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# The minimal risk of hedging with a convex risk measure

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**Abstract**

We study on the minimal hedging risk for a bounded European contingent claim when we use a convex risk measure. We find the infimum of hedging risk by using a kind of min-max theorem, Also we show that this infimum is again regarded as a convex risk measure.

## 1 Introduction

Let $(\Omega, \mathcal{F}, P)$ be a probability space. For $1 \leq q \leq \infty$, We denote $L^q(\Omega, \mathcal{F}, P)$ by $L^q$, and its norm by $\| \cdot \|_q$. Let $\mathcal{P}$ be the set of probability measures on $(\Omega, \mathcal{F})$ that are absolutely continuous with respect to $P$. Föllmer and Schied [2] introduce the following notation.

**Definition 1.1.** *We say that a mapping $\rho : L^\infty \to \mathbb{R}$ is a convex risk measure, if the following three conditions are satisfied* :

(1) $X \geq Y \Longrightarrow \rho(X) \leq \rho(Y)$,

(2) $\rho(\lambda X + (1-\lambda)Y) \leq \lambda\rho(X) + (1-\lambda)\rho(Y), \quad \lambda \in (0,1)$,

(3) $\rho(X + c) = \rho(X) - c, \quad c \in \mathbb{R}$.

For a convex risk measure $\rho$, $\tilde{\rho} : L^\infty \to \mathbb{R}$, $\tilde{\rho}(X) = \rho(X) - \rho(0)$ is also a convex risk measure, and $\tilde{\rho}(0) = 0$. So we may assume $\rho(0) = 0$ in the following discussions.

Föllmer and Schied [3] proved the following.

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**Theorem 1.2.** *For a convex risk measure $\rho : L^\infty \to \mathbb{R}$, the following properties are equivalent.*

(1) *There exists a penalty function $\alpha : \mathcal{P} \to \mathbb{R} \cup \{+\infty\}$, which is bounded from below such that $\rho(X) = \sup_{Q \in \mathcal{P}} (E^Q[-X] - \alpha(Q))$.*

(2) *( Fatou Property ) $\rho(X) \leq \liminf_{n \to \infty} \rho(X_n)$ for any sequence $(X_n)_{n \in \mathbb{N}}$ of random variable which is uniformly bounded by 1 and converges to $X \in L^\infty$ in probability.*

(3) *$\rho$ is continuous from above, i.e., if a sequence $(X_n)_{n \in \mathbb{N}}$ of random variable in $L^\infty$ decreasing to $X \in L^\infty$ a.s., then $\rho(X_n)$ converges to $\rho(X)$.*

Let $\alpha_{min}(Q) = \sup_{Y \in \mathcal{A}_\rho} E^Q[-Y]$, where $\mathcal{A}_\rho = \{X \in L^\infty \mid \rho(X) \leq 0\}$, then we have $\alpha_{min}(Q) \leq \alpha(Q), \quad Q \in \mathcal{P}$ for any penalty function $\alpha$ satisfying the equation in (1). Note that $\alpha_{min}(Q) \geq 0$ for $Q \in \mathcal{P}$ by the assumption $\rho(0) = 0$.

Now we state our main theorem. Let $\mathcal{C} \subset L^\infty$ be a nonempty convex subset, and $\mathcal{M}(\mathcal{C}) = \{Q \in \mathcal{P} \mid \sup_{Z \in \mathcal{C}} E^Q[Z] < \infty\}$.

