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**Cohomological aspects
of Magnus expansions**

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COHOMOLOGICAL ASPECTS OF MAGNUS EXPANSIONS.

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Dedicated to Professor Tatsuo Suwa on his sixtieth birthday

ABSTRACT. We generalize the notion of a Magnus expansion of a free group in order to extend each of the Johnson homomorphisms defined on a decreasing filtration of the Torelli group for a surface with one boundary component to the whole of the automorphism group of a free group $\text{Aut}(F_n)$. The extended ones are *not* homomorphisms, but satisfy an infinite sequence of coboundary relations, so that we call them *the Johnson maps*. In this paper we confine ourselves to studying the first and the second relations, which have cohomological consequences about the group $\text{Aut}(F_n)$ and the mapping class groups for surfaces. The first one means that the first Johnson map is a twisted 1-cocycle of the group $\text{Aut}(F_n)$. Its cohomology class coincides with “the unique elementary particle” of all the Morita-Mumford classes on the mapping class group for a surface [Ka1] [KM1]. The second one restricted to the mapping class group is equal to a fundamental relation among twisted Morita-Mumford classes proposed by Garoufalidis and Nakamura [GN] and established by Morita and the author [KM2]. This means we give a coherent proof of the fundamental relation. The first Johnson map gives the abelianization of the induced automorphism group IA_n of a free group in an explicit way.

INTRODUCTION

In the cohomological study of the mapping class group for a surface, or equivalently, in that of the moduli space of Riemann surfaces, the Morita-Mumford class, $e_m = (-1)^{m+1}\kappa_m$, $m \geq 1$, [Mu] [Mo1] plays an essential role. Harer [Har] established that the i -th cohomology group of the mapping class group of genus g does not depend on the genus g if $i < g/3$. This enables us to consider the stable cohomology algebra of the mapping class groups, $H^*(\mathcal{M}_\infty; \mathbb{Z})$. Recently Madsen and Weiss [MW] proved that the rational stable cohomology algebra of the mapping class groups, $H^*(\mathcal{M}_\infty; \mathbb{Q})$, is generated by the Morita-Mumford classes.

As was shown by Miller [Mi], each of the Morita-Mumford classes is indecomposable in the algebra $H^*(\mathcal{M}_\infty; \mathbb{Q})$. But, if we consider the twisted cohomology of the mapping class groups, then the Morita-Mumford class decomposes itself into

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an algebraic combination of some copies of the extended first Johnson homomorphism \tilde{k} , introduced by Morita [Mo3], by means of the intersection product of the surface [Mo4] [KM1] [KM2]. In short, the unique elementary particle for all the Morita-Mumford classes is the extended first Johnson homomorphism \tilde{k} , which is a crossed homomorphism defined on $\mathcal{M}_{g,1}$ [Mo3] and equals to $-\frac{1}{6}m_{0,3}$. It is the $(0, 3)$ -twisted (or generalized) Morita-Mumford class [KM2] in a terminology of the author's work [Ka1]. There are many nontrivial relations among different combinations of \tilde{k} . This is the reason why all the cohomology classes with trivial coefficients we obtain from \tilde{k} are just the polynomials of the Morita-Mumford classes [KM1] [KM2].

For simplicity, we often confine ourselves to the mapping class group for a compact surface of genus g with 1 boundary component, $\mathcal{M}_{g,1}$, throughout this paper. The natural action of the mapping class group on the fundamental group of the compact surface, which is isomorphic to a free group of rank $2g$, defines a homomorphism $\mathcal{M}_{g,1} \rightarrow \text{Aut}(F_{2g})$. Here F_n is a free group of rank $n \geq 2$.

Andreadakis [An] introduced a decreasing filtration of the group $\text{Aut}(F_n)$ by using the action of $\text{Aut}(F_n)$ on the lower central series of the group F_n . Its pullback to the mapping class group $\mathcal{M}_{g,1}$ coincides with the decreasing filtration $\mathcal{M}_{g,1}(p)$, $p \geq 0$, introduced by Johnson [J2]. Moreover Johnson [J1] [J2] defined a sequence of injective homomorphisms, $\tau_p : \mathcal{M}_{g,1}(p)/\mathcal{M}_{g,1}(p+1) \rightarrow H^* \otimes \mathcal{L}_{p+1}$, $p \geq 1$. The homomorphism τ_p is called the p -th Johnson homomorphism. Here H denotes the first integral homology group for the surface, H^* its dual, namely, the first integral cohomology group of the surface, and \mathcal{L}_p the p -th component of the free Lie algebra generated by H . The group $\mathcal{M}_{g,1}(0)$ is the full mapping class group $\mathcal{M}_{g,1}$, and $\mathcal{M}_{g,1}(1)$ the Torelli group $\mathcal{I}_{g,1}$. Johnson [J3] proved τ_1 induces a surjection $\mathcal{I}_{g,1}^{\text{abel}} \rightarrow \Lambda^3 H$, and that its kernel is 2-torsion. The extended one \tilde{k} coincides with τ_1 on the Torelli group $\mathcal{M}_{g,1}(1) = \mathcal{I}_{g,1}$.

The purpose of the present paper is to give a coherent point of view about all the Johnson homomorphisms, the twisted Morita-Mumford classes and their higher relations. It should be remarked that Hain [Hai] established a remarkable theory about all the Johnson homomorphisms on the Torelli groups by means of Hodge theory. On the other hand we focus our study on the automorphism group $\text{Aut}(F_n)$ instead of the mapping class group $\mathcal{M}_{g,1}$, and so all the tools we will use are contained in elementary algebra, except the Lyndon-Hochschild-Serre spectral sequences in group cohomology [HS]. As a consequence all the Johnson homomorphisms extend to the whole automorphism group $\text{Aut}(F_n)$ in a natural way. But they are *not* homomorphisms any longer. They satisfy an infinite sequence of coboundary relations, so that we call them *the Johnson maps*. In a forthcoming paper [Ka3] we will show that how far they are from true homomorphisms is measured by the Stasheff associahedrons [S] in an infinitesimal way.

The key to our consideration is the notion of the Magnus expansion of a free group. In the classical context it is an embedding of the free group into the multiplicative group of the completed tensor algebra generated by the abelianization of the free group, F_n^{abel} . Magnus [M1] constructed it by an explicit use of the generators. It can be defined also by using Fox' free differentials. See, e.g., [F]. Throughout this paper we call the classical one *the standard Magnus expansion*. It

is remarkable that Kitano [Ki] described explicitly the Johnson homomorphism τ_p on the group $\mathcal{M}_{g,1}(p)$ in terms of the standard Magnus expansion. On the other hand, Bourbaki [Bou] gave a general theory on an embedding of the free group into the multiplicative group of the completed tensor algebra. Our consideration is a refinement of Kitano's description and Bourbaki's theory. More precisely, we generalize the notion of a Magnus expansion. We call a map of the free group F_n into the completed tensor algebra with a property enough to define the Johnson homomorphisms a *Magnus expansion* (§1). The set of all the Magnus expansions in our generalized sense, Θ_n , admits two kinds of group actions. The automorphism group $\text{Aut}(F_n)$ and a certain Lie group $IA(\widehat{T})$ act on it in a natural way. The latter action is free and transitive. So, if we fix a Magnus expansion $\theta \in \Theta_n$, then we can write the action of $\text{Aut}(F_n)$ on the set Θ_n in terms of the action of $IA(\widehat{T})$. This gives us the p -th Johnson map $\tau_p^\theta : \text{Aut}(F_n) \rightarrow H^* \otimes H^{\otimes p+1}$ for each $p \geq 1$, which is equal to the original τ_p on the subgroup $\mathcal{M}_{g,1}(p)$. On the mapping class group $\mathcal{M}_{g,1}$ the first one τ_1^θ coincides with \tilde{k} , and τ_2^θ gives a fundamental relation among combinations of \tilde{k} 's. The cohomology class of τ_1^θ is equal to the Gysin image of the square of a certain twisted first cohomology class k_0 on the semi-direct product $F_n \rtimes \text{Aut}(F_n)$. The second Johnson map τ_2^θ gives a fundamental relation among twisted Morita-Mumford classes proposed by Garoufalidis and Nakamura [GN] and established by Morita and the author [KM2]. Recently Akazawa [Ak] gave an alternative proof of it by using representation theory of the symplectic group. This means we give a coherent proof of the fundamental relation (Theorem 5.5). In §5 we show a proper subset of the twisted Morita-Mumford classes extends to the automorphism group $\text{Aut}(F_n)$.

In a similar way to the Torelli group $\mathcal{T}_{g,1}$ the IA-automorphism group IA_n is defined to be the kernel of the homomorphism $\text{Aut}(F_n) \rightarrow GL_n(\mathbb{Z})$ induced by the natural action on the abelianization $H = F_n^{\text{abel}} \cong \mathbb{Z}^n$. In §6 we prove the map τ_1^θ induces an isomorphism $IA_n^{\text{abel}} \cong H^* \otimes \Lambda^2 H \cong \mathbb{Z}^{\frac{1}{2}n^2(n-1)}$ using Magnus' generators of the group IA_n [M2].

Some of the results in this paper were announced in [Ka2]. In a forthcoming paper [Ka3] we will investigate the Teichmüller space $\mathcal{T}_{g,1}$ of triples (C, P_0, v) , where C is a compact Riemann surface of genus g , $P_0 \in C$, and $v \in T_{P_0}C \setminus \{0\}$. There we will define and study a canonical map of $\mathcal{T}_{g,1}$ into Θ_{2g} , which will make the advantage of our generalization of Magnus expansions clearer.

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1. Magnus Expansions.

In our generalized sense we define the notion of a Magnus expansion of a free group of rank n , F_n . It is defined to be a group homomorphism of the free group into the multiplicative group of the completed tensor algebra \widehat{T} generated by the first homology group of the free group with a certain condition (Definition 1.1). An automorphism group $\text{IA}(\widehat{T})$ of the algebra \widehat{T} acts on the set Θ_n consisting of all the Magnus expansions as well as the automorphism group $\text{Aut}(F_n)$ of the free group. In the latter half of this section we introduce and study the group $\text{IA}(\widehat{T})$. We prove the group $\text{IA}(\widehat{T})$ acts on the set Θ_n in a free and transitive way (Theorem 1.3). Our construction of Johnson maps is based on this free and transitive action.

Let $n \geq 2$ be an integer, F_n a free group of rank n with free basis x_1, x_2, \dots, x_n

$$F_n = \langle x_1, x_2, \dots, x_n \rangle,$$

and R a commutative ring with a unit element 1. We denote by $H = H_R$ the first homology group of the free group F_n with coefficients in R

$$H = H_R := H_1(F_n; R) = F_n^{\text{abel}} \otimes_{\mathbb{Z}} R \cong R^{\oplus n}.$$

Here G^{abel} is the abelianization of a group G , $G^{\text{abel}} = G/[G, G]$. We denote

$$[\gamma] := (\gamma \bmod [F_n, F_n]) \otimes_{\mathbb{Z}} 1 \in H$$

for $\gamma \in F_n$, and $X_i := [x_i] \in H$ for i , $1 \leq i \leq n$. The set $\{X_1, X_2, \dots, X_n\}$ is an R -free basis of H .

For the rest of the paper we write simply \otimes and Hom for the tensor product and the homomorphisms over the ring R , respectively. The completed tensor algebra generated by H

$$\widehat{T} = \widehat{T}(H) := \prod_{m=0}^{\infty} H^{\otimes m}$$

is equal to the ring of noncommutative formal power series $R\langle\langle X_1, X_2, \dots, X_n \rangle\rangle$, which Bourbaki [Bou] denotes by $\widehat{A}(\{x_1, x_2, \dots, x_n\})$. The two-sided ideals

$$\widehat{T}_p := \prod_{m \geq p} H^{\otimes m}, \quad p \geq 1,$$

give a decreasing filtration of the algebra \widehat{T} . For each m we regard $H^{\otimes m}$ as a subspace of \widehat{T} in an obvious way. So we can write

$$z = \sum_{m=0}^{\infty} z_m = z_0 + z_1 + z_2 + \dots + z_m + \dots$$

for $z = (z_m) \in \widehat{T}$, $z_m \in H^{\otimes m}$. It should be remarked that the subset $1 + \widehat{T}_1$ is a subgroup of the multiplicative group of the algebra \widehat{T} , which Bourbaki [Bou] denotes by $\Gamma(\{x_1, x_2, \dots, x_n\})$ and calls *the Magnus group* over the set $\{x_1, x_2, \dots, x_n\}$. Now we can define a Magnus expansion of the free group F_n in our generalized sense.

Definition 1.1. A map $\theta : F_n \rightarrow 1 + \widehat{T}_1$ is an R -valued Magnus expansion of the free group F_n , if

- (1) $\theta : F_n \rightarrow 1 + \widehat{T}_1$ is a group homomorphism, and
- (2) $\theta(\gamma) \equiv 1 + [\gamma] \pmod{\widehat{T}_2}$ for any $\gamma \in F_n$.

We write $\theta(\gamma) = \sum_{m=0}^{\infty} \theta_m(\gamma)$, $\theta_m(\gamma) \in H^{\otimes m}$. The m -th component $\theta_m : F_n \rightarrow H^{\otimes m}$ is a map, but *not* a group homomorphism. The condition (2) is equivalent to the two conditions $\theta_0(\gamma) = 1$ and $\theta_1(\gamma) = [\gamma]$ for any $\gamma \in F_n$.

From the universal mapping property of the free group F_n , for any $\xi_i \in \widehat{T}_2$, $1 \leq i \leq n$, there exists a unique Magnus expansion θ satisfying $\theta(x_i) = 1 + X_i + \xi_i$ for each i . In other words, when we denote by $\Theta_n = \Theta_{n,R}$ the set of all the R -valued Magnus expansions, we have a bijection

$$\Theta_n \cong (\widehat{T}_2)^n, \quad \theta \mapsto (\theta(x_i) - 1 - X_i). \quad (1.1)$$

The standard Magnus expansion Magnus [M1] introduced corresponds to $(0, 0, \dots, 0) \in (\widehat{T}_2)^n$, which we denote

$$\text{std} : F_n \rightarrow 1 + \widehat{T}_1, \quad x_i \mapsto 1 + X_i.$$

We denote by $\text{Aut}(\widehat{T})$ the group of all the filtration-preserving R -algebra automorphisms of the algebra \widehat{T} . Here an R -algebra automorphism U of \widehat{T} is defined to be *filtration-preserving* if $U(\widehat{T}_p) = \widehat{T}_p$ for each $p \geq 1$. It is easy to see whether an R -algebra endomorphism of \widehat{T} is a filtration-preserving automorphism or not.

Lemma 1.2. An R -algebra endomorphism U of \widehat{T} is a filtration-preserving R -algebra automorphism, $U \in \text{Aut}(\widehat{T})$, if and only if it satisfies the conditions

- (i) $U(\widehat{T}_p) \subset \widehat{T}_p$ for each $p \geq 1$, and
- (ii) the induced endomorphism $|U|$ of $\widehat{T}_1/\widehat{T}_2 = H$ is an isomorphism.

