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Abstract

In this paper we consider the vector-valued extension of the Fefferman-Stein-Stronberg sharp maximal inequality under growth condition. As an application we obtain the vector-valued extension of the boundedness of the commutator. Furthermore we prove the boundedness of the commutator.

Keywords weighted norm inequality, non-doubling(nonhomogeneous), sharp-maximal function, commutator

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1 Introduction

In this paper we obtain the vector-valued extension of the sharp-maximal inequality and develop its applications.

We denote M as the Hardy-Littlewood maximal operator and $M^\sharp$ as the sharp maximal operator. We mean the sharp maximal inequality by the one of the form:

$$\|Mf \,:\, L^p(\mathbf{R}^d)\| \le C \, \|M^\sharp f \,:\, L^p(\mathbf{R}^d)\| \quad (1 < p < \infty),$$

which appeared in the paper in [1]. The primary aim of this paper is to obtain the inequality of the form

$$\left\| \left(\sum_{j=1}^{\infty} M f_j{}^q \right)^{\frac{1}{q}} \,:\, L^p(\mu) \right\| \le C \left\| \left(\sum_{j=1}^{\infty} M^\sharp f_j{}^q \right)^{\frac{1}{q}} \,:\, L^p(\mu) \right\| \quad (1 < p, q < \infty). \tag{1}$$

Throughout this paper μ will be a (positive) Radon measure on $\mathbf{R}^d$ satisfying the growth condition:

$$\mu(B(x,l)) \le C_0 \, l^n \text{ for all } x \in \operatorname{supp}(\mu) \text{ and } l > 0, \tag{2}$$

where C_0 and n, $0 < n \le d$, are some fixed numbers and $B(x,r)$ means a ball with its center x and its radius $r > 0$. We do not assume the doubling condition $\mu(B(x,2r)) \le C\mu(B(x,r))$ $(x \in \operatorname{supp}(\mu), r > 0)$. We are going to obtain (1) with the underlying measure μ satisfying only the growth condition and the function space will be the Morrey space, which is an extension of L^p space.

By "cube" $Q \subset \mathbf{R}^d$ we mean a compact cube whose edges are parallel to the coordinate axes. Its side length will be denoted by $\ell(Q)$. For $c > 0$, $c\,Q$ will denote a cube concentric to Q with its sidelength $c\,\ell(Q)$. The set of all cubes $Q \subset \mathbf{R}^d$ with positive μ-measure will be denoted by $\mathcal{Q}(\mu)$. We recall the definition of the Morrey spaces with non-doubling measures.

Let $k > 1$ and $1 \le q \le p < \infty$. We define a Morrey space $\mathcal{M}^p_q(k,\mu)$ as

$$\mathcal{M}^p_q(k,\mu) := \left\{ f \in L^q_{loc}(\mu) \,:\, \|f \,|\, \mathcal{M}^p_q(k,\mu)\| < \infty \right\},$$

where the norm $\|f \,|\, \mathcal{M}^p_q(k,\mu)\|$ is given by

$$\|f \,|\, \mathcal{M}^p_q(k,\mu)\| := \sup_{Q \in \mathcal{Q}(\mu)} \mu(k\,Q)^{\frac{1}{p}-\frac{1}{q}} \left(\int_Q |f(x)|^q \, d\mu(x) \right)^{\frac{1}{q}}. \tag{3}$$

By using Hölder's inequality to (3), it is easy to see that

$$L^p(\mu) = \mathcal{M}^p_p(k,\mu) \subset \mathcal{M}^p_{q_1}(k,\mu) \subset \mathcal{M}^p_{q_2}(k,\mu) \tag{4}$$

for $1 \le q_2 \le q_1 \le p < \infty$. The definition of the spaces does not depend on the constant $k > 1$. The norms for different choices of $k > 1$ are equivalent. For details we refer [12]. Nevertheless, for definiteness, we will assume $k = 2$ in the definition and denote $\mathcal{M}^p_q(2,\mu)$ by $\mathcal{M}^p_q(\mu)$.

Our BMO here is a RBMO (regular bounded mean oscillation) introduced by X. Tolsa which are the suitable substitutes for the classical spaces [15]. We adopt the notation due to Sawyer and Wendu, who modified the notion of Tolsa in order to develop the theory of commutators of the fractional integral operator.

Definition 1.1. (1) Let $0 \le \alpha < n$. Given two cubes $Q \subset R \in \mathcal{Q}(\mu)$, we set

$$K^{(\alpha)}_{Q,R} := 1 + \sum_{k=0}^{N_{Q,R}} \left(\frac{\mu(2^k Q)}{\ell(2^k Q)^n} \right)^{1-\alpha/n},$$

where $N_{Q,R}$ is the least integer $k \ge 0$ such that $2^k Q \supset R$. For simplicity we denote $K_{Q,R} = K^{(0)}_{Q,R}$.

(2) We say that $Q \in \mathcal{Q}(\mu)$ is a doubling cube if $\mu(2Q) \le 2^{d+1}\mu(Q)$. We denote $\mathcal{Q}(\mu,2)$ as the set of all doubling cubes.

(3) Given $Q \in \mathcal{Q}(\mu)$, we set Q^* as the smallest doubling cube R of the form $R = 2^j Q$ with $j \in \mathbf{N}_0 := \{0\} \cup \mathbf{N}$.

(4) We say that $f \in L^1_{loc}(\mu)$ is an element of RBMO if it satisfies

$$\sup_{Q \in \mathcal{Q}(\mu)} \frac{1}{\mu\left(\frac{3}{2}Q\right)} \int_Q |f(y) - m_{Q^*}(f)| \, d\mu(y) + \sup_{\substack{Q \subset R \\ Q,R \in \mathcal{Q}(\mu,2)}} \frac{|m_Q(f) - m_R(f)|}{K_{Q,R}} < \infty,$$

where $m_Q(f) := \dfrac{1}{\mu(Q)} \displaystyle\int_Q f(x) \, d\mu(x)$. We denote this quantity by $\|f\|_*$.

By the growth condition (2) there are a lot of big doubling cubes. Precisely speaking, given any cube $Q \in \mathcal{Q}(\mu)$, we can find $j \in \mathbf{N}$ with $2^j Q \in \mathcal{Q}(\mu, 2)$. Meanwhile, for μ-a.e. $x \in \mathbf{R}^d$, there exists a sequence of doubling cubes $\{Q_k\}_k$ centered at x with $\ell(Q_k) \to 0$ as $k \to \infty$. So we can say that there are a lot of small doubling cubes, too. (See [15].)

