UTMS 2005–25 July 13, 2005

The conditional stability in line unique continuation for a wave equation and an inverse wave source problem

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THE CONDITIONAL STABILITY IN LINE UNIQUE CONTINUATION FOR A WAVE EQUATION AND AN INVERSE WAVE SOURCE PROBLEM

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Abstract. In this paper, we prove a conditional stability estimate of the logarithmic type for a wave equation on a line in $\mathbb{R}^n$, $2 \le n \le 3$ by combining the Fourier-Bros-Iagolnitzer transformation. Then we apply it to an inverse wave source problem of determining a spatially varying source term on its extended line by observations of a segment and establish the conditional stability.

1. Introduction and the main results

The unique continuation is a fundamental topic for partial differential equations and there are a vast of references (e.g., Carleman [2], Hörmander [11], Isakov [12], Robbiano [15] and the references therein). On the other hand, Cheng, Ding and Yamamoto [3] consider a unique continuation property for a wave equation along a segment over a time interval and apply it to prove the uniqueness in determining a wave source term along an extension of the segment for the observation. Our main concern for such a special continuation, is to discuss on how long extension we can determine the solution to a wave equation or a wave source if we can know observation data of values of the solution on a segment. This segment can be interpreted as a probe where we can make spatial one-dimensional observations.

In this paper, under an a priori assumption on boundedness of solutions, we will establish the conditional stability in the line unique continuation for a wave equation and prove a stability estimate for the inverse wave source problem.

1991 *Mathematics Subject Classification.* 35B60, 35L05, 35R30.

Key words and phrases. inverse source problem, unique continuation, uniqueness, Fourier-Bros-Iagolnitzer transformation.

Jin Cheng is partly supported by the National Science Foundation of China (No. 10271032, No. 10431030), the Shuguang Project and E-Institute of Shanghai Municipal Education Commission (N.E03004). Masahiro Yamamoto is supported partly by Grant 15340027 from the Japan Society for the Promotion of Science and Grant 15654015 from the Ministry of Education, Cultures, Sports and Technology. This paper was completed when Jin Cheng was a guest professor of the University of Tokyo.

It has been shown that the conditional stability is very useful and it has a close relation with the Tikhonov regularization. Actually, the conditional stability results imply the convergence rate of the regularized solutions (e.g. Cheng, Yamamoto [7], Cheng, Yamamoto, Zou [9]).

In order to state our main results, we introduce notations. Let $2 \le n \le 3$ and $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary $\partial\Omega$, and $x = (x_1, x')$ where $x' = (x_2, ..., x_n) \in \mathbb{R}^{n-1}$. We set

$$\Box = \partial_t^2 - \sum_{j=1}^{n} \partial_{x_j}^2.$$

Set

$$B(x, r) = \{y \in \mathbb{R}^n; |y - x| < r\}, \quad B'(0, r) = B(0, r) \cap \{x' = 0\}$$

for fixed $r > 0$ and $x \in \mathbb{R}^n$.

Then we can state the unique continuation for the wave equation $\Box u = 0$ along a segment over a time interval: Can we determine $u(x_1, 0, t)$, $|x_1| < R$, $|t| < t_0$ by $u(x_1, 0, t)$, $|x_1| < r$, $|t| < T$ with some $R > 0$ and $t_0 > 0$? Here $R > r$. This continuation has a different character from a usual unique continuation where for suitable open sets $\mathcal{U} \subset \mathcal{U}'$, we are required to determine u in $\mathcal{U}'$ by $u|_{\mathcal{U}}$. In our continuation, the information is restricted to a set on the (x_1, t)-space and we will determine the solution in a wider set in x_1. At the expense of determination on a longer x_1-segment, we can expect that $t_0 < T$, which means that the time interval is shrunk in the determination.

Our first main result asserts the stability.

Theorem 1.1. *Let $0 < r < R$, $\kappa > \frac{n}{2}$ and $B'(0, R) \subset \Omega$. Suppose that $u \in C^1([-T, T]; H^\kappa(\Omega))$ satisfies $(\partial_t^2 - \Delta)u = 0$ in $\Omega \times (-T, T)$ in the sense of the distribution. Let $s_0 \in (0, T)$ be fixed, and let*

$$\|u\|_{C^1([-T,T];H^\kappa(\Omega))} \le M. \tag{1.1}$$

Then

$$|u(x_1, 0, t)| \le \frac{CM}{\sqrt{\log \frac{1}{\varepsilon}}} \tag{1.2}$$

for $(x_1, t) \in (-R, R) \times (-T + s_0, T - s_0)$ satisfying

$$|t| + K(R - |x_1|)^{-1/2}(|x_1| - r)^{1/2} < T - \sqrt{3}s_0.$$

Here we set

$$\varepsilon = \sup_{-T \le t \le T} \|u(\cdot, 0, t)\|_{L^2(-r,r)},$$

and $C = C(r, T, s_0) > 0$, $K = K(r, R, s_0) > 0$ *are constants.*

Estimate (1.2) shows the stability of the logarithmic order which is conditional under a priori assumption (1.1), while in a usual continuation, we can prove the conditional stability of the Hölder type (e.g., Isakov [12]) which is stronger than (1.2).

Next we consider the following initial/boundary value problem for a wave equation with a source term:

$$\partial_t^2 u(x,t) = \Delta u(x,t) + \sigma(t) f(x), \quad x \in \Omega,\ t > 0, \tag{1.3}$$

$$u(x,0) = \partial_t u(x,0) = 0, \qquad x \in \Omega \tag{1.4}$$

and

$$u(x,t) = 0, \qquad x \in \partial\Omega,\ t > 0. \tag{1.5}$$

Here the source term $\sigma(t)f(x)$ is assumed to cause the vibration. This kind of source term in the form of a product of a spatial function and a temporal function, is commonly used in modelling vibration phenomena. Henceforth we fix $\sigma = \sigma(t) \in C^2[0,\infty)$. Then, for $f \in H_0^{1+\kappa}(\Omega)$ with $\kappa > \frac{n}{2}$, there exists a unique weak solution $u \in C^1([0,\infty); H_0^{2+\kappa}(\Omega)) \cap C^2([0,\infty); H^{1+\kappa}(\Omega))$ (e.g., Lions and Magenes [14]). Therefore, for any segment $\ell \subset \mathbb{R}^n$, by the Sobolev embedding (e.g., Adams [1]), we can regard $u(x,t)$, $x \in \ell$, $0 < t < T$, as a function in $L^2(\ell \times (0,T))$. We denote it by $u(f) = u(f)(x,t)$.