Proof. We may suppose the R -algebra endomorphism U satisfies the condition (i). We have $\text{Ker } U = \text{Ker} \left(U|_{\widehat{T}_1} \right)$ and $\text{Coker } U = \text{Coker} \left(U|_{\widehat{T}_1} \right)$, since the algebra endomorphism U induces the identity map on $R = \widehat{T}/\widehat{T}_1$. Hence the homomorphism of short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \widehat{T}_2 & \longrightarrow & \widehat{T}_1 & \longrightarrow & H & \longrightarrow & 0 \\ & & \downarrow U & & \downarrow U & & \downarrow |U| & & \\ 0 & \longrightarrow & \widehat{T}_2 & \longrightarrow & \widehat{T}_1 & \longrightarrow & H & \longrightarrow & 0 \end{array}$$

induces a long exact sequence

$$\begin{aligned} 0 \rightarrow \text{Ker} \left(U|_{\widehat{T}_2} \right) \rightarrow \text{Ker } U \rightarrow \text{Ker} |U| \rightarrow \text{Coker} \left(U|_{\widehat{T}_2} \right) \\ \rightarrow \text{Coker } U \rightarrow \text{Coker} |U| \rightarrow 0, \quad (\text{exact}). \end{aligned} \quad (1.2)$$

Here we prove

$$\text{Coker} \left(U|_{\widehat{T}_2} \right) = 0 \quad \text{if } \text{Coker}|U| = 0. \quad (1.3)$$

It suffices to show that, under the condition $|U|$ is surjective, for any given $w = (w_m) \in \widehat{T}_2$, there exists a solution $z = (z_m) \in \widehat{T}_2$, $z_m \in H^{\otimes m}$, for the equation

$$Uz = w. \quad (1.4)$$

The m -th component is

$$\begin{aligned} |U|^{\otimes 2} z_2 &= w_2, \quad \text{for } m = 2, \\ |U|^{\otimes m} z_m + \text{terms in } z_2, z_3, \dots, z_{m-1} &= w_m, \quad \text{for } m \geq 3. \end{aligned}$$

For each m , $|U|^{\otimes m}$ is surjective. Hence we can find a tensor z_m satisfying the equation by induction on degree m . This proves (1.3).

Now suppose $U \in \text{Aut}(\widehat{T})$. Then the sequence (1.2) implies $\text{Coker}|U| = 0$ and $\text{Ker}|U| \cong \text{Coker} \left(U|_{\widehat{T}_2} \right)$. The latter one vanishes from (1.3). Hence $|U|$ is an isomorphism, that is, U satisfies the condition (ii).

Conversely suppose U is an R -algebra endomorphism satisfying the conditions (i) and (ii). Using the sequence (1.2) and (1.3), we obtain $\text{Coker} U = 0$. Let $z = (z_m)$ be an element of $\text{Ker} \left(U|_{\widehat{T}_2} \right)$. The equation $Uz = 0$ is equivalent to

$$\begin{aligned} |U|^{\otimes 2} z_2 &= 0 \in H^{\otimes 2}, \\ |U|^{\otimes m} z_m + \text{terms in } z_2, z_3, \dots, z_{m-1} &= 0 \in H^{\otimes m}, \quad \text{for } m \geq 3. \end{aligned}$$

We deduce $z_m = 0$ by induction on degree m , since $|U|^{\otimes m}$ is an isomorphism for each m . Hence $U : \widehat{T} \rightarrow \widehat{T}$ is an R -algebra isomorphism. If w in the equation (1.4) satisfies $w_2 = w_3 = \dots = w_{p-1} = 0$, then the solution z also satisfies $z_2 = z_3 = \dots = z_{p-1} = 0$ because $|U|^{\otimes m}$ is an isomorphism. This means $U^{-1}(\widehat{T}_p) \subset \widehat{T}_p$ for each $p \geq 2$. It is also proved $U^{-1}(\widehat{T}_1) \subset \widehat{T}_1$ in a similar way. Consequently U is a filtration-preserving R -algebra automorphism of the algebra \widehat{T} , that is, $U \in \text{Aut}(\widehat{T})$.

This completes the proof of the lemma. \blacksquare

As is the lemma, we denote by $|U| \in \text{GL}(H)$ the automorphism of $H = \widehat{T}_1/\widehat{T}_2$ induced by the automorphism $U \in \text{Aut}(\widehat{T})$. This defines a group homomorphism

$$|\cdot| : \text{Aut}(\widehat{T}) \rightarrow \text{GL}(H), \quad U \mapsto |U|,$$

whose kernel we denote by

$$\text{IA}(\widehat{T}) := \text{Ker}|\cdot| \subset \text{Aut}(\widehat{T}). \quad (1.5)$$

Any element $A \in \text{GL}(H)$ can be regarded as a filtration-preserving automorphism of \widehat{T} by $A(z_m) = (A^{\otimes m} z_m)$, $z_m \in H^{\otimes m}$. Hence we have a semi-direct product decomposition

$$\text{Aut}(\widehat{T}) = \text{IA}(\widehat{T}) \rtimes \text{GL}(H). \quad (1.6)$$

From Lemma 1.2 we have a natural bijection

$$E : \text{IA}(\widehat{T}) \rightarrow \text{Hom}(H, \widehat{T}_2), \quad U \mapsto U_H - 1_H. \quad (1.7)$$

We often identify $\text{IA}(\widehat{T})$ and $\text{Hom}(H, \widehat{T}_2)$ by the map E .

The group $\text{IA}(\widehat{T})$ acts on the set Θ_n of all the Magnus expansions in a natural way. If $U \in \text{IA}(\widehat{T})$ and $\theta \in \Theta_n$, then the composite $U \circ \theta : F_n \xrightarrow{\theta} 1 + \widehat{T}_1 \xrightarrow{U} 1 + \widehat{T}_1$ is also a Magnus expansion. To study the action, recall the completed group ring $\widehat{R[F_n]}$ of the group F_n . By definition, it is the completion of the group ring $R[F_n]$ with respect to the augmentation ideal $I_R[F_n] = \{\sum_{\gamma \in F_n} r_\gamma \gamma \in R[F_n]; \sum r_\gamma = 0\}$

$$\widehat{R[F_n]} := \varprojlim_{m \rightarrow +\infty} R[F_n]/I_R[F_n]^m,$$

which has a decreasing filtration of the ideals

$$I_R[\widehat{R[F_n]}]^p := \varprojlim_{p \leq m \rightarrow +\infty} I_R[F_n]^p/I_R[F_n]^m, \quad p \geq 1.$$

Any Magnus expansion $\theta \in \Theta_n$ induces an R -algebra homomorphism $\theta : R[F_n] \rightarrow \widehat{T}$ in an obvious way. Since $\theta(I_R[F_n]) \subset \widehat{T}_1$, we obtain an R -algebra homomorphism $\theta : \widehat{R[F_n]} \rightarrow \widehat{T}$, which maps the ideal $I_R[\widehat{R[F_n]}]^p$ into \widehat{T}_p for each $p \geq 1$.

Our construction of Johnson maps of the automorphism group $\text{Aut}(F_n)$ is based on

Theorem 1.3. (1) For any $\theta \in \Theta_n$ the homomorphism

$$\theta : \widehat{R[F_n]} \rightarrow \widehat{T}$$

is an R -algebra isomorphism, which maps $I_R[\widehat{R[F_n]}]^p$ onto \widehat{T}_p for each $p \geq 1$.

(2) If θ' and $\theta'' \in \Theta_n$, then there exists a unique $U \in \text{IA}(\widehat{T})$ such that

$$\theta'' = U \circ \theta' \in \Theta_n.$$

In other words, the action of the group $\text{IA}(\widehat{T})$ on the set Θ_n is free and transitive.

Proof. First we prove the assertion (1) for the standard Magnus expansion $\text{std} : F_n \rightarrow 1 + \widehat{T}_1$, $\text{std}(x_i) = 1 + X_i$, $1 \leq i \leq n$. The R -algebra homomorphism $\kappa : \widehat{T} \rightarrow \widehat{R[F_n]}$ given by $\kappa(X_i) = x_i - 1$ maps \widehat{T}_p into $I_R[\widehat{R[F_n]}]^p$ for each p . Now we have

$$\begin{aligned} \text{std} \circ \kappa(X_i) &= \text{std}(x_i - 1) = 1 + X_i - 1 = X_i, \quad \text{and} \\ \kappa \circ \text{std}(x_i) &= \kappa(X_i) = 1 + x_i - 1 = x_i. \end{aligned}$$

These imply $\text{std} \circ \kappa = 1_{\widehat{T}}$ and $\kappa \circ \text{std} = 1_{\widehat{R[F_n]}}$, respectively. Hence $\text{std} : \widehat{R[F_n]} \rightarrow \widehat{T}$ is an R -algebra isomorphism, which maps $I_R[\widehat{R[F_n]}]^p$ onto \widehat{T}_p for each p .

Next we consider an arbitrary Magnus expansion $\theta \in \Theta_n$. The R -algebra endomorphism $\theta \circ \kappa : \widehat{T} \xrightarrow{\kappa} \widehat{R[F_n]} \xrightarrow{\theta} \widehat{T}$ satisfies the conditions in (i) and (ii) in Lemma 1.2. In fact, $|\theta \circ \kappa| = 1_H$. Hence $\theta \circ \kappa \in \text{IA}(\widehat{T})$, which we denote by U . Especially $\theta = U \circ \text{std} : \widehat{R[F_n]} \rightarrow \widehat{T}$ is an R -algebra isomorphism, which maps $\widehat{I_R[F_n]^p}$ onto \widehat{T}_p . This implies the action of $\text{IA}(\widehat{T})$ on Θ_n is transitive.

Finally we prove the action is free. Suppose $U \in \text{IA}(\widehat{T})$ satisfies $U \circ \text{std} = \text{std} : F_n \rightarrow 1 + \widehat{T}_1$. Then we have $U \circ \text{std} = \text{std} : \widehat{R[F_n]} \xrightarrow{\cong} \widehat{T}$, and so $U = U \circ \text{std} \circ \kappa = \text{std} \circ \kappa = 1_{\widehat{T}}$, as was to be shown.

This completes the proof of the theorem. \blacksquare

We conclude the section by writing down the group structure on the set $\text{Hom}(H, \widehat{T}_2) \times \text{GL}(H)$ induced by the decomposition (1.6) and the bijection (1.7) in low degree. We denote $((u, A)) := (E^{-1}u) \circ A \in \text{Aut}(\widehat{T})$ for $u \in \text{Hom}(H, \widehat{T}_2)$ and $A \in \text{GL}(H)$. We have

$$((u, A))a = Aa + \sum_{m=1}^{\infty} u_m(Aa)$$

for any $a \in H$. By straightforward computation we obtain

Lemma 1.4. *Suppose $((w, C)) = ((u, A))((v, B)) \in \text{Aut}(\widehat{T})$ for (u, A) , (v, B) and $(w, C) \in \text{Hom}(H, \widehat{T}_2) \times \text{GL}(H)$. Then we have*

$$\begin{aligned} C &= AB, \\ w_1 &= u_1 + Av_1, \quad \text{and} \\ w_2 &= u_2 + (u_1 \otimes 1 + 1 \otimes u_1)Av_1 + Av_2, \end{aligned}$$

where u_p , v_p and $w_p \in \text{Hom}(H, H^{\otimes(p+1)})$ are the p -th components of u , v and w , respectively, and

$$Av_p := \overbrace{(A \otimes \cdots \otimes A)}^{p+1} v_p A^{-1} \in \text{Hom}(H, H^{\otimes(p+1)}).$$

Proof. For any $a \in H$ we have

$$\begin{aligned} Ca + w_1(Ca) + w_2(Ca) &\equiv ((w, C))a = ((u, A))((v, B))a \\ &\equiv ((u, A))(Ba + v_1(Ba) + v_2(Ba)) \\ &\equiv ABa + u_1(ABa) + u_2(ABa) \\ &\quad + (A \otimes A)v_1(Ba) + (u_1 \otimes 1 + 1 \otimes u_1)(A \otimes A)v_1(Ba) \\ &\quad + (A \otimes A \otimes A)v_2(Ba) \\ &= ABa + (u_1 + Av_1)(ABa) + (u_2 + (u_1 \otimes 1 + 1 \otimes u_1)Av_1 + Av_2)(ABa) \end{aligned}$$

modulo \widehat{T}_4 , as was to be shown. \blacksquare

2. Johnson Maps.

As was shown in Theorem 1.3(2), the group $\text{IA}(\widehat{T})$ acts on the set Θ_n of all the R -valued Magnus expansions of the free group F_n in a free and transitive way. We denote by H^* the dual of H , $H^* := \text{Hom}(H, R)$. We often identify

$$\text{IA}(\widehat{T}) \stackrel{E}{\cong} \text{Hom}(H, \widehat{T}_2) = \prod_{p=1}^{\infty} \text{Hom}(H, H^{\otimes(p+1)}) = \prod_{p=1}^{\infty} H^* \otimes H^{\otimes(p+1)}. \quad (2.1)$$

Now we consider the automorphism group of the group F_n , $\text{Aut}(F_n)$. It acts on the set Θ_n in a natural way. In fact, we define

$$\varphi \cdot \theta := |\varphi| \circ \theta \circ \varphi^{-1}$$

for $\varphi \in \text{Aut}(F_n)$ and $\theta \in \Theta_n$. Here $|\varphi| \in \text{GL}(H)$ is the induced map on $H = H_1(F_n; R)$ by the automorphism φ . From the free and transitive action of $\text{IA}(\widehat{T})$, there exists a unique automorphism $\tau^\theta(\varphi) \in \text{IA}(\widehat{T})$ such that

$$\varphi \cdot \theta = \tau^\theta(\varphi)^{-1} \circ \theta \in \Theta_n. \quad (2.2)$$

If we fix a Magnus expansion $\theta \in \Theta_n$, it defines a map

$$\tau^\theta : \text{Aut}(F_n) \rightarrow \text{IA}(\widehat{T}), \quad \varphi \mapsto \tau^\theta(\varphi),$$

which we call *the total Johnson map induced by the Magnus expansion θ* . Immediately from (2.2) we have a commutative diagram

$$\begin{array}{ccc} \widehat{R[F_n]} & \xrightarrow{\theta} & \widehat{T} \\ \varphi \downarrow & \tau^\theta(\varphi) \circ |\varphi| \downarrow & \\ \widehat{R[F_n]} & \xrightarrow{\theta} & \widehat{T}. \end{array} \quad (2.3)$$

Hence we obtain

$$\tau^\theta(\varphi\psi) = \tau^\theta(\varphi) \circ |\varphi| \circ \tau^\theta(\psi) \circ |\varphi|^{-1} \quad (2.4)$$

for any φ and $\psi \in \text{Aut}(F_n)$.