For $f \in L^1_{loc}(\mu)$ we define two maximal operators mainly due to Tolsa (see [15]): Let $0 \le \alpha < n$. The sharp maximal operator $M^{\sharp,\alpha} f(x)$ is defined as

$$M^{\sharp,\alpha} f(x) := \sup_{x \in Q \in \mathcal{Q}(\mu)} \frac{1}{\mu\left(\frac{3}{2}Q\right)} \int_Q |f(y) - m_{Q^*}(f)|\, d\mu(y) + \sup_{\substack{x \in Q \subset R \\ Q,R \in \mathcal{Q}(\mu,2)}} \frac{|m_Q(f) - m_R(f)|}{K^{(\alpha)}_{Q,R}}$$

and $Nf(x)$ is defined as $Nf(x) := \sup_{x \in Q \in \mathcal{Q}(\mu,2)} m_Q(|f|)$. The modification parameter α is introduced by Sawyer and Wendu. We also introduce a κ-times maximal operator:

$$M_\kappa f(x) := \sup_{x \in Q \in \mathcal{Q}(\mu)} \frac{1}{\mu(\kappa Q)} \int_Q |f(x)|\, d\mu(x), \ (\kappa > 1).$$

Since there are a lot of doubling cubes, we have a pointwise estimate $|f(x)| \le Nf(x)$, $|f(x)| \le \kappa^{d+1} M_\kappa f(x)$ for μ-almost all $x \in \mathbf{R}^d$. It is known that $M_\kappa : \mathcal{M}^p_q(\mu) \to \mathcal{M}^p_q(\mu)$ is a bounded operator (c.f. [12]), if $\kappa > 1$. If μ is a finite measure, we denote $m_{\mathbf{R}^d}(f) := \dfrac{1}{\mu(\mathbf{R}^d)} \displaystyle\int_{\mathbf{R}^d} f(x)\, d\mu(x)$.

Proposition 1.2. [13] *Suppose that* $1 < q \le p < \infty$, $0 \le \alpha < n$.

(1) *For any* $f \in L^1_{loc}(\mu)$, *there exists a constant* $C > 0$ *independent on* f *such that*

$$\|Nf : \mathcal{M}^p_q(\mu)\| \le C \left(\|M^{\sharp,\alpha} f : \mathcal{M}^p_q(\mu)\| + \|f : \mathcal{M}^p_1(\mu)\| \right).$$

(2) *Suppose that there exists an increasing sequence of concentric doubling cubes* $I_0 \subset I_1 \subset \ldots \subset I_k \subset \ldots$ *such that*

$$\lim_{k \to \infty} m_{I_k}(f) = 0 \ \text{and} \ \bigcup_k I_k = \mathbf{R}^d. \tag{5}$$

Then there exists a constant $C > 0$ *independent on* f *such that*

$$\|Nf : \mathcal{M}^p_q(\mu)\| \le C \, \|M^{\sharp,\alpha} f : \mathcal{M}^p_q(\mu)\|. \tag{6}$$

In particular, if μ *is finite,* (6) *is available for all* f *with* $m_{\mathbf{R}^d}(f) = 0$.

(3) *Suppose that* $\mu(\mathbf{R}^d) < \infty$. *Then we have*

$$\|Nf : \mathcal{M}^p_q(\mu)\| \le C \left(\|M^{\sharp,\alpha} f : \mathcal{M}^p_q(\mu)\| + \|f : L^1(\mu)\| \right).$$

In this paper we prove the vector-valued extension of (2) and (3) of the previous proposition. Since $\mathcal{M}^p_1(\mu)$ is contains $\mathcal{M}^p_q(\mu)$, the norms in (1) is equivalent. The condition (5) will be a key to our argument. Here we list our main results.

Theorem 1.3. *Suppose that* $1 \le q \le p < \infty$, $1 < r < \infty$, $\kappa > 1$ *and* $0 \le \alpha < n$. *Let* $f_j \in \mathcal{M}^p_q(\mu)$ *with* $j = 1, 2, \ldots$.

(1) *Assume that* $\mu(\mathbf{R}^d) = \infty$. *Then we have*

$$\left\| \left(\sum_{j=1}^\infty N f_j{}^r \right)^{\frac{1}{r}} : \mathcal{M}^p_q(\mu) \right\| \le \left\| \left(\sum_{j=1}^\infty M^{\sharp,\alpha} f_j{}^r \right)^{\frac{1}{r}} : \mathcal{M}^p_q(\mu) \right\|. \tag{7}$$

(2-a) *Assume that* $\mu(\mathbf{R}^d) < \infty$. *If* $m_{\mathbf{R}^d}(f_j) = 0$ *for all* $j = 1, 2, \ldots$, *then we have* (7).

(2-b) *Assume that* $\mu(\mathbf{R}^d) < \infty$. *Then we have for all* $\{f_j\}_{j=1}^\infty \subset \mathcal{M}_q^p(\mu)$

$$\left\| \left(\sum_{j=1}^\infty N f_j{}^r \right)^{\frac{1}{r}} : \mathcal{M}_q^p(\mu) \right\| \leq C \left\| \left(\sum_{j=1}^\infty M^{\sharp,\alpha} f_j{}^r \right)^{\frac{1}{r}} : \mathcal{M}_q^p(\mu) \right\| + C \left\{ \sum_{j=1}^\infty \left(\int_{\mathbf{R}^d} |f_j(x)| \, d\mu(x) \right)^r \right\}^{\frac{1}{r}}. \quad (8)$$

At first glance the assumption that the integrability condition $f_j \in \mathcal{M}_q^p(\mu)$ seems to be superfluous. But this assumption can be verified by using Proposition 1.2. So this seemingly strong assumption suffices. It is easy to see that $M^{\sharp,\alpha}$ is bounded pointwise by M_2. Since we have the Fefferman-Stein type inequality for M_2 on $L^p(\mu)$ spaces and on Morrey spaces $\mathcal{M}_q^p(\mu)$ (See [10] and [12].), it follows that the right-hand side and the left-hand side of the formulae of Theorem 1.3 are equivalent. By using Minkowski's inequality and $\mu(\mathbf{R}^d) < \infty$ the equivalence in (2-b) also holds. Theorem 1.3 (2-b) can be obtained from Theorem 1.3 (2-a) easily. In fact given a system of functions $f_j \in \mathcal{M}_q^p(\mu)$ $(j = 0, 1, \ldots)$, we have $f_j - m_{\mathbf{R}^d}(f_j)$ $(j = 0, 1, \ldots)$ satisfies the assumption of Theorem 1.3 (2-a). Thus we have

$$\left\| \left(\sum_{j=1}^\infty (N(f_j - m_{\mathbf{R}^d}(f_j)))^r \right)^{\frac{1}{r}} : \mathcal{M}_q^p(\mu) \right\| \leq C \left\| \left(\sum_{j=1}^\infty M^{\sharp,\alpha} f_j{}^r \right)^{\frac{1}{r}} : \mathcal{M}_q^p(\mu) \right\|,$$

which yields Theorem 1.3 (2-b). It follows from this remark that we have only to prove Theorem 1.3 (1) and (2-a). In both cases we have (5).

