappa = -\kappa$, and let γ be the Ricci-flat Kähler form representing κ. Then*

$$\sigma^*\gamma = -\gamma. \tag{3.7}$$

(3) *There exists a σ-invariant Ricci-flat Kähler metric on Y.*

Proof. **(1)** Let ξ be a nowhere vanishing holomorphic 2-form on Y. Since σ is anti-holomorphic, $\sigma^*\bar{\xi}$ is a holomorphic 2-form on Y. Then either $\xi + \sigma^*\bar{\xi}$ or $(\xi - \sigma^*\bar{\xi})/\sqrt{-1}$ is a nowhere vanishing holomorphic 2-form on Y satisfying (3.6).
(2) Let g be the Riemannian metric on Y whose Kähler form is γ. By (3.5), $-\sigma^*\gamma$ is the Kähler form of $\sigma^* g$ representing κ. By the Ricci-flatness of γ, there exists a real non-zero constant C with $C\,\gamma^2 = \eta \wedge \bar{\eta}$. This, together with (3.6), yields that

$$C\,(-\sigma^*\gamma)^2 = \sigma^*\eta \wedge \sigma^*\bar{\eta} = \bar{\eta} \wedge \eta = \eta \wedge \bar{\eta}.$$

This implies the Ricci-flatness of $-\sigma^*\gamma$. By the uniqueness of the Ricci-flat Kähler form in the Kähler class κ, we get (3.7).
(3) By (2), there exists a Ricci-flat Kähler metric g on Y whose Kähler form satisfies (3.7). Since σ is anti-holomorphic, we get $\sigma^* g = g$ by (3.7). □

Definition 3.13. A holomorphic 2-form η on a real $K3$ surface (Y, σ) is *defined over* $\mathbb{R}$ if Eq. (3.6) holds.

Proposition 3.14. *Let (Y, σ) be a real $K3$ surface equipped with a σ-invariant Ricci-flat Kähler metric g. Let J be the complex structure on X with $Y = X_J$, let η be a holomorphic 2-form on Y defined over $\mathbb{R}$ with $\eta \wedge \bar{\eta} = 2\gamma_J^2$, and let $I, K \in \mathcal{C}_g$ be the complex structures with $\gamma_I = -\mathrm{Re}\,\eta$ and $\gamma_K = \mathrm{Im}\,\eta$. Then*
(1) σ is a hyperbolic involution and $g \in \mathcal{E}^\sigma$;
(2) the hyper-Kähler structure (I, J, K) is compatible with (g, σ).

Proof. By (3.6) and (3.7), we get

$$\sigma^*\gamma_I = \gamma_I, \qquad \sigma^*\gamma_J = -\gamma_J, \qquad \sigma^*\gamma_K = -\gamma_K, \tag{3.8}$$

which, together with $\sigma^* g = g$, implies (3.1). Hence it suffices to verify the hyperbolicity of σ. Consider the $K3$ surface X_I. By (3.1) and (3.8), $\sigma\colon X_I \to X_I$ is an anti-symplectic holomorphic involution. Hence σ is hyperbolic. □

Proposition 3.15. *Let $\iota\colon X \to X$ be a hyperbolic involution, and let $g \in \mathcal{E}^\iota$. Let (I, J, K) be a hyper-Kähler structure on (X, g) compatible with ι. Then*
(1) (X_I, ι) is a 2-elementary $K3$ surface, and $\gamma_J + \sqrt{-1}\gamma_K$ is a holomorphic 2-form on X_I;
(2) (X_J, ι) is a real $K3$ surface, and $\gamma_I + \sqrt{-1}\gamma_K$ is a holomorphic 2-form on X_J defined over $\mathbb{R}$.

Proof. The result follows from (3.1) and Propositions 3.9 and 3.14. □

3.4. The period map for Ricci-flat metrics compatible with involution

Let $M \subset \mathbb{L}_{K3}$ be a sublattice.