Under the identification (2.1), for each $p \geq 1$, we define *the p -th Johnson map induced by the R -valued Magnus expansion θ*

$$\tau_p^\theta : \text{Aut}(F_n) \rightarrow H^* \times H^{\otimes(p+1)}, \quad \varphi \mapsto \tau_p^\theta(\varphi)$$

by the p -th component of the total Johnson map $E\tau^\theta$. We have

$$E\tau^\theta(\varphi) = \sum_{p=1}^{\infty} \tau_p^\theta(\varphi) \in \text{Hom}(H, \widehat{T}_2) = \prod_{p=1}^{\infty} H^* \otimes H^{\otimes(p+1)}$$

for $\varphi \in \text{Aut}(F_n)$.

The map τ_p^θ is not a group homomorphism. The relation (2.4) means an infinite sequence of coboundary relations. In this paper we confine ourselves to studying

the first and the second relations, which have cohomological consequences about the group $\text{Aut}(F_n)$ and the mapping class groups for surfaces. In the case $p = 1$ and 2, from Lemma 1.4, we have

$$\begin{aligned}\tau_1^\theta(\varphi\psi) &= \tau_1^\theta(\varphi) + |\varphi|\tau_1^\theta(\psi) \\ \tau_2^\theta(\varphi\psi) &= \tau_2^\theta(\varphi) + (\tau_1^\theta(\varphi) \otimes 1 + 1 \otimes \tau_1^\theta(\varphi))|\varphi|\tau_1^\theta(\psi) + |\varphi|\tau_2^\theta(\psi)\end{aligned}$$

for any φ and $\psi \in \text{Aut}(F_n)$. In the succeeding sections we will show these elementary formulae have some significant consequences in the cohomology of the group $\text{Aut}(F_n)$. Throughout this paper we denote by $C^*(G; M)$ the normalized standard complex of a group G with values in a G -module M , and use the Alexander-Whitney cup product $\cup : C^*(G; M_1) \otimes C^*(G; M_2) \rightarrow C^*(G; M_1 \otimes M_2)$. For details, see [HS] ch.II. The formulae are equivalent to

Lemma 2.1.

$$-d\tau_1^\theta = 0 \in C^*(\text{Aut}(F_n); H^* \otimes H^{\otimes 2}), \quad (2.5)$$

$$-d\tau_2^\theta = (\tau_1^\theta \otimes 1 + 1 \otimes \tau_1^\theta) \cup \tau_1^\theta \in C^*(\text{Aut}(F_n); H^* \otimes H^{\otimes 3}). \quad (2.6)$$

In (2.6) we drop the composite map $\text{Hom}(H^{\otimes 2}, H^{\otimes 3}) \otimes \text{Hom}(H, H^{\otimes 2}) \rightarrow \text{Hom}(H, H^{\otimes 3}) = H^* \otimes H^{\otimes 3}$, $f \otimes g \mapsto f \circ g$, for simplicity.

From (2.5) the map τ_1^θ is a 1-cocycle of the group $\text{Aut}(F_n)$ with values in the $\text{Aut}(F_n)$ -module $H^* \otimes H^{\otimes 2}$. The cohomology class $[\tau_1^\theta] \in H^1(\text{Aut}(F_n); H^* \otimes H^{\otimes 2})$ is independent of the choice of a Magnus expansion θ . It can be proved directly from Theorem 1.3(2). As will be shown in §4, the class $[\tau_1^\theta]$ is the Gysin image of a certain cohomology class in $H^2(F_n \rtimes \text{Aut}(F_n); H^{\otimes 2})$ independent of the choice of a Magnus expansion.

Lemma 2.2. *We have*

$$\tau_1^\theta(\varphi)|\varphi|[\gamma] = \theta_2(\varphi(\gamma)) - |\varphi|^{\otimes 2}\theta_2(\gamma) \quad (2.7)$$

$$\tau_1^\theta(\varphi)[\gamma] = \theta_2(\gamma) - |\varphi|^{\otimes 2}\theta_2(\varphi^{-1}(\gamma)) \quad (2.8)$$

for any $\gamma \in F_n$ and $\varphi \in \text{Aut}(F_n)$.

In fact, from (2.3), we have

$$\begin{aligned}1 + [\varphi(\gamma)] + \theta_2(\varphi(\gamma)) &\equiv \theta(\varphi(\gamma)) = \tau^\theta(\varphi)|\varphi|\theta(\gamma) \\ &\equiv \tau^\theta(\varphi)(1 + |\varphi|[\gamma] + |\varphi|^{\otimes 2}\theta_2(\gamma)) \\ &\equiv 1 + |\varphi|[\gamma] + \tau_1^\theta(\varphi)|\varphi|[\gamma] + |\varphi|^{\otimes 2}\theta_2(\gamma)\end{aligned}$$

modulo \widehat{T}_3 .

Finally we compute the Johnson maps on the inner automorphisms of the group F_n . The image of the homomorphism

$$\iota : F_n \rightarrow \text{Aut}(F_n), \quad \gamma \mapsto (\iota(\gamma) : \delta \mapsto \gamma\delta\gamma^{-1}) \quad (2.9)$$

is, by definition, the inner automorphism group of F_n , and often denoted by $\text{Inn}(F_n)$. The quotient $\text{Out}(F_n) := \text{Aut}(F_n)/\text{Inn}(F_n)$ is called the outer automorphism group of F_n .

Lemma 2.3. For $p \geq 1$, $\gamma \in F_n$ and $a \in H$ we have

$$\tau_p^\theta(\iota(\gamma))a = \theta_p(\gamma)a + \sum_{j=1}^m \sum_{\substack{q_0+q_1+\dots+q_j=p \\ q_0 \geq 0, q_1, \dots, q_j \geq 1}} (-1)^j \theta_{q_0}(\gamma) a \theta_{q_1}(\gamma) \cdots \theta_{q_j}(\gamma). \quad (2.10)$$

Proof. Recall

$$\theta(\gamma)^{-1} = \left(1 + \sum_{s=1}^{\infty} \theta_s(\gamma)\right)^{-1} = 1 + \sum_{j=1}^{\infty} (-1)^j \left(\sum_{s=1}^{\infty} \theta_s(\gamma)\right)^j.$$

The map $U_\gamma : \widehat{T} \rightarrow \widehat{T}$, $z \mapsto \theta(\gamma)z\theta(\gamma)^{-1}$, is an element of $\text{Aut}(\widehat{T})$. Now, since $|\iota(\gamma)| = 1$, we have

$$\begin{aligned} \tau^\theta(\iota(\gamma))a &= U_\gamma a = \theta(\gamma)a\theta(\gamma)^{-1} \\ &= \left(\sum_{q_0=0}^{\infty} \theta_{q_0}(\gamma)\right) a \left(1 + \sum_{j=1}^{\infty} (-1)^j \sum_{q_1, \dots, q_j=1}^{\infty} \theta_{q_1}(\gamma) \cdots \theta_{q_j}(\gamma)\right). \end{aligned}$$

Taking the $(p+1)$ -st components in $H^{\otimes(p+1)}$, we obtain the lemma. ■

In the case $p = 1$ and 2 , we have

$$\tau_1^\theta(\iota(\gamma))a = [\gamma]a - a[\gamma], \quad (2.11)$$

$$\tau_2^\theta(\iota(\gamma))a = \theta_2(\gamma)a - a\theta_2(\gamma) + a[\gamma][\gamma] - [\gamma]a[\gamma]. \quad (2.12)$$

3. Lower Central Series.

In this section we suppose that the natural ring homomorphism $\nu \in \mathbb{Z} \mapsto \nu \cdot 1 \in R$ is injective. We denote by $\Gamma_m = \Gamma_m(F_n)$, $m \geq 1$, the lower central series of the group F_n

$$\Gamma_1 := F_n, \quad \Gamma_{m+1} := [\Gamma_m, F_n], \quad m \geq 1.$$

The group $\text{Aut}(F_n)$ acts on the subgroup Γ_m and the quotient Γ_1/Γ_m in a natural way. S. Andreadakis [An] introduced a decreasing filtration $\{A(m)\}_{m=1}^\infty$ of $\text{Aut}(F_n)$ by

$$A(m) := \text{Ker}(\text{Aut}(F_n) \rightarrow \text{Aut}(\Gamma_1/\Gamma_{m+1})), \quad m \geq 0.$$

In [An] he wrote K_m for $A(m)$. The m -th Johnson homomorphism $\tau_m = \tau_m^J$ describes the quotient $A(m)/A(m+1)$, which was introduced by D. Johnson [J1]. The homomorphism τ_m can be regarded as an embedding of the quotient $A(m)/A(m+1)$ into the module $H^* \otimes H^{\otimes(m+1)}$. We prove the restriction of τ_m^θ to $A(m)$ coincides with τ_m (Theorem 3.1). Especially $\tau_m^\theta|_{A(m)}$ is a homomorphism independent of the choice of a Magnus expansion θ .

Choose a Magnus expansion $\theta \in \Theta_n$. As was proved by Magnus [M3], we have

$$\theta^{-1}(1 + \widehat{T}_m) = \Gamma_m \tag{3.1}$$

for each $m \geq 1$. See [Bou] ch. 2, §5, no. 4, Theorem 2. This implies the m -th component θ_m gives an injective homomorphism

$$\theta_m : \Gamma_m/\Gamma_{m+1} \hookrightarrow H^{\otimes m}.$$

Here the restriction $\theta_m|_{\Gamma_m}$ is independent of the choice of θ . We prove it by induction on m . From Definition 1.1, θ_1 is independent of θ . Assume $m \geq 2$. We have $\Gamma_m = [\Gamma_{m-1}, \Gamma_1]$. Let $\gamma \in \Gamma_{m-1}$. From (3.1) follows $\theta(\gamma) \equiv 1 + \theta_{m-1}(\gamma) \pmod{\widehat{T}_m}$. For $\delta \in \Gamma_1$, we have, modulo \widehat{T}_{m+1} ,

$$\begin{aligned} & \theta(\gamma)\theta(\delta)\theta(\gamma^{-1})\theta(\delta^{-1}) \\ &= \theta(\gamma)(1 + (\theta(\delta) - 1))\theta(\gamma^{-1})\theta(\delta^{-1}) \\ &= \theta(\delta^{-1}) + \theta(\gamma)(\theta(\delta) - 1)\theta(\gamma^{-1})\theta(\delta^{-1}) \\ &\equiv \theta(\delta^{-1}) + (1 + \theta_{m-1}(\gamma))(\theta(\delta) - 1)(1 - \theta_{m-1}(\gamma))\theta(\delta^{-1}) \\ &\equiv \theta(\delta^{-1}) + (\theta(\delta) - 1)\theta(\delta^{-1}) + \theta_{m-1}(\gamma)(\theta(\delta) - 1)\theta(\delta^{-1}) - (\theta(\delta) - 1)\theta_{m-1}(\gamma)\theta(\delta^{-1}) \\ &\equiv 1 + \theta_{m-1}(\gamma)[\delta] - [\delta]\theta_{m-1}(\gamma). \end{aligned}$$

Hence

$$\theta_m(\gamma\delta\gamma^{-1}\delta^{-1}) = \theta_{m-1}(\gamma)[\delta] - [\delta]\theta_{m-1}(\gamma). \tag{3.2}$$

From the inductive assumption $\theta_{m-1}(\gamma)$ is independent of the choice of θ . This completes the induction.

We denote the image $\theta_m(\Gamma_m/\Gamma_{m+1})$ by $\mathcal{L}_m \subset H^{\otimes m}$. We identify Γ_m/Γ_{m+1} and \mathcal{L}_m by the isomorphism θ_m . As is known, the sum $\bigoplus_{m=1}^\infty \mathcal{L}_m \subset \widehat{T}$ is a Lie subalgebra of the associative algebra \widehat{T} , and naturally isomorphic to the free Lie algebra $\mathcal{L}(H_{\mathbb{Z}})$ generated by $H_{\mathbb{Z}} = F_n^{\text{abel}}$. See [Bou] loc. cit.

Now we recall the definition of the Johnson homomorphisms [J1][J2]. If $\varphi \in A(m)$ and $\gamma \in F_n$, then we have $\gamma^{-1}\varphi(\gamma) \in \Gamma_{m+1}$. This allows us to consider

$$\tau_m(\varphi, \gamma) := \gamma^{-1}\varphi(\gamma) \pmod{\Gamma_{m+2}} \in \mathcal{L}_{m+1}.$$

It is easy to prove the map $\gamma \in F_n \mapsto \tau_m(\varphi, \gamma) \in \mathcal{L}_{m+1}$ is a group homomorphism. Hence it can be regarded as an element $\tau_m(\varphi) \in \text{Hom}_{\mathbb{Z}}(H_{\mathbb{Z}}, \mathcal{L}_{m+1})$, and induces a map

$$\tau_m = \tau_m^J : A(m) \rightarrow \text{Hom}_{\mathbb{Z}}(H_{\mathbb{Z}}, \mathcal{L}_{m+1}), \quad \varphi \mapsto \tau_m(\varphi).$$

Moreover one can easily prove τ_m is a group homomorphism. The homomorphism τ_m is, by definition, the m -th Johnson homomorphism [J2]. Immediately from the definition we have

$$\text{Ker } \tau_m = A(m+1), \tag{3.3}$$

so that it can be regarded as an embedding

$$\tau_m : A(m)/A(m+1) \hookrightarrow \text{Hom}_{\mathbb{Z}}(H_{\mathbb{Z}}, \mathcal{L}_{m+1}).$$

Since the integers \mathbb{Z} is a subring of R , we may regard $\text{Hom}_{\mathbb{Z}}(H_{\mathbb{Z}}, \mathcal{L}_{m+1})$ as a \mathbb{Z} -submodule $H^* \otimes H^{\otimes(m+1)}$ in an obvious way. Then

Theorem 3.1. *We have*

$$\tau_m = \tau_m^\theta|_{A(m)} : A(m) \rightarrow H^* \otimes H^{\otimes(m+1)}$$

for each $m \geq 1$. Especially the restriction $\tau_m^\theta|_{A(m)}$ is a group homomorphism independent of the choice of the Magnus expansion θ .

Proof. We prove the theorem by induction on $m \geq 1$. For any $\varphi \in A(m) \subset A(1)$ we have $|\varphi| = 1$. If $m = 1$, then we have, modulo \widehat{T}_3 , $\theta(\varphi(\gamma)) = \tau^\theta(\varphi)\theta(\gamma) \equiv \theta(\gamma) + \tau_1^\theta(\varphi)[\gamma]$, and so

$$\begin{aligned} 1 + \tau_1(\varphi, \gamma) &= 1 + \theta_2(\gamma^{-1}\varphi(\gamma)) \equiv \theta(\gamma^{-1}\varphi(\gamma)) = \theta(\gamma)^{-1}\theta(\varphi(\gamma)) \\ &\equiv \theta(\gamma)^{-1}(\theta(\gamma) + \tau_1^\theta(\varphi)[\gamma]) \equiv 1 + \tau_1^\theta(\varphi)[\gamma]. \end{aligned}$$

This implies we have $\tau_1^\theta|_{A(2)} = 0$ from (3.3).