Remark 1.4. It is worth restating Theorem 1.3 in the case of the Lebesgue space $L^p(dx)$. Notice that if $\mu = dx$ then $M^\sharp f(x)$ is equivalent to the usual one in [1]. Applying our result with $\mu = dx$ and $1 < p = q < \infty$ and using the Fefferman-Stein vector-valued inequality, we have a norm equivalence for any countable subset $\{f_j\}_{j=1}^\infty \subset L^p(dx)$

$$\left\| \left(\sum_{j=1}^\infty M f_j{}^r \right)^{\frac{1}{r}} : L^p(dx) \right\| \sim \left\| \left(\sum_{j=1}^\infty M^{\sharp,\alpha} f_j{}^r \right)^{\frac{1}{r}} : L^p(dx) \right\|. \quad (9)$$

As an application of Theorem 1.3 we obtain the vector-valued extension of the boundedness of commutators. We mean a commutator by an operator of the form $[a, T]f(x) = a(x)Tf(x) - T(af)(x)$, where a is a function and T is a bounded operator. The classical results say that $[a, T]$ is a bounded operator from $L^p(dx)$ to $L^p(dx)$ if $a \in$BMO and T is a Carderón-Zygmund operator and that $[a, T]$ is a bounded operator from $L^p(dx)$ to $L^q(dx)$ if $a \in$BMO and T is a fractional integral operator, where p and q are suitable real numbers. Fazio and Ragusa [8] extended these results to the classical Morrey spaces. The definition will be given in the next section.

2 Preliminaries

The letter C will be used for constants that may change from one occurrence to another. Constants with subscripts, such as C_1, C_2, do not change in different occurrences. In this section we collect the known facts on the maximal operators, weighted norm inequalities and commutator operators.

In what follows we will use the notation due to Hans Triebel [16] to state the vector-valued inequality. Let X be $\mathcal{M}^p_q(\mu)$ or $L^p(\mu)$ with $1 \le q \le p < \infty$ and let $\|\cdot : X\|$ be its norm. We shall denote

$$\|f_j : X(l^r)\| := \left\| \left(\sum_{j=1}^{\infty} |f_j(\cdot)|^r \right)^{\frac{1}{r}} : X \right\|. \tag{10}$$

Thus we are going to prove that $\|M_\kappa f_j : X(l^r)\| \le \|M^{\sharp,\alpha} f_j : X(l^r)\|$.

Maximal operators For $f \in L^1_{loc}(\mu)$, $\kappa > 1$ and $0 \le \alpha < n$, a fractional maximal operator $M^\alpha_\kappa f(x)$ is defined as

$$M^\alpha_\kappa f(x) := \sup_{x \in Q \in \mathcal{Q}(\mu)} \frac{1}{\mu(\kappa Q)^{1-\frac{\alpha}{n}}} \int_Q |f(x)|\, d\mu(x).$$

It follows that by definition $M^0_\kappa = M_\kappa$. As for the boundedness of M^α_κ on the Morrey spaces, the vector-valued inequality of Fefferman-Stein type is known.

Lemma 2.1. [12] *Suppose that $\kappa > 1$, $0 \le \alpha < n$, $1 < q \le p < \infty$, $1 < t \le s < \infty$, $1 < r < \infty$, $1/s = 1/p - \alpha/n$ and $t/s = q/p$. Then we have*

$$\|M^\alpha_\kappa f_j : \mathcal{M}^s_t(l^r)\| \le C \left\| f_j : \mathcal{M}^p_q(l^r) \right\|.$$

In particular we have (noting $Nf(x) \le CM_2 f(x)$ μ-a.e.) $\|N f_j : \mathcal{M}^s_t(l^r)\| \le C \left\| f_j : \mathcal{M}^p_q(l^r) \right\|$.

We use a covering lemma of Besicovitch type:

Lemma 2.2. *Let $\kappa > 1$. Suppose that $\{R_j\}_{j \in J}$ is a family of cubes with bounded diameters. Then we can find a subset J_0 in J such that*

$$\bigcup_{j \in J} R_j \subset \bigcup_{j \in J_0} \kappa R_j, \ \sum_{j \in J_0} \chi_{R_j}(x) \le C_\kappa. \tag{11}$$

Here we used χ_A to denote the indicator function of $A \subset \mathbf{R}^d$.

As for the weak-type assertion we have the following proposition, which will be obtained easily by using Lemma 2.2.

Lemma 2.3. *Suppose that $\kappa > 1$. Then we have*

$$\mu\{x \in \mathbf{R}^d : M_\kappa f(x) > \lambda\} \le \frac{C_\kappa}{\lambda} \int_{\mathbf{R}^d} |f(x)|\, d\mu(x),$$

where C_κ is the same constant as that in Lemma 2.2.

Weighted norm inequality

In order to prove Theorem 1.3 we use the technique of the weighted norm inequality. In [5], Komori considered weighted norm inequalities with respect to M_κ. He considered weights with Radon measure μ, where μ does not necessarily satisfy the doubling condition nor the growth condition. In this paper we consider a class of weight which is "almost" in $A_1(\mu)$. Here we consider the double-weighted norm inequality:

$$\int_{\{x\in\mathbf{R}^d\,;\,M_\kappa f(x)>\lambda\}} u(x)\,d\mu(x) \le \frac{C}{\lambda}\int_{\mathbf{R}^d} |f(x)|v(x)\,d\mu(x). \tag{12}$$

Our problem is that for given $\kappa > 1$ we have to find a pair (u, v) with (12). If the measure μ is doubling, as in [3], this is equivalent to $M_\kappa u(x) \le Cv(x)$. The following result is due to Komori [5]. His result contains L^p-version but here we need L^1-assertion only :

Proposition 2.4. *Let $\kappa > \kappa' > 1$. Suppose that a locally μ-integrable function w satisfies $M_{\kappa'}w(x) \le Cw(x)$ for some positive constant C. Then it holds that*

$$\int_{\{x\in\mathbf{R}^d\,:\,M_\kappa f(x)>\lambda\}} w(x)\,d\mu(x) \le \frac{C}{\lambda}\int_{\mathbf{R}^d} |f(x)|w(x)d\mu(x). \tag{13}$$

Although he considered single weighted norm inequalities, this results can be readily translated into the double-weighted norm inequality. With minor modification of the proof of the previous proposition, we can prove the following proposition which will be used later.