We discuss

Inverse wave source problem on a segment: Let $\ell \subset L$ be two segments in Ω. Determine $f|_L$ from $u(f)|_{\ell\times(0,T)}$.

In Cheng, Ding and Yamamoto [3], the uniqueness is proved: If $\sigma \not\equiv 0$ and $u(f)|_{\ell\times(0,T)} = 0$ with sufficiently large $T > 0$, then $f = 0$ on L. Our second main result in this paper is the conditional stability.

Theorem 1.2. *We assume* $\kappa > \frac{n}{2}$,

$$\sigma \in C^2[0,\infty), \qquad \sigma(0) \neq 0 \tag{1.6}$$

and

$$\|f\|_{H_0^{1+\kappa}(\Omega)} \le M \tag{1.7}$$

with some constant $M > 0$. *Then there exists a constant* $T_0 = T_0(\ell, L) > 0$ *such that if* $T > T_0$, *then we can choose a constant* $C = C(\Omega, \sigma, T, \ell, L, M) > 0$ *such that*

$$\|f\|_{L^\infty(L)} \le \frac{C}{\left(\log \frac{1}{\|u(f)\|_{L^2(\ell\times(0,T))}}\right)^{\frac{1}{4}}}. \tag{1.8}$$

We can represent $C(\Omega, \sigma, T, \ell, L, M) > 0$ *as follows: for any* $\nu > 0$, *there exists a constant* $\widetilde{C}_0 = \widetilde{C}_0(\Omega, \sigma, T, \ell, L, \nu) > 0$ *such that*

$$C(\Omega, \sigma, T, \ell, L, M) = M^{1+\nu}\widetilde{C}_0(\Omega, \sigma, T, \ell, L, \nu).$$

Moreover, for a fixed segment ℓ, *we can give an estimate for the critical observation time* T_0 *as follows; For sufficiently small* $\mu > 0$ *and* $\delta > 0$, *there exists a constant* $\widetilde{C} = \widetilde{C}(\delta, \mu) > 0$ *such that if*

$$T_0 > \mu + \widetilde{C}(|L| - |\ell|)^{\frac{1}{2}}, \tag{1.9}$$

then estimate (1.8) *holds for* $T > T_0$, *provided that* $dist(L, \partial\Omega) > \delta$.

In the case of $L = \ell$, we can take any short observation time T:

Corollary 1.3. *We assume* (1.6) *and* (1.7). *For any* $\mu > 0$, *there exists a constant* $C = C(\Omega, \sigma, \mu, \ell, M) > 0$ *such that*

$$\|f\|_{L^\infty(\ell)} \le \frac{C}{\left(\log \frac{1}{\|u(f)\|_{L^2(\ell\times(0,\mu))}}\right)^{\frac{1}{4}}}.$$

Our proof relies on the analyticity of the harmonic function which is related by the Fourier-Bros-Iagolnizter transformation (see (2.1) below) to the wave equation, so that it is essential that all the coefficients of hyperbolic equations under consideration are constant.

By the finiteness of the propagation speed, we should observe $u(f)|_\ell$ over a sufficiently large time T, which is estimated by (1.9). Our observation is only on $\ell \times (0, T)$ and we can determine f on the extended segment L of ℓ, and such observations do not give any information of $u(f)$ outside ℓ. Moreover the stability rate is of the logarithmic rate. For similar inverse wave source problems, we refer to Yamamoto [16].

The proof is based on the Fourier-Bros-Iagolnitzer transformation and a line unique continuation for the Laplace equation (Cheng, Hon and Yamamoto [4], Cheng and Yamamoto [6]). The methodology here is similar to Cheng, Ding and Yamamoto [3], Cheng, Lin and Nakamura [5], Cheng, Yamamoto and Zhou [8], but for proving the stability results, more independent analysis is required. The Fourier-Bros-Iagolnitzer transformation is used in Lerner [13], Robbiano [15]] for proving sharp results on the unique continuation for a hyperbolic equation.

The paper is composed of four sections. In Section 2, we show key lemmata and Section 3 is devoted to the proof of Theorem 1.1. In Section 4, we complete the proof of Theorem 1.2.

2. Key lemmata

We set $i = \sqrt{-1}$. For $\lambda > 0$ and $a \in \mathbb{R}$, we define a transformation by

$$v_{a,\lambda}(x,s) = \sqrt{\frac{\lambda}{2\pi}} \int_{-T}^{T} e^{-\frac{\lambda}{2}(is+a-t)^2} u(x,t)dt, \tag{2.1}$$

which we call the Fourier-Bros-Iagolnitzer transformation (FBI transformation for short). Henceforth we fix $s_1 > 0$ such that

$$0 < s_1 < s_0 < T \quad \text{and } s_0 - s_1 > 0 \text{ is sufficiently small.}$$

Moreover for $a \in (-T,T)$ and $s_1 \in (0, T-|a|)$, we set

$$\lambda = \lambda(\varepsilon) = \frac{2}{(T-|a|)^2 - s_1^2} \log \frac{1}{\varepsilon}, \tag{2.2}$$

$$\varphi_1(\varepsilon) = \frac{\sqrt{(T-|a|)^2 - s_1^2}}{\sqrt{\log 1/\varepsilon}} + \varepsilon^{1/2} + \varepsilon^{\frac{(T-|a|)^2}{2(T+|a|)^2}}, \tag{2.3}$$

and

$$\begin{aligned} \varphi_2(\varepsilon) \quad = \quad & \left(\frac{1}{(T-|a|)^2 - s_1^2}\right)^{3/2} \left(\log \frac{1}{\varepsilon}\right)^{3/2} \\ & \times \varepsilon^{\frac{s_0^2}{(T-|a|)^2 - s_1^2} \quad \frac{(T-|a|)^2}{[K(R-|x_1|)^{-1/2}(|x_1|-r)^{1/2} + s_1\sqrt{3}]^2} - 1} \end{aligned} \tag{2.4}$$

where $K = K(r, R, s_1) > 0$ is chosen later. We note that

$$\varepsilon = e^{-\frac{\lambda}{2}((T-|a|)^2 - s_1^2)}.$$

Henceforth the constants C_j depends on s_0, s_1, T, but independent of ε.