Suppose $m \geq 2$. From the inductive assumption and (3.3) we have $\tau_1^\theta(\varphi) = \dots = \tau_{m-1}^\theta(\varphi) = 0$ for any $\varphi \in A(m)$, and so $\theta(\varphi(\gamma)) \equiv \theta(\gamma) + \tau_m^\theta(\varphi)[\gamma] \pmod{\widehat{T}_{m+2}}$. Hence we have

$$\begin{aligned} 1 + \tau_m(\varphi, \gamma) &= 1 + \theta_{m+1}(\gamma^{-1}\varphi(\gamma)) \equiv \theta(\gamma^{-1}\varphi(\gamma)) = \theta(\gamma)^{-1}\theta(\varphi(\gamma)) \\ &\equiv \theta(\gamma)^{-1}(\theta(\gamma) + \tau_m^\theta(\varphi)[\gamma]) \equiv 1 + \tau_m^\theta(\varphi)[\gamma] \end{aligned}$$

modulo \widehat{T}_{m+2} . This completes the induction and the proof of the theorem. \blacksquare

4. Twisted Cohomology Classes.

In this section we introduce two series of twisted cohomology classes

$$h_p \in H^p(\mathrm{Aut}(F_n); H^* \otimes H^{\otimes(p+1)}) \quad \text{and} \quad \bar{h}_p \in H^p(\mathrm{Aut}(F_n); H^{\otimes p})$$

for $p \geq 1$ by an analogous construction to the Morita-Mumford classes on the mapping class groups for surfaces [Mu] [Mo1]. Restricted to the mapping class group $\mathcal{M}_{g,1}$ of genus g with 1 boundary component, they coincide with the twisted Morita-Mumford classes [Ka] [KM1] [KM2]

$$(p+2)! h_p|_{\mathcal{M}_{g,1}} = -m_{0,p+2} \in H^p(\mathcal{M}_{g,1}; H^{\otimes(p+2)}), \quad \text{and} \quad (5.8)$$

$$p! \bar{h}_p|_{\mathcal{M}_{g,1}} = m_{1,p} \in H^p(\mathcal{M}_{g,1}; H^{\otimes p}), \quad (5.18)$$

as will be shown in §5. We prove a suitable algebraic combination of p copies of the 1-cocycle τ_1^θ introduced in §2 represents the cohomology class h_p for each $p \geq 1$ (Theorem 4.1). In the case $p = 1$, τ_1^θ represents the class h_1 . Furthermore we describe some contraction formulae deduced from the relation (2.6).

In order to define the cohomology classes h_p and \bar{h}_p , we consider the semi-direct product

$$\overline{A}_n := F_n \rtimes \mathrm{Aut}(F_n)$$

and the map

$$k_0 : \overline{A}_n \rightarrow H, \quad (\gamma, \varphi) \mapsto [\gamma], \quad (4.1)$$

introduced in [Mo3]. The group \overline{A}_n is, by definition, the product set $F_n \times \mathrm{Aut}(F_n)$ with the group law

$$(\gamma_1, \varphi_1)(\gamma_2, \varphi_2) := (\gamma_1\varphi_1(\gamma_2), \varphi_1\varphi_2), \quad (\gamma_i \in F_n, \varphi_i \in \mathrm{Aut}(F_n)).$$

We often write simply $\gamma\varphi$ for (γ, φ) . It is easy to prove k_0 satisfies the cocycle condition. We write also k_0 for the cohomology class $[k_0] \in H^1(\overline{A}_n; H)$. Consider the $(p+1)$ -st power of k_0

$$k_0^{\otimes(p+1)} \in H^{p+1}(\overline{A}_n; H^{\otimes(p+1)})$$

for each $p \geq 0$.

The group \overline{A}_n admits a group extension

$$F_n \xrightarrow{\iota} \overline{A}_n \xrightarrow{\pi} \mathrm{Aut}(F_n) \quad (4.2)$$

given by $\iota(\gamma) = \gamma$ and $\pi(\gamma\varphi) = \varphi$ for $\gamma \in F_n$ and $\varphi \in \mathrm{Aut}(F_n)$. It induces the Gysin map

$$\pi_\sharp : H^{p+1}(\overline{A}_n; H^{\otimes(p+1)}) \rightarrow H^p(\mathrm{Aut}(F_n); H^* \otimes H^{\otimes(p+1)}).$$

Here we identify

$$H^1(F_n; H^{\otimes(p+1)}) = \mathrm{Hom}(H, H^{\otimes(p+1)}) = H^* \otimes H^{\otimes(p+1)} \quad (4.3)$$

in a natural way. For each $p \geq 1$ we define

$$h_p := \pi_{\sharp}(k_0^{\otimes(p+1)}) \in H^p(\text{Aut}(F_n); H^* \otimes H^{\otimes(p+1)}). \quad (4.4)$$

In the case $p = 0$ we have

$$\pi_{\sharp}(k_0) = \iota^* k_0 = 1_H \in H^0(\text{Aut}(F_n); H^* \otimes H). \quad (4.5)$$

Contracting the coefficients by the $\text{GL}(H)$ -homomorphism

$$r_p : H^* \otimes H^{\otimes(p+1)} \rightarrow H^{\otimes p}, \quad f \otimes v_0 \otimes v_1 \otimes \cdots \otimes v_p \mapsto f(v_0)v_1 \otimes \cdots \otimes v_p, \quad (4.6)$$

we define

$$\bar{h}_p := r_{p*}(h_p) \in H^p(\text{Aut}(F_n); H^{\otimes p}). \quad (4.7)$$

We introduce a $\text{GL}(H)$ -homomorphism

$$\varsigma_p : (H^* \otimes H^{\otimes 2})^{\otimes p} = \text{Hom}(H, H^{\otimes 2})^{\otimes p} \rightarrow \text{Hom}(H, H^{\otimes(p+1)}) = H^* \otimes H^{\otimes(p+1)}$$

for each $p \geq 1$. If $p \geq 2$, we define

$$\begin{aligned} & \varsigma_p(u_{(1)} \otimes u_{(2)} \otimes \cdots \otimes u_{(p-1)} \otimes u_{(p)}) \\ & := \left(u_{(1)} \otimes 1_{H^{\otimes(p-1)}} \right) \circ \left(u_{(2)} \otimes 1_{H^{\otimes(p-2)}} \right) \circ \cdots \circ \left(u_{(p-1)} \otimes 1_H \right) \circ u_{(p)}, \end{aligned} \quad (4.8)$$

where $u_{(i)} \in \text{Hom}(H, H^{\otimes 2}) = H^* \otimes H^{\otimes 2}$, $1 \leq i \leq p$. In the case $p = 1$, we define $\varsigma_1 := 1_{H^* \otimes H^{\otimes 2}}$. Now we prove

Theorem 4.1.

$$h_p = \varsigma_{p*}([\tau_1^\theta]^{\otimes p}) \in H^p(\text{Aut}(F_n); H^* \otimes H^{\otimes(p+1)})$$

for any Magnus expansion $\theta \in \Theta_n$ and each $p \geq 1$. In the case $p = 1$ we have $[\tau_1^\theta] = h_1 \in H^1(\text{Aut}(F_n); H^* \otimes H^{\otimes 2})$, which is independent of the choice of θ .

Proof. We define a 1-cochain $\tilde{\theta}_2 \in C^1(\overline{A}_n; H^{\otimes 2})$ by $\tilde{\theta}_2(\gamma\varphi) := \theta_2(\gamma)$ for $\gamma \in F_n$ and $\varphi \in \text{Aut}(F_n)$. Then we have

$$d\tilde{\theta}_2 = -(\tau_1^\theta \circ k_0 + k_0^{\otimes 2}) \in C^2(\overline{A}_n; H^{\otimes 2}). \quad (4.9)$$

In fact, it follows from (2.7)

$$\begin{aligned} & d\tilde{\theta}_2(\gamma_1\varphi_1, \gamma_2\varphi_2) = |\varphi_1|^{\otimes 2}\tilde{\theta}_2(\gamma_2\varphi_2) - \tilde{\theta}_2(\gamma_1\varphi_1(\gamma_2)\varphi_1\varphi_2) + \tilde{\theta}_2(\gamma_1\varphi_1) \\ & = |\varphi_1|^{\otimes 2}\theta_2(\gamma_2) - \theta_2(\gamma_1\varphi_1(\gamma_2)) + \theta_2(\gamma_1) \\ & = |\varphi_1|^{\otimes 2}\theta_2(\gamma_2) - \theta_2(\varphi_1(\gamma_2)) - [\gamma_1] \otimes [\varphi_1(\gamma_2)] \\ & = -\tau_1^\theta(\varphi_1)|\varphi_1|k_0(\gamma_2\varphi_2) - k_0(\gamma_1\varphi_1) \otimes |\varphi_1|k_0(\gamma_2\varphi_2) \\ & = -(\tau_1^\theta \circ k_0 + k_0^{\otimes 2})(\gamma_1\varphi_1, \gamma_2\varphi_2). \end{aligned}$$

Consider a $(p+1)$ -cocycle $f_p \in C^{p+1}(\overline{A}_n; H^{\otimes(p+1)})$ defined by

$$\begin{aligned} f_p &:= (\varsigma_{p*}(\tau_1^\theta)^{\otimes p}) \circ k_0 \\ &= \left(\tau_1^\theta \otimes 1_H^{\otimes(p-1)} \right) \circ \left(\tau_1^\theta \otimes 1_H^{\otimes(p-2)} \right) \circ \cdots \circ \left(\tau_1^\theta \otimes 1_H \right) \circ \tau_1^\theta \circ k_0 \end{aligned}$$

and a p -cochain $g_p \in C^p(\overline{A}_n; H^{\otimes(p+1)})$ defined by

$$g_p := \left(\tau_1^\theta \otimes 1_H^{\otimes(p-1)} \right) \circ \left(\tau_1^\theta \otimes 1_H^{\otimes(p-2)} \right) \circ \cdots \circ \left(\tau_1^\theta \otimes 1_H \right) \circ \tilde{\theta}_2$$

for $p \geq 1$. If $p = 0$, we define $f_0 := k_0$. From (4.9) follows

$$\begin{aligned} dg_p &= (-1)^{p-1} \left(\tau_1^\theta \otimes 1_H^{\otimes(p-1)} \right) \circ \cdots \circ \left(\tau_1^\theta \otimes 1_H \right) \circ d\tilde{\theta}_2 \\ &= (-1)^p \left(\tau_1^\theta \otimes 1_H^{\otimes(p-1)} \right) \circ \cdots \circ \left(\tau_1^\theta \otimes 1_H \right) \circ (\tau_1^\theta \circ k_0 + k_0^{\otimes 2}) \\ &= (-1)^p (f_p + f_{p-1} \otimes k_0). \end{aligned}$$

Hence we obtain

$$\begin{aligned} [(\varsigma_{p*}(\tau_1^\theta)^{\otimes p}) \circ k_0] &= [f_p] = -[f_{p-1} \otimes k_0] = \cdots \\ &= (-1)^{p-1} [f_1 \otimes k_0^{\otimes(p-1)}] = (-1)^p [k_0^{\otimes(p+1)}], \end{aligned}$$

that is,

$$(\varsigma_{p*}(\tau_1^\theta)^{\otimes p}) \circ k_0 = (-1)^p k_0^{\otimes(p+1)} \in H^{p+1}(\overline{A}_n; H^{\otimes(p+1)}), \quad (4.10)$$

for each $p \geq 1$. We have

$$h_p = \pi_{\#}(k_0^{\otimes(p+1)}) = \varsigma_{p*}(\tau_1^\theta)^{\otimes p} \circ \pi_{\#}k_0 = \varsigma_{p*}(\tau_1^\theta)^{\otimes p} \in H^p(\text{Aut}(F_n); H^* \otimes H^{\otimes(p+1)}). \quad (4.11)$$

In fact, the cocycle $f_p \in C^{p+1}(\overline{A}_n; H^{\otimes(p+1)})$ is contained in the p -th filter A_p introduced in [HS] ch.II, p.118. Therefore, from [HS] ch.II, Proposition 3, p.125, its Gysin image is given by

$$\begin{aligned} (\pi_{\#}f_p)(\varphi_1, \dots, \varphi_p) [\gamma] &= (f_p)_1(\gamma, \varphi_1, \dots, \varphi_p) \\ &= (-1)^p f_p(\varphi_1, \dots, \varphi_p, (\varphi_1 \cdots \varphi_p)^{-1}(\gamma)) \\ &= (-1)^p \left(\left(\left(\tau_1^\theta \otimes 1_H^{\otimes(p-1)} \right) \circ \cdots \circ \left(\tau_1^\theta \otimes 1_H \right) \circ \tau_1^\theta \right) (\varphi_1, \dots, \varphi_p) \right) \circ |\varphi_1 \cdots \varphi_p| \\ &\quad \circ |\varphi_1 \cdots \varphi_p|^{-1} [\gamma] \\ &= (-1)^p \left((\varsigma_{p*}(\tau_1^\theta)^{\otimes p}) (\varphi_1, \dots, \varphi_p) \right) [\gamma]. \end{aligned}$$

See also [HS] ch.II, Theorem 3, p.126. This completes the proof. \blacksquare

We conclude this section by describing contraction formulae deduced from the relation (2.8). As was proved in the previous theorem for $p = 2$,

$$h_2 = \varsigma_{2*}h_1^{\otimes 2} = (h_1 \otimes 1_H) \circ \tau_1^\theta = (h_1 \otimes 1_H) \circ h_1 \in H^2(\text{Aut}(F_n); H^{\otimes 2}). \quad (4.10)$$

We can consider other ways than ς_p of contracting the coefficients of the cohomology class $h_1^{\otimes p}$. For example, we may consider the $\mathrm{GL}(H)$ -homomorphism

$$\varsigma'_2 : (H^* \otimes H^{\otimes 2})^{\otimes 2} \rightarrow H^* \otimes H^{\otimes 3}, \quad u_{(1)} \otimes u_{(2)} \mapsto (1_H \otimes u_{(1)}) \circ u_{(2)}.$$

But we have $\varsigma'_{2*}(h_1^{\otimes 2}) = -\varsigma_{2*}(h_1^{\otimes 2}) = -h_2$. In fact, from (2.6), the second Johnson map τ_2^θ gives the relation

$$(h_1 \otimes 1_H) \circ h_1 + (1_H \otimes h_1) \circ h_1 = 0 \in H^2(\mathrm{Aut}(F_n); H^* \otimes H^{\otimes 3}). \quad (4.11)$$

The relation (4.11) is the IH relation among twisted Morita-Mumford classes proposed by Garoufalidis and Nakamura [GN] and established by Morita and the author [KM2]. Recently Akazawa [Aka] gave an alternative proof of it by using representation theory of the symplectic group. In [KM2] a more precise formula was proved. In §5 we will give a simple proof of the precise formula using the second Johnson map τ_2^θ (Theorem 5.5).