Definition 3.16. A hyperbolic involution $\iota\colon X \to X$ is *of type M* if there exists an isometry of lattices $\alpha\colon H^2(X, \mathbb{Z}) \cong \mathbb{L}_{K3}$ such that $M = \alpha(H^2_+(X, \mathbb{Z}))$. An isometry $\alpha\colon H^2(X, \mathbb{Z}) \cong \mathbb{L}_{K3}$ with this property is called a *marking of type M*.

Let ι be a hyperbolic involution of type M, and let $\alpha\colon H^2(X, \mathbb{Z}) \cong \mathbb{L}_{K3}$ be a marking of type M. Then $M \subset \mathbb{L}_{K3}$ is a primitive, 2-elementary, hyperbolic sublattice by [13, Cor 1.5.2]. The orthogonal complement of M in $\mathbb{L}_{K3}$ is denoted by $M^\perp$. Then $M^\perp = \alpha(H^2_-(X, \mathbb{Z}))$. We set $r(M) := \mathrm{rank}_\mathbb{Z} M$ and

$$\Omega_M := \{[\eta] \in \mathbb{P}(M^\perp \otimes \mathbb{C});\ \langle \eta, \eta \rangle = 0,\ \langle \eta, \bar{\eta} \rangle > 0\}.$$

Since $M^\perp$ has signature $(2, 20 - r(M))$, Ω_M consists of two connected components, each of which is isomorphic to a symmetric bounded domains of type IV of dimension $20 - r(M)$ (cf. [1, p.282, Lemma 20.1]). Then Ω_M is the period domain for 2-elementary $K3$ surfaces of type M by [18, Sect. 1.4]. Notice that the two connected components of Ω_M is exchanged by the complex conjugation on $\mathbb{P}(M^\perp \otimes \mathbb{C})$.

Lemma 3.17. *Let $\iota\colon X \to X$ be a hyperbolic involution of type M, and let α be a marking of type M. Let $g \in \mathcal{E}^\iota$, and let (I, J, K) be a hyper-Kähler structure on (X, g) compatible with ι. Then the pair of conjugate points $[\alpha(\gamma_J \pm \sqrt{-1}\gamma_K)] \in \Omega_M$ is independent of the choice of (I, J, K) compatible with ι.*

Proof. By Proposition 3.15 (1), $[\alpha(\gamma_J + \sqrt{-1}\gamma_K)]$ is the period of a marked 2-elementary $K3$ surface of type M. Hence $[\alpha(\gamma_J + \sqrt{-1}\gamma_K)] \in \Omega_M$ by [18, Sect. 1.4]. Since the complex conjugation preserves Ω_M, we get $[\alpha(\gamma_J \pm \sqrt{-1}\gamma_K)] \in \Omega_M$.

Let (I', J', K') be an arbitrary hyper-Kähler structure on (X, g) compatible with ι. By Proposition 3.6 (2), there exists $\psi \in \mathbb{R}$ such that

$$\gamma_{J'} + \sqrt{-1}\gamma_{K'} = e^{\sqrt{-1}\psi}(\gamma_J \pm \sqrt{-1}\gamma_K).$$

Hence $[\alpha(\gamma_J \pm \sqrt{-1}\gamma_K)] = [\alpha(\gamma_{J'} \pm \sqrt{-1}\gamma_{K'})] \in \Omega_M$. □

Definition 3.18. With the same notation as in Lemma 3.17, the pair of conjugate points $[\alpha(\gamma_J \pm \sqrt{-1}\gamma_K)] \in \Omega_M$ is called the *period* of (g, α) and is denoted by

$$\varpi_M(g, \alpha) := [\alpha(\gamma_J \pm \sqrt{-1}\gamma_K)].$$

4. An invariant of Ricci-flat metric compatible with involution

Throughout this section, we fix the following notation. Let $\iota\colon X \to X$ be a hyperbolic involution of type M, and let $\alpha\colon H^2(X, \mathbb{Z}) \cong \mathbb{L}_{K3}$ be a marking of type M. Let $\mathbb{Z}_2 = \langle \iota \rangle$ be the group of diffeomorphisms of X generated by ι. Let $g \in \mathcal{E}^\iota$.