**Theorem 1.3.** *Let $\rho : L^\infty \to \mathbb{R}$ be a convex risk measure which is continuous from above. Suppose that $\rho$ is continuous from below. i.e., if a sequence $(X_n)_{n \in \mathbb{N}}$ of random variable in $L^\infty$ increases to $X \in L^\infty$ a.s., then $\rho(X_n)$ converges to $\rho(X)$. Then we have*

$$\inf_{Z \in \mathcal{C}} \rho(Z + H) = \sup_{Q \in \mathcal{P}} (E^Q[-H] - \tilde{\alpha}(Q)), \tag{1}$$

*for any $H \in L^\infty$, where*

$$\tilde{\alpha}(Q) = \alpha_{min}(Q) + \sup_{Z \in \mathcal{C}} E^Q[Z], \quad Q \in \mathcal{P}. \tag{2}$$

**Remark .** *Roorda [5] showed a simple version of this result in the case that $\rho$ is a coherent risk measure.*

We give a proof of this theorem in Section 3.

Now let us consider the following mathematical financial market model. Let $(\Omega, \mathcal{F}, P; \{\mathcal{F}(t)\}_{t \in [0,T]})$ be a filtered probability space. We assume that the filtration $\{\mathcal{F}(t)\}_{t \in [0,T]}$ satisfies the usual conditions, i.e., $\{\mathcal{F}(t)\}_{t \in [0,T]}$ is right-continuous and $\mathcal{F}(0)$ contains all P-negligible sets in $\mathcal{F}$. We also assume that $\mathcal{F}(0)$ is trivial and $\mathcal{F}(T) = \mathcal{F}$. Let $S(t) = (S^i(t)),\ 1 \le i \le d$, be an $\{\mathcal{F}(t)\}$-adapted, RCLL, and locally bounded $d$ dimensional process. This process is interpreted as the discount price processes of $d$ risky assets.

We say that a $d$ dimensional process $\xi(t) = (\xi^i(t)),\ 1 \le i \le d$ is a strategy if $\xi$ is $\{\mathcal{F}(t)\}$-predictable and $S$-integrable. We define an appropriate class $\mathcal{A}d$ of strategies by the following.

$$\mathcal{A}d = \{\xi = (\xi^i) \mid \xi \text{ is a strategy and } \int_0^{\cdot} \xi(u)dS(u) \text{ is bounded}\}. \tag{3}$$

For a pair $(v, \xi),\ v \in \mathbb{R}^+ \cup \{0\},\ \xi \in \mathcal{A}d$, we define a process $\{V(t)\}_{t\in[0,T]}$ by

$$V(t) = V(t; (v, \xi)) = v + \int_0^t \xi(u)dS(u), \quad t \in [0, T] \tag{4}$$

This process $V(t; (v, \xi))$ is interpreted as the value of self-financing portfolio strategy $(v, \xi)$ at time $t \in [0, T]$.

We denote by $\mathcal{M}(S)$ the set of probability measures $Q \in \mathcal{P}$ such that the components $S^i(t),\ 1 \le i \le d$ are local martingales under $Q$. We assume that $\mathcal{M}(S) \ne \phi$. Then we have the following.

**Corollary 1.4.** *Let $\rho : L^\infty \to \mathbb{R}$ be a convex risk measure which is continuous from above and below. Then we have*

$$\inf_{\xi\in\mathcal{A}d} \rho(V(T; (0, \xi)) + H) = \inf_{Q\in\mathcal{P}} (E^Q[-H] - \tilde{\alpha}(Q)), \tag{5}$$

*for $H \in L^\infty$, where*

$$\tilde{\alpha}(Q) = \begin{cases} \alpha_{min}(Q), & \text{if } Q \in \mathcal{M}(S) \cap \{Q \ll P \mid \alpha_{min}(Q) < \infty\} \\ +\infty, & \text{otherwise.} \end{cases} \tag{6}$$

**Remark .** *Delbaen [1] showed this result in the case that $\rho$ is a coherent risk measure and $H = 0$.*

# 2 Remarks on a Convex Risk Measure

We prove the following in this section.