Proposition 2.5. *Let $\kappa > \kappa' > 1$ and u, v be μ-locally integrable functions. Suppose that $M_{\kappa'}u(x) \le Cv(x)$. Then we have*

$$\int_{\{x\in\mathbf{R}^n\,:\,M_\kappa f(x)>\lambda\}} u(x)d\mu(x) \le \frac{C}{\lambda}\int_{\mathbf{R}^n} |f(x)|v(x)d\mu(x). \tag{14}$$

Komori showed that in his class of weights, there is no equivalence as in the classical case. The rate of modification parameter κ is a barrier for it. We wish to obtain an estimate such as

$$\int_{\{x\in\mathbf{R}^d\,:\,M_\kappa f(x)>\lambda\}} w(x)d\mu(x) \le \frac{C}{\lambda}\int_{\mathbf{R}^d} |f(x)|M_\kappa w(x)d\mu(x) \quad (\kappa > 1).$$

But this is false. In fact we can show that it is false by constructing a counterexample for which we cannot take $\kappa = 3$. It can be found in [11], for completeness we gave the counterexample and the proof in Appendix.

Commutator operators Here we list some definitions and known results needed to state our commutator theorems.

Definition 2.6. ([7] p466) The singular integral operator T is a bounded linear operator on $L^2(\mu)$ with a kernel function K that satisfies the following three properties :

(1) For some appropriate constant $C > 0$, we have $|K(x, y)| \le \dfrac{C}{|x-y|^n}$, where n is a constant in the growth condition (2).

(2) There exist constants $\varepsilon > 0$ and $C > 0$ such that

$$|K(x, y) - K(z, y)| + |K(y, x) - K(y, z)| \le C\,\frac{|x-z|^\varepsilon}{|x-y|^{n+\varepsilon}} \text{ if } |x-y| > 2|x-z|.$$

(3) If f is a bounded measurable function with a compact support, then we have

$$Tf(x) = \int_{\mathbf{R}^d} K(x, y)f(y)\,d\mu(y) \text{ for all } x \notin \operatorname{supp}(f).$$

Definition 2.7. ([4] Definition 3.1) Let $0 < \alpha < n$. Then we define a fractional integral operator I_α by

$$I_\alpha f(x) := \int_{\mathbf{R}^d} \frac{f(y)}{|x-y|^{n-\alpha}} \, d\mu(y),$$

where n is a constant in the growth condition (2).

It is well-known that T is a bounded operator on $L^p(\mu)$ if $1 < p < \infty$ (see [7]) and I_α is a bounded operator from $L^p(\mu)$ to $L^q(\mu)$ if $1 < p < q \le \infty$ and $1/q = 1/p - \alpha/n$ (see [4]). In [12] it is also proved that T is a bounded operator on $\mathcal{M}^p_q(\mu)$ if $1 < q \le p < \infty$ and I_α is a bounded operator from $\mathcal{M}^p_q(\mu)$ to $\mathcal{M}^s_t(\mu)$ if

$$1 < q \le p < \infty, \quad 1 < t \le s < \infty, \quad 1/s = 1/p - \alpha/n \text{ and } t/s = q/p. \tag{15}$$

Next we shall introduce the commutator results for these operators.

Proposition 2.8. *Suppose that $a \in$RBMO. Let $1 < q \le p < \infty$ and T be a singular integral operator with associated kernel K. Then*

$$[a,T]f(x) := \lim_{\varepsilon \to 0} \int_{|x-y|>\varepsilon} (a(x) - a(y))K(x,y)f(y) \, d\mu(y)$$

can be extended to a bounded operator on $\mathcal{M}^p_q(\mu)$.

Proposition 2.9. *Suppose that $a \in$RBMO. If the parameters satisfy* (15) *and that $1 < r < \infty$, then*

$$[a,I_\alpha]f(x) := \lim_{\varepsilon \to 0} \int_{|x-y|>\varepsilon} \frac{(a(x) - a(y))}{|x-y|^{n-\alpha}} f(y) \, d\mu(y)$$

can be extended to a bounded operator from $\mathcal{M}^p_q(\mu)$ to $\mathcal{M}^s_t(\mu)$.

In proving the vector-valued estimate, we will need the ones for I_α and a singular integral T.

Proposition 2.10. [12] *Suppose that the parameters satisfy* (15) *and that $1 < r < \infty$. Let T be a singular integral operator. Then we have*

$$\|I_\alpha f_j \,:\, \mathcal{M}^s_t(l^r)\| \le C \left\| f_j \,:\, \mathcal{M}^p_q(l^r) \right\| \text{ and } \left\| T f_j \,:\, \mathcal{M}^p_q(l^r) \right\| \le C \left\| f_j \,:\, \mathcal{M}^p_q(l^r) \right\|.$$

3 Weighted norm estimates

3.1 A_1-properties

In this section we prove estimates with weights. In the first subsection we remove the growth condition on μ. We consider the following problem:

Problem 3.1. *Given $\kappa > 1$, find a condition for locally integrable functions (u,v) such that $M_\kappa u(x) \le C v(x)$.*

Set $w = (M_a g)^\delta$, where $a > 1$ and δ is a real number slightly less than 1. If the measure μ satisfies the doubling condition, it is well-known that $w \in A_1(\mu)$, that is, $M_\kappa w(x) \le C w(x)$ for

μ-a.e.. But now we are in the non-doubling situation, so that unfortunately we have to modify the notion of $A_1(\mu)$ weights. Only in this subsection don't we have to pose the condition on μ: it suffices to assume that μ is just a Radon measure on $\mathbf{R}^d$.

Simple calculation yields the following lemma.

Lemma 3.2. *Let $\kappa > b > 1$. Then $C_{b,\kappa} := b + \dfrac{4b^2}{\kappa - b}$ satisfies the following property.*
Property: Let $a > C_{b,\kappa}$ and $Q, R \in \mathcal{Q}(\mu)$. If R meets both Q and $\mathbf{R}^d \setminus \kappa b^{-1} Q$, then we have $Q \subset ab^{-1}R$.