4.1. Equivariant determinant of the Laplacian

Let $d^*\colon A^1(X) \to C^\infty(X)$ be the formal adjoint of the exterior derivative $d\colon C^\infty(X) \to A^1(X)$ with respect to the L^2-inner product induced by g. The Laplacian of (X, g) is defined as $\Delta_g = \frac{1}{2}d^*d$. We define

$$C^\infty_\pm(X) := \{f \in C^\infty(X);\ \iota^* f = \pm f\}.$$

Since ι preserves g, Δ_g commutes with the ι-action on $C^\infty(X)$. Hence Δ_g preserves the subspaces $C^\infty_\pm(X)$. We set

$$\Delta_{g,\pm} := \Delta_g|_{C^\infty_\pm(X)}.$$

Define the spectral zeta function of $\Delta_{g,\pm}$ as

$$\zeta_{g,\pm}(s) := \mathrm{Tr}\left\{\Delta_{g,\pm}|_{(\ker \Delta_g)^\perp}\right\}^{-s} = \mathrm{Tr}\left[\frac{1 \pm \iota^*}{2} \circ \left(\Delta_g|_{(\ker \Delta_g)^\perp}\right)^{-s}\right], \qquad \mathrm{Re}\, s \gg 0.$$

Then $\zeta_{g,\pm}(s)$ converges absolutely for $\mathrm{Re}\, s \gg 0$, it extends meromorphically to the complex plane $\mathbb{C}$, and it is holomorphic at $s = 0$.

Definition 4.1. (1) The equivariant determinant of Δ_g with respect to $\mathbb{Z}_2 = \langle \iota \rangle$ is defined by

$$\det{}^*_{\mathbb{Z}_2}\Delta_g(\iota) := \exp[-\zeta'_{g,+}(0) + \zeta'_{g,-}(0)].$$

(2) For a real $K3$ surface (Y, σ) and a σ-invariant Ricci-flat Kähler metric g, set

$$\det{}^*_{\mathbb{Z}_2}\Delta_{Y,g}(\sigma) := \det{}^*_{\mathbb{Z}_2}\Delta_g(\sigma).$$

4.2. **Equivariant determinant of the Laplacian and equivariant analytic torsion.** Let (I, J, K) be a hyper-Kähler structure on (X, g) compatible with ι. By Proposition 3.15 (1), ι is an anti-symplectic holomorphic involution on X_I.

Let $\square_{g,I}^{0,q}$ be the $\bar{\partial}$-Laplacian acting on $(0, q)$-forms on the Kähler manifold (X_I, γ_I). By the definition of Δ_g and the Kähler identities, one has $\Delta_g = \square_{g,I}^{0,0}$. We set

$$\zeta^{0,q}(g, I, \iota)(s) := \mathrm{Tr}\left[\iota^*(\square_{g,I}^{0,q}|_{(\ker \square_{g,I}^{0,q})^\perp})^{-s}\right], \qquad \mathrm{Re}\, s \gg 0.$$

Then

$$\zeta^{0,1}(g, I, \iota)(s) = \zeta^{0,0}(g, I, \iota)(s) + \zeta^{0,2}(g, I, \iota)(s), \tag{4.1}$$

$$\zeta^{0,0}(g, I, \iota)(s) = \zeta_g^+(s) - \zeta_g^-(s). \tag{4.2}$$

After [2] and [11], we make the following:

Definition 4.2. The equivariant analytic torsion of (X_I, γ_I, ι) is defined by

$$\tau_{\mathbb{Z}_2}(g, I, \iota) := \exp\left[\zeta^{0,1}(g, I, \iota)'(0) - 2\zeta^{0,2}(g, I, \iota)'(0)\right].$$