**Theorem 2.1.** *For a convex risk measure $\rho$ which is continuous from above, the following properties are equivalent.*

(1) *$\rho$ is continuous from below.*

(2) *For arbitrary $c > 0$, the set $\{Q \in \mathcal{P} \mid \alpha_{min}(Q) \leq c\}$ is $L^1(P)$-weakly compact convex subset.*

We make some preparation. Let $\rho : L^\infty \to \mathbb{R}$ be a convex risk measure which is continuous from above. Let $\Lambda_c$ and $\Lambda_\infty$ denote

$$\begin{aligned} &\Lambda_c = \{Q \in \mathcal{P} \mid \alpha_{min}(Q) \leq c\} \quad c > 0, \\ &\Lambda_\infty = \{Q \in \mathcal{P} \mid \alpha_{min}(Q) < \infty\}. \end{aligned} \tag{7}$$

We note that

$$\rho(X) = \sup_{Q \in \mathcal{Q}} (E^Q[-X] - \alpha_{min}(Q)), \quad \mathcal{Q} \supset \Lambda_\infty,\ X \in L^\infty. \tag{8}$$

**Lemma 2.2.** *We have $\rho(X) = \sup_{Q \in \Lambda_c} (E^Q[-X] - \alpha_{min}(Q))$ for $X \in L^\infty$ and $c > 2\|X\|_\infty$.*

*Proof.* $\rho(X) \geq \sup_{Q \in \Lambda_c} (E^Q[-X] - \alpha_{min}(Q))$ is obvious. We show the inverse inequality. For each $n \in \mathbb{N}$, there exists $Q_n \in \mathcal{P}$ such that $\rho(X) - 1/n \leq E^{Q_n}[-X] - \alpha_{min}(Q_n)$. We can easily see that $\rho(X) \geq -\|X\|_\infty$ by the monotonicity of $\rho$. Then for $n \geq 1/(c - 2\|X\|_\infty)$ we see that

$$\alpha_{min}(Q_n) \leq E^{Q_n}[-X] - \rho(X) + 1/n \leq 2\|X\|_\infty + (c - 2\|X\|_\infty) = c. \tag{9}$$

And so $Q_n \in \Lambda_c$. This implies that

$$\rho(X) - 1/n \leq E^{Q_n}[-X] - \alpha_{min}(Q_n) \leq \sup_{Q \in \Lambda_c} (E^Q[-X] - \alpha_{min}(Q)). \tag{10}$$

Letting $n \to \infty$, we have $\rho(X) \leq \sup_{Q \in \Lambda_c} (E^Q[-X] - \alpha_{min}(Q))$. $\square$

Now we prove Theorem 2.1. Assume that the Assertion (1) holds. Since the mapping $Q \mapsto E^Q[-Y]$ is continuous for any $Y \in L^\infty$, we can immediately see that $\alpha_{min} : Q \mapsto \sup_{Y \in \mathcal{A}_\rho} E^Q[-Y]$ is lower semicontinuous with respect to the $L^1$-weak topology. Hence $\Lambda_c$ is closed for $c > 0$.

Let $(B_n)_{n \in \mathbb{N}}$ be a decreasing sequence of measurable sets such that $\bigcap_n B_n = \phi$. Take $Q \in \Lambda_c$. Then we have $c \geq \alpha_{min}(Q) \geq E^Q[-\lambda 1_{B_n^c}] - \rho(\lambda 1_{B_n^c})$

for $\lambda > 0$, and so $c/\lambda + \rho(\lambda 1_{B_n^c})/\lambda + 1 \geq Q[B_n]$. Since $\rho(\lambda 1_{B_n^c}) \to -\lambda$ by the assumption, we have

$$c/\lambda \geq \lim_{n\to\infty} \sup_{Q\in\Lambda_c} Q[B_n], \quad \lambda > 0. \tag{11}$$

Letting $\lambda \to \infty$, we have $\lim_{n\to\infty} \sup_{Q\in\Lambda_c} Q[B_n] = 0$ for any $c > 0$, and this implies that the set $\Lambda_c$ is uniformly $P$-integrable. Hence we obtain the assertion (2) by Dunford-Pettis theorem,