Lemma 2.1. *Let $\kappa > \frac{n}{2}$, $0 < \varepsilon < 1$, $|a| < T$ and $s_1 \in (0, T-|a|)$ be fixed. We further assume that*

$$\|u\|_{C^1([-T,T];H^\kappa(\Omega))} \le M. \tag{2.5}$$

Then

$$\|v_{a,\lambda}(\cdot,0)-u(\cdot,a)\|_{L^\infty(\Omega)}\le C_1M\varphi_1(\varepsilon),\qquad -T<a<T \tag{2.6}$$

and

$$\begin{aligned}&\|v_{a,\lambda}(\cdot,s)\|_{L^\infty(\Omega)},\quad \|\partial_s v_{a,\lambda}(\cdot,s)\|_{L^\infty(\Omega)},\quad \|\partial_{x_j} v_{a,\lambda}(\cdot,s)\|_{L^\infty(\Omega)}\\ &\le C_2M\lambda^{\frac12}e^{\frac{\lambda}{2}s_1^2},\qquad 1\le j\le n,\ -s_1<s<s_1.\end{aligned} \tag{2.7}$$

Proof of Lemma 2.1: According to the definition of the FBI transformation, we have

$$\begin{aligned}v_{a,\lambda}(x,0) &= \sqrt{\frac{\lambda}{2\pi}}\int_{-T}^{T} e^{-\frac{\lambda}{2}(a-t)^2}u(x,t)dt\\ &= \sqrt{\frac{1}{2\pi}}\int_{\sqrt{\lambda}(-T-a)}^{\sqrt{\lambda}(T-a)} e^{-\frac{t^2}{2}}u\left(x,\frac{t}{\sqrt{\lambda}}+a\right)dt.\end{aligned} \tag{2.8}$$

Noting that

$$\int_{-\infty}^{\infty} e^{-\frac{t^2}{2}}dt=\sqrt{2\pi},$$

by (2.5) and (2.8) we obtain

$$\begin{aligned}&|v_{a,\lambda}(x,0)-u(x,a)|\\ &=\sqrt{\frac{1}{2\pi}}\left|\int_{\sqrt{\lambda}(-T-a)}^{\sqrt{\lambda}(T-a)} e^{-\frac{t^2}{2}}u\left(x,\frac{t}{\sqrt{\lambda}}+a\right)dt-\int_{-\infty}^{\infty}e^{-\frac{t^2}{2}}u(x,a)dt\right|\\ &\le\sqrt{\frac{1}{2\pi}}\left\{\int_{\sqrt{\lambda}(-T-a)}^{\sqrt{\lambda}(T-a)} e^{-\frac{t^2}{2}}\left|u\left(x,\frac{t}{\sqrt{\lambda}}+a\right)-u(x,a)\right|dt\right.\\ &\left.+\int_{\sqrt{\lambda}(T-a)}^{\infty}e^{-\frac{t^2}{2}}|u(x,a)|dt+\int_{-\infty}^{\sqrt{\lambda}(-T-a)}e^{-\frac{t^2}{2}}|u(x,a)|dt\right\}\\ &\le\sqrt{\frac{1}{2\pi}}\left\{\int_{-\infty}^{\infty}\frac{|t|}{\sqrt{\lambda}}e^{-\frac{t^2}{2}}dt\times\|\partial_t u(x,\cdot)\|_{L^\infty(-T,T)}\right.\\ &\left.+\int_{\sqrt{\lambda}(T-a)}^{\infty}e^{-\frac{t^2}{2}}|u(x,a)|dt+\int_{-\infty}^{\sqrt{\lambda}(-T-a)}e^{-\frac{t^2}{2}}|u(x,a)|dt\right\}\\ &\le\frac{\max_{-T\le t\le T}(|u(x,t)|+|\partial_t u(x,t)|)}{\sqrt{2\pi}}\left\{\frac{2}{\sqrt{\lambda}}+\sqrt{\pi}e^{-\frac{\lambda(T-a)^2}{4}}+\sqrt{\pi}e^{-\frac{\lambda(T+a)^2}{4}}\right\}.\end{aligned}$$

At the second term, we have estimated:

$$\begin{aligned}\int_{\sqrt{\lambda}(T-a)}^{\infty}e^{-\frac{t^2}{2}}|u(x,a)|dt &\le \max_{-T\le t\le T}|u(x,t)|e^{-\frac14(\lambda(T-a)^2)}\int_0^\infty e^{-\frac{t^2}{4}}dt\\ &= \max_{-T\le t\le T}|u(x,t)|\times\sqrt{\pi}e^{-\frac{\lambda(T-a)^2}{4}}.\end{aligned}$$

As for the third term, we estimate similarly.

Hence, by the Sobolev embedding $H^\kappa(\Omega) \subset C(\overline{\Omega})$ by $\kappa > \frac{n}{2}$ (e.g., [1]),

$$\|v_{a,\lambda}(\cdot,0) - u(\cdot,a)\|_{L^\infty(\Omega)} \le C_1 M \varphi_1(\varepsilon),$$

which completes the proof of (2.6). Since

$$\partial_s \left(\exp\left(-\frac{\lambda}{2}(is+a-t)^2 \right) \right) = -i\partial_t \left(\exp\left(-\frac{\lambda}{2}(is+a-t)^2 \right) \right),$$

the integration by parts yields

$$\begin{aligned}
|\partial_s v_{a,\lambda}(x,s)| &= \sqrt{\frac{\lambda}{2\pi}} \int_{-T}^{T} \left(\partial_s e^{-\frac{\lambda}{2}(is+a-t)^2} \right) u(x,t)dt \\
&= \sqrt{\frac{\lambda}{2\pi}} \int_{-T}^{T} -i\partial_t (e^{-\frac{\lambda}{2}(is+a-t)^2}) u(x,t)dt \\
&= \sqrt{\frac{\lambda}{2\pi}} \left\{ iu(x,-T)e^{-\frac{\lambda}{2}(is+a+T)^2} - iu(x,T)e^{-\frac{\lambda}{2}(is+a-T)^2} \right\} \\
&\quad + i\sqrt{\frac{\lambda}{2\pi}} \int_{-T}^{T} e^{-\frac{\lambda}{2}(is+a-t)^2} \partial_t u(x,t)dt.
\end{aligned}$$

Hence

$$\|\partial_s v_{a,\lambda}(\cdot,s)\|_{L^\infty(\Omega)} \le C_2 M \lambda^{\frac{1}{2}} e^{\frac{\lambda}{2}s_1^2}.$$

The rest estimates are proved in the same way, and thus the proof of Lemma 2.1 is complete.