Let S_p be the set of all the maximal ways of inserting parentheses into the word $12 \cdots (p+1)$ of $p+1$ letters. We have $S_1 = \{(12)\}$ and $S_2 = \{((12)3), (1(23))\}$. S_p is equal to the vertices of the Stasheff associahedron K_{p+1} [St]. If we define $S_0 := \{1\}$, then we have

$$S_p = \coprod_{q=0}^{p-1} S_q \times S_{p-q-1} \quad (4.12)$$

for $p \geq 1$. We define a map

$$h : S_p \rightarrow H^p(\mathrm{Aut}(F_n); H^* \otimes H^{\otimes (p+1)})$$

by $h(1) := 1_H$, $h((12)) := h_1$ and

$$h((x_1, x_2)) := (h(x_1) \otimes h(x_2)) \circ h_1, \quad x_1 \in S_q, \quad x_2 \in S_{p-q-1}.$$

Here $(x_1, x_2) \in S_q \times S_{p-q-1}$ is regarded as an element of S_p by (4.12). Immediately from (4.11) we have

Theorem 4.2. $h(x) = \pm h_p \in H^p(\mathrm{Aut}(F_n); H^* \otimes H^{\otimes (p+1)})$ for any $x \in S_p$.

In a forthcoming paper [Ka3] we will discuss more about the relation between the Stasheff associahedron K_{p+1} and the cohomology class h_p .

5. Mapping Class Groups.

Let $g \geq 1$ be a positive integer, $\Sigma_{g,1}$ a 2-dimensional oriented compact connected C^∞ manifold of genus g with 1 boundary component. We choose a basepoint $*$ on the boundary $\partial\Sigma_{g,1}$. The fundamental group $\pi_1(\Sigma_{g,1}, *)$ is a free group of rank $2g$. Taking a symplectic generator system $\{x_i\}_{i=1}^{2g} \subset \pi_1(\Sigma_{g,1}, *)$, we identify $\pi_1(\Sigma_{g,1}, *) = F_{2g}$. This induces a natural isomorphism

$$H = H_1(F_{2g}; R) = H_1(\Sigma_{g,1}; R).$$

A simple loop parallel to the boundary gives a word

$$w_0 := \prod_{i=1}^g x_i x_{g+i} x_i^{-1} x_{g+i}^{-1}.$$

The intersection form on the surface $\Sigma_{g,1}$ is given by

$$I := \sum_{i=1}^g (X_i X_{g+i} - X_{g+i} X_i) \in H^{\otimes 2}.$$

The Poincaré duality $\vartheta := \cap[\Sigma_{g,1}, \partial\Sigma_{g,1}] : H^* = H^1(\Sigma_{g,1}, \partial\Sigma_{g,1}; R) \rightarrow H_1(\Sigma_{g,1}; R) = H$ is given by

$$H^* \xrightarrow{\cong} H, \quad \ell \mapsto (\ell \otimes 1_H)(I). \quad (5.1)$$

In fact, $\xi_i \cap [\Sigma_{g,1}, \partial\Sigma_{g,1}] = X_{g+i}$ and $\xi_{g+i} \cap [\Sigma_{g,1}, \partial\Sigma_{g,1}] = -X_i$, for any i , $1 \leq i \leq g$. Here $\{\xi_i\}_{i=1}^{2g} \subset H^*$ is the dual basis of $\{X_i\} \subset H$. In this section we identify $H^* = H$ by the Poincaré duality (5.1), which is equivariant under the action of the mapping class group $\mathcal{M}_{g,1} := \pi_0 \text{Diff}(\Sigma_{g,1} \text{ rel } \partial\Sigma_{g,1})$.

The mapping class group $\mathcal{M}_{g,1}$ acts on the fundamental group $\pi_1(\Sigma_{g,1}, *) = F_{2g}$. This induces a group homomorphism $\mathcal{M}_{g,1} \rightarrow \text{Aut}(F_{2g})$. We prove the pull-back of the cohomology classes h_p and \bar{h}_p to the group $\mathcal{M}_{g,1}$ are twisted Morita-Mumford classes [Ka1] (Theorem 5.1 and Corollary 5.4). Furthermore we give a simple proof of a precise version [KM2] of the IH-relation [GN].

We introduce some variants of the mapping class group $\mathcal{M}_{g,1}$ in order to recall the definition of the Morita-Mumford classes [Mo1] [Mu] and that of the twisted ones [Ka1] [KM1]. Collapsing the boundary $\partial\Sigma_{g,1}$ into a single point $*$, we obtain a 2-dimensional oriented closed connected C^∞ manifold of genus g , $\Sigma_g = \Sigma_{g,1}/\partial\Sigma_{g,1}$. We write simply $\pi_1 := \pi_1(\Sigma_g, *)$ and $\pi_1^0 := \pi_1(\Sigma_{g,1}, *)$. Let N_g be the kernel of the collapsing homomorphism $c : \pi_1^0 \rightarrow \pi_1$. Then we have a group extension

$$N_g \rightarrow \pi_1^0 \xrightarrow{c} \pi_1. \quad (5.2)$$

Let \mathcal{M}_g and $\mathcal{M}_{g,*}$ be the mapping class groups for the surface Σ_g and the pointed one $(\Sigma_g, *)$, respectively, that is, $\mathcal{M}_g := \pi_0 \text{Diff}_+(\Sigma_g)$ and $\mathcal{M}_{g,*} := \pi_0 \text{Diff}_+(\Sigma_g, *)$. Forgetting the basepoint induces a group extension

$$\pi_1 \rightarrow \mathcal{M}_{g,*} \xrightarrow{\pi} \mathcal{M}_g, \quad (5.3)$$

and collapsing the boundary a central extension of groups

$$\mathbb{Z} \rightarrow \mathcal{M}_{g,1} \xrightarrow{\varpi} \mathcal{M}_{g,*}. \quad (5.4)$$

The kernel $\text{Ker } \varpi \cong \mathbb{Z}$ is generated by the Dehn twist along a simple loop parallel to the boundary, which acts on the fundamental group π_1^0 by conjugation by the word w_0 .

We denote the Euler class of the central extension (5.4) by $e := \text{Euler}(\mathbb{Z} \rightarrow \mathcal{M}_{g,1} \xrightarrow{\varpi} \mathcal{M}_{g,*}) \in H^2(\mathcal{M}_{g,*}; \mathbb{Z})$. The extension (5.3) gives the Gysin map

$$\pi_! : H^*(\mathcal{M}_{g,*}; M) \rightarrow H^{*-2}(\mathcal{M}_g; M)$$

for any \mathcal{M}_g -module M . The Morita-Mumford class e_i , $i \geq 1$, is defined to be the Gysin image of the $(i+1)$ -st power of the Euler class e

$$e_i := \pi_!(e^{i+1}) \in H^{2i}(\mathcal{M}_g; \mathbb{Z}).$$

The fiber product $\mathcal{M}_{g,1} \times_{\mathcal{M}_g} \mathcal{M}_{g,*}$ is identified with the semi-direct product $\pi_1 \rtimes \mathcal{M}_{g,1}$ by the isomorphism $(\varphi, \psi) \in \mathcal{M}_{g,1} \times_{\mathcal{M}_g} \mathcal{M}_{g,*} \mapsto (\psi\varpi(\varphi)^{-1}, \varphi) \in \pi_1 \rtimes \mathcal{M}_{g,1}$. We denote the first and the second projections of the product $\mathcal{M}_{g,1} \times_{\mathcal{M}_g} \mathcal{M}_{g,*}$ by $\pi : \pi_1 \rtimes \mathcal{M}_{g,1} \rightarrow \mathcal{M}_{g,1}$ and $\bar{\pi} : \pi_1 \rtimes \mathcal{M}_{g,1} \rightarrow \mathcal{M}_{g,*}$, respectively, and write $\bar{e} := \bar{\pi}^*(e) \in H^2(\pi_1 \rtimes \mathcal{M}_{g,1}; \mathbb{Z})$. Then the pullback $e_i = \varpi^* \pi^* e_i \in H^{2i}(\mathcal{M}_{g,1}; \mathbb{Z})$ is given by

$$e_i = \pi_!(\bar{e}^{i+1}) \in H^{2i}(\mathcal{M}_{g,1}; \mathbb{Z}). \quad (5.5)$$

As in §4, we can consider the 1-cocycle $k_0 : \pi_1 \rtimes \mathcal{M}_{g,1} \rightarrow H$, $(\gamma, \varphi) \mapsto [\gamma]$. The twisted Morita-Mumford class $m_{i,j}$, $i, j \geq 0$, $2i+j \geq 2$, is defined to be the Gysin image of $\bar{e}^i k_0^j$

$$m_{i,j} := \pi_!(\bar{e}^i k_0^j) \in H^{2i+j-2}(\mathcal{M}_{g,1}; \Lambda^j H). \quad (5.6)$$

Here k_0^j is the j -th exterior power of k_0 , and so we have

$$k_0^j = j! k_0^{\otimes j} \in H^p(\mathcal{M}_{g,1}; H^{\otimes j}). \quad (5.7)$$

Now we have

Theorem 5.1.

$$h_p = -\pi_! \left(k_0^{\otimes(p+2)} \right) \in H^p(\mathcal{M}_{g,1}; H^{\otimes(p+2)})$$

for each $p \geq 1$.

It is an immediate consequence of Lemma 5.4 in [KM2]. But we will give a self-contained proof of the theorem. From (5.6) and (5.7) we obtain

$$(p+2)! h_p = -m_{0,p+2}. \quad (5.8)$$

We should remark on the identification of $\text{Hom}(H, H^{\otimes(p+1)})$ with $H^{\otimes(p+2)}$. We denote it by

$$t_p : \text{Hom}(H, H^{\otimes(p+1)}) = H^* \otimes H^{\otimes(p+1)} \xrightarrow{\vartheta \otimes 1} H^{\otimes(p+2)}.$$

From (5.1) we have

$$t_p(u) = (1, 2, \dots, p+1, p+2)(u \otimes 1_H)(I) \quad (5.9)$$

for any $u \in \text{Hom}(H, H^{\otimes(p+1)})$. Here the cyclic permutation $(1, 2, \dots, p+1, p+2)$ acts on $H^{\otimes(p+2)}$ by permuting the components of the tensors in $H^{\otimes(p+2)}$.

In order to prove the theorem we construct a cohomology class introduced in [Mo2]

$$\nu \in H^2(\pi_1 \rtimes \mathcal{M}_{g,1}; \mathbb{Z})$$

in an algebraic way similar to [KM2]. In this section we write simply

$$\overline{\mathcal{M}} := \pi_1 \rtimes \mathcal{M}_{g,1} \quad \text{and} \quad \overline{\mathcal{M}}^0 := \pi_1^0 \rtimes \mathcal{M}_{g,1}.$$

The collapsing homomorphism gives a group extension

$$N_g \rightarrow \overline{\mathcal{M}}^0 \xrightarrow{c} \overline{\mathcal{M}}. \quad (5.10)$$

The Lyndon-Hochschild-Serre spectral sequence of the extension (5.2) gives

$$H^p(\pi_1; H^1(N_g; \mathbb{Z})) = 0, \quad \text{if } p \geq 1, \quad (5.11)$$

and an $\mathcal{M}_{g,1}$ -invariant isomorphism

$$d_2 : H^1(N_g; \mathbb{Z})^{\pi_1} \xrightarrow{\cong} H^2(\Sigma_g; \mathbb{Z}) \cong \mathbb{Z}.$$

Choose a Magnus expansion $\theta \in \Theta_{2g, \mathbb{Z}}$. We have $\theta_2(w_0) = I$. From (3.2) for $m = 2$ follows

$$\theta_2(\gamma w_0 \gamma^{-1}) = \theta_2(\gamma w_0 \gamma^{-1} w_0^{-1} w_0) = \theta_2(\gamma w_0 \gamma^{-1} w_0^{-1}) + \theta_2(w_0) = \theta_2(w_0)$$

for any $\gamma \in \pi_1^0$. Since N_g is the normal closure of the word w_0 , we can define an $\overline{\mathcal{M}}$ -invariant homomorphism $\nu_0 : N_g \rightarrow \mathbb{Z}$ by

$$\nu_0(\delta)I = -\theta_2(\delta) \in H^{\otimes 2}$$

for $\delta \in N_g$. Consider the transgression of the Lyndon-Hochschild-Serre spectral sequence of the extension (5.10)

$$d_2 : H^0(\overline{\mathcal{M}}; H^1(N_g; \mathbb{Z})) \rightarrow H^2(\overline{\mathcal{M}}; \mathbb{Z}).$$

If we choose a map $\widehat{\cdot} : \pi_1 \rightarrow \pi_1^0$, $\gamma \mapsto \widehat{\gamma}$, satisfying $c(\widehat{\gamma}) = \gamma$ for any $\gamma \in \pi_1$ and $\widehat{1} = 1$, then the 2-cochain $\widehat{\nu}$ defined by

$$\widehat{\nu}(\gamma_1 \varphi_1, \gamma_2 \varphi_2) := \nu_0(\widehat{\gamma_1 \varphi_1} \widehat{\gamma_2} \varphi_1 \widehat{\gamma_2}^{-1} \widehat{\gamma_1}^{-1}) \quad (5.12)$$

for $\gamma_1 \varphi_1$ and $\gamma_2 \varphi_2 \in \overline{\mathcal{M}}$ represents

$$\nu := d_2 \nu_0 \in H^2(\overline{\mathcal{M}}; \mathbb{Z}).$$

Moreover we have

$$\pi_!(\nu) = \langle \widehat{\nu}, [\Sigma_g] \rangle = 1. \quad (5.13)$$

To prove (5.13) we consider the homomorphism $\phi : \pi_1^0 \rightarrow F_2$ given by $\phi(x_1) = x_1$, $\phi(x_{g+1}) = x_2$ and $\phi(x_j) = \phi(x_{g+j}) = 1$ for $j \geq 2$, which induces a homomorphism of group extensions

$$\begin{array}{ccccc} N_g & \longrightarrow & \pi_1^0 & \xrightarrow{c} & \pi_1 \\ \phi \downarrow & & \phi \downarrow & & \phi \downarrow \\ [F_2, F_2] & \longrightarrow & F_2 & \longrightarrow & F_2^{\text{abel}}. \end{array} \quad (5.14)$$

We regard F_2^{abel} as the fundamental group of the 2-dimensional (real) torus T^2 . Then ϕ preserves the orientations, that is, $\phi_*[\Sigma_g] = [T^2] \in H_2(T^2; \mathbb{Z})$. The fundamental class is given by a normalized bar 2-chain

$$[x_1|x_2] + [x_1x_2|x_1^{-1}] - [x_1|x_1^{-1}].$$

See, e.g., [Me] p.245. If we use the map $\widehat{\cdot} : F_2^{\text{abel}} \rightarrow F_2$, $x_1^a x_2^b \mapsto x_1^a x_2^b$, then we have $\langle \widehat{\nu}, [T^2] \rangle = \nu_0(x_2 x_1 x_2^{-1} x_1^{-1}) = 1$. From (5.14) we have $\langle \widehat{\nu}, [\Sigma_g] \rangle = \langle \widehat{\nu}, \phi_*[\Sigma_g] \rangle = \langle \widehat{\nu}, [T^2] \rangle = 1$ for any $g \geq 1$. This proves (5.13). ■

The following is the key to the proof of Theorem 5.1.