Proof of Lemma 3.2. By considering each component, we may assume that $d = 1$. Since in this lemma we don't have to consider the measure μ, we may normalize Q to have $Q = [-1, 1]$. In this case we have $\ell(R) > \kappa b^{-1} - 1$. Thus if $(\kappa b^{-1} - 1)(ab^{-1} - 1) > 4$, that is, $a > C_{b,\kappa}$, we have $Q \subset ab^{-1}R$. □

Using this lemma, we will prove

Theorem 3.3. *Let $\kappa > b > 1$, $a > C_{b,\kappa}$, $0 < \delta < 1$ and $0 \le \varepsilon < \dfrac{1}{\delta} - 1$. For a locally integrable function f with $M_a f(x) < \infty$ μ-a.e. we have*

$$M_\kappa\{(M_a f)^{\delta(1+\varepsilon)}\}(x)^{\frac{1}{1+\varepsilon}} \le C_{\delta,a,b,\varepsilon,\kappa} M_b f(x)^\delta. \tag{16}$$

Here C is a constant depending only on $\delta, a, b, \varepsilon, \kappa$ not on f.

Remark 3.4. Before proving this theorem, we collect some corollaries of this theorem.

(1) Letting $\varepsilon = 0$, we obtain $M_\kappa\{(M_a f)^\delta\}(x) \le C_{a,b,\delta,\kappa} M_b f(x)^\delta$. This is a substitute of $A_1(\mu)$ weight in our theory. Since we can take κ' so that $\kappa > \kappa' > 1$ and $a > C_{\kappa',b}$, we can apply Proposition 2.5 with $u = (M_a w)^\delta$ and $v = (M_b w)^\delta$, where h is a μ-locally integrable function. Then we obtain

$$\int_{\{x \in \mathbf{R}^n \,:\, M_\kappa f(x) > \lambda\}} (M_a w)^\delta(x) d\mu(x) \le \frac{C}{\lambda} \int_{\mathbf{R}^n} |f(x)| (M_b w)^\delta(x) d\mu(x). \tag{17}$$

(2) Suppose that Q satisfies $\mu(\kappa Q) \le \kappa^{d+1} \mu(Q)$. Then this theorem yields

$$\begin{aligned}\left(\frac{1}{\mu(\kappa Q)} \int_Q M_a f(x)^{\delta(1+\varepsilon)} d\mu(x)\right)^{\frac{1}{1+\varepsilon}} &\le \left(C_{\delta,a,b,\varepsilon} \inf_Q M_b f(x)^\delta\right) \\ \le \frac{C_{\delta,a,b,\varepsilon}}{\mu(Q)} \int_Q M_b f(x)^\delta d\mu(x) &\le \frac{C_{\delta,a,b,\varepsilon,\kappa}}{\mu(\kappa Q)} \int_Q M_b f(x)^\delta d\mu(x).\end{aligned}$$

Thus it follows that

$$\left(\frac{1}{\mu(\kappa Q)} \int_Q M_a f(x)^{\delta(1+\varepsilon)} d\mu(x)\right)^{\frac{1}{1+\varepsilon}} \le \frac{C_{\delta,a,b,\varepsilon,\kappa}}{\mu(\kappa Q)} \int_Q M_b f(x)^\delta d\mu(x). \tag{18}$$

This will be a substitute of the Reverse Hölder's inequality.

Proof of Theorem 3.3. By putting $\delta' = \delta(1+\varepsilon)$ and rewriting δ' by δ, we may assume $\varepsilon = 0$. Fix a cube $Q \in \mathcal{Q}(\mu)$. We decompose f according to $\kappa b^{-1} Q$: $f = f_1 + f_2$, where $f_1 = f 1_{\kappa b^{-1} Q}$.

Noting that M_a is weak-$(1,1)$ bounded (c.f. Lemma 2.3), we are in the position of using Kolmogorov's lemma. Thus it follows that

$$\frac{1}{\mu(\kappa Q)} \int_Q (M_a f_1)^\delta(y)\, d\mu(y) \le C \left(\frac{1}{\mu(\kappa Q)} \int_{\kappa b^{-1} Q} |f(y)|\, d\mu(y) \right)^\delta \le C(M_b f(x))^\delta. \tag{19}$$

Since Q is arbitrary as long as $x \in Q$, it follows that $M_\kappa\{(M_a f_1)^\delta\}(x) \le C M_b f(x)^\delta$. Hence the estimate for f_1 is over. Let us turn our attention to the estimate of f_2. By Lemma 3.2, $R \cap (\kappa b^{-1} Q)^c \ne \emptyset$ implies that $Q \subset a b^{-1} R$. Thus we have for all $y \in Q$

$$\begin{aligned} M_a f_2(y) &= \sup_{\substack{R \in \mathcal{Q}(\mu) \\ y \in bR}} \frac{1}{\mu(aR)} \int_R |f_2(z)|\, d\mu(z) \\ &\le \sup_{\substack{R \in \mathcal{Q}(\mu) \\ Q \subset a b^{-1} R}} \frac{1}{\mu(aR)} \int_{a b^{-1} R} |f_2(z)|\, d\mu(z) \\ &\le \sup_{\substack{S \in \mathcal{Q}(\mu) \\ Q \subset bS}} \frac{1}{\mu(bS)} \int_S |f(z)|\, d\mu(z) \le M_b f(x). \end{aligned}$$

Thus it follows that

$$\frac{1}{\mu(\kappa Q)} \int_Q (M_a f_2)^\delta(y)\, d\mu(y) \le C M_b f(x)^\delta. \tag{20}$$

This is what we want for f_2. Combining (19) and (20), we obtain Theorem 3.3. □

Before finishing this section, we state one more corollary of this theorem.

Corollary 3.5. *Suppose that the parameters a, b, κ, δ and function f are the same as those in Theorem 3.3. $Q \in \mathcal{Q}(\mu)$ satisfies $\mu(\alpha\kappa Q) \le K\mu(Q)$ for some $\alpha \ge 1$ and $K > 0$. Then we have for any μ-measurable subset $A \subset \alpha Q$*

$$\int_A M_a f(x)^\delta\, d\mu(x) \le C \left(\frac{\mu(A)}{\mu(Q)} \right)^{\frac{\varepsilon}{1+\varepsilon}} \int_Q M_b f(x)^\delta\, d\mu(x), \tag{21}$$

where $\varepsilon = (1-\delta)/2\delta$.