Remark 2.2. As is seen by the proof, Lemma 2.1 holds for any $\lambda > 0$.

Define an elliptic operator by

$$\Delta_{x,s} := \partial_s^2 + \sum_{j=1}^n \partial_{x_j}^2, \quad x \in \mathbb{R}^n, \quad s \in \mathbb{R}$$

and set

$$\chi_{a,\lambda} := \triangle_{x,s} v_{a,\lambda} \tag{2.9}$$

where $v_{a,\lambda}$ is defined by (2.1). Then, by the same way as Lemma 2 in Cheng, Ding and Yamamoto [3], we can prove:

Lemma 2.3. *Let $p > 1$. Suppose that $u \in C([-T,T]; L^p(\Omega)) \cap C^1([-T,T]; L^p(\Omega))$ satisfies $\Box u = 0$. Then there exists a positive number C_3 such that*

$$\|\chi_{a,\lambda}(\cdot,s)\|_{L^p(\Omega)} \le C_3 M_1 \lambda^{\frac{3}{2}} \exp\left(-\frac{\lambda}{2}((T-|a|)^2 - s_1^2) \right), \qquad -s_1 < s < s_1. \tag{2.10}$$

Here we set $M_1 = \|u\|_{C([-T,T];L^p(\Omega))} + \|\partial_t u\|_{C([-T,T];L^p(\Omega))}$.

Our main result relies on the conditional stability in the line unique continuation for the Laplace equation.

Lemma 2.4. *Let $\varphi \in W^{2,p}(B(0,R)\times(-s_1,s_1))$ satisfy $\Delta_{x,s}\varphi = 0$ in $B(0,R)\times(-s_1,s_1)$ and $\| \varphi \|_{L^1(B(0,R)\times(-s_1,s_1))} \le M_1$. We fix $\nu > 0$ sufficiently small. Then for $\rho \in [r, R-\nu]$, there exist positive constants $C_4 = C_4(r,R,s_1,\nu)$ and $\alpha = \alpha(\rho,r,R,s_1) \in (0,1)$ such that*

$$\|\varphi(\cdot,0,0)\|_{L^\infty(-\rho,\rho)} \le C_4 M_1^{1-\frac{\alpha}{3}} \| \varphi(\cdot,0,0) \|_{L^1(-r,r)}^{\frac{\alpha}{3}} . \tag{2.11}$$

Moreover

$$\lim_{\rho\uparrow R} \alpha = 0, \qquad \lim_{\rho\downarrow r} \alpha = 1 \tag{2.12}$$

and for $\rho \in (r,R)$

$$\alpha(\rho,r,R,s_1) \ge C_5(R-\rho), \quad 1-\alpha(\rho,r,R,s_1) \le C_6(\rho-r)^{\frac{1}{2}}, \tag{2.13}$$

where the constants $C_5 > 0$ and $C_6 > 0$ depend on r, R, s_1.

For the proof, see [6] or Corollary in [3].

3. Proof of Theorem 1.1

Without loss of generality, we may assume that $0 < \varepsilon < 1$. First we recall that

$$\chi_{a,\lambda} = \triangle_{x,s} v_{a,\lambda} = \partial_s^2 v_{a,\lambda} + \sum_{j=1}^n \partial_{x_j}^2 v_{a,\lambda}, \quad (x,s) \in \Omega\times(-s_1,s_1). \tag{3.1}$$

We set

$$\varphi_{a,\lambda} = v_{a,\lambda} - N\chi_{a,\lambda}. \tag{3.2}$$

Here $N\chi_{a,\lambda}$ is the Newtonian potential of $\chi_{a,\lambda}$ in $\Omega\times(-s_1,s_1)$, that is,

$$N\chi_{a,\lambda}(\xi) := \int_{\Omega\times(-s_1,s_1)} \Gamma(\xi-\eta)\chi_{a,\lambda}(\eta)d\eta, \qquad \xi = (x,s) \in \mathbb{R}^{n+1},$$

where Γ is the fundamental solution of the Laplace equation given by

$$\Gamma(\xi-\eta) = \frac{1}{(n+1)(1-n)\omega_{n+1}} |\xi-\eta|^{1-n}, \qquad n+1 \ge 3$$

$$\Gamma(\xi-\eta) = \frac{1}{2\pi}\log|\xi-\eta|, \qquad n+1 = 2$$

and ω_{n+1} is the volume of the unit ball in $\mathbb{R}^{n+1}$ (see e.g., DiBenedetto [10]). Since $\chi_{a,\lambda} \in L^2(\Omega\times(-s_1,s_1))$ by Lemma 2.3, applying the property of the Newtonian potential (e.g., Section 12 of Chapter II in [10]) and approximating $\chi_{a,\lambda}$ by functions in $C_0^\infty(\Omega \times (-s_1, s_1))$, we see that $\Delta_{x,s}(N\chi_{a,\lambda}) = \chi_{a,\lambda}$ in the sense of distribution. Therefore $\Delta_{x,s}\varphi_{a,\lambda} = 0$ in $\Omega \times (-s_1, s_1)$.

