UTMS 2005–26 July 15, 2005

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July 15, 2005

Abstract

We give some stability estimates for the inverse problem consisting in the determination of source term and coefficients which appear in an elliptic or parabolic equations and depend fully on all the components of variables, from boundary measurements.

Key words: Inverse source problem, stability estimate.

AMS subject classifications: 35R30.

1 Introduction

Let Ω be a bounded domain in $\mathbb{R}^n$. Let $T > 0$ be given and $Q = \Omega \times (0,T)$. We consider the following initial-boundary value problem.

$$\begin{cases} \partial_t u - \Delta u = f \text{ in } Q \\ u = 0 \text{ on } \Sigma_0 \cup \Sigma. \end{cases} \tag{1.1}$$

Here and henceforth, $\Sigma = \partial\Omega \times (0,T)$ and $\Sigma_0 = \Omega \times \{0\}$.

We deal with the inverse problem consisting in the determination of the source term f by measurements of the Neumann data

$$\partial_\nu u = g, \text{ on } \Sigma,$$

where ν is the unit outward normal vector to $\partial\Omega$.

In general there is no uniqueness for this inverse problem as long as unknown f depends both on x and t. Indeed any $f = (\partial_t - \Delta)u$, $u \in C_c^\infty(Q)$, is such that $\partial_\nu u = 0$. In other words, the kernel of the mapping $f \to \partial_\nu u_f$, where u_f is the solution of the initial-boundary value problem (1.1), is never reduced to $\{0\}$.

Inverse heat source problems in the case $f = \sigma(t)\varphi(x)$, with σ known, were already considered in [CY1], [CY2], [Y1] and [Y2]. In these papers stability estimates of logarithmic type were established. When $\Omega = \mathbb{R}^n$ and $f = \sigma(t)\varphi(x)$, where φ is known, S. Saitoh, V. K. Tuan and M. Yamamoto [STY] obtain also logarithmic type estimate by using some reverse convolution inequalities. An uniqueness result with singular heat source was proved by A. El Badia and T. Ha Duong in [EH2].

In the present work we prove some new stability estimates for inverse problems for parabolic and elliptic equations where unknown functions depend on all the components of the variables but are subject to some constraints.

Throughout this paper, we shall assume, even if it is not necessary, that Ω is of class C^2.

2 Parabolic equations

We start with the general case where the source term depends both on x and t.

In the sequel, we exclusively treat real-valued functions, and we will use some properties of the unbounded operator A given by

$$Au = \Delta u \text{ and } D(A) = \{u \in C_0(\overline{\Omega}) \cap H_0^1(\Omega);\ \Delta u \in C_0(\overline{\Omega})\},$$

where $C_0(\overline{\Omega}) = \{w \in C(\overline{\Omega});\ w = 0 \text{ on } \partial\Omega\}$.

We recall the following well known properties:

(i) A generates a C_0-semigroup e^{tA} in $C_0(\overline{\Omega}) \cap H_0^1(\Omega)$.

(ii) e^{tA} may be extended to a semigroup, still denoted by e^{tA}, on $L^p(\Omega)$, $1 \leq p \leq \infty$.

(iii) e^{tA} is a C_0-semigroup in $L^p(\Omega)$, if $1 \leq p < \infty$.

(iv) For each $1 \leq q \leq p \leq \infty$, $t > 0$ and $u \in L^p(\Omega)$,

$$\|e^{tA}u\|_{L^p(\Omega)} \leq (4\pi t)^{-\frac{n}{2}(\frac{1}{q}-\frac{1}{p})}\|u\|_{L^q(\Omega)}.$$

We refer, for instance, to [CH] and [D] for more details and the proof of these properties.

As usual, we rewrite the initial-boundary value problem (1.1) as an abstract differential equation in $L^p(\Omega)$, $1 \leq p < \infty$:

$$\begin{cases} \frac{d}{dt}u(\cdot,t) = Au(\cdot,t) + f(.,t) \text{ in } (0,T) \\ u(\cdot,0) = 0. \end{cases}$$

Consequently (1.1) has a unique solution $u_f \in C([0,T]; L^p(\Omega))$, for any $1 \leq p < \infty$, given by

$$u_f(\cdot,t) = \int_0^t e^{(t-s)A} f(\cdot,s)ds.$$

In addition, from a theorem in [LM] (see also [LSU]) $u_f \in H^{2,1}(Q)$, where

$$H^{2,1}(Q) = L^2(0,T;H^2(\Omega)) \cap H^1(0,T;L^2(\Omega)),$$

and $\partial_\nu u_f \in L^2(\Sigma)$ (in fact $\partial_\nu u_f$ is in a better space than $L^2(\Sigma)$, see [LM] for precise trace theorems for $H^{2,1}(Q)$).

We recall

$$f^+ = \max\{f, 0\} = \frac{1}{2}(|f| + f), \quad f^- = -\min\{f, 0\} = \frac{1}{2}(|f| - f).$$

Then $f = f^+ - f^-$.

If $\alpha > 0$ and $0 \le \theta < 1$, we set

$$\begin{aligned} \mathcal{A}_{\alpha,\theta} &= \{f \in L^\infty(Q);\ \|f\|_{L^\infty(Q)} \le \alpha \|f\|_{L^1(Q)}, \\ &\qquad \text{and} \int_Q f^- \le \theta \int_Q f^+ \text{ or } \int_Q f^+ \le \theta \int_Q f^-\}. \end{aligned}$$

Proposition 2.1 *We assume that $\alpha T < (\frac{1-\theta}{1+\theta})^2$. Then for all $f \in \mathcal{A}_{\alpha,\theta}$,*

$$\|f\|_{L^\infty(Q)} \le \left(\frac{\alpha}{\frac{1+\theta}{1-\theta} - \sqrt{\alpha T}}\right) \|\partial_\nu u_f\|_{L^1(\Sigma)}.$$

The admissible set $\mathcal{A}_{\alpha,\theta}$ of unknown source terms is not general but is a sufficiently large set. For example,

$$\begin{aligned} \mathcal{A}_{\alpha,\theta} &\supset \{\varphi_0 - \theta\varphi_1;\ \varphi_0, \varphi_1 \in C_c^\infty(Q),\ \operatorname{supp}\varphi_0 \cap \operatorname{supp}\varphi_1 = \emptyset, \\ &\qquad \int_Q \varphi_0 = \int_Q \varphi_1 = 1, \quad 0 \le \varphi_0, \varphi_1 \le 1\}, \end{aligned}$$

for any $\alpha \ge 1$. In fact, a straightforward computation shows that $f = \varphi_0 - \theta\varphi_1$ belongs to $\mathcal{A}_{\alpha,\theta}$. More precisely, we have $f^+ = \varphi_0$, $f^- = \theta\varphi_1$ (this implies $\int_Q f^- = \theta \int_Q f^+$), $\|f\|_{L^\infty(Q)} \le 1$ and $\|\varphi\|_{L^1(Q)} = (1+\theta)$.

We need the following lemma in the proof of Proposition 2.1.