Proof of Corollary 3.5. In fact we have by Remark 3.4

$$\begin{aligned} \int_A M_a f(x)^\delta\, d\mu(x) &\le \left(\int_A M_a f(x)^{\delta(1+\varepsilon)}\, d\mu(x) \right)^{\frac{1}{1+\varepsilon}} \mu(A)^{\frac{\varepsilon}{1+\varepsilon}} \\ &\le \left(\int_{\alpha Q} M_a f(x)^{\delta(1+\varepsilon)}\, d\mu(x) \right)^{\frac{1}{1+\varepsilon}} \mu(A)^{\frac{\varepsilon}{1+\varepsilon}} \\ &\le C \left(\frac{1}{\mu(\alpha\kappa Q)} \int_{\alpha Q} M_a f(x)^{\delta(1+\varepsilon)}\, d\mu(x) \right)^{\frac{1}{1+\varepsilon}} \mu(Q)^{\frac{1}{1+\varepsilon}} \mu(A)^{\frac{\varepsilon}{1+\varepsilon}} \\ &\le C \inf_{x \in Q} M_\kappa \left(M_a f^{\delta(1+\varepsilon)} \right)(x)^{\frac{1}{1+\varepsilon}} \mu(Q)^{\frac{1}{1+\varepsilon}} \mu(A)^{\frac{\varepsilon}{1+\varepsilon}} \\ &\le C \left(\int_Q M_b f(x)^\delta\, d\mu(x) \right) \left(\frac{\mu(A)}{\mu(Q)} \right)^{\frac{\varepsilon}{1+\varepsilon}}. \end{aligned}$$

This is the desired. □

3.2 Good λ-inequality

In this subsection we apply the results in the previous subsection with $\kappa = \frac{9}{5}$. In the sequel we assume that μ satisfies the growth condition again. Let $C_0 := C_{\frac{9}{5},\frac{3}{2}} = \dfrac{63}{2}$, where $C_{b,\kappa}$ is a constant appearing in Lemma 3.2. To simplify the notation of the weighted measure, we use $w(A)$ to denote $\int_A w(x)\,d\mu(x)$, where w is a positive measurable function and A is μ-measurable. This is the main theorem of this section.

Theorem 3.6. *Suppose that the parameters satisfy* $0 \le \alpha < n,\ \lambda > 0,\ 0 < \varepsilon < \dfrac{1}{\delta} - 1$. *Assume that a μ-locally integrable function f and an increasing sequence of doubling cube $\{Q_j\}$ satisfy* (5). *Then there is a constant $C > 0$ and η_0 depending only on the parameters such that if $0 < \eta < \eta_0$ we have*

$$\begin{aligned}&(M_{36}w)^{\delta}\left(\{x \in \mathbf{R}^d \,:\, M^{\sharp,\alpha}f(x) \le \eta\lambda, Nf(x) > 2\lambda\}\right)\\ \le\ & C_1\eta^{\frac{\varepsilon}{1+\varepsilon}}\left(M_{\frac{3}{2}}w\right)^{\delta}\left(\{x \in \mathbf{R}^d \,:\, Nf(x) > \lambda\}\right).\end{aligned}$$

Proof of Theorem 3.6. The proof will be similar to that of Theorem 6.2 in [15] except that we are considering the weighted norm inequality. So that we omit some details. For the proof we set

$$E_\lambda := \{x \in \mathbf{R}^d \,:\, M^{\sharp,\alpha}f(x) \le \delta\lambda, Nf(x) > 2\lambda\} \text{ and } \Omega_\lambda := \{x \in \mathbf{R}^d \,:\, Nf(x) > \lambda\}.$$

We may assume that $E_\lambda \ne \emptyset$, otherwise we have nothing to prove.

Let $x \in E_\lambda$. Then by the definition of E_λ, there exists a doubling cube $R = R_x \in \mathcal{Q}(\mu, 2)$ such that $m_R(|f|) > 2\lambda$.

Claim 3.7. *By taking η sufficiently small, we can even arrange that the following condition holds: Suppose that S_x is a doubling cube containing $2R_x$. Then we have* $m_{S_x}(|f|) \le \dfrac{5}{4}\lambda$.

Proof of Claim 3.7. Suppose otherwise. Then defining inductively $R_k (k = 0, 1, 2, \ldots)$ by $R_k = (2R_{k-1})^*, R_0 = R_x$, we can find a sequence of concentric doubling cubes $R_1, R_2, \ldots$ such that $\lim\limits_{k\to\infty} \ell(R_k) = \infty$ and $m_{R_k}(|f|) > \dfrac{5}{4}\lambda$. If j is large, x is contained in Q_j appearing in the assumption (5). Suppose that $\ell(R_k) \le \ell(Q_j) \le \ell(R_{k+1})$. Then $K_{R_k,Q_j} \le C$. Thus we have $|m_{R_k}(f) - m_{Q_j}(f)| \le CM^{\sharp,\alpha}f(x) \le C\eta\lambda$. If η is sufficiently small, we have $|m_{R_k}(f) - m_{Q_j}(f)| < \dfrac{1}{4}\lambda$. By assumption we have $\lim\limits_{j\to\infty} m_{Q_j}(f) = 0$ thus if j is large enough, we have $|m_{Q_j}(f)| \le \dfrac{1}{8}\lambda$, which implies that $|m_{R_k}(f)| \le \dfrac{3}{8}\lambda$. Thus it follows that

$$m_{R_k}(|f|) \le m_{R_k}(|f - m_{R_k}(f)|) + |m_{R_k}(f)| \le \frac{3}{8}\lambda + 2^{d+1}\eta\lambda.$$

Combining them, if η is small enough and k is large enough, we have $m_{R_k}(|f|) \le \lambda$. This contradicts to the construction of R_k, in fact we have taken R_k so that $m_{R_k}(|f|) > 2\lambda$. Thus our claim is proved. $\square$

Let us return to the proof of the theorem. By Linderöf's covering lemma we can find a countable subset $E_{\lambda,0}$ such that $E_\lambda \subset \bigcup\limits_{x\in E_{\lambda,0}} R_x$. Thus we have $(M_a w)^\delta(E_\lambda) \le (M_a w)^\delta\left(\bigcup\limits_{x\in E_{\lambda,0}} R_x\right)$.