We have

$$\|N\chi_{a,\lambda}\|_{W^{1,q}(\Omega\times(-s_1,s_1))} \le C_7(\mu)\|\chi_{a,\lambda}\|_{L^p(\Omega\times(-s_1,s_1))} \tag{3.3}$$

with

$$q = \frac{(n+1)p}{n+1-p} - \mu,$$

for sufficiently small $\mu > 0$ (e.g., Lemma 10.1 (pp.85-86) in [10]). By assumption (1.1) and the Sobolev embedding, we can choose $2 < p < 3$ such that

$$\|u\|_{C^1([-T,T];L^p(\Omega))} \le C_8 M$$

Hence (3.3) and (2.10) yield

$$\|N\chi_{a,\lambda}\|_{L^1(\Omega\times(-s_1,s_1))} \le C_3C_7C_8M\lambda^{\frac{3}{2}}e^{-\frac{\lambda}{2}((T-|a|)^2-s_1^2)}.$$

Therefore, by (3.2) and (2.7), we obtain

$$\begin{aligned}&\|\varphi_{a,\lambda}\|_{L^1(\Omega\times(-s_1,s_1))} \le \|v_{a,\lambda}\|_{L^1(\Omega\times(-s_1,s_1))} + \|N\chi_{a,\lambda}\|_{L^1(\Omega\times(-s_1,s_1))}\\ &\le C_2M\lambda^{\frac{1}{2}}e^{\frac{\lambda}{2}s_1^2} + C_9M\lambda^{\frac{3}{2}}e^{-\frac{\lambda}{2}((T-|a|)^2-s_1^2)}.\end{aligned}$$

Since

$$\sup_{0<\varepsilon<1} \lambda(\varepsilon)e^{-\frac{\lambda(\varepsilon)}{2}((T-|a|)^2-s_1^2)} \equiv C_{10}(s_1, T), \tag{3.4}$$

if $T - |a| > s_1\sqrt{3}$, then we have

$$\|\varphi_{a,\lambda}\|_{L^1(\Omega\times(-s_1,s_1))} \le C_{11}M\lambda^{\frac{1}{2}}e^{\frac{\lambda}{2}s_1^2}. \tag{3.5}$$

Next, since $q > n + 1$ by $n = 2, 3$ and $p > 2$, we apply Lemma 2.3 and the Sobolev embedding (e.g., [1]) to obtain

$$\begin{aligned}\|N\chi_{a,\lambda}\|_{L^\infty(\Omega\times(-s_1,s_1))} &\le C_8\|\chi_{a,\lambda}\|_{L^p(\Omega\times(-s_1,s_1))}\\ &\le C_{12}M\lambda^{\frac{3}{2}}\exp\left(-\frac{\lambda}{2}((T-|a|)^2-s_1^2)\right).\end{aligned} \tag{3.6}$$

Since $\sup_{-T\le t\le T}\|u(\cdot,0,t)\|_{L^2(-r,r)} \le \varepsilon$, by (2.1) we obtain

$$|v_{a,\lambda}(x_1,0,0)| \le \sqrt{\frac{\lambda}{2\pi}}\int_{-T}^{T} e^{-\frac{\lambda}{2}(t-a)^2}dt \times \sup_{-T\le t\le T}|u(x_1,0,t)|,$$

and so

$$\|v_{a,\lambda}(\cdot,0,0)\|_{L^2(-r,r)} \le \sup_{-T\le t\le T}|u(\cdot,0,t)| \le \varepsilon.$$

Therefore (3.2) and (3.6) yield

$$
\begin{aligned}
\|\varphi_{a,\lambda}(\cdot,0,0)\|_{L^2(-r,r)} &\le \|v_{a,\lambda}(\cdot,0,0)\|_{L^2(-r,r)} + \|N\chi_{a,\lambda}(\cdot,0,0)\|_{L^2(-r,r)} \\
&\le \varepsilon + C_8 M\lambda^{\frac{3}{2}} \exp\left(-\frac{\lambda}{2}((T-|a|)^2 - s_1^2)\right) \\
&\le C_{13} M\lambda^{\frac{3}{2}} \exp\left(-\frac{\lambda}{2}((T-|a|)^2 - s_1^2)\right), \tag{3.7}\\
&\qquad \text{if } |a| < T - s_1\sqrt{3}.
\end{aligned}
$$

At the last inequality, by (2.2) we used

$$\varepsilon = e^{-\frac{\lambda}{2}((T-|a|)^2 - s_1^2)} \le C_{13}'\lambda^{\frac{3}{2}} \exp\left(-\frac{\lambda}{2}((T-|a|)^2 - s_1^2)\right).$$

Applying Lemma 2.4 in terms of (3.5) and (3.7), we obtain

$$|\varphi_{a,\lambda}(x,0)| \le C_{14} M\lambda^{\frac{3}{2}} e^{-\frac{\lambda}{2}[(T-|a|)^2\alpha/3 - s_1^2]}, \qquad x \in B'(0,\rho), \tag{3.8}$$

where $r \le \rho \le R$, $C_{14} > 0$ is dependent on r, T, a, but independent of $\lambda > 0$. Here we choose $\widetilde{R} > R > 0$ such that $B(0,\widetilde{R}) \subset \Omega$, so that in place of R in Lemma 2.4, we take $\widetilde{R}$ to apply the lemma. Therefore we note that ρ can vary over $[r, R]$, not $[r, R-\nu]$.

Note that $\alpha = \alpha(\rho, r, R, s_1)$ satisfies (2.12) and (2.13). Henceforth we simply write $\alpha = \alpha(\rho)$, omitting the dependency on r, R, and s_1.

Consequently, by (3.2), (3.6) and (3.8), it follows that

$$
\begin{aligned}
&|v_{a,\lambda}(x,0)| \le |\varphi_{a,\lambda}(x,0)| + |N\chi_{a,\lambda}(x,0)| \\
&\le C_{15} M \left\{ \lambda^{\frac{3}{2}} e^{-\frac{\lambda}{2}[(T-|a|)^2 \frac{\alpha(|x_1|)}{3} - s_1^2]} + \lambda^{\frac{3}{2}} e^{-\frac{\lambda}{2}[(T-|a|)^2 - s_1^2]} \right\}, \quad x \in B'(0,\rho),
\end{aligned}
$$

that is,

$$
\begin{aligned}
|v_{a,\lambda}(x,0)| &\le C_{15} M\lambda^{\frac{3}{2}} e^{-\frac{\lambda}{2}[(T-|a|)^2\alpha(|x_1|)/3 - s_1^2]} \\
&\quad \text{for } x \in B^{'}(0,\rho) \text{ and } -T + s_1\sqrt{3} < a < T - s_1\sqrt{3}. \tag{3.9}
\end{aligned}
$$