Lemma 2.1 *Let $1 \le p < \infty$ and $f \in L^\infty(Q)$. Then*

$$\|u_f(\cdot,t)\|_{L^p(\Omega)} \le \sqrt{t} \|f\|^{\frac{1}{2}}_{L^\infty(\Omega\times(0,t))} \|f\|^{\frac{1}{2}}_{L^1((0,t);L^p(\Omega))},\ 0 < t \le T.$$

Proof. As we have seen before,

$$u_f(\cdot,t) = \int_0^t e^{(t-s)A} f(\cdot,s)\,ds.$$

Therefore, for each $r \ge 1$,

$$\begin{aligned} \|u_f(\cdot,t)\|_{L^p(\Omega)} &\le \int_0^t \|e^{(t-s)A} f(\cdot,s)\|_{L^p(\Omega)} ds \\ &\le |\Omega|^{\frac{1}{pr^*}} \int_0^t \|e^{(t-s)A} f(\cdot,s)\|_{L^{pr}(\Omega)} ds, \end{aligned}$$

where $|\Omega|$ is the Lebesgue measure of Ω and r^* is the conjugate exponent of r, $\frac{1}{r}+\frac{1}{r^*}=1$.

In view of property (iv) of e^{tA}, we deduce

$$\|u_f(\cdot,t)\|_{L^p(\Omega)} \le (|\Omega|(4\pi)^{-\frac{n}{2}})^{\frac{1}{pr^*}} \int_0^t (t-s)^{-\frac{n}{2pr^*}} \|f(\cdot,s)\|_{L^p(\Omega)} ds.$$

Hence, if r is such that $\frac{n}{pr^*}<1$ then

$$\begin{aligned}\|u_f(\cdot,t)\|_{L^p(\Omega)} &\le (|\Omega|(4\pi)^{-\frac{n}{2}})^{\frac{1}{pr^*}} [\int_0^t (t-s)^{-\frac{n}{pr^*}}]^{\frac{1}{2}} [\int_0^t \|f(\cdot,s)\|^2_{L^p(\Omega)} ds]^{\frac{1}{2}} \\ &\le \frac{(|\Omega|(4\pi)^{-\frac{n}{2}})^{\frac{1}{pr^*}}}{(1-\frac{n}{pr^*})^{\frac{1}{2}}} t^{\frac{1}{2}-\frac{n}{2pr^*}} \|f\|^{\frac{1}{2}}_{L^\infty(\Omega\times(0,t))} \|f\|^{\frac{1}{2}}_{L^1((0,t);L^p(\Omega))}.\end{aligned}$$

The desired inequality follows by letting r^* tend to $+\infty$. □

Proof of Proposition 2.1. By the Green theorem, we have

$$\begin{aligned}\int_Q f &= \int_Q (\partial_t u_f - \Delta u_f) \\ &= \int_\Omega u_f(\cdot,T) - \int_\Sigma \partial_\nu u_f \\ &\le \|u(\cdot,T)\|_{L^1(\Omega)} + \|\partial_\nu u_f\|_{L^1(\Sigma)}.\end{aligned}$$

Similarly we have

$$-\int_Q f \le \|u(\cdot,T)\|_{L^1(\Omega)} + \|\partial_\nu u_f\|_{L^1(\Sigma)}.$$

First let $\int_Q f^- \le \theta \int_Q f^+$. Then, by $|f| = f^+ + f^-$ and $f = f^+ - f^-$, one can easily show that

$$\int_Q |f| \le \frac{1+\theta}{1-\theta} \int_Q f.$$

Second, in the case where $\int_Q f^+ \le \theta \int_Q f^-$, we can similarly prove

$$\int_Q |f| \le \frac{1+\theta}{1-\theta} \int_Q -f.$$

Therefore

$$\frac{1-\theta}{1+\theta}\|f\|_{L^1(Q)} \le \|u(\cdot,T)\|_{L^1(\Omega)} + \|\partial_\nu u_f\|_{L^1(\Sigma)}.$$

A combination of this inequality and the previous lemma (with $p=1$) gives

$$\frac{1-\theta}{1+\theta}\|f\|_{L^1(Q)} \le \sqrt{T}\|f\|^{\frac{1}{2}}_{L^\infty(Q)}\|f\|^{\frac{1}{2}}_{L^1(Q)} + \|\partial_\nu u_f\|_{L^1(\Sigma)}.$$

By the assumption, we have $\|f\|_{L^\infty(Q)} \le \alpha\|f\|_{L^1(Q)}$. Hence

$$\frac{1-\theta}{1+\theta}\|f\|_{L^1(Q)} \le \sqrt{\alpha T}\|f\|_{L^1(Q)} + \|\partial_\nu u_f\|_{L^1(\Sigma)},$$

and then

$$\|f\|_{L^\infty(Q)} \le \alpha\|f\|_{L^1(Q)} \le \frac{\alpha}{\frac{1+\theta}{1-\theta} - \sqrt{\alpha T}} \|\partial_\nu u_f\|_{L^1(\Sigma)}.$$

$\square$

If we observe also $u(\cdot, T)$, then we can take a more general admissible set of f's. That is, the following estimate is seen directly from the proof of Proposition 2.1.

Proposition 2.2 *For* $0 \le \theta < 1$, *we set*

$$\widetilde{\mathcal{A}}_\theta = \{f \in L^1(Q);\ \int_Q f^- \le \theta \int_Q f^+ \ or\ \int_Q f^+ \le \theta \int_Q f^-\}.$$

Then for all $f \in \widetilde{\mathcal{A}}_\theta$, *we have*

$$\|f\|_{L^\infty(Q)} \le \frac{1+\theta}{1-\theta}(\|u(\cdot, T)\|_{L^1(\Omega)} + \|\partial_\nu u_f\|_{L^1(\Sigma)}).$$

We now apply the preceding result to the problem of determining the zeroth order coefficient in a parabolic equation.

We consider then the following initial-boundary value problem

$$\begin{cases} \partial_t u - \Delta u + cu = 0 \text{ in } Q \\ u = \psi \text{ on } \Sigma_0 \\ u = g \text{ on } \Sigma. \end{cases} \tag{2.1}$$

From here and until the end of Corollary 2.1, we assume that Ω is of class $C^{2,\gamma}$ for some $\gamma \in (0,1)$.

Let $\psi \in C^{2+\gamma}(\overline{\Omega})$, $g \in C^{2+\gamma, 1+\frac{\gamma}{2}}(\overline{\Sigma})$ be given such that

$$\partial_t g - \Delta\psi = 0 \text{ on } \partial\Omega \times \{0\}$$

and

$$\psi \ge \delta \text{ in } \overline{\Omega}, \quad g \ge \delta \text{ in } \overline{\Sigma}.$$

Let

$$\tilde{C}^{\gamma,\frac{\gamma}{2}}(\overline{Q}) = \{c \in C^{\gamma,\frac{\gamma}{2}}(\overline{Q});\ c = 0 \text{ on } \partial\Omega \times \{0\}\}.$$

It is well known (see for instance [LSU]) that, for each $c \in \tilde{C}^{\gamma,\frac{\gamma}{2}}(\overline{Q})$, the initial-boundary value problem (2.1) has a unique solution $u = u_c \in C^{2+\gamma,1+\frac{\gamma}{2}}(\overline{Q})$. Moreover, the weak maximum principle applied to $v = e^{-\lambda t}u$ with $\lambda \ge \|c\|_{L^\infty(Q)}$ and $-v$ gives

$$e^{-\lambda T}\delta \le u \le e^{\lambda T} M \text{ in } \overline{Q} \text{ for all } c \in \tilde{C}^{\gamma,\frac{\gamma}{2}}(\overline{Q}), \tag{2.2}$$

where $M = \max(\|\psi\|_{L^\infty(\Omega)}, \|g\|_{L^\infty(\Sigma)})$.