Hence it suffices to estimate $\int_{\bigcup_{x\in F} R_x} M_a w(x)^\delta \, d\mu(x)$ for a finite subset F in $E_{\lambda,0}$ independently on F. By using Lemma 2.2, we can take a subset $F_0 \subset F$ satisfying

$$\bigcup_{x\in F} R_x \subset \bigcup_{x\in F_0} \frac{10}{9} R_x, \ \sum_{x\in F_0} \chi_{R_x}(x) \le C\chi_{\Omega_\lambda}(x). \tag{22}$$

By using the similar argument in [15] it follows that $\mu\left(E_\lambda \cap \frac{10}{9}R_x\right) \le C\eta\mu\left(\frac{10}{9}R_x\right)$ for all $x \in E_{\lambda,0}$, if η is sufficiently small.

By Corollary 3.5 with $\kappa = \dfrac{9}{5}, \alpha = \dfrac{10}{9}$ we have

$$(M_{36}w)^\delta \left(E_\lambda \cap \frac{10}{9}R_x\right) \le C\eta^{\frac{\varepsilon}{1+\varepsilon}} (M_{\frac{3}{2}}w)^\delta(R_x). \tag{23}$$

Combining (22) and (23), we have the desired result. $\square$

As a corollary, by distribution formula, we have the following results. Here we replaced $\eta^{\frac{\varepsilon}{1+\varepsilon}}$ by η and used $M_{36}f(x) \le M_{\frac{3}{2}}f(x)$ for all μ-measurable function f.

Corollary 3.8. *Let $1 < p < \infty$. Under the same assumption of the previous theorem for small $\eta > 0$, we have*

$$\begin{aligned}
&\int_{\mathbf{R}^d} Nf(x)^p M_{36}w(x)^\delta \, d\mu(x) \\
\le\ & C_{\eta,\delta} \int_{\mathbf{R}^d} M^{\sharp,\alpha} f(x)^p M_{\frac{3}{2}} w(x)^\delta d\mu(x) + C_2 \cdot \eta \int_{\mathbf{R}^d} Nf(x)^p M_{\frac{3}{2}} w(x)^\delta d\mu(x),
\end{aligned}$$

where C_2 is dependent not on η but on δ.

4 Proof of Theorem 1.3

4.1 A technical lemma

To prove Theorem 1.3 we need the following lemma. In what follows we usually use the letters u, v, w to indicate the parameters with $1 < u, v, w < \infty$: We do not use the letter u, v, w to denote weight functions.

Lemma 4.1. *Suppose that the parameters satisfy $1 < v < \infty, \ 1 < u \le w, \ \max\left(\dfrac{1}{v'}, \dfrac{1}{w'}\right) < \delta < 1$. Then there exists a constant C independent of Q such that*

$$\mu(100Q)^{\frac{1}{w}-\frac{1}{u}} \int_{\mathbf{R}^d} \left(\sum_{j=1}^\infty |h_j(x)| M_{\frac{3}{2}}(|g_j|^{\frac{1}{\delta}})(x)^\delta\right) d\mu(x) \le C \sup_{R\in\mathcal{Q}(\mu)} \mu(2R)^{\frac{1}{w}-\frac{1}{u}} \|\chi_R h_j \ : \ L^u(l^v)\|$$

for all μ-measurable functions g_j, h_j $(j = 1, 2, \dots)$ with $\operatorname{supp}(g_j) \subset Q$ *and* $\|g_j \ : \ L^{u'}(l^{v'})\| \le 1$.

Proof of Lemma 4.1. Firstly, we estimate $\mu(100Q)^{\frac{1}{w}-\frac{1}{u}} \int_{50Q} \left(\sum_{j=1}^{\infty} |h_j(x)| M_{\frac{3}{2}} |g_j|^{\frac{1}{\delta}}(x)^{\delta} \right) d\mu(x)$. This is easily done by using Hölder's inequality and Lemma 2.1.

$$\begin{aligned} &\mu(100Q)^{\frac{1}{w}-\frac{1}{u}} \int_{50Q} \left(\sum_{j=1}^{\infty} |h_j(x)| M_{\frac{3}{2}} |g_j|^{\frac{1}{\delta}}(x)^{\delta} \right) d\mu(x) \\ \le\; & \mu(100Q)^{\frac{1}{w}-\frac{1}{u}} \|\chi_{50Q} h_j \,:\, L^u(l^v)\| \cdot \|\chi_{50Q} M_{\frac{3}{2}}(|g_j|^{\frac{1}{\delta}}) \,:\, L^{\delta u'}(l^{\delta v'})\|^{\delta} \\ \le\; & C \sup_{R \in \mathcal{Q}(\mu)} \mu(2R)^{\frac{1}{w}-\frac{1}{u}} \|\chi_R h_j \,:\, L^u(l^v)\|. \end{aligned}$$

Thus the estimate of the integral on $50Q$ is finished.

In what follows we concentrate on the integral over $\mathbf{R}^d \setminus 50Q$:

$$\mathrm{I} := \mu(100Q)^{\frac{1}{w}-\frac{1}{u}} \int_{\mathbf{R}^d \setminus 50Q} \left(\sum_{j=1}^{\infty} |h_j(x)| \{M_{\frac{3}{2}}(|g_j|^{\frac{1}{\delta}})(x)\}^{\delta} \right) d\mu(x).$$

By using Hölder's inequality once and noticing $u \le w$, we have

$$\mathrm{I} \le \mu(Q)^{\frac{1}{w}-\frac{1}{u}} \left(\int_{\mathbf{R}^d \setminus 50Q} \left(\sum_{j=1}^{\infty} |h_j(x)|^v \right)^{\frac{1}{v}} \cdot \left(\sum_{j=1}^{\infty} \{M_{\frac{3}{2}}(|g_j|^{\frac{1}{\delta}})(x)\}^{\delta v'} \right)^{\frac{1}{v'}} d\mu(x) \right) := \mathrm{II}.$$