Lemma 5.2. $\nu k_0 = 0 \in H^3(\overline{\mathcal{M}}; H)$.

Proof. The lemma is an immediate consequence of Theorem 5.1 (ii) in [KM2]. But we give a self-contained proof of it. The Lyndon-Hochschild-Serre spectral sequence of the semi-direct product $\overline{\mathcal{M}} = \pi_1 \rtimes \mathcal{M}_{g,1}$ gives an isomorphism

$$\pi^* : H^1(\mathcal{M}_{g,1}; H^1(N_g)^{\pi_1} \otimes H) \xrightarrow{\cong} H^1(\overline{\mathcal{M}}; H^1(N_g) \otimes H), \quad (5.15)$$

since $H^1(\pi_1; H^1(N_g)) = 0$ (5.11). Consider the homomorphism $s : \mathcal{M}_{g,1} \rightarrow \overline{\mathcal{M}}$, $\varphi \mapsto (1, \varphi)$. Then

$$s^* : H^1(\overline{\mathcal{M}}; H^1(N_g) \otimes H) \rightarrow H^1(\mathcal{M}_{g,1}; H^1(N_g) \otimes H)$$

is an isomorphism. In fact, $\pi^* : H^1(\mathcal{M}_{g,1}; H^1(N_g) \otimes H) \rightarrow H^1(\overline{\mathcal{M}}; H^1(N_g) \otimes H)$ is surjective from the isomorphism (5.15). Clearly we have $s^* \pi^* = 1$ on $H^1(\mathcal{M}_{g,1}; H^1(N_g) \otimes H)$. Hence s^* is the inverse of the isomorphism π^* .

Since $s^* k_0 = 0$, we have $\nu_0 k_0 = \pi^* s^*(\nu_0 k_0) = 0 \in H^1(\overline{\mathcal{M}}; H^1(N_g) \otimes H)$. Consequently $\nu k_0 = d_2(\nu_0 k_0) = 0$, as was to be shown. ■

Proof of Theorem 5.1. We denote the Gysin map of the semi-direct product $\overline{\mathcal{M}}^0 = \pi_1^0 \rtimes \mathcal{M}_{g,1}$ by $\pi_{\sharp}^* : H^*(\overline{\mathcal{M}}^0; M) \rightarrow H^{*-1}(\mathcal{M}_{g,1}; H \otimes M)$ for any $\mathcal{M}_{g,1}$ -module M . From the definition of h_p , we have $h_p = \pi_{\sharp}^*(k_0^{\otimes(p+1)})$. Consider the 1-cochain $\widehat{\theta}_2 \in C^1(\overline{\mathcal{M}}; H^{\otimes 2})$ defined by

$$\widehat{\theta}_2(\gamma\varphi) := \theta_2(\widehat{\gamma})$$

for $\gamma\varphi \in \pi_1 \times \mathcal{M}_{g,1} = \overline{\mathcal{M}}$. Then we have

$$\begin{aligned}
(d\widehat{\theta}_2)(\gamma_1\varphi_1, \gamma_2\varphi_2) &= \varphi_1\theta_2(\widehat{\gamma}_2) - \theta_2(\widehat{\gamma_1\varphi_1(\gamma_2)}) + \theta_2(\widehat{\gamma}_1) \\
&= \varphi_1\theta_2(\widehat{\gamma}_2) - \theta_2(\widehat{\gamma_1\varphi_1(\gamma_2)})\varphi_1(\widehat{\gamma}_2)^{-1}\widehat{\gamma}_1^{-1}\widehat{\gamma}_1\varphi_1(\widehat{\gamma}_2) + \theta_2(\widehat{\gamma}_1) \\
&= \varphi_1\theta_2(\widehat{\gamma}_2) - \theta_2(\varphi_1(\widehat{\gamma}_2)) + \theta_2(\varphi_1(\widehat{\gamma}_2)) - \theta_2(\widehat{\gamma_1\varphi_1(\gamma_2)})\varphi_1(\widehat{\gamma}_2)^{-1}\widehat{\gamma}_1^{-1} \\
&\quad - \theta_2(\widehat{\gamma}_1\varphi_1(\widehat{\gamma}_2)) + \theta_2(\widehat{\gamma}_1) \\
&= -\tau_1^\theta(\varphi_1)|\varphi_1|[\gamma_2] - [\gamma_1] \otimes |\varphi_1|[\gamma_2] + \widehat{\nu}(\gamma_1\varphi_1, \gamma_2\varphi_2)I
\end{aligned}$$

for any $\gamma_1\varphi_1$ and $\gamma_2\varphi_2 \in \overline{\mathcal{M}}$. This means

$$\tau_1^\theta \circ k_0 + k_0^{\otimes 2} = \nu I \in H^2(\overline{\mathcal{M}}; H^{\otimes 2}). \quad (5.16)$$

Hence we have

$$\pi_!(k_0^{\otimes 2}) = I \in H^0(\mathcal{M}_{g,1}; H^{\otimes 2}) \quad (5.17)$$

from (5.13). Moreover, from Lemma 5.2 and (5.16), we have

$$(h_1 \otimes 1_H^{\otimes p}) \circ k_0^{\otimes(p+1)} + k_0^{\otimes(p+2)} = 0$$

for each $p \geq 1$. If we denote $h'_p := \pi_!(k_0^{\otimes(p+2)}) \in H^p(\mathcal{M}_{g,1}; H^{\otimes(p+2)})$, then $(h_1 \otimes 1_H^{\otimes p}) \circ h'_{p-1} + h'_p = 0$, and so

$$\begin{aligned}
h'_p &= (-1)^p (h_1 \otimes 1_H^{\otimes p}) \circ (h_1 \otimes 1_H^{\otimes(p-1)}) \circ \cdots \circ (h_1 \otimes 1_H) \circ h'_0, \\
&= (-1)^p (h_p \otimes 1_H)(I)
\end{aligned}$$

from Theorem 4.1 and (5.17). Consequently, from (5.9) and the commutativity of the cup product, we obtain

$$\begin{aligned}
t_{p*}(h_p) &= (1, 2, \dots, p+1, p+2)_*(h_p \otimes 1_H)(I) \\
&= (-1)^p (1, 2, \dots, p+1, p+2)_* h'_p \\
&= (-1)^p (1, 2, \dots, p+1, p+2)_* \pi_!(k_0^{\otimes(p+2)}) \\
&= -\pi_!(k_0^{\otimes(p+2)}),
\end{aligned}$$

as was to be shown. \blacksquare

Next we study the cohomology class \overline{h}_p . We denote by $\mu : H^{\otimes 2} \rightarrow \mathbb{Z}$ the intersection product on the surface $\Sigma_{g,1}$. Recall the following theorem due to Morita.

Theorem 5.3. (Morita [Mo2], Theorem 1.3.)

$$\mu_*(k_0^{\otimes 2}) = 2\nu - \bar{e} \in H^2(\pi_1 \times \mathcal{M}_{g,1}; \mathbb{Z}).$$

An algebraic proof of it is given in [KM2], Theorem 6.1.

From Lemma 5.2 we have $\mu_*(k_0^{\otimes 2}) \otimes k_0^{\otimes p} = -\bar{e} \otimes k_0^{\otimes p}$. Theorem 5.1 implies

$$\overline{h}_p = (\mu \otimes 1_H^{\otimes p})_* h_p = -(\mu \otimes 1_H^{\otimes p})_* \pi_!(k_0^{\otimes(p+2)}) = \pi_!(\bar{e} \otimes k_0^{\otimes p}).$$

Hence we obtain

Corollary 5.4.

$$\bar{h}_p = \pi_!(\bar{e} \otimes k_0^{\otimes p}) \in H^p(\mathcal{M}_{g,1}; H^{\otimes p}).$$

From (5.7) follows

$$p! \bar{h}_p = m_{1,p}. \quad (5.18)$$

If we contract the coefficients of h_p by an iteration of the product μ , then we obtain the (original) Morita-Mumford class e_i . See [KM1] [KM2]. Hence each of the e_i 's is given by a certain algebraic combination of copies of h_1 . See also Theorems 4.1 and 5.3. So we may consider the cohomology class h_1 as “the unique elementary particle” for all the Morita-Mumford classes.

Finally we study a consequence of the relation (2.6) on the mapping class group $\mathcal{M}_{g,*}$. The conjugation by the word w_0 , $\iota(w_0)$, corresponds to a generator of the kernel $\text{Ker}(\varpi : \mathcal{M}_{g,1} \rightarrow \mathcal{M}_{g,*})$ and we have $\tau_1^\theta \iota(w_0) = 0$ from (2.11). Hence the first Johnson map τ_1^θ can be regarded as a 1-cocycle τ_1^θ on the group $\mathcal{M}_{g,*}$. We denote $h_1 := [\tau_1^\theta] \in H^1(\mathcal{M}_{g,*}; H^* \otimes H^{\otimes 2})$. The following is a precise version of the IH-relation [GN].

Theorem 5.5. ([KM2], Theorem 1.3 (iii))

$$(h_1 \otimes 1_H) \circ h_1 + (1_H \otimes h_1) \circ h_1 = e(I \otimes 1_H - 1_H \otimes I) \in H^2(\mathcal{M}_{g,*}; H^* \otimes H^{\otimes 3}).$$

Proof. From (2.6) we have

$$-d\tau_2^\theta = (\tau_1^\theta \otimes 1_H + 1_H \otimes \tau_1^\theta) \cup \tau_1^\theta$$

on the group $\mathcal{M}_{g,1}$. The Gysin sequence of the extension (5.4)

$$H^0(\mathcal{M}_{g,*}; H^* \otimes H^{\otimes 3}) \xrightarrow{\cup e} H^2(\mathcal{M}_{g,*}; H^* \otimes H^{\otimes 3}) \xrightarrow{\varpi^*} H^2(\mathcal{M}_{g,1}; H^* \otimes H^{\otimes 3})$$

implies

$$(h_1 \otimes 1_H) \circ h_1 + (1_H \otimes h_1) \circ h_1 = e\tau_2^\theta(\iota(w_0)) = e(I \otimes 1_H - 1_H \otimes I)$$

from (2.12). This proves the theorem. ■

As was stated above, the coefficients $H^* \otimes H^{\otimes 3}$ in the theorem are identified with $H^{\otimes 4}$ by the map t_3 in (5.9). In [KM2] a closed trivalent graph describes an $Sp_{2g}(\mathbb{Q})$ -invariant of the algebra $H^*(\Lambda^3 H_{\mathbb{Z}}; \mathbb{Q}) \cong \Lambda^*(\Lambda^3 H_{\mathbb{Q}})$. In this context the cohomology class $h_1 = [\tau_1^\theta] \in H^1(\mathcal{M}_{g,1}; H^{\otimes 3})$ corresponds to the open star of each vertex on the graph. A subgraph shaped like the letter H means the twisted cohomology class $(1_H \otimes h_1) \circ h_1$, while one like the letter I means $-(h_1 \otimes 1_H) \circ h_1$ because of the commutativity of the cup product. Similarly $I \otimes 1_H$ and $1_H \otimes I$ are interpreted as suitable edges in $\Gamma_1 \setminus \tau_1$ and $\Gamma_2 \setminus \tau_2$ in [KM2], respectively. Hence Theorem 1.3 (iii) in [KM2] follows from our theorem.

6. The Abelianization of IA_n .

The group IA_n is defined to be the kernel of the homomorphism $|\cdot| : \text{Aut}(F_n) \rightarrow \text{GL}(H_{\mathbb{Z}})$ induced by the abelianization $F_n \rightarrow F_n^{\text{abel}} = H_{\mathbb{Z}}$. In other words, $IA_n = A(1)$ in §3. Classically it is called *the induced automorphism group*. In this section we compute the abelianization of IA_n by evaluating the first Johnson map on the generators of the group IA_n given by Magnus [M2], and give some consequences of the computation. Here it should be remarked that S. Andreadakis has already studied the abelianization IA_n^{abel} in [Ad].

First recall the second exterior power $\Lambda^2 H_{\mathbb{Z}}$ of $H_{\mathbb{Z}} = F_n^{\text{abel}}$. Let $S^2(H_{\mathbb{Z}}) \subset H_{\mathbb{Z}}^{\otimes 2}$ be the \mathbb{Z} -submodule generated by the set $\{Y \otimes Y; Y \in H_{\mathbb{Z}}\}$. By definition we have $\Lambda^2 H_{\mathbb{Z}} = H_{\mathbb{Z}}^{\otimes 2} / S^2(H_{\mathbb{Z}})$. We define an homomorphism $\alpha_2 : H_{\mathbb{Z}}^{\otimes 2} \rightarrow H_{\mathbb{Z}}^{\otimes 2}$ by $\alpha_2(X_i \otimes X_j) = X_i \otimes X_j - X_j \otimes X_i$ for $1 \leq i, j \leq n$. We have $\text{Ker } \alpha_2 = S^2(H_{\mathbb{Z}})$. This induces an injective homomorphism $\overline{\alpha}_2 : \Lambda^2 H_{\mathbb{Z}} \rightarrow H_{\mathbb{Z}}^{\otimes 2}$. Throughout this section we regard $\Lambda^2 H_{\mathbb{Z}}$ as a submodule of $H_{\mathbb{Z}}^{\otimes 2}$ by the injection $\overline{\alpha}_2$.

Now fix a \mathbb{Z} -valued Magnus expansion $\theta \in \Theta_{n, \mathbb{Z}}$. It gives the first Johnson map $\tau_1^\theta : \text{Aut}(F_n) \rightarrow H_{\mathbb{Z}}^* \otimes H_{\mathbb{Z}}^{\otimes 2}$. As was proved in Theorem 3.1, the restriction of τ_1^θ to $IA_n = A(1)$ is equal to the first Johnson homomorphism τ_1 , which is independent of the choice of θ .