If $0 \le \theta < 1$ and $\alpha > 0$, we set

$$\mathcal{C}_{\alpha,\theta} = \{c \in \tilde{C}^{\gamma,\frac{\gamma}{2}}(\overline{Q});\ \|c\|_{L^\infty(\overline{Q})} \le \alpha\|c\|_{L^1(\overline{Q})} \text{ and } \int_Q c^- \le \theta \int_Q c^+\}$$

Since $u_c - u_0$ is the solution of the initial-boundary value problem

$$\begin{cases} \partial_t u - \Delta u = -cu_c \text{ in } Q \\ u = 0 \text{ on } \Sigma_0 \cup \Sigma, \end{cases}$$

and in view of (2.2), we obtain as a consequence of Proposition 2.1:

Corollary 2.1 *Let $T_0 > 0$ be given. We assume that $0 < T \le T_0$, $\theta < \frac{\delta}{M}$,*

$$0 < \lambda < \frac{1}{2T_0} \ln \frac{\delta}{M\theta} \quad \text{and} \quad \beta T \le (\frac{1+\mu}{1-\mu})^2,$$

where $\beta = \frac{e^{2\lambda T_0} M\alpha}{\delta}$ and $\mu = \frac{e^{2\lambda T_0} M\theta}{\delta}$. Then for each $c \in \mathcal{C}_{\alpha,\theta}$, with $\|c\|_{L^\infty(Q)} \le \lambda$,

$$\|c\|_{L^\infty(Q)} \le (\frac{\beta}{\frac{1+\mu}{1-\mu} - \sqrt{\beta T}})\|\partial_\nu u_c - \partial_\nu u_0\|_{L^1(\Sigma)}.$$

Similarly as for $\mathcal{A}_{\alpha,\theta}$, we can prove that $\mathcal{C}_{\alpha,\theta} \ne \{0\}$. Also, since $\mathbb{R}_+\mathcal{C}_{\alpha,\theta} \subset \mathcal{C}_{\alpha,\theta}$, $\mathcal{C}_{\alpha,\theta}$ contain non zero function with arbitrary small L^∞-norm.

Next, we show how Proposition 2.1 and its corollary can be extented to the case $\partial_t - \Delta + c(x,t)$ instead of $\partial_t - \Delta$. Let us then start with the following initial-boundary value problem.

$$\begin{cases} \partial_t u - \Delta u + c(x,t)u = f \text{ in } Q \\ u = 0 \text{ on } \Sigma_0 \cup \Sigma. \end{cases} \tag{2.3}$$

Lemma 2.2 *Assume that $c \in C([0,T]; L^\infty(\Omega))$ and $f \in L^\infty(Q)$.*

(i) (2.3) has a unique solution $u \in C([0,T]; L^p(\Omega))$ such that $\partial_t u$, $\partial_i u$, $\partial^2_{ij} u \in L^p(Q)$, For any p, $1 \le p < \infty$.

(ii) Let $1 \le p < \infty$ and $\mu = \|c\|_{C([0,T];L^p(\Omega))}$. Then

$$\|u(\cdot,t)\|_{L^p(\Omega)} \le \sqrt{t} e^{\mu t} \|f\|^{\frac{1}{2}}_{L^\infty(\Omega\times(0,t))} \|f\|^{\frac{1}{2}}_{L^1(0,t;L^p(\Omega))},\ 0 \le t \le T.$$

Proof. (i) First, as a consequence of a classical theorem by J. L. Lions (see for instance [LM]), we know that (2.3) has a unique solution $u \in L^2(0,T; H_0^1(\Omega)) \cap C([0,T]; L^2(\Omega))$. On the other hand, we see, by using Duhamel's principle, that u must be also a solution of the following integral equation

$$u(\cdot,t) = \int_0^t e^{(t-s)A} B(s) u(\cdot,s) ds + \int_0^t e^{(t-s)A} f(\cdot,s) ds,\ 0 \le t \le T. \tag{2.4}$$

Here A is the operator defined in the begining of this section, and $B(t) : L^p(\Omega) \to L^p(\Omega)$ is the multiplication operator by $c(\cdot,t)$.

By the Banach fixed point theorem we see that (2.4) has a unique solution in $C([0,T]; L^p(\Omega))$. Furthermore, as

$$\partial_t u - \Delta u = -c(x,t)u + f \in L^p(Q),$$

$\partial_t u$, $\partial_i u$, $\partial^2_{ij} u \in L^p(Q)$ from a classical L^p-regularity theorem (see for instance [LSU]).

(ii) Let $s \leq t$. Then (2.4) implies

$$\|u(\cdot,s)\|_{L^p(\Omega)} = \int_0^s \|B(\tau)\|_{B(L^p(\Omega))}\|u(\cdot,\tau)\|_{L^p(\Omega)}d\tau + \|\int_0^s e^{(s-\tau)A}f(\cdot,\tau)d\tau\|_{L^p(\Omega)}.$$

Here and henceforth $\|\cdot\|_{B(L^p(\Omega))}$ denotes the operator norm on $L^p(\Omega)$.

By Lemma 2.1,

$$\begin{aligned}\|\int_0^s e^{(s-\tau)A}f(\cdot,\tau)d\tau\|_{L^p(\Omega)} &\leq \sqrt{s}\|f\|^{\frac{1}{2}}_{L^\infty(\Omega\times(0,s))}\|f\|^{\frac{1}{2}}_{L^1(0,s;L^p(\Omega))}\\ &\leq \sqrt{t}\|f\|^{\frac{1}{2}}_{L^\infty(\Omega\times(0,t))}\|f\|^{\frac{1}{2}}_{L^1(0,t;L^p(\Omega))}.\end{aligned}$$

Hence

$$\|u(\cdot,s)\|_{L^p(\Omega)} = M_p\int_0^s \|u(\cdot,\tau)\|_{L^p(\Omega)}d\tau + \sqrt{t}\|f\|^{\frac{1}{2}}_{L^\infty(\Omega\times(0,t))}\|f\|^{\frac{1}{2}}_{L^1(0,t;L^p(\Omega))},$$

where we used $\|B(\tau)\|_{B(L^p(\Omega))} \leq \mu$. An application of Gronwall's lemma leads to

$$\|u(\cdot,s)\| \leq e^{\mu s}\sqrt{t}\|f\|^{\frac{1}{2}}_{L^\infty(\Omega\times(0,t))}\|f\|^{\frac{1}{2}}_{L^1(0,t;L^p(\Omega))}.$$

The desired estimate follows by taking $s = t$ in the last inequality. □

In the present case, a result similar to Proposition 2.1 is the following

Proposition 2.3 *Let $c \in C([0,T];L^\infty(\Omega))$, $\alpha > 0$, $0 \leq \theta < 1$, be given and let $M = \|c\|_{C([0,T];L^1(\Omega))}$. If $\alpha T e^{MT} < (\frac{1-\theta}{1+\theta})^2$, then for each $f \in \mathcal{A}_{\alpha,\theta}$,*

$$\|f\|_{L^\infty(Q)} \leq \frac{\alpha}{\frac{1-\theta}{1+\theta} - \sqrt{\alpha T e^{MT}}}\|\partial_\nu u_f\|_{L^1(\Sigma)},$$

where u_f is the solution of the initial boundary value problem (2.3).