Next we will estimate $(T-|a|)^2\frac{\alpha(|x_1|)}{3} - s_1^2 > 0$. Let us set $\beta(\rho) = \frac{s_1\sqrt{3}}{\sqrt{\alpha(\rho)}}$. Then for $r \le \rho \le R$, by (2.12) and (2.13), we see that $s_1\sqrt{3} \le \beta(\rho) < \infty$ and

$$
\begin{aligned}
0 &\le \beta(\rho) - s_1\sqrt{3} \le s_1\sqrt{3}\left|\frac{1}{\sqrt{\alpha(\rho)}} - \frac{1}{\sqrt{\alpha(r)}}\right| \\
&\le s_1\sqrt{3}|\alpha(\rho) - \alpha(r)|\alpha(\rho)^{-\frac{1}{2}}\alpha(r)^{-\frac{1}{2}}(\sqrt{\alpha(\rho)} + \sqrt{\alpha(r)})^{-1} \\
&\le s_1\sqrt{3}|\alpha(\rho) - 1|\alpha(\rho)^{-\frac{1}{2}} \le \frac{s_1\sqrt{3}C_6(\rho - r)^{\frac{1}{2}}}{\sqrt{C_5}(R-\rho)^{\frac{1}{2}}}.
\end{aligned}
$$

Set

$$K = K(r,R,s_1) = \frac{s_1\sqrt{3}C_6(r,R,s_1)}{\sqrt{C_5(r,R,s_1)}}. \tag{3.10}$$

Hence

$$0 \le \beta(|x_1|) - s_1\sqrt{3} \le K(R-|x_1|)^{-\frac{1}{2}}(|x_1|-r)^{\frac{1}{2}}, \quad r \le |x_1| < R. \tag{3.11}$$

Let $|a| + K(R-|x_1|)^{-\frac{1}{2}}(|x_1|-r)^{\frac{1}{2}} < T - s_1\sqrt{3}$. Then

$$[K(R-|x_1|)^{-\frac{1}{2}}(|x_1|-r)^{\frac{1}{2}} + s_1\sqrt{3}]^2 < (T-|a|)^2.$$

Hence (3.11) yields

$$\frac{\alpha(|x_1|)}{3s_1^2} = \frac{1}{\beta^2(|x_1|)} \ge \frac{1}{[K(R-|x_1|)^{-\frac{1}{2}}(|x_1|-r)^{\frac{1}{2}} + s_1\sqrt{3}]^2},$$

which implies that

$$\begin{aligned}&(T-|a|)^2\frac{\alpha(|x_1|)}{3} - s_1^2 \\ &\ge s_1^2\left\{\frac{(T-|a|)^2}{[K(R-|x_1|)^{-\frac{1}{2}}(|x_1|-r)^{\frac{1}{2}} + s_1\sqrt{3}]^2} - 1\right\},\end{aligned}$$

so that

$$\begin{aligned}&\lambda^{\frac{3}{2}}\exp\left(-\frac{\lambda}{2}\left((T-|a|)^2\frac{\alpha(|x_1|)}{3} - s_1^2\right)\right) \\ &\le \lambda^{\frac{3}{2}}\exp\left(-\frac{\lambda s_1^2}{2}\left\{\frac{(T-|a|)^2}{[K(R-|x_1|)^{-\frac{1}{2}}(|x_1|-r)^{\frac{1}{2}} + s_1\sqrt{3}]^2} - 1\right\}\right).\end{aligned}$$

By (2.4), (3.9) and $s_0 > s_1$, we have

$$|v_{a,\lambda}(x,0)| \le C_{15}M\varphi_2(\varepsilon), \qquad x \in B'(0,\rho). \tag{3.12}$$

Therefore from (2.6) and (3.9), we obtain

$$\begin{aligned}|u(x,a)| &\le |v_{a,\lambda}(x,0) - u(x,a)| + |v_{a,\lambda}(x,0)| \\ &\le C_1M\varphi_1(\varepsilon) + C_{15}M\varphi_2(\varepsilon).\end{aligned}$$

Therefore, replacing a by t, we have proved:

$$\begin{aligned}|u(x,t)| \le C_{16}M\Bigg(&\frac{\sqrt{(T-|t|)^2 - s_1^2}}{\sqrt{\log\frac{1}{\varepsilon}}} + \varepsilon^{\frac{1}{2}} + \varepsilon^{\frac{(T-|t|)^2}{2(T+|t|)^2}} \\ &+ \left(\frac{1}{(T-|t|)^2 - s_1^2}\right)^{\frac{3}{2}}\left(\log\frac{1}{\varepsilon}\right)^{\frac{3}{2}}\varepsilon^{\frac{s_0^2}{(T-|t|)^2-s_1^2}\left\{\frac{(T-|t|)^2}{[K(R-|x_1|)^{-\frac{1}{2}}(|x_1|-r)^{\frac{1}{2}}+s_1\sqrt{3}]^2}-1\right\}}\Bigg)\end{aligned}$$

if $x \in B'(0,R)$ and $|t| + K(R-|x_1|)^{-\frac{1}{2}}(|x_1|-r)^{\frac{1}{2}} < T - s_1\sqrt{3}$.

Now we will complete the proof of Theorem 1.1. We recall that $0 < s_1 < s_0 < T$ and $s_0 - s_1 > 0$ is sufficiently small. Let

$$x \in B'(0,R), \quad |t| + K(R-|x_1|)^{-\frac{1}{2}}(|x_1|-r)^{\frac{1}{2}} < T - s_0\sqrt{3}. \tag{3.13}$$

In particular, $|t| < T - s_0\sqrt{3}$. Therefore we can directly verify that

$$\frac{(T-|t|)^2}{2(T+|t|)^2} \ge \frac{3s_0^2}{2(2T-s_0\sqrt{3})^2}, \quad \frac{1}{(T-|t|)^2 - s_1^2} \le \frac{1}{s_0^2 - s_1^2}$$

and

$$\begin{aligned}&\frac{s_0^2}{(T-|t|)^2-s_1^2}\left\{\frac{(T-|t|)^2}{[K(R-|x_1|)^{-\frac{1}{2}}(|x_1|-r)^{\frac{1}{2}}+s_1\sqrt{3}]^2}-1\right\}\\ \ge\ & \frac{s_1^2}{T^2}\left(\frac{(T-|t|)^2}{(T-|t|-\sqrt{3}(s_0-s_1))^2}-1\right) \ge \frac{s_1^2}{T^2}\frac{3(s_0^2-s_1^2)}{T^2}.\end{aligned}$$