Theorem 6.1. *The first Johnson homomorphism τ_1 induces an isomorphism*

$$\tau_1 : IA_n^{\text{abel}} \xrightarrow{\cong} H_{\mathbb{Z}}^* \otimes \Lambda^2 H_{\mathbb{Z}},$$

which is equivariant under the action of $\text{GL}(H_{\mathbb{Z}}) = \text{Aut}(F_n)/IA_n$. Especially the abelianization IA_n^{abel} is free abelian of rank $n^2(n-1)/2$, and the commutator subgroup of IA_n coincides with $\text{Ker } \tau_1$

$$[IA_n, IA_n] = \text{Ker } \tau_1 = A(2). \quad (6.1)$$

Andreadakis [An] proved the theorem for the case $n = 3$. All we need to prove it are due to W. Magnus. So it had been likely proved by someone contemporary with Magnus or Andreadakis. Comparing it with Johnson's result computing the abelianization of the Torelli groups [J3], the reader would find how simpler the automorphism groups of free groups are than the mapping class groups for surfaces.

Proof of Theorem 6.1. According to Magnus [M2], the group IA_n is generated by the following automorphisms

$$\begin{aligned} K_{i,l} &: x_i \mapsto x_l x_i x_l^{-1}, \quad x_j \mapsto x_j \quad (j \neq i) \\ K_{i,l,s} &: x_i \mapsto x_i x_l x_s x_l^{-1} x_s^{-1}, \quad x_j \mapsto x_j \quad (j \neq i). \end{aligned}$$

Here the indices run over the sets $\{(i, l); 1 \leq i, l \leq n, i \neq l\}$ and $\{(i, l, s); 1 \leq i, l, s \leq n, i \neq l < s \neq i\}$ respectively. The number of the generators is $n(n-1) + n \frac{1}{2}(n-1)(n-2) = \frac{1}{2}n^2(n-1)$. Hence we have a surjection $p_n : \mathbb{Z}^{n^2(n-1)/2} \rightarrow IA_n^{\text{abel}}$.

Now we denote by $\theta_2 : F_n \rightarrow H_{\mathbb{Z}}^{\otimes 2}$ the second component of the expansion θ as before. From (3.2) for $m = 1$ we have

$$\theta_2(\gamma \delta \gamma^{-1} \delta^{-1}) = [\gamma][\delta] - [\delta][\gamma] \quad (6.2)$$

for any γ and $\delta \in F_n$. Since $\theta_2(\gamma\delta\gamma^{-1}) = \theta_2(\gamma\delta\gamma^{-1}\delta^{-1}\delta) = \theta_2(\gamma\delta\gamma^{-1}\delta^{-1}) + \theta_2(\delta)$, we have

$$\theta_2(\gamma\delta\gamma^{-1}) - \theta_2(\delta) = [\gamma][\delta] - [\delta][\gamma]. \quad (6.3)$$

Let $\{\ell_i\}_{i=1}^n \subset H_{\mathbb{Z}}^*$ be the dual basis of the basis $\{X_i\}_{i=1}^n \subset H_{\mathbb{Z}}$. From (2.7), for any $\varphi \in IA_n$ and $\gamma \in F_n$, we have $\tau_1^\theta(\varphi)[\gamma] = \theta_2(\varphi(\gamma)) - \theta_2(\gamma)$. From (6.2) and (6.3) we obtain

$$\begin{aligned} \tau_1^\theta(K_{i,l}) &= \ell_i \otimes \tau_1^\theta(K_{i,l})(X_i) = \ell_i \otimes (\theta_2(x_l x_i x_l^{-1}) - \theta_2(x_i)) \\ &= \ell_i \otimes (X_l X_i - X_i X_l) \\ \tau_1^\theta(K_{i,l,s}) &= \ell_i \otimes \tau_1^\theta(K_{i,l,s})(X_i) = \ell_i \otimes (\theta_2(x_i x_l x_s x_l^{-1} x_s^{-1}) - \theta_2(x_i)) \\ &= \ell_i \otimes (X_l X_s - X_s X_l). \end{aligned} \quad (6.4)$$

These form exactly a \mathbb{Z} -free basis of $H_{\mathbb{Z}}^* \otimes \Lambda^2 H_{\mathbb{Z}}$. Therefore the composite

$$\tau_1^\theta \circ p_n : \mathbb{Z}^{n^2(n-1)/2} \xrightarrow{p_n} IA_n^{\text{abel}} \xrightarrow{\tau_1^\theta} H_{\mathbb{Z}}^* \otimes \Lambda^2 H_{\mathbb{Z}}$$

is an isomorphism. Since p_n is surjective, the homomorphism $\tau_1 = \tau_1^\theta : IA_n^{\text{abel}} \rightarrow H_{\mathbb{Z}}^* \otimes \Lambda^2 H_{\mathbb{Z}}$ is an isomorphism.

The isomorphism $\tau_1^\theta : IA_n^{\text{abel}} \rightarrow H_{\mathbb{Z}}^* \otimes \Lambda^2 H_{\mathbb{Z}}$ is $\text{GL}(H_{\mathbb{Z}})$ -equivariant, because we have

$$\begin{aligned} \tau_1^\theta(\varphi\psi\varphi^{-1}) &= \tau_1^\theta(\varphi\psi\varphi^{-1}) - \tau_1^\theta(\varphi\varphi^{-1}) \\ &= |\varphi|\tau_1^\theta(\psi) + |\varphi\psi|\tau_1^\theta(\varphi^{-1}) - |\varphi|\tau_1^\theta(\varphi^{-1}) = |\varphi|\tau_1^\theta(\psi) \end{aligned}$$

for any $\varphi \in \text{Aut}(F_n)$ and $\psi \in IA_n$. This completes the proof of Theorem 6.1. \blacksquare

We define the group IO_n to be the kernel of the homomorphism $|\cdot| : \text{Out}(F_n) \rightarrow \text{GL}(H_{\mathbb{Z}})$ induced by the abelianization. We have $IO_n = IA_n / \text{Inn}(F_n)$. Here $\text{Inn}(F_n)$ is, by definition, the image of the homomorphism ι in (2.9). From (2.11) the induced homomorphism

$$\iota_* : F_n^{\text{abel}} = H_{\mathbb{Z}} \rightarrow IA_n^{\text{abel}} \xrightarrow{\tau_1} H_{\mathbb{Z}}^* \otimes \Lambda^2 H_{\mathbb{Z}}$$

is given by

$$\iota_*(Y)Z = YZ - ZY \in \Lambda^2 H_{\mathbb{Z}} \quad (6.5)$$

for $Y, Z \in H_{\mathbb{Z}}$, which is an injection, and whose image is a direct summand of $H_{\mathbb{Z}}^* \otimes \Lambda^2 H_{\mathbb{Z}}$ as a \mathbb{Z} -module. Hence

Theorem 6.2. *We have a $\text{GL}(H_{\mathbb{Z}})$ -equivariant isomorphism*

$$IO_n^{\text{abel}} \cong (H_{\mathbb{Z}}^* \otimes \Lambda^2 H_{\mathbb{Z}}) / \iota_*(H_{\mathbb{Z}}),$$

where ι_* is the homomorphism given in (6.5). Especially the abelianization IO_n^{abel} is free abelian of rank $(n+1)n(n-2)/2$.

Let q be a prime integer. The congruence IA-automorphism group $IA_{n,q}$ is defined to be the kernel of the natural homomorphism $\text{Aut}(F_n) \rightarrow \text{GL}(H_{\mathbb{Z}/q})$.

Using the first Johnson map τ_1^θ for a \mathbb{Z}/q -valued Magnus expansion θ , T. Satoh [Sa2] computes the abelianization of $IA_{n,q}$.

Theorem 6.1 has an application to twisted cohomology of the group $\text{Aut}(F_n)$ with values in a $GL(H_{\mathbb{Z}})$ -module. Let $\bar{\Gamma}$ be a subgroup of $GL(H_{\mathbb{Z}})$, $\Gamma \subset \text{Aut}(F_n)$ the preimage of $\bar{\Gamma}$, and M a $\mathbb{Z}[\frac{1}{2}][\bar{\Gamma}]$ -module. We denote by $\pi : \Gamma \rightarrow \bar{\Gamma}$ the natural projection, and by j the inclusion $j : IA_n \hookrightarrow \Gamma$. We have the Lyndon-Hochschild-Serre spectral sequence

$$E_2^{p,q} = E_2^{p,q}(M) = H^p(\bar{\Gamma}; H^q(IA_n; M)) \Rightarrow H^{p+q}(\Gamma; M) \quad (6.6)$$

of the group extension $IA_n \xrightarrow{j} \Gamma \xrightarrow{\pi} \bar{\Gamma}$.

Proposition 6.3. *For any $p \geq 0$ we have*

$$d_2^{p,1} = 0 : H^p(\bar{\Gamma}; H^1(IA_n; M)) \rightarrow H^{p+2}(\bar{\Gamma}; M).$$

In the case $p = 0$ we have a natural decomposition

$$H^1(\Gamma; M) = \text{Hom}(H_{\mathbb{Z}}^* \otimes \Lambda^2 H_{\mathbb{Z}}, M)^{\bar{\Gamma}} \oplus H^1(\bar{\Gamma}; M).$$

Proof. We denote by $\varpi_2 : H_{\mathbb{Z}}^{\otimes 2} \rightarrow \Lambda^2 H_{\mathbb{Z}}$ the natural projection. Twice the abelianization $\tau_1^\theta : IA_n \rightarrow H_{\mathbb{Z}}^* \otimes \Lambda^2 H_{\mathbb{Z}}$ extends to the crossed homomorphism

$$\tilde{\tau} := (1 \otimes \varpi_2) \circ \tau_1^\theta : \text{Aut}(F_n) \rightarrow H_{\mathbb{Z}}^* \otimes H_{\mathbb{Z}}^{\otimes 2} \rightarrow H_{\mathbb{Z}}^* \otimes \Lambda^2 H_{\mathbb{Z}}$$

defined on the whole automorphism group $\text{Aut}(F_n)$. Since the action of the subgroup IA_n on the module M is trivial, we have the Kronecker product

$$\kappa : H_{\mathbb{Z}}^* \otimes \Lambda^2 H_{\mathbb{Z}} \otimes H^1(IA_n; M) = H_1(IA_n) \otimes H^1(IA_n; M) \rightarrow M.$$

For any p -cocycle $f \in Z^p(\bar{\Gamma}; H^1(IA_n; M))$ the $(p+1)$ -cocycle

$$\kappa_*(\tilde{\tau} \cup \pi^*(\frac{1}{2}f)) \in Z^{p+1}(\Gamma; M)$$

is an element of the p -th filter $A^{p+1} \cap A_p^*$ in [HS] ch.II, p.119. It is clear that the cocycle $\kappa_*(\tilde{\tau} \cup \pi^*(\frac{1}{2}f))$ induces the cocycle f in $E_1^{*p,1} = C^p(\bar{\Gamma}; H^1(IA_n; M))$. Thus the cocycle f extends to a cocycle defined on the whole Γ , so that $d_2^{p,1}[f] = 0 \in E_2^{p+2,0}$. In the case $p = 0$ we have an exact sequence

$$0 \rightarrow H^1(\bar{\Gamma}; M) \xrightarrow{\pi^*} H^1(\Gamma; M) \xrightarrow{j^*} H^0(\bar{\Gamma}; H^1(IA_n; M)).$$

The homomorphism

$$[f] \in H^0(\bar{\Gamma}; H^1(IA_n; M)) \mapsto [\kappa_*(\tilde{\tau} \cup \pi^*(\frac{1}{2}f))] \in H^1(\Gamma; M)$$

is a right inverse of the homomorphism j^* . Hence we have

$$\begin{aligned} H^1(\Gamma; M) &= H^0(\bar{\Gamma}; H^1(IA_n; M)) \oplus H^1(\bar{\Gamma}; M) \\ &= \text{Hom}(H_{\mathbb{Z}}^* \otimes \Lambda^2 H_{\mathbb{Z}}, M)^{\bar{\Gamma}} \oplus H^1(\bar{\Gamma}; M). \end{aligned}$$

This completes the proof. ■

When M is a non-trivial irreducible $\mathbb{Q}[\text{GL}(H_{\mathbb{Z}})]$ -module, as was proved by Borel [B], the cohomology group $H^*(\text{GL}(H_{\mathbb{Z}}); M)$ vanishes in a stable range, so that we obtain *stably*

$$H^1(\text{Aut}(F_n); M) = \text{Hom}(H_{\mathbb{Z}}^* \otimes \Lambda^2 H_{\mathbb{Z}}, M)^{\text{GL}(H_{\mathbb{Z}})}. \quad (6.7)$$

In the simplest case $M = H_{\mathbb{Q}}$, we have $H^1(\text{Aut}(F_n); H_{\mathbb{Q}}) = \mathbb{Q}$ for any *sufficiently large* n . It is generated by the class $\bar{h}_1 = r_{1*}[\tau_1^\theta]$. On the other hand, Satoh [Sa1] used a direct method involved with a presentation of the group $\text{Aut}(F_n)$ given by Gersten [G] to prove that

$$H^1(\text{Aut}(F_n); H_{\mathbb{Z}}) = \mathbb{Z}$$

for $n \geq 4$, and that it is generated by \bar{h}_1 . Moreover he proved

$$H^1(\text{Aut}(F_3); H_{\mathbb{Z}}) = \mathbb{Z}/2 \oplus \mathbb{Z}.$$

7. Decomposition of Cohomology Groups.

In this section we prove the cohomology class $\bar{h}_1 = r_{1*}[\tau_1^\theta] \in H^1(\text{Aut}(F_n); H)$ gives a canonical decomposition of cohomology groups of $\text{Aut}(F_n)$, and discuss the rational cohomology of $\text{Aut}(F_n)$ with trivial coefficients.

Recall the homomorphism $\iota : F_n \rightarrow \text{Aut}(F_n)$, $\gamma \mapsto (\iota(\gamma) : \delta \mapsto \gamma\delta\gamma^{-1})$ in (2.9). We have a group extension

$$F_n \xrightarrow{\iota} \text{Aut}(F_n) \xrightarrow{\pi} \text{Out}(F_n). \quad (7.1)$$

Theorem 7.1. *Suppose $1 - n$ is invertible in the ring R . Then we have a natural decomposition of the cohomology group*

$$H^*(\text{Aut}(F_n); M) = H^*(\text{Out}(F_n); M) \oplus H^{*-1}(\text{Out}(F_n); H^* \otimes M)$$

for any $R[\text{Out}(F_n)]$ -module M . Especially, $\pi^* : H^*(\text{Out}(F_n); M) \rightarrow H^*(\text{Aut}(F_n); M)$ is an injection.