We leave to the reader to write down the statement of the corresponding corollary when we take $\partial_t - \Delta + c(x,t)$ in place of $\partial_t - \Delta$.

Let us point out that M. V. Klibanov [Kl] proves uniqueness theorems for one broad class of coefficient inverse problems. His method is based on Carleman estimates.

We now give another result. As we seen before, the mapping $f \in L^2(Q) \to u_f \in H^{2,1}(Q)$ defines a bounded operator. This and a trace theorem in [LM] give

$$T : f \in L^2(Q) \to \partial_\nu u_f \in H^{\frac{1}{2},\frac{1}{4}}(\Sigma)$$

is bounded, where

$$H^{\frac{1}{2},\frac{1}{4}}(\Sigma) = L^2(0,T;H^{\frac{1}{2}}(\partial\Omega)) \cap H^{\frac{1}{4}}(0,T;L^2(\partial\Omega)).$$

We consider the following sets.

$$\begin{array}{rcl} H & = & \{w \in H^{2,1}(Q);\ (\partial_t + \Delta)w = 0 \text{ in } Q \text{ and } w = 0 \text{ on } \Sigma_T\} \\ K & = & \{u \in H^{2,1}(Q);\ u = 0 \text{ on } \Sigma_0 \cup \Sigma \text{ and } \partial_\nu u = 0 \text{ on } \Sigma\} \end{array}$$

and $L = (\partial_t - \Delta)K$, where $\Sigma_T = \Omega \times \{T\}$. If $N(T) = \mathrm{Ker}(T)$, then

Proposition 2.4 *$N(T) = H^\perp$ and $L^2(Q) = \overline{H} \oplus L$, where $\overline{H}$ is the closure of H in $L^2(Q)$, and $H^\perp$ denotes the orthogonal of H in $L^2(Q)$.*

Proof. Let $f \in N(T)$. Noting that $\partial_\nu u_f = 0$ on Σ, an integration by parts gives

$$\int_Q fv = \int_Q (\partial_t - \Delta)u_f v = 0,$$

for all $v \in H$. That is $f \in H^\perp$.

Conversely, if $f \in H^\perp$ then, using once again an integration by parts, we discover

$$0 = \int_Q fv = \int_Q (\partial_t - \Delta)u_f v = -\int_\Sigma \partial_\nu u_f v,$$

for each $v \in H$. Hence $\partial_\nu u_f = 0$ on Σ. In other words $f \in N(T)$.

To finish the proof, we observe that $f \in N(T)$ if and only if $u_f \in K$. That is, since $f = (\partial_t - \Delta)u_f$, $f \in N(T)$ if and only if $f \in L$. $\square$

On $X = \overline{H}$, the closure of H in $L^2(Q)$, we define the mapping $\mathcal{N}$ as follows:

$$\mathcal{N}(f) = \sup\{\int_Q f v_\varphi;\ \varphi \in \Phi\},$$

where

$$\Phi = \{\varphi \in C^\infty(\overline{\Sigma}); \varphi(\cdot, T) = \partial_t\varphi(\cdot, T) = 0 \text{ and } \|\varphi\|_{L^\infty(\Sigma)} = 1\},$$

and $v_\varphi \in H^{2,1}(Q)$ is the unique solution of the following initial-boundary value problem

$$\begin{cases} \partial_t v + \Delta v = 0 \text{ in } Q \\ u = 0 \text{ on } \Sigma_T \\ u = \varphi \text{ on } \Sigma. \end{cases}$$

We note that $\varphi(\cdot, T) = \partial_t\varphi(\cdot, T) = 0$ is the compatibility condition needed in the existence of a solution in $H^{2,1}(Q)$ of the above initial-boundary value problem (see for instance [LM] for more details).

Lemma 2.3 *$\mathcal{N}$ defines a norm on X.*

Proof. As the supremum of a set of real numbers that is symmetric about 0, $N(f)$ is non negative. On the other hand, we have trivially $\mathcal{N}(\lambda f) = |\lambda|\mathcal{N}(f)$, for all $f \in X$ and $\lambda \in \mathbb{R}$; $\mathcal{N}(f+g) \leq \mathcal{N}(f) + \mathcal{N}(g)$, for all f, $g \in X$. Also, if $\mathcal{N}(f) = 0$ then, using $\mathcal{N}(f) = \mathcal{N}(-f)$,

$$0 = \int_Q f v_\varphi \text{ for all } \varphi \in \Phi.$$

Therefore we obtain by using an integration by parts

$$0 = \int_Q f v_\varphi = \int_Q (\partial_t - \Delta) u_f v_\varphi = -\int_\Sigma \partial_\nu u_f \varphi, \text{ for all } \varphi \in \Phi.$$

Hence $\partial_\nu u_f = 0$ on Σ, and then $f \in L = X^\perp$. Consequently $f = 0$. □

As we have seen in the preceding proof, we have

$$\int_Q f v_\varphi = \int_Q (\partial_t - \Delta) u_f v_\varphi = -\int_\Sigma \partial_\nu u_f \varphi \leq \|\partial_\nu u_f\|_{L^1(\Sigma)}, \text{ for all } \varphi \in \Phi.$$

From this we derive the following stability in detrmining f with the norm $\mathcal{N}(f)$.

Proposition 2.5

$$\mathcal{N}(f) \leq \|\partial_\nu u_f\|_{L^1(\Sigma)}, \textit{ for all } f \in X.$$

We note that the norm $\mathcal{N}$ is weaker than the L^1-norm on X. This is a consequence of the fact that

$$0 \leq v_\varphi \leq \|\varphi\|_{L^\infty(\Sigma)} = 1,$$

for all $\varphi \in \Phi$, which is obtained from an application of the maximum principle for weak solutions (see for instance [Li]).