At the second last inequality, we used (3.13). Therefore

$$|u(x,t)| \le C_{16}M\left(\frac{1}{\sqrt{\log\frac{1}{\varepsilon}}} + \varepsilon^{\frac{1}{2}} + \varepsilon^{\frac{3s_0^2}{2(2T-s_0\sqrt{3})^2}} + \left(\frac{1}{s_0^2-s_1^2}\right)^{\frac{3}{2}}\left(\log\frac{1}{\varepsilon}\right)^{\frac{3}{2}}\varepsilon^{\frac{3s_1^2(s_0^2-s_1^2)}{T^4}}\right)$$

under (3.10). Since $\varepsilon^{\frac{1}{2}} = O\left(\frac{1}{\sqrt{\log\frac{1}{\varepsilon}}}\right)$, $\varepsilon^{\frac{3s_0^2}{2(2T-s_0\sqrt{3})^2}} = O\left(\frac{1}{\sqrt{\log\frac{1}{\varepsilon}}}\right)$ and

$$\left(\log\frac{1}{\varepsilon}\right)^{\frac{3}{2}}\varepsilon^{\frac{3s_1^2(s_0^2-s_1^2)}{T^4}} = O\left(\varepsilon^{\frac{s_1^2(s_0^2-s_1^2)}{T^4}}\right) = O\left(\frac{1}{\sqrt{\log\frac{1}{\varepsilon}}}\right)$$

as $\varepsilon \longrightarrow 0$, we obtain (1.2) under (3.13). Thus the proof of Theorem 1.1 is complete.

4. Proof of Theorem 1.2

Let $v = v(x,t)$ satisfy

$$\begin{cases}\partial_t^2 v(x,t) = \Delta v(x,t), & x\in\Omega,\ t\in\mathbb{R},\\ v(x,0) = 0, \quad \partial_t v(x,0) = f(x), & x\in\Omega,\\ v(x,t) = 0, & x\in\partial\Omega,\ t\in\mathbb{R}.\end{cases} \tag{4.1}$$

Then by $f \in H_0^{1+\kappa}(\Omega)$, applying the regularity property (e.g., Lions and Magenes [14]), we see that

$$v \in C^1(\mathbb{R}; H_0^{\kappa+1}(\Omega)). \tag{4.2}$$

Next by a congruent transformation, we can assume that $B(0,R) \subset \Omega$,

$$L = B'(0, R-\delta) = \{(x_1, 0) \in \mathbb{R}^n; |x_1| < R-\delta\}$$

and

$$\ell = B'(0,r) = \{(x_1,0) \in \mathbb{R}^n; |x_1| < r\}$$

with some $0 < r < R - \delta$. Then $\operatorname{dist}(L, \partial L) > \delta$.

For sufficiently small $\mu > 0$, we set $(1+\sqrt{3})s_0 = \mu$, and we define $K = K(r_0, R-\delta_0, s_0)$ by (3.10). We fix $r_0 > 0$ and $\delta_0 > 0$ such that $0 < r_0 < r < R - \delta < R - \delta_0$, both $r - r_0 > 0$ and $\delta - \delta_0$ are sufficiently small. Then, setting $\widetilde{C}(\delta,\mu) = K\delta^{-\frac{1}{2}}$ in (1.9), we have

$$(1+\sqrt{3})s_0 + K\delta^{-\frac{1}{2}}(R-r-\delta)^{\frac{1}{2}} < T_0. \tag{4.3}$$

By Theorem 1.1, we obtain that if $T > T_0$, then

$$\|v\|_{L^\infty(B'(0,R-\delta_0)\times(-s_0,s_0))} \le \frac{CM_2}{\left(\log \frac{1}{\sup_{-T\le t\le T}\|v(\cdot,0,t)\|_{L^2(-r_0,r_0)}}\right)^{\frac{1}{2}}},$$

that is,

$$\|v\|_{L^\infty(L\times(-s_0,s_0))} \le \frac{CM_2}{\left(\log \frac{1}{\sup_{-T\le t\le T}\|v(\cdot,0,t)\|_{L^2(-r,r)}}\right)^{\frac{1}{2}}}. \tag{4.4}$$

Here we set $M_2 = \|v\|_{C([-T,T];H^\kappa(\Omega))} + \|v\|_{C^1([-T,T];H^\kappa(\Omega))}$.

On the other hand, by [14], we have

$$\begin{aligned} M_2 \;&\le\; \|v\|_{C([-T,T];H_0^{2+\kappa}(\Omega))} + \|\partial_t v\|_{C([-T,T];H_0^{1+\kappa}(\Omega))} \\ &+\; \|\partial_t^2 v\|_{C([-T,T];H_0^{\kappa}(\Omega))} \le C_{17}\|f\|_{H_0^{1+\kappa}(\Omega)} \le C_{17}M. \end{aligned} \tag{4.5}$$

By the Sobolev embedding, $\kappa > \frac{n}{2}$ and (1.7), we have

$$\sup_{x\in L, -s_0<t<s_0} |\partial_t^2 v(x,t)| \le C_{18}\|f\|_{H_0^{1+\kappa}(\Omega)} \le C_{18}M.$$

Hence, by the interpolation inequality (e.g., [1]) and (4.4), we obtain

$$\sup_{x\in L, -s_0<t<s_0} |\partial_t v(x,t)| \le \frac{C_{19}M}{\left(\log \frac{1}{\sup_{-T\le t\le T}\|v(\cdot,0,t)\|_{L^2(-r,r)}}\right)^{\frac{1}{4}}}. \tag{4.6}$$

We take the even extensions of $\sigma = \sigma(t)$ and $u(f) = u(f)(x,t)$ for $t < 0$ and we use the same notations: $\sigma(t) = \sigma(-t)$ and $u(f)(x,-t) = u(f)(x,t)$ for $t \in \mathbb{R}$ and $x \in \Omega$. Then we readily see that $u(f) \in C(\mathbb{R}; H_0^{\kappa+1}(\Omega))$ satisfies (1.3) - (1.5) also for $t < 0$. By the Duhamel principle, we obtain

$$u(f)(x,t) = \int_0^t \sigma(t-s)v(x,s)ds, \quad -T < t < T,\ x \in B'(0,r). \tag{4.7}$$

Moreover we can prove (4.7) by verifying that the right hand side of (4.7) satisfies (1.3) - (1.5).