Lemma 4.3. *The following identity holds*

$$\tau_{\mathbb{Z}_2}(g, I, \iota) = \left(\det\nolimits_{\mathbb{Z}_2}^* \Delta_g(\iota)\right)^{-2}.$$

Proof. Let K_{X_I} be the canonical line bundle of X_I, and set $\eta_I = \gamma_J + \sqrt{-1}\gamma_K \in H^0(X_I, K_{X_I})$. Since γ_J and γ_K are parallel with respect to the Levi-Civita connection of (X, g), so is η_I. The isomorphism of complex line bundles $\otimes\bar{\eta}\colon \mathcal{O}_{X_I} \cong \overline{K}_{X_I}$ induces an isometry with respect to the L^2-inner products:

$$\otimes\bar{\eta}/\sqrt{2}\colon C^\infty(X) \ni f \to f \cdot \bar{\eta}/\sqrt{2} \in A^{0,2}(X_I).$$

Let $E_g(\lambda)$ (resp. $E_{g,I}^{0,2}(\lambda)$) be the eigenspace of Δ_g (resp. $\square_{g,I}^{0,2}$) with respect to the eigenvalue $\lambda \in \mathbb{R}$. Then ι preserves $E_g(\lambda)$ and $E_{g,I}^{0,2}(\lambda)$. Let $E_g(\lambda)_\pm$ and $E_{g,I}^{0,2}(\lambda)_\pm$ be the ± 1-eigenspaces of the ι-actions on $E_g(\lambda)$ and $E_{g,I}^{0,2}(\lambda)$, respectively. Since $\iota^*\bar{\eta} = -\bar{\eta}$ and

$$\square_{g,I}^{0,2}(f \cdot \bar{\eta}) = (\Delta_g f) \cdot \bar{\eta}, \qquad f \in C^\infty(X),$$

we get the isomorphism $\otimes\bar{\eta}/\sqrt{2}\colon E_g(\lambda)_\pm \cong E_{g,I}^{0,2}(\lambda)_\mp$ for all $\lambda \in \mathbb{R}$, which yields that

$$\zeta^{0,2}(g, I, \iota)(s) = -\zeta_g^+(s) + \zeta_g^-(s), \qquad s \in \mathbb{C}. \tag{4.3}$$

By (4.1), (4.2) and (4.3), we get

$$\begin{aligned} \log \tau_{\mathbb{Z}_2}(g, I, \iota) &= \zeta^{0,1}(g, I, \iota)'(0) - 2\zeta^{0,2}(g, I, \iota)'(0) \\ &= \zeta^{0,0}(g, I, \iota)'(0) - \zeta^{0,2}(g, I, \iota)'(0) \\ &= 2\left.\frac{d}{ds}\right|_{s=0} (\zeta_g^+(s) - \zeta_g^-(s)) = -2 \log \det\nolimits_{\mathbb{Z}_2}^* \Delta_g(\iota). \end{aligned} \tag{4.4}$$

This completes the proof of Lemma 4.3. ∎

4.3. **A function τ_ι on $\mathcal{E}^\iota$**

Let X^ι be the set of fixed points of ι:

$$X^\iota := \{x \in X;\, \iota(x) = x\}.$$

By [13, Th. 3.10.6] or [14, Th. 4.2.2], X^ι is either the empty set or the disjoint union of finitely many compact, connected, orientable two-dimensional manifolds. Moreover, $r(\iota) = 10$ when $X^\iota = \emptyset$.