We now show an L^2-stability estimate in a smaller subspace than X. Let Y be the closure of H in $H^{\frac{1}{2}}(Q)$ and we recall that the trace operator $f \to f_{|\Sigma}$ is bounded from $H^{\frac{1}{2}}(Q)$ into $L^2(\Sigma)$. Fix $f \in Y$ and let (f_n) be a sequence in H converging to f in Y. As above, an integration by parts leads to

$$\int_Q f f_n = \int_Q (\partial_t - \Delta) u_f f_n = -\int_\Sigma \partial_\nu u_f f_n.$$

Passing to the limit, we find

$$\int_Q f^2 = -\int_\Sigma \partial_\nu u_f f,$$

and then

$$\|f\|^2_{L^2(Q)} \leq \|f_{|\Sigma}\|_{L^2(\Sigma)} \|\partial_\nu u_f\|_{L^2(\Sigma)}. \tag{2.5}$$

As a consequence of this estimate, we have

Proposition 2.6 *Let $M > 0$. Then there exists a positive constant C such that*

(i) If $f \in \{q \in Y;\ \|g_{|\Sigma}\|_{L^2(\Sigma)} \le M\|g\|_{L^2(Q)}\}$ then

$$\|f\|_{L^2(Q)} \le C\|\partial_\nu u_f\|_{L^2(\Sigma)}.$$

(ii) If $f \in \{g \in Y;\ \|f\|_{H^{\frac{1}{2}}(Q)} \le M\}$ then

$$\|f\|_{L^2(Q)} \le C\|\partial_\nu u_f\|^{\frac{1}{2}}_{L^2(\Sigma)}.$$

(iii) If $f \in Y \cap \{g \in H^1(Q);\ \|g\|_{H^1(Q)} \le M\|g\|_{L^2(Q)}\}$ then

$$\|f\|_{L^2(Q)} \le C\|\partial_\nu u_f\|^{\frac{2}{3}}_{L^2(\Sigma)}.$$

Proof. (i) and (ii) are immediate from (2.5). *(iii)* follows also from (2.5), the continuity of the trace operator $w \in H^{\frac{1}{2}}(Q) \to w_{|\Sigma} \in L^2(\Sigma)$ and the interpolation inequality

$$\|f\|_{H^{\frac{1}{2}}(Q)} \le K\|f\|^{\frac{1}{2}}_{H^1(Q)}\|f\|^{\frac{1}{2}}_{L^2(Q)},$$

where K is some positive constant. □

All details concerning interpolation inequalities between Sobolev spaces can be found for instance in [LM].

In the rest of this section we prove a logarithmic stability estimate for a wide class of time-independent source term.

Let $\lambda_1 < \lambda_2 \le \ldots \le \lambda_n$ be the sequence of eigenvalues of the operator $A = -\Delta$ with $D(A) = H_0^1(\Omega) \cap H^2(\Omega)$. Let (φ_n) be an orthonormal basis in $L^2(\Omega)$ consisting of eigenfunctions of the operator A.

For $f \in L^2(\Omega)$, let $u_f \in H^{2,1}(Q)$ denote the solution of the following initial-boundary value problem

$$\begin{cases} \partial_t u - \Delta u = f(x) \text{ in } Q \\ u = 0 \text{ on } \Sigma_0 \cup \Sigma. \end{cases}$$

We first note that $v = \partial_t u_f$ is the solution of the boundary value problem

$$\begin{cases} \partial_t v - \Delta v = 0 \text{ in } Q \\ v = f \text{ on } \Sigma_0 \\ v = 0 \text{ on } \Sigma. \end{cases}$$

It is well known that

$$v(\cdot, t) = \sum_{k \ge 1} e^{-\lambda_k t}(f, \varphi_k)_{L^2(\Omega)}\varphi_k,$$

where $(\cdot,\cdot)_{L^2(\Omega)}$ is the usual scalar product on $L^2(\Omega)$. ¿From this formula, we derive

$$(v(\cdot, t), \varphi_k)_{L^2(\Omega)} = e^{-\lambda_k t}(f, \varphi_k)_{L^2(\Omega)}, \text{ for all } t \ge 0,$$

and then

$$(f,\varphi_k)_{L^2(\Omega)}=(v(\cdot,0),\varphi_k)_{L^2(\Omega)}=e^{\lambda_k T}(v(\cdot,T),\varphi_k)_{L^2(\Omega)}.$$

This identity gives

$$|(f,\varphi_k)_{L^2(\Omega)}|\le e^{\lambda_k T}|(v(\cdot,T),\varphi_k)_{L^2(\Omega)}|. \tag{2.6}$$

We recall that

$$H_0^\beta(\Omega)=\{h\in H^\beta(\Omega),\ h=0 \text{ on } \partial\Omega\} \text{ if } \frac{1}{2}<\beta<\frac{3}{2}.$$

In the sequel $\frac{1}{4}<\alpha<\frac{3}{4}$ is fixed. Following Fujiwara [Fu], we have

$$H_0^{2\alpha}(\Omega)=D(A^\alpha)=\{h\in L^2(\Omega);\ \sum_{k\ge 1}\lambda_k^{2\alpha}(\varphi_k,h)^2_{L^2(\Omega)}<\infty\}.$$

and

$$|||h|||_{H_0^{2\alpha}(\Omega)}=(\sum_{k\ge 1}\lambda_k^{2\alpha}(\varphi_k,h)^2_{L^2(\Omega)})^{\frac{1}{2}}$$

is an equivalent norm to the original norm on $H_0^{2\alpha}(\Omega)$.

We assume

$$f\in Z(M):=\{h\in H_0^{2\alpha}(\Omega);\ |||h|||_{H_0^{2\alpha}(\Omega)}\le M\},$$

where M is a given positive constant.

Let $\lambda\ge\lambda_1$ and $N=N(\lambda)$ be the integer such that $\lambda_N\le\lambda<\lambda_{N+1}$. Then

$$\begin{aligned}\|f\|^2_{L^2(\Omega)} &= \sum_{k\ge 1}(f,\varphi_k)^2_{L^2(\Omega)}\\ &= \sum_{k\le N}(f,\varphi_k)^2_{L^2}+\sum_{k>N}(f,\varphi_k)^2_{L^2(\Omega)}\\ &\le \sum_{k\le N}(f,\varphi_k)^2_{L^2(\Omega)}+\frac{1}{\lambda^{2\alpha}}\sum_{k>N}\lambda_k^{2\alpha}(f,\varphi_k)^2_{L^2(\Omega)}\\ &\le \sum_{k\le N}(f,\varphi_k)^2_{L^2(\Omega)}+\frac{M^2}{\lambda^{2\alpha}}.\end{aligned}$$

A combination of this estimate and (2.6) gives

$$\begin{aligned}\|f\|^2_{L^2(\Omega)} &\le e^{2\lambda T}\sum_{k\le N}(v(\cdot,T),\varphi_k)^2_{L^2}+\frac{M^2}{\lambda^{2\alpha}}\\ &\le e^{2\lambda T}\|v(\cdot,T)\|^2_{L^2(\Omega)}+\frac{M^2}{\lambda^{2\alpha}}.\end{aligned} \tag{2.7}$$

The term $\|w(\cdot,T)\|_{L^2(\Omega)}$ can be estimated in terms of Neumann boundary data by using the following observability inequality :

Theorem 2.1 *Let S be a closed subset of $\partial\Omega$ with nonempty interior and $\Gamma = S \times (0,T)$. Then there exists positive constant C depending only on Ω, T and S such that : For all $h \in H_0^1(\Omega)$, the solution of the initial-boundary value problem*

$$\begin{cases} \partial_t w - \Delta w = 0 \text{ in } Q \\ w = h \text{ on } \Sigma_0 \\ w = 0 \text{ on } \Sigma. \end{cases}$$

satisfies

$$\|w(\cdot,T)\|_{H_0^1(\Omega)} \leq C\|\partial_\nu w\|_{L^2(\Gamma)}. \tag{2.8}$$

This theorem is a simple consequence of a global Carleman estimate. We give its proof in the appendix.