Assume that the Assertion (2) holds. Let $\{X_n\}_{n\in\mathbb{N}}$ be random variables in $L^\infty$ such that $X_n$ increases to $X$ as $n \to \infty$. Then there exists a positive number $M > 0$ such that $\|X_n\|_\infty \leq M,\ n \in \mathbb{N}$ and $\|X\|_\infty \leq M$. We have

$$\begin{aligned} \rho(X_n) &= \sup_{Q\in\Lambda_{2M+1}} (E^Q[-X_n] - \alpha_{min}(Q)), \quad n \in \mathbb{N}, \\ \rho(X) &= \sup_{Q\in\Lambda_{2M+1}} (E^Q[-X] - \alpha_{min}(Q)). \end{aligned} \tag{12}$$

by Lemma 2.2. Since $\Lambda_{2M+1}$ is $L^1$-weakly compact by assumption, Dini's theorem implies that

$$|(E^Q[-X_n] - \alpha_{min}(Q)) - (E^Q[-X] - \alpha_{min}(Q))| = |E^Q[X] - E^Q[X_n]| \tag{13}$$

converges to 0 uniformly in $Q \in \Lambda_{2M+1}$ as $n \to \infty$. Hence we have the assertion (1). This completes the proof.

# 3 The Proof of the Main Theorem

Before we start the proof, we prepare a version of minimax theorem due to Kim [4]. For a convenience, we set the conditions a little stronger than the original.

**Lemma 3.1.** *Let $\mathcal{X}$ be a nonempty convex subset of some locally convex linear topological space, $\mathcal{Y}$ be a non-empty subset of a vector space ( not necessarily topologized ) , and $f$ be a real-valued function on $\mathcal{X} \times \mathcal{Y}$ such that*

(1) *$x \mapsto f(x, y)$ is convex and lower semicontinuous for any $y \in \mathcal{Y}$,*

(2) *There exists $y_0 \in \mathcal{Y}$ such that $(1-\lambda)f(x, y_1) + \lambda f(x, y_2) \leq f(x, y_0),\ x \in \mathcal{X}$ for any $y_1,\ y_2 \in \mathcal{Y}$ and $\lambda \in [0, 1]$,*

(3) *The mapping*

$$\lambda \in [0,1] \ \mapsto f(x, \lambda y_1 + (1-\lambda)y_2) \tag{14}$$

*is continuous for any* $x \in \mathcal{X}$ *and* $y_1, y_2 \in \mathcal{Y}$,

*and*

(4) *There exists a non-empty compact subset* $C_F$ *of* $\mathcal{X}$ *such that*

$$\inf_{x\in\mathcal{X}\setminus C_F} f(x,y_0) \geq \max\{\inf_{x\in C_F} f(x,y_0), \inf_{x\in\mathcal{X}}\sup_{y\in\mathcal{Y}} f(x,y)\}, \quad y_0 \in \mathrm{co}(F), \tag{15}$$

*for any non-empty finite set* $F$ *of* $Y$, *where* $\mathrm{co}(F)$ *is the minimal convex set which contains all elements of* $F$.

*Then we have* $\sup_{y\in\mathcal{Y}}\inf_{x\in\mathcal{X}} f(x,y) \geq \inf_{x\in\mathcal{X}}\sup_{y\in\mathcal{Y}} f(x,y)$.

Now we prove Theorem 1.3.

Step1. First we consider the case that $\mathcal{M}(\mathcal{C}) \cap \{Q \ll P \mid \alpha_{min}(Q) < \infty\} \neq \phi$. We can easily see that

$$\begin{aligned}
&\inf_{Z\in\mathcal{C}} \rho(Z+H) \\
&= \inf_{Z\in\mathcal{C}}\sup_{Q\in\mathcal{P}}(E^Q[-Z-H] - \alpha_{min}(Q)) \\
&\geq \sup_{Q\in\mathcal{P}}\inf_{Z\in\mathcal{C}}(E^Q[-Z-H] - \alpha_{min}(Q)) \\
&= \sup_{Q\in\mathcal{P}}(E^Q[-H] - \tilde{\alpha}(Q)).
\end{aligned} \tag{16}$$