When $X^\iota \neq \emptyset$, the Riemannian metric $g|_{X^\iota}$ induces a complex structure on X^ι such that $g|_{X^\iota}$ is Kähler. Equipped with this complex structure, X^ι is a complex submanifold of X_I, since ι is holomorphic with respect to I. Let

$$X^\iota = \amalg_i C_i$$

be the decomposition into the connected components. Let $\Delta_{(C_i, g|_{C_i})} := \frac{1}{2} d^* d$ be the Laplacain of the Riemannian manifold $(C_i, g|_{C_i})$, and let

$$\zeta_{(C_i, g|_{C_i})}(s) := \mathrm{Tr} \left[\Delta_{(C_i, g|_{C_i})}|_{(\ker \Delta_{(C_i, g|_{C_i})})^\perp} \right]^{-s}$$

be the spectral zeta function of $\Delta_{(C_i, g|_{C_i})}$. The regularized determinant of $\Delta_{(C_i, g|_{C_i})}$ is defined as

$$\det{}^* \Delta_{(C_i, g|_{C_i})} := \exp\left(-\zeta'_{(C_i, g|_{C_i})}(0) \right).$$

Similarly, let $\tau(C_{i,I}, \gamma_I|_{C_i})$ be the analytic torsion of the trivial Hermitian line bundle on the Kähler manifold $(C_i, I, \gamma_I|_{C_i})$ (cf. [16]). For all i, one has

$$\tau(C_{i,I}, \gamma_I|_{C_i}) = (\det{}^* \Delta_{(C_i, g|_{C_i})})^{-1}. \tag{4.5}$$

We define a function τ_ι on $\mathcal{E}^\iota$ and a function τ_M on the moduli space of 2-elementary $K3$ surfaces of type M (cf. [18, Def. 5.1]) as follows:

Definition 4.4. Let (I, J, K) be a hyper-Kähler structure on (X, g) compatible with ι. When $X^\iota \neq \emptyset$, set

$$\tau_\iota(g) := \left\{ \det{}^*_{\mathbb{Z}_2} \Delta_g(\iota) \right\}^{-2} \prod_i \mathrm{Vol}(C_i, g|_{C_i})\, (\det{}^* \Delta_{(C_i, g|_{C_i})})^{-1},$$

$$\tau_M(X_I, \iota) := \tau_{\mathbb{Z}_2}(X_I, \gamma_I)(\iota) \prod_i \mathrm{Vol}(C_i, \gamma_I|_{C_i})\, \tau(C_{i,I}, \gamma_I|_{C_i}).$$

When $X^\iota = \emptyset$, set

$$\tau_\iota(g) := \left\{ \det{}^*_{\mathbb{Z}_2} \Delta_g(\iota) \right\}^{-2}, \qquad \tau_M(X_I, \iota) := \tau_{\mathbb{Z}_2}(X_I, \gamma_I)(\iota).$$

Notice that (X, g) has volume 1 for $g \in \mathcal{E}^\iota$. By [18, Th. 5.7], $\tau_M(X_I, \iota)$ is independent of the choice of an ι-invariant Ricci-flat Kähler metric on X_I.

Lemma 4.5. *If the hyper-Kähler structure (I, J, K) on (X, g) is compatible with ι, then*

$$\tau_\iota(g) = \tau_M(X_I, \iota). \tag{4.6}$$

In particular, one has

$$\tau_M(X_I, \iota) = \tau_M(X_{-I}, \iota). \tag{4.7}$$

Proof. The first result follows from Lemma 4.3 and (4.5). If (I, J, K) is compatible with ι, so is $(-I, J, -K)$. Hence the second result follows from the first one. $\square$

In the next theorem, we shall use the notion of automorphic forms on Ω_M, for which we refer to [18, Sect. 3]. For an automorphic form Ψ on Ω_M, its norm $\|\Psi\|$ is a function on Ω_M defined in [18, Def. 3.16]. If $X^\iota = \emptyset$ or if every connected component of X^ι is diffeomorphic to a 2-sphere, then Ψ is an automorphic form in the classical sense and $\|\Psi\|$ coincides with the Petersson norm of Ψ.