Now (2.7) and (2.8) imply

$$\begin{aligned} \|f\|^2_{L^2(\Omega)} &\leq e^{2\lambda T}C^2\|\partial_\nu v\|^2_{L^2(\Gamma)} + \frac{M^2}{\lambda^{2\alpha}} \\ &\leq e^{2\lambda T}C^2\|\partial_t\partial_\nu u_f\|^2_{L^2(\Gamma)} + \frac{M^2}{\lambda^{2\alpha}} \\ &\leq e^{2\lambda T}C^2\|\partial_\nu u_f\|^2_{H^1(0,T;L^2(S))} + \frac{M^2}{\lambda^{2\alpha}}. \end{aligned}$$

That is,

$$\|f\|^2_{L^2(\Omega)} \leq \min_{\lambda\geq\lambda_1}(C^2e^{2T\lambda}\gamma^2 + \frac{M^2}{\lambda^{2\alpha}}), \tag{2.9}$$

where we set $\gamma = \|\partial_\nu u_f\|_{H^1(0,T;S)}$. The function $\lambda \to C^2e^{2T\lambda}\gamma^2 + \frac{M^2}{\lambda^{2\alpha}}$ attains its minimum at λ_* such that

$$2TC^2e^{2T\lambda_*}\gamma^2 - 2\alpha\frac{M^2}{{\lambda_*}^{2\alpha+1}} = 0. \tag{2.10}$$

Hence

$$e^{(2\alpha+1+2T)\lambda_*} \geq {\lambda_*}^{2\alpha+1}e^{2T\lambda_*} = \frac{\alpha M^2}{TC^2\gamma^2},$$

and then

$$\lambda_* \geq \frac{1}{2\alpha+1+2T}\ln(\frac{\alpha M^2}{TC^2\gamma^2}). \tag{2.11}$$

If we assume that γ is small enough in such a way that $\lambda_* \geq \max(\lambda_1, 1)$, then (2.9) and (2.10) give

$$\|f\|^2_{L^2(\Omega)} \leq \frac{\alpha M^2}{T{\lambda_*}^{2\alpha+1}} + \frac{M^2}{{\lambda_*}^{2\alpha}} \leq (\frac{\alpha M^2}{T} + M^2)\frac{1}{{\lambda_*}^{2\alpha}}. \tag{2.12}$$

In view of inequalities (2.11) and (2.12), we can state the following theorem.

Theorem 2.2 *Let S be a closed subset of $\partial\Omega$ with nonempty interior. Then we find three positive constants ϵ, A and B, depending only on Ω, T, M and S, such that*

$$\|f\|_{L^2(\Omega)} \leq \frac{A}{(\ln(\frac{B}{\|\partial_\nu u_f\|_{H^1(0,T;L^2(S))}}))^\alpha},$$

for each $f \in Z(M)$, $\|f\|_{L^2(\Omega)} \leq \epsilon$.

Remark. The second author has established in [Y2] a similar stability estimate to the one in Theorem 2.2 when the source term is of the form $\sigma(t)f(x)$ and $S = \partial\Omega$. The key step in the proof of [Y2] is the construction of a biorthogonal system to $(\psi_k(t)\varphi_k(x))$, where $\psi_k(t) = \int_0^t e^{-\lambda_k(t-s)}\sigma(s)ds$.

Let us see that the approach developed in this section can be extended to the case studied in [Y2]. That is, when the source term is of the form $\sigma(t)f(x)$, where σ is a given smooth function. To this end we assume that $\sigma \in C^1[0,T]$ and $\sigma(0) \neq 0$. As before the solution of the following boundary value problem

$$\begin{cases} \partial_t u - \Delta u = \sigma(t)f(x) \text{ in } Q \\ u = 0 \text{ on } \Sigma_0 \cup \Sigma \end{cases}$$

will be denoted by u_f. Let $v(\cdot,t) = e^{tA}f$, where e^{tA} is the semigroup generated by $A = \Delta$ with $D(A) = H_0^1(\Omega) \cap H^2(\Omega)$. We find by applying Duhamel's formula

$$u(x,t) = \int_0^t \sigma(t-s)v(x,s)ds.$$

Let us define the operator $K : L^2(0,T) \to H^1(0,T)$ by

$$(Kp)(t) = \int_0^t \sigma(t-s)p(s)ds,\ 0 < t < T.$$

We set $Y_0 = \{q \in H^1(0,T),\ q(0) = 0\}$. We can prove (e.g. [Y1])that K defines an isomorphism from $L^2(0,T)$ onto Y_0. Moreover, since

$$\partial_t u(x,t) = \sigma(0)v(x,t) + \int_0^t \sigma'(t-s)v(x,s)ds,\ 0 < t < T, \text{ a.e. } x \in \Omega,$$

we have

$$v(x,t) = \int_0^t \tau(t,s)\partial_t u(x,s)ds,\ 0 < t < T, \text{ a.e. } x \in \Omega,$$

where τ is a continuous function on $[0,T]^2$. From the last identity we derive

$$\|\partial_\nu v\|_{L^2(\Gamma)} \leq M\|\partial_\nu u\|_{H^1(0,T;L^2(S))}, \tag{2.13}$$

for some positive constant M depending only on σ, where Γ and S are the same as in Theorem 2.2.

On the other hand, by the inequality after Theorem 2.1 we have

$$\|f\|^2_{L^2(\Omega)} \leq e^{2\lambda T}C^2\|\partial_\nu v\|^2_{L^2(\Gamma)} + \frac{M^2}{\lambda^{2\alpha}}. \tag{2.14}$$

A combination of (2.13) and (2.14) gives

$$\|f\|^2_{L^2(\Omega)} \leq e^{2\lambda T}C^2\|\partial_\nu u\|_{H^1(0,T;L^2(S))} + \frac{M^2}{\lambda^{2\alpha}}.$$

¿From this we obtain an estimate similar to (2.9) which leads to the estimate in Theorem 2.2 when $f(x)$ is replaced by $\sigma(t)f(x)$.

3 Elliptic equations

In this section we consider inverse problems of determining a source function f by a single measurement of Neumann data coming from the Dirichlet boundary value problem. The unknown function f has n independent variables in general, and our inverse problems are underdetermined. Hence for stability, as well as for the uniqueness, we discuss two cases of unknown f :

(i) harmonic f.

(ii) $f(x_1, \dots, x_n)$ is independent of x_n.