By [14], we see that

$$\|\partial_t u(f)\|_{C([-T,T];H_0^{2+\kappa}(\Omega))} + \|\partial_t^2 u(f)\|_{C([-T,T];H_0^{1+\kappa}(\Omega))} \le C_{20}M. \tag{4.8}$$

Hence

$$\partial_t u(f)(x,t) = \sigma(0)v(x,t) + \int_0^t \sigma'(t-s)v(x,s)ds, \quad x \in B'(0,r),\ -T < t < T.$$

By $\sigma(0) \neq 0$, this is a Volterra equation of the second kind, so that

$$\|v(x_1,0,\cdot)\|_{C[-T,T]} \le C_{21}\|\partial_t u(f)(x_1,0,\cdot)\|_{C[-T,T]}, \quad x = (x_1,0) \in B'(0,r),$$

that is,

$$\sup_{-T\le t\le T} \|v(\cdot,0,t)\|_{L^2(-r,r)} \le C_{22}\|\partial_t u(f)(\cdot,0,\cdot)\|_{C([-T,T];L^2(-r,r))}. \tag{4.9}$$

We note by $\kappa > \frac{n}{2}$ and the Sobolev embedding that $H_0^\kappa(\Omega) \subset C(\overline{\Omega})$. By means of (4.8) and the interpolation inequality, we have

$$\begin{aligned}
&\|\partial_t u(f)(\cdot,0,\cdot)\|_{C([-T,T];L^2(-r,r))}\\
&\le C'_{23}\|\partial_t^2 u(f)\|^{\frac{1}{2}}_{C([-T,T];H_0^\kappa(\Omega))}\|u(f)(\cdot,0,\cdot)\|^{\frac{1}{2}}_{C([-T,T];L^2(-r,r))}\\
&\le C_{23}M^{\frac{1}{2}}\|u(f)(\cdot,0,\cdot)\|^{\frac{1}{2}}_{H^1(-T,T;L^2(-r,r))}.
\end{aligned}$$

Again application of the interpolation inequality yields

$$\|u(f)(\cdot,0,\cdot)\|_{H^1(-T,T;L^2(-r,r))} \le C_{24}\|u(f)\|^{\frac{1}{2}}_{L^2(\ell\times(-T,T))}\|u(f)\|^{\frac{1}{2}}_{C^2([-T,T];H_0^\kappa(\Omega))}$$

in terms of $H_0^\kappa(\Omega) \subset C(\overline{\Omega})$. Thus, noting that $u(f)(\cdot,-t) = u(f)(\cdot,t)$, we have

$$\|\partial_t u(f)(\cdot,0,\cdot)\|_{C([-T,T];L^2(-r,r))} \le C_{25}M^{\frac{3}{4}}\|u(f)\|^{\frac{1}{4}}_{L^2(2\ell\times(0,T))},$$

with which we combine (4.9) to obtain

$$\begin{aligned}
\sup_{-T\le t\le T}\|v(\cdot,0,t)\|_{L^2(-r,r)} &\le C_{22}C_{25}M^{\frac{3}{4}}\|u(f)\|^{\frac{1}{4}}_{L^2(\ell\times(0,T))}\\
&\equiv C_{26}M^{\frac{3}{4}}\|u(f)\|^{\frac{1}{4}}_{L^2(\ell\times(0,T))}.
\end{aligned} \tag{4.10}$$

We consider two cases separately:

$$\text{(a)} \quad \|u(f)\|^{\frac{1}{4}}_{L^2(\ell\times(0,T))} \le \frac{1}{C_{26}M^{\frac{3}{4}}}.$$

$$\text{(b)} \quad \|u(f)\|^{\frac{1}{4}}_{L^2(\ell\times(0,T))} \ge \frac{1}{C_{26}M^{\frac{3}{4}}}.$$

Case (a): We have

$$\log\frac{1}{C_{26}M^{\frac{3}{4}}} + \frac{1}{4}\log\frac{1}{\|u(f)\|_{L^2(\ell\times(0,T))}} \ge \frac{1}{8}\log\frac{1}{\|u(f)\|_{L^2(\ell\times(0,T))}}.$$

Therefore (4.10) implies

$$\begin{aligned}\frac{1}{\log \frac{1}{\sup_{-T\le t\le T}\|v(\cdot,0,t)\|_{L^2(-r,r)}}} &\le \frac{1}{\log \frac{1}{C_{26}M^{\frac{3}{4}}\|u(f)\|^{\frac{1}{4}}_{L^2(\ell\times(0,T))}}}\\ &\le \frac{8}{\log \frac{1}{\|u(f)\|_{L^2(\ell\times(0,T))}}}.\end{aligned}$$

Hence (4.6) yields

$$\sup_{x\in L,-s_0\le t\le s_0}|\partial_t v(x,t)| \le \frac{C_{27}M}{\left(\log \frac{1}{\|u(f)\|_{L^2(\ell\times(0,T))}}\right)^{\frac{1}{4}}}.$$

Since $\partial_t v(x,0)=f(x)$ by (4.1), we see (1.8).

Case (b):

$$\frac{1}{\left(\log \frac{1}{\|u(f)\|_{L^2(\ell\times(0,T))}}\right)^{\frac{1}{4}}} \ge \frac{1}{(\log C_{26}^4 M^3)^{\frac{1}{4}}}.$$

By (1.7) and the Sobolev embedding, we have

$$\|f\|_{L^\infty(L)} \le M \le \frac{M(\log C_{26}^4M^3)^{\frac{1}{4}}}{\left(\log \frac{1}{\|u(f)\|_{L^2(\ell\times(0,T))}}\right)^{\frac{1}{4}}}.$$

Thus the proof of Theorem 1.2 is complete.

5. Some remarks

Remark 5.1. Our results heavily depends on the conditional stability in the line unique continuation for the elliptic equations with the analytic coefficients. It can be seen that our results can be easily extended to a hyperbolic equation with time-independent analytic coefficients.

Remark 5.2. The results in this paper are the local stability result in the sense that we can not obtain any information or estimate about the value of the solution outside the hyperplane.

Remark 5.3. The lines ℓ and L can be replaced by some analytic curves or analytic surfaces. The stability results are same.

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