We show the inverse inequality. We apply Lemma 3.1 for $\mathcal{X} = \mathcal{P}, \ \mathcal{Y} = \mathcal{C}$. To show the inverse inequality, it is sufficient that the mapping

$$f : (Q, Z) \mapsto E^Q[Z+H] + \alpha_{min}(Q) \tag{17}$$

satisfies the conditions in Lemma 3.1. Clearly Conditions (1), (2), (3) are satisfied ( It is already shown in the proof of theorem 2.1 that the mapping $Q \mapsto \alpha_{min}(Q)$ is lower semicontinuous with respect to $L^1$-weak topology ). We will verify that $f$ satisfies Condition (4). Let $F = \{Z_1, Z_2, \ldots, Z_m\}$, $m < \infty$, $Z_0 \in \mathrm{co}(F)$, and

$$M = \max_{1\leq i\leq m} \|Z_i\|_\infty \vee \{\inf_{Q\in\Lambda_\infty\cap\mathcal{M}(\mathcal{C})}(\alpha_{min}(Q) + \sup_{Z\in\mathcal{C}} E^Q[Z]) + 2\|H\|_\infty\} < \infty. \tag{18}$$

We show that $C_F = \Lambda_{2M+1}$ satisfies Condition (4). We see that

$$\begin{aligned} & \inf_{Q\in\Lambda_{2M+1}} (E^Q[Z_0+H]+\alpha_{min}(Q)) \\ &= \inf_{Q\in\mathcal{P}} (E^Q[Z_0+H]+\alpha_{min}(Q)) \\ &\le \inf_{Q\in\mathcal{P}\setminus\Lambda_{2M+1}} (E^Q[Z_0+H]+\alpha_{min}(Q)). \end{aligned} \tag{19}$$

by Lemma 2.2. And we see that

$$\begin{aligned} & E^Q[Z_0+H]+\alpha_{min}(Q) \\ &\ge -\|Z_0\|_\infty - \|H\|_\infty + 2M + 1 \\ &\ge -\|H\|_\infty + M + 1 \\ &\ge \|H\|_\infty + \inf_{Q\in\Lambda_\infty\cap\mathcal{M}(C)} (\alpha_{min}(Q) + \sup_{Z\in\mathcal{C}} E^Q[Z]) \\ &\ge \inf_{Q\in\mathcal{P}} \sup_{Z\in\mathcal{C}} (E^Q[Z+H]+\alpha_{min}(Q)). \end{aligned} \tag{20}$$

for $Q \in \mathcal{P}\setminus\Lambda_{2M+1}$. Hence we have

$$\begin{aligned} & \inf_{Q\in\mathcal{P}} \sup_{Z\in\mathcal{C}} (E^Q[Z+H]+\alpha_{min}(Q)) \\ &\le \inf_{Q\in\mathcal{P}\setminus\Lambda_{2M+1}} (E^Q[Z+H]+\alpha_{min}(Q)). \end{aligned} \tag{21}$$

So we verify that $f$ satisfies Condition (4).

Step2. We consider the case that $\mathcal{M}(\mathcal{C}) \cap \{Q \ll P \mid \alpha_{min}(Q) < \infty\} = \phi$. In this case, it is sufficient to show that $\inf_{Z\in\mathcal{C}} \rho(Z+H) = -\infty$.
Let $\mathcal{C}_n = \{Z \in \mathcal{C} \mid \|Z\|_\infty \le n\}$ for each $n \in \mathbb{N}$. We can easily see that $\mathcal{C}_n$ is convex and

$$\mathcal{M}(\mathcal{C}_n) \cap \{Q \ll P \mid \alpha_{min}(Q) < \infty\} = \{Q \ll P \mid \alpha_{min}(Q) < \infty\} \neq \phi. \tag{22}$$