We first consider the case (i). Let the space $H_\Delta(\Omega)$ be given by

$$H_\Delta(\Omega) = \{u \in L^2(\Omega);\ \Delta u \in L^2(\Omega)\}.$$

The vector space $H_\Delta(\Omega)$ endowed with the norm

$$\|u\|_{H_\Delta(\Omega)} = (\|u\|^2_{L^2(\Omega)} + \|\Delta u\|^2_{L^2(\Omega)})^{\frac{1}{2}}$$

is a Hilbert space. The principle properties of this space are collected in the following lemma whose proof can be found in [LM] (see also [BU]).

Lemma 3.1 *(1) (trace theorem) The mapping* $u \in C^\infty(\overline{\Omega}) \to (u_{|\partial\Omega}, \partial_\nu u_{|\partial\Omega})$ *can be extended to a bounded operator from* $H_\Delta(\Omega)$ *into* $H^{-\frac{1}{2}}(\partial\Omega) \times H^{-\frac{3}{2}}(\partial\Omega)$.
(2) (Green's formula) for each $u \in H_\Delta(\Omega)$ *and* $v \in H^2(\Omega)$ *we have*

$$\int_\Omega [\Delta uv - u\Delta v] = \langle \partial_\nu u, v\rangle_{H^{-\frac{3}{2}}(\partial\Omega), H^{\frac{3}{2}}(\partial\Omega)} - \langle u, \partial_\nu v\rangle_{H^{-\frac{1}{2}}(\partial\Omega), H^{\frac{1}{2}}(\partial\Omega)},$$

where $\langle\cdot,\cdot\rangle_{X,X'}$ *is the duality mapping between the Banach space* X *and its dual space* X'.

In what follows $\mathcal{H}(\Omega)$ denotes the closed subspace of $H_\Delta(\Omega)$ consisting in harmonic functions in Ω, i.e.

$$\mathcal{H}(\Omega) = \{u \in H_\Delta(\Omega);\ \Delta u = 0 \text{ in } \Omega\},$$

and let λ_1 denote the first eigenvalue of the $-\Delta$ with Dirichlet boundary condition.

Let $q \in L^\infty(\Omega)$ be such that if 0 is not in the spectrum of the operator $-\Delta + q$ with Dirichlet boundary condition, It is shown in [LM] that for any $f \in L^2(\Omega)$ the boundary value problem

$$\begin{cases} (-\Delta + q)u = f \text{ in } \Omega \\ u = 0 \text{ on } \partial\Omega \end{cases}$$

has a unique solution $u_f \in H^2(\Omega)$. Moreover

$$f \in L^2(\Omega) \to u_f \in H_0^1(\Omega) \cap H^2(\Omega)$$

is an isomorphism. From this and the continuity of the trace operator $w \in H^2(\Omega) \to H^{\frac{1}{2}}(\partial\Omega)$ we deduce that the mapping

$$\Phi : f \in \mathcal{H}(\Omega) \to \partial_\nu u_f \in H^{\frac{1}{2}}(\partial\Omega)$$

is bounded when $\mathcal{H}(\Omega)$ is endowed with the L^2-norm.

Within $\mathcal{H}(\Omega)$, we have the following Lipschitz stability estimate.

Theorem 3.1 *Φ is an isomorphism. In particular, there exists $C > 0$ depending on q such that*

$$\|f\|_{L^2(\Omega)} \leq C\|\partial_\nu u_f\|_{H^{\frac{1}{2}}(\partial\Omega)}, \textit{ for all } f \in \mathcal{H}(\Omega).$$

Proof. We suppose that 0 is not in the spectrum of the operator $-\Delta^2 + \Delta(q\cdot)$ with Dirichlet boundary condition. Then it is known (see for instance [LM]) that for all $\varphi \in H^{\frac{1}{2}}(\partial\Omega)$ the boundary value problem

$$\begin{cases} -\Delta^2 v + \Delta(qv) = 0 \text{ in } \Omega \\ v = 0 \text{ on } \partial\Omega \\ \partial_\nu v = \varphi \text{ on } \partial\Omega \end{cases} \tag{3.1}$$

has a unique weak solution $v_\varphi \in H^2(\Omega)$.

Let $\varphi \in H^{\frac{1}{2}}(\partial\Omega)$ and $f = -\Delta v_\varphi + qv_\varphi \in L^2(\Omega)$. Then one can easily see that $f \in \mathcal{H}(\Omega)$ and $u_f = v_\varphi$. That is, $\Phi(f) = \varphi$. On the other hand $\Phi(g) = 0$ implies u_g is the solution of the boundary value problem (3.2) with $\varphi = 0$. Hence $u_g = 0$ and then $g = 0$. In the other words, we proved that Φ is one to one, and since Φ is bounded Φ^{-1} is also bounded according to Banach's theorem.

We complete the proof by showing that 0 is not in the spectrum of the operator $-\Delta^2 + \Delta(q\cdot)$ with Dirichlet boundary condition. We proceed by contradiction. So we assume that there exists $u \in H^2(\Omega)$, non identically equal to zero, satisfying (3.2) with $\varphi = 0$. From this it follows that $v = -\Delta u + qu$ is in $\mathcal{H}(\Omega)$ and $\Phi(v) = 0$. Or we know that $N(\Phi) = \mathcal{H}(\Omega)^\perp$, the orthogonal of $\mathcal{H}(\Omega)$ in $L^2(\Omega)$ (the proof of this fact is similar to that of Proposition 2.1, see also [EH1]). Hence $v = 0$ and then $-\Delta u + qu = 0$ in Ω. That is 0 is an eigenvalue of $-\Delta u + qu = 0$ with Dirichlet boundary condition. This leads to the desired contradiction. □

We now consider the case (ii) when f depends only on $n - 1$ variables. To this end we assume that $\Omega = \Omega' \times (a, b) \subset \mathbb{R}^{n-1} \times \mathbb{R}$. For simplicity, we suppose that a is non negative.

In the sequel if w is a function defined in Ω, w_0 will denote its extension by 0 outside Ω.

Let $f \in L^2(\Omega')$. Then from the Green's formula in Theorem 3.1 we have

$$\int_\Omega fv = -\langle \partial_\nu u_f, v\rangle_{H^{-\frac{1}{2}}(\partial\Omega), H^{\frac{1}{2}}(\partial\Omega)} \text{ for all } v \in \mathcal{H}(\Omega).$$

In this identity if we take $v = e^{-ix'\cdot\xi'}e^{|\xi'|x_n}$, where $\xi' \in \mathbb{R}^{n-1}$ and $x = (x', x_n) \in \Omega' \times (a, b)$, then we obtain

$$\left(\int_a^b e^{|\xi'|x_n} dx_n\right)\hat{f}_0(\xi') = \int_{\partial\Omega} v\partial_\nu u_f.$$

Here and henceforth $\hat{f}_0$ is the Fourier transform of f_0.