Theorem 4.6. *There exist $\nu(M) \in \mathbb{N}$ and an automorphic form Φ_M on Ω_M of weight $((r(M)-6)\nu(M), 4\nu(M))$ for some cofinite subgroup of $O(M^\perp)$ satisfying*
(1) $\|\Phi_M([\eta])\| = \|\Phi_M([\overline{\eta}])\|$ *for all* $[\eta] \in \Omega_M$*;*
(2) *For all* $g \in \mathcal{E}^\iota$*,*

$$\tau_\iota(g) = \|\Phi_M(\varpi_M(g,\alpha))\|^{-\frac{1}{2\nu(M)}}. \tag{4.8}$$

Proof. Let Φ_M be the automorphic form as in [18, Th. 5.2]. Let (I,J,K) be a hyper-Kähler structure on (X,g) compatible with ι. Let (X_I,ι) be a 2-elementary $K3$ surface of type M. Then so is (X_{-I},ι). Since an anti-holomorphic 2-form on X_I is a holomorphic 2-form on X_{-I}, the Griffiths period of (X_{-I},ι) in the sense of [18, (1.11)] is the complex conjugate of the Griffiths period of (X_I,ι). This, together with [18, Th. 5.2] and (4.7), implies the first assertion. Since $\varpi_M(g,\alpha) = \alpha(\gamma_J \pm \sqrt{-1}\gamma_K)$ and since $\gamma_J + \sqrt{-1}\gamma_K \in H^0(X_I, \Omega^2_{X_I})$, the second assertion follows from [18, Th. 5.2] and (4.6). □

We assume that ι has no fixed points. By Proposition 3.15 (1), ι is a holomorphic involution on X_I without fixed points, so that the quotient X_I/ι is an Enriques surface by [1, Chap. 8, Lemma 15.1]. By [1, Chap. 8, Lemma 19.1], there exists an isometry $\alpha\colon H^2(X,\mathbb{Z}) \cong \mathbb{L}_{K3}$ such that

$$\alpha\,\iota^*\alpha^{-1}(a,b,c,x,y) = (b,a,-c,y,x), \qquad a,b,c \in \mathbb{U}, \quad x,y \in \mathbb{E}_8.$$

Set $\mathfrak{L} := \alpha(H^2_+(X,\mathbb{Z}))$. Then ι is of type $\mathfrak{L}$. We refer to [1, Chap. 8, Sects. 15-21] for more details about Enriques surfaces.

Let Φ be the *Borcherds Φ-function*, which is an automorphic form of weight 4 on the period domain for Enriques surfaces by [3]. By [18, Th. 8.2], there exists a constant $C_{\mathfrak{L}} \neq 0$ such that

$$\Phi_{\mathfrak{L}} = C_{\mathfrak{L}}\,\Phi. \tag{4.9}$$

Since ι has no fixed points, we may choose $\nu(\mathfrak{L}) = 1$ in Theorem 4.6 by the definition of $\nu(M)$ in [18, pp. 79].

Corollary 4.7. *Let (Y,σ) be a real $K3$ surface without real points. Let g be a σ-invariant Ricci-flat Kähler metric on Y with volume 1. Let ω_g be the Kähler form of g, and let η_g be a holomorphic 2-form on Y defined over $\mathbb{R}$ such that $\eta_g \wedge \overline{\eta}_g = 2\omega_g^2$. Let α be a marking of type $\mathfrak{L}$. Under the identifications of ω_g and η_g with their cohomology classes, the following identity holds:*

$$\det\nolimits^*_{\mathbb{Z}_2}\Delta_{Y,g}(\sigma) = C_{\mathfrak{L}}^{\frac{1}{4}}\,\|\Phi([\alpha(\gamma_g + \sqrt{-1}\mathrm{Im}\,\eta_g)])\|^{\frac{1}{4}}.$$

Proof. By Proposition 3.14 and Definition 3.18, we get $\varpi_{\mathfrak{L}}(g,\alpha) = [\alpha(\gamma_g + \sqrt{-1}\mathrm{Im}\,\eta_g)]$. Substituting this equality and (4.9) into (4.8), we get the result. □

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