Proof. We denote by $\{\ell_i\}_{i=1}^n \subset H^*$ the dual basis of $\{X_i\}_{i=1}^n \subset H$. From (2.11) we have

$$\tau_1^\theta(\iota(x_i)) = \sum_{j=1}^n \ell_j \otimes (X_i X_j - X_j X_i) \in H^* \otimes H^{\otimes 2},$$

and so $r_1(\tau_1^\theta(\iota(x_i))) = (1 - n)X_i$. Hence

$$\iota^* \bar{h}_1 = (1 - n)1_H \in H^1(F_n; H) = \text{Hom}(H, H). \quad (7.2)$$

The Lyndon-Hochschild-Serre spectral sequence of the extension (7.1) induces an exact sequence

$$\cdots \rightarrow H^p(\mathrm{Out}(F_n); M) \xrightarrow{\pi^*} H^p(\mathrm{Aut}(F_n); M) \xrightarrow{\pi_\#} H^{p-1}(\mathrm{Out}(F_n); H^* \otimes M) \rightarrow \cdots \quad (7.3)$$

Here $\pi_\#$ is the Gysin map, and we have a natural isomorphism $H^1(F_n; M) = H^* \otimes M$. Consider the contraction map

$$C : H \otimes H^* \otimes M \rightarrow M, \quad Y \otimes f \otimes m \mapsto f(Y)m.$$

By (7.2) we have $\pi_\#(\bar{h}_1) = (1-n)1_H \in H^0(\mathrm{Out}(F_n); H^* \otimes H)$, and so

$$\pi_\#(C(\bar{h}_1 \cup \pi^*u)) = (1_{H^*} \otimes C)(\pi_\#(\bar{h}_1) \cup u) = (1_{H^*} \otimes C)((1-n)1_H \cup u) = (1-n)u$$

for any $u \in H^{p-1}(\mathrm{Out}(F_n); H^* \otimes M)$. This implies the map

$$H^{p-1}(\mathrm{Out}(F_n); H^* \otimes M) \rightarrow H^p(\mathrm{Aut}(F_n); M), \quad u \mapsto \frac{1}{1-n}C(k \cup \pi^*(u))$$

is a right inverse of the map $\pi_\#$. Hence the exact sequence (7.3) splits. This completes the proof. ■

We conclude the paper by discussing the rational cohomology of the group $\mathrm{Aut}(F_n)$ with trivial coefficients. Similar results hold for the group $\mathrm{Out}(F_n)$ because $\pi^* : H^*(\mathrm{Out}(F_n); \mathbb{Q}) \rightarrow H^*(\mathrm{Aut}(F_n); \mathbb{Q})$ is injective by Theorem 7.1.

As was shown by Morita [Mo4], we obtain any of the Morita-Mumford classes on the mapping class group for a surface by contracting the coefficient of a power of the cohomology class h_1 in a suitable way involved with the intersection form $H_{\mathbb{Z}}^{\otimes 2} \rightarrow \mathbb{Z}$. Thus the class h_1 yields rich nontrivial classes in the rational cohomology of the mapping class group. For details, see also [KM][KM2].

For the group $\mathrm{Aut}(F_n)$, in contrast, we have

Theorem 7.2.

$$f_*(h_1^{\otimes m}) = 0 \in H^m(\mathrm{Aut}(F_n); \mathbb{Q})$$

for any $m \geq 1$ and any $\mathrm{GL}(H_{\mathbb{Z}})$ -invariant linear form $f : (H_{\mathbb{Z}}^* \otimes \Lambda^2 H_{\mathbb{Z}})^{\otimes m} \rightarrow \mathbb{Q}$.

Proof. We denote by M the $\mathrm{GL}(H_{\mathbb{C}})$ -module $\mathrm{Hom}_{\mathbb{C}}(H_{\mathbb{C}}^* \otimes \Lambda^2 H_{\mathbb{C}}, \mathbb{C})$. Since the special linear group $\mathrm{SL}(H_{\mathbb{Z}})$ is Zariski dense in $\mathrm{SL}(H_{\mathbb{C}})$, we have

$$\mathrm{Hom}_{\mathbb{Z}}((H_{\mathbb{Z}}^* \otimes \Lambda^2 H_{\mathbb{Z}})^{\otimes m}, \mathbb{Q})^{\mathrm{GL}(H_{\mathbb{Z}})} \subset (M^{\otimes m})^{\mathrm{GL}(H_{\mathbb{Z}})} \subset (M^{\otimes m})^{\mathrm{SL}(H_{\mathbb{C}})}. \quad (7.4)$$

First we prove

$$(M^{\otimes m})^{\mathrm{GL}(H_{\mathbb{Z}})} = 0, \quad \text{if } m \leq 2n-1. \quad (7.5)$$

Consider the tori

$$\begin{aligned} \mathbb{T} &:= \{\mathrm{diag}(\zeta_1, \zeta_2, \dots, \zeta_n); \zeta_i \in \mathbb{C}, |\zeta_i| = 1, 1 \leq i \leq n\}, \quad \text{and} \\ \mathbb{T}_0 &:= \{\mathrm{diag}(\zeta_1, \zeta_2, \dots, \zeta_n) \in \mathbb{T}; \zeta_1 \zeta_2 \cdots \zeta_n = 1\} \end{aligned}$$

in $\mathrm{GL}(H_{\mathbb{C}}) = \mathrm{GL}_n(\mathbb{C})$. The representation ring RT is isomorphic to the Laurant polynomials

$$\mathrm{RT} = \mathbb{Z}[t_1^{\pm 1}, t_2^{\pm 1}, \dots, t_n^{\pm 1}].$$

Here t_i is the 1-dimensional $\mathbb{C}[\mathbb{T}]$ -module on which $\mathrm{diag}(\zeta_1, \zeta_2, \dots, \zeta_n)$ acts by multiplication by ζ_i . Then M is equal to

$$g(t) := (n-1) \sum_{j=1}^n t_j^{-1} + \sum_{i \neq j \neq k \neq i} t_i t_j^{-1} t_k^{-1} \in \mathrm{RT}$$

as a \mathbb{T} -module, which is homogeneous of degree -1 . A monomial, or a 1-dimensional representation, $h(t) \in \mathrm{RT}$ with $\nu_i := \deg_{t_i} h(t) \neq \nu_j := \deg_{t_j} h(t)$ for some $i < j$ is not \mathbb{T}_0 -invariant. In fact, $\mathrm{diag}(1, \dots, 1, \zeta, 1, \dots, 1, \zeta^{-1}, 1, \dots, 1) \in \mathbb{T}_0$, where ζ is in the i -th component, and ζ^{-1} in the j -th, acts on $h(t)$ by multiplication by $\zeta^{\nu_i - \nu_j}$. Hence, if n does not divide m , then $g(t)^m$ has no \mathbb{T}_0 -invariant part. Moreover, if $m = n$, $(M^{\otimes n})^{\mathbb{T}_0}$ is equal to $ct_1^{-1} t_2^{-1} \dots t_n^{-1} \in \mathrm{RT}$ for some $c \geq 0$. But $t_1^{-1} t_2^{-1} \dots t_n^{-1}$ is not invariant under the action of $\mathrm{diag}(-1, 1, \dots, 1) \in \mathrm{GL}(H_{\mathbb{Z}}) \cap \mathbb{T}$. This proves (7.5).

Now, if $m \leq 2n - 1$, the theorem follows from (7.5). On the other hand, as was established by Culler and Vogtmann [CV], the virtual cohomology dimension of $\mathrm{Aut}(F_n)$ is $2n - 2$. Hence, if $m \geq 2n - 1$, then $H^m(\mathrm{Aut}(F_n); \mathbb{Q}) = 0$. This completes the proof of the theorem. ■

Igusa [I], Theorem 8.5.3, p.333, proved the homomorphism of reduced cohomology

$$\tilde{H}^*(\mathrm{GL}(H_{\mathbb{Z}}); \mathbb{Q}) \rightarrow \tilde{H}^*(\mathrm{Aut}(F_n); \mathbb{Q})$$

induced by the abelianization map $\mathrm{Aut}(F_n) \rightarrow \mathrm{GL}(H_{\mathbb{Z}})$ vanishes in the stable range $* \leq \frac{1}{2}(n-1)$. From Igusa's result we obtain

Corollary 7.3. *The homomorphism*

$$\tilde{H}^*(\mathrm{Aut}(F_n)/[IA_n, IA_n]; \mathbb{Q}) \rightarrow \tilde{H}^*(\mathrm{Aut}(F_n); \mathbb{Q})$$

induced by the natural projection vanishes in some stable range $ \ll n$.*

Proof. From Theorem 6.1 we have the group extension

$$H_{\mathbb{Z}}^* \otimes \Lambda^2 H_{\mathbb{Z}} \rightarrow \mathrm{Aut}(F_n)/[IA_n, IA_n] \rightarrow \mathrm{GL}(H_{\mathbb{Z}}),$$

which induces the Lyndon-Hochschild-Serre spectral sequence

$$E_2^{p,q} = H^p(\mathrm{GL}(H_{\mathbb{Z}}); H^q(H_{\mathbb{Z}}^* \otimes \Lambda^2 H_{\mathbb{Z}}; \mathbb{Q})) \Rightarrow H^{p+q}(\mathrm{Aut}(F_n)/[IA_n, IA_n]; \mathbb{Q}).$$

In view of a theorem of Borel [B] we have

$$E_2^{p,q} = H^p(\mathrm{GL}(H_{\mathbb{Z}}); \mathbb{Q}) \otimes H^q(H_{\mathbb{Z}}^* \otimes \Lambda^2 H_{\mathbb{Z}}; \mathbb{Q})^{\mathrm{GL}(H_{\mathbb{Z}})} = E_2^{p,0} \otimes E_2^{0,q}$$

in some stable range $p+q \ll n$. From Theorem 7.2, $E_2^{0,q}$ has no nontrivial contributions in $H^*(\mathrm{Aut}(F_n); \mathbb{Q})$. On the other hand, $E_2^{p,0} \rightarrow H^p(\mathrm{Aut}(F_n); \mathbb{Q})$ vanishes for $1 \leq p \leq \frac{1}{2}(n-1)$ from Igusa's result. This completes the proof. ■

REFERENCES

- [Ak] H. Akazawa, *Symplectic invariants arising from a grassmannian quotient and trivalent graphs*, Doctoral Dissertation, Okayama University (2005).
- [An] S. Andreadakis, *On the automorphisms of free groups and free nilpotent groups*, Proc. London Math. Soc. **15** (1965), 239–268.
- [B] A. Borel, *Stable real cohomology of arithmetic groups II*, in: *Manifolds and Lie Groups*, Progr. Math. **14** (1981), 21–55.
- [Bou] N. Bourbaki, *Groupes et Algèbres de Lie*, Ch. 2, Hermann, Paris, 1972.
- [CV] M. Culler and K. Vogtmann, *Moduli of graphs and automorphisms of free groups*, Invent. math. **84** (1986), 91–119.
- [F] R. Fenn, *Techniques of Geometric Topology*, London Math. Soc. Lect. Note, 57, Cambridge Univ. Press, Cambridge, 1983.
- [G] S. M. Gersten, *A presentation for the special automorphism group of a free group*, J. Pure Appl. Algebra **33** (1984), 269–279.
- [Hai] R. Hain, *Infinitesimal presentations of the Torelli groups*, J. Amer. Math. Soc. **10** (1997), 597–651.
- [Har] J. Harer, *Stability of the homology of the mapping class group of orientable surfaces*, Ann. Math. **121** (1985), 215–249.
- [HS] G. Hochschild and J.-P. Serre, *Cohomology of group extensions*, Trans. Amer. Math. Soc. **74** (1953), 110–134.
- [I] K. Igusa, *Higher Franz-Reidemeister Torsion*, American Mathematical Society and International Press, Providence, 2002.
- [J1] D. Johnson, *An abelian quotient of the mapping class group \mathcal{I}_g* , Math. Ann. **249** (1980), 225–242.
- [J2] ———, *A survey of the Torelli group*, Contemporary Math. **20** (1983), 165–179.
- [J3] ———, *The structure of the Torelli group, III*, Topology **24** (1985), 127–144.
- [Ka1] N. Kawazumi, *A generalization of the Morita-Mumford classes to extended mapping class groups for surfaces*, Invent. math. **131** (1998), 137–149.
- [Ka2] ———, *Certain cohomology classes on the automorphism groups of free groups*, in: ‘Analysis of discrete groups, II’, Sūrikaiseikikenkyūsho Kōkyūroku **1022** (1997), 35–42. (Japanese)
- [Ka3] ———, *Harmonic Magnus Expansion on the Universal Family of Riemann Surfaces*, in preparation.
- [KM1] N. Kawazumi and S. Morita, *The primary approximation to the cohomology of the moduli space of curves and cocycles for the stable characteristic classes*, Math. Research Lett. **3** (1996), 629–641.
- [KM2] ———, *The primary approximation to the cohomology of the moduli space of curves and cocycles for the Mumford-Morita-Miller classes*, preprint, University of Tokyo. **UTMS 2001-13** (2001).
- [Ki] T. Kitano, *Johnson’s homomorphisms of subgroups of the mapping class group, the Magnus expansion and Massey higher products of mapping tori*, Topology Appl. **69** (1996), 165–172.
- [MW] I. Madsen and M. Weiss, *The stable moduli space of Riemann surfaces: Mumford’s conjecture*, preprint, arXiv:math.AT/0212321.

- [M1] W. Magnus, *Beziehungen zwischen Gruppen und Idealen in einem speziellen Ring*, Math. Annalen **111** (1935), 259–280.
- [M2] ———, *Über n -dimensionale Gittertransformationen*, Acta Math. **64** (1934), 353–367.
- [M3] ———, *Über Beziehungen zwischen höheren Kommutatoren*, J. reine angew. Math. **177** (1937), 105–115.
- [Me] W. Meyer, *Die Signatur von Flächenbündeln*, Math. Ann. **201** (1973), 239–264.
- [Mi] E.Y. Miller, *The homology of the mapping class group*, J. Diff. Geom. **24** (1986), 1–14.
- [Mo1] S. Morita, *Characteristic classes of surface bundles*, Inventiones math. **90** (1987), 551–577.
- [Mo2] ———, *Families of Jacobian manifolds and characteristic classes of surface bundles, I*, Ann. Inst. Fourier **39** (1989), 777–810.
- [Mo3] ———, *Families of Jacobian manifolds and characteristic classes of surface bundles, II*, Math. Proc. Camb. Phil. Soc. **105** (1989), 79–101.
- [Mo4] ———, *The extension of Johnson’s homomorphism from the Torelli group to the mapping class group*, Invent. math. **111** (1993), 197–224.
- [Mo5] ———, *A linear representation of the mapping class group of orientable surfaces and characteristic classes of surface bundles*, in: *Topology and Teichmüller Spaces* (1996), World Scientific, Singapore, 159–186.
- [Mu] D. Mumford, *Towards an enumerative geometry of the moduli space of curves*, Arithmetic and Geometry, Progr. Math. **36** (1983), 271–328.
- [Sa1] T. Satoh, *Twisted first homology group of the automorphism group of a free group*, J. Pure Appl. Alg. (to appear).
- [Sa2] ———, *The abelianization of the congruence IA -automorphism group of a free group*, preprint, University of Tokyo. **UTMS 2005-1** (2005).
- [S] J.D. Stasheff, *Homotopy Associativity of H -Spaces, I*, Trans. Amer. Math. Soc. **108** (1963), 275–292.

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