Then using the result of Step1 we have

$$\inf_{Z\in\mathcal{C}_n} \rho(Z+H) = \sup_{Q\in\mathcal{P}} \{E^Q[-H] - (\alpha_{min}(Q) + \sup_{Z\in\mathcal{C}_n} E^Q[Z])\}. \tag{23}$$

Assume that $\inf_{Z\in\mathcal{C}} \rho(Z+H) = \gamma > -\infty$. Since $\inf_{Z\in\mathcal{C}_n} \rho(Z+H) \downarrow \gamma$ as $n \to \infty$, there exists $Q_n \in \mathcal{P}$ such that

$$\gamma - 1/n \le E^{Q_n}[-H] - (\alpha_{min}(Q_n) + \sup_{Z\in\mathcal{C}_n} E^{Q_n}[Z]) \tag{24}$$

for $n \in \mathbb{N}$. Then we see that

$$\begin{aligned}
&\alpha_{min}(Q_n) \\
&\leq E^{Q_n}[-H] - \gamma + 1/n - \sup_{Z \in \mathcal{C}_n} E^{Q_n}[Z] \\
&\leq \|H\|_\infty - \gamma + 1 - \sup_{Z \in \mathcal{C}_1} E^{Q_n}[Z] \\
&\leq (\|H\|_\infty - \gamma + 2) \vee 1.
\end{aligned} \tag{25}$$

Since the set $\{Q \ll P \mid \alpha_{min}(Q) \leq (\|H\|_\infty - \gamma + 2) \vee 1\}$ is $L^1$-weakly compact by Theorem 2.1, there exist a subsequence $\{Q_{n_k}\}$ of $\{Q_n\}_{n \in \mathbb{N}}$ and $\bar{Q} \in \{Q \ll P \mid \alpha_{min}(Q) \leq (\|H\|_\infty - \gamma + 2) \vee 1\}$ such that $Q_k \to \bar{Q}$ as $k \to \infty$.

We note that $Q \mapsto \sup_{Z \in \mathcal{C}_m} E^Q[Z]$ is lower semicontinuous for fixed $m \in \mathbb{N}$. Then we see that

$$\begin{aligned}
&\sup_{Z \in \mathcal{C}_m} E^{\bar{Q}}[Z] \\
&\leq \alpha_{min}(\bar{Q}) + \sup_{Z \in \mathcal{C}_m} E^{\bar{Q}}[Z] \\
&\leq \liminf_{k \to \infty} \alpha_{min}(Q_{n_k}) + \liminf_{k \to \infty} \sup_{Z \in \mathcal{C}_m} E^{Q_{n_k}}[Z] \\
&\leq \liminf_{k \to \infty} (\alpha_{min}(Q_{n_k}) + \sup_{Z \in \mathcal{C}_{n_k}} E^{Q_{n_k}}[Z]) \\
&\leq \liminf_{k \to \infty} (E^{Q_{n_k}}[-H] - \gamma + 1/n_k) \\
&\leq \|H\|_\infty - \gamma.
\end{aligned} \tag{26}$$

for $n_k \geq m$. Letting $m \to \infty$, we have $\sup_{Z \in \mathcal{C}} E^{\bar{Q}}[Z] \leq \|H\|_\infty - \gamma < \infty$. Then we have $\bar{Q} \in \mathcal{M}(\mathcal{C}) \cap \{Q \ll P \mid \alpha_{min}(Q) < \infty\}$. This is a contradiction. Hence we have $\inf_{Z \in \mathcal{C}} \rho(Z + H) = -\infty$. This completes the proof.

We can prove Corollary 1.4 by applying Theorem 1.3 for $\mathcal{C} = \{V(T; (0, \xi)) \mid \xi \in \mathcal{A}d\}$, since we can easily see that $\mathcal{M}(\mathcal{C}) = \mathcal{M}(S)$.

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