¿From $\|v\|_{L^\infty(\partial\Omega)} \leq e^{b|\xi'|}$ we deduce

$$|\hat{f}_0(\xi')| \leq (b-a)e^{(b-a)|\xi'|}\|\partial_\nu u_f\|_{L^1(\partial\Omega)}. \tag{3.2}$$

Moreover we have

Lemma 3.2 *Let D be a bounded domain of $\mathbb{R}^d$, $\rho : L^2(D) \to \mathbb{R}$ be a continuous mapping satisfying $\rho(0) = 0$ and there exists three positive constants α, β and m such that*

$$|\hat{f}_0(\xi)| \leq \alpha\rho(f)e^{\beta|\xi|^m} \text{ for all } \xi \in \mathbb{R}^d \text{ and } f \in L^2(D). \tag{3.3}$$

Then for each $s > 0$ and $M > 0$ we can find three positive constants ϵ, A and B (depending on α, β, m, s, M and d) such that

$$\|f\|_{L^2(\Omega)} \leq \frac{A}{(\ln \frac{B}{\rho(f)})^{\frac{1}{sm}}},$$

for all $f \in L^2(D)$ such that $f_0 \in H^s(\mathbb{R}^d)$, $\|f\|_{L^2(D)} \leq \epsilon$ and $\|f_0\|_{H^s(\mathbb{R}^d)} \leq M$.

Lemma 3.2 and (3.2) yield

Theorem 3.2 *Let $s > 0$ and $M > 0$. Then there exists three positive constants ϵ, A and B (depending on a, b, s, M and n) such that*

$$\|f\|_{L^2(\Omega')} \leq \frac{A}{(\ln \frac{B}{\|\partial_\nu u_f\|_{L^1(\partial\Omega)}})^{\frac{1}{s}}},$$

for all $f \in L^2(\Omega')$ such that $f_0 \in H^s(\mathbb{R}^{n-1})$, $\|f\|_{L^2(\Omega')} \leq \epsilon$ and $\|f_0\|_{H^s(\mathbb{R}^{n-1})} \leq M$.

Proof of Lemma 3.2. Let $s > 0$, $M > 0$ be given and let $X(M)$ be the set of functions $f \in L^2(D)$ satisfying $f_0 \in H^s(\mathbb{R}^d)$ and $\|f_0\|_{H^s(\mathbb{R}^d)} \leq M$.

Let $f \in X(M)$ and $r > 0$. Then

$$\int_{|\xi|\geq r} |\hat{f}_0|^2 \leq \frac{1}{r^{2s}}\int_{|\xi|\geq r} |\xi|^{2s}|\hat{f}_0|^2 \leq \frac{\|f_0\|^2_{H^s(\mathbb{R}^d)}}{r^{2s}} \leq \frac{M^2}{r^{2s}}. \tag{3.4}$$

On the other hand

$$\|f\|^2_{L^2(D)} = \|f_0\|^2_{L^2(\mathbb{R}^d)} = \|\hat{f}_0\|^2_{L^2(\mathbb{R}^d)} = \int_{|\xi|\leq r} |\hat{f}_0|^2 + \int_{|\xi|\geq r} |\hat{f}_0|^2.$$

Inequalities (3.3) and (3.4) imply

$$\|f\|^2_{L^2(D)} \leq \alpha^2 \rho(f)^2 e^{2\beta r^m} r^d \omega_d + \frac{M^2}{r^{2s}}, \tag{3.5}$$

where ω_d is the measure of the unit ball of $\mathbb{R}^d$.

Let $c = \frac{\alpha^2(2d)!}{(2\beta)^{2d}}\omega_d$ and $d = 4\beta$. Then, we easily obtain from (3.5)

$$\|f\|^2_{L^2(D)} \leq \min_{r\geq 1}(c\rho(f)^2 e^{dr^m} r^d \omega_d + \frac{M^2}{r^{2s}}).$$

The rest of the proof is similar to that used for establishing the estimate in Theorem 2.2 from (2.7). □

4 Appendix : proof of Theorem 2.1

We give a proof based on a global Carleman estimate.

We shall need some notations. Let S and Γ be as in the statement of Theorem 2.2. That is S is a closed subset of $\partial\Omega$ with nonempty interior and $\Gamma = S \times (0, T)$. Following [CIK] we can find a function $\psi \in C^4(\mathbb{R}^n)$ with the following properties

(i) $\psi(x) > 0$ in $\overline{\Omega}$,

(ii) there exists $\alpha > 0$ such that $|\nabla\psi(x)| \geq \alpha$ for all $x \in \overline{\Omega}$,

(iii) $\partial_\nu\psi \geq 0$ on $\partial\Omega\backslash S$.

Let $g(t) = \frac{1}{t(T-t)}$ and set

$$\varphi = \varphi(x,t) = g(t)(e^{\rho(\psi(x)+a)} - e^{\rho(\|\psi\|_\infty + \tilde{a})}),$$

where $a > \|\psi\|_\infty$ et $a < \tilde{a} < 2a - \|\psi\|_\infty$.

The following Carleman estimate is proved in [Fe] (see also [CIK] or [Ch]).

Theorem 4.1 *There exists three positive constants C, ρ_0 and λ_0, depending on α, Ω, S and T such that*

$$\begin{aligned}\int_Q e^{2\lambda\varphi}[(\lambda g)^{-1}(\Delta u)^2 \quad &+ \quad (\lambda g)^{-1}(\partial_t u)^2 + (\lambda g)|\nabla u|^2 + (\lambda g)^3 u^2] \\ &\leq \quad C(\int_Q e^{2\lambda\varphi}[(\partial_t - \Delta)u]^2 + \int_\Gamma e^{2\lambda\varphi}(\lambda g)(\partial_\nu u)^2),\end{aligned}$$

for each $\lambda \geq \lambda_0$, $\rho \geq \rho_0$ and $u \in C^{2,1}(\overline{Q})$, $u = 0$ on Σ.

Proof of Theorem 2.1. From the last theorem we easily have the following estimate

$$\|w\|_{L^2(\Omega\times(\frac{T}{4},\frac{3T}{4}))} \leq C_0\|\partial_\nu w\|_{L^2(\Gamma)}, \tag{4.1}$$

for some positive constant C_0 depending only on Ω and T.

Next, let $\varphi \in C^\infty[0,T]$ such that $0 \leq \varphi \leq 1$, $\varphi = 0$ on $[0, \frac{T}{4}]$ and $\varphi = 1$ on $[\frac{3T}{4}, T]$. Then $v = \varphi w$ is the solution of the initial-boundary value problem

$$\begin{cases} \partial_t v - \Delta v = \varphi' w \text{ in } Q \\ v = 0 \text{ on } \Sigma_0 \cup \Sigma. \end{cases}$$

A well known estimate (see for instance [CH]) gives

$$\|v(T)\|_{H_0^1(\Omega)} \leq C_1 \|\varphi' w\|_{L^2(Q)},$$

where C_1 is some positive constant depending only on Ω and T. However φ' is zero outside $[\frac{T}{4}, \frac{3T}{4}]$. Hence

$$\|w(T)\|_{H_0^1(\Omega)} = \|v(T)\|_{H_0^1(\Omega)} \leq C_1 \|\varphi'\|_{L^\infty(0,T)} \|w\|_{L^2(\Omega \times (\frac{T}{4}, \frac{3T}{4}))}.$$

This estimate together with (4.1) imply (2.8). □

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