By the definition of $M_{\frac{3}{2}}$, we have $M_{\frac{3}{2}}(|g_j|^{\frac{1}{\delta}})(x)$ is less than or equal to

$$\sup_{\substack{R \in \mathcal{Q}(\mu) \\ \{x\} \cup Q \subset R}} \frac{1}{\mu\left(\frac{7}{5}R\right)} \int_Q |g_j(x)|^{\frac{1}{\delta}} d\mu(x) = \left(\int_Q |g_j(z)|^{\frac{1}{\delta}} d\mu(z) \right) \cdot \sup_{\substack{R \in \mathcal{Q}(\mu) \\ \{x\} \cup Q \subset R}} \frac{1}{\mu\left(\frac{7}{5}R\right)}$$

for all $x \in \mathbf{R}^d \setminus 50Q$. Setting $T_Q(x) := \sup_{\substack{R \in \mathcal{Q}(\mu) \\ \{x\} \cup Q \subset R}} \frac{\mu(Q)}{\mu\left(\frac{7}{5}R\right)}$ for $x \in \mathbf{R}^d \setminus 50Q$, it follows that $M_{\frac{3}{2}}(|g_j|^{\frac{1}{\delta}})(x) \le \left(\frac{1}{\mu(Q)} \int_Q |g_j(z)|^{\frac{1}{\delta}} d\mu(z) \right) T_Q(x)$ holds. Plugging this into II, we have

$$\begin{aligned} \mathrm{II} \;\le\; & \mu(Q)^{\frac{1}{w}-\frac{1}{u}} \cdot \int_{\mathbf{R}^d \setminus 50Q} \left(\sum_{j=1}^{\infty} |h_j(x)|^v \right)^{\frac{1}{v}} \left(\sum_{j=1}^{\infty} \left(\frac{T_Q(x)}{\mu(Q)} \int_Q |g_j(z)|^{\frac{1}{\delta}} d\mu(z) \right)^{\delta v'} \right)^{\frac{1}{v'}} d\mu(x) \\ \le\; & \cdot \left(\mu(Q)^{\frac{1}{w}-\frac{1}{u}} \int_{\mathbf{R}^d \setminus 50Q} \left(\sum_{j=1}^{\infty} |h_j(x)|^v \right)^{\frac{1}{v}} \cdot T_Q(x)^{\delta} \, d\mu(x) \right) \\ & \times \left(\sum_{j=1}^{\infty} \left(\frac{1}{\mu(Q)} \int_Q |g_j(z)|^{\frac{1}{\delta}} d\mu(z) \right)^{\delta v'} \right)^{\frac{1}{v'}}. \end{aligned}$$

By Minkowski's inequality and the assumption on g_j's, we have

$$\left\{ \sum_{j=1}^{\infty} \left(\frac{1}{\mu(Q)} \int_Q |g_j(z)|^{\frac{1}{\delta}} d\mu(z) \right)^{\delta v'} \right\}^{\frac{1}{v'}} \le \left(\frac{1}{\mu(Q)} \int_Q \left(\sum_{j=1}^{\infty} |g_j(z)|^{v'} \right)^{\frac{1}{\delta v'}} d\mu(x) \right)^{\delta} \le \mu(Q)^{-\frac{1}{u'}}.$$

Thus it follows that $\mathrm{II} \le \mu(Q)^{\frac{1}{w}-1} \cdot \left(\int_{\mathbf{R}^d \setminus 50Q} \left(\sum_{j=1}^{\infty} |h_j(x)|^v \right)^{\frac{1}{v}} \cdot T_Q(x)^\delta \, d\mu(x) \right)$. Denoting O as an origin, we obtain by monotone convergence theorem

$$\mathrm{II} \le \lim_{r \to \infty} \mu(Q)^{\frac{1}{w}-1} \cdot \left(\int_{B(O,r) \setminus 50Q} \left(\sum_{j=1}^{\infty} |h_j(x)|^v \right)^{\frac{1}{v}} \cdot T_Q(x)^\delta \, d\mu(x) \right).$$

We define $S_{l,r}$ for $r \gg 1$ by $S_{l,r} := \{x \in B(O,r) \setminus 50Q \mid 2^{-l} < T_Q(x) \le 2^{-l+1}\}$, where $l = 1, 2, \ldots$. Notice that $B(O,r) \setminus 50Q$ can be separated into a disjoint union of $\{S_{l,r}\}_{l=1}^{\infty}$, since $T_Q(x) \le 1$ for all $x \in \mathbf{R}^d \setminus 50Q$. Using this partition, we have

$$\begin{aligned}
\mathrm{II} &\le C \lim_{r \to \infty} \mu(Q)^{\frac{1}{w}-1} \left(\int_{B(O,r) \setminus 50Q} \left(\sum_{j=1}^{\infty} |h_j(x)|^v \right)^{\frac{1}{v}} T_Q(x)^\delta \, d\mu(x) \right) \\
&\le C \lim_{r \to \infty} \sum_{l=1}^{\infty} 2^{-l\delta} \mu(Q)^{\frac{1}{w}-1} \left(\int_{S_{l,r}} \left(\sum_{j=1}^{\infty} |h_j(x)|^v \right)^{\frac{1}{v}} d\mu(x) \right) \\
&\le C \lim_{r \to \infty} \sum_{l=1}^{\infty} 2^{-l\delta} \mu(Q)^{\frac{1}{w}-1} \mu(S_{l,r})^{\frac{1}{u'}} \| \chi_{S_{l,r}} h_j \, : \, L^u(l^v) \|.
\end{aligned}$$

By using Lemma 2.2 and the definition of $S_{l,r}$, there are cubes $R_{l,r}^{(m)} (m = 1, \ldots, N)$ such that $S_{l,r} \subset \bigcup_{m=1}^{N} \frac{6}{5} R_{l,r}^{(m)}$, and that $\mu\left(\frac{7}{5} R_{l,r}^{(m)}\right) \sim 2^{-l} \mu(Q)$, where N is a number independent of l and r. Using this covering, we can proceed

$$\mathrm{I} \le C_N \lim_{r \to \infty} \sum_{l=1}^{\infty} 2^{-l(\delta + \frac{1}{w} - 1)} \mu\left(\frac{7}{5} R_{l,r}^{(\alpha)}\right)^{\frac{1}{w} - \frac{1}{u}} \left\| \chi_{\frac{6}{5} R_{l,r}^{(\alpha)}} h_j \, : \, L^u(l^v) \right\|.$$

By assumption $1/w' < \delta < 1$, the series converges so we have the desired.

□

4.2 Proof of 1.3 (1),(2-a)

Proof of Theorem 1.3 (1),(2-a). Throughout the proof we may assume that $f_n \equiv 0$ with finite exception due to the monotone convergence theorem. In the proof we can always use (5) because we limit ourselves to the proof of Theorem 1.3 (1) and (2-a).

We take an auxiliary t such that $1 < t < \min(q, r)$. And we fix a cube Q. Then we are to estimate $\mathrm{I} := \mu(100Q)^{\frac{1}{p} - \frac{1}{q}} \left(\int_Q \left(\sum_{j=1}^{\infty} N f_j(x)^r \right)^{\frac{q}{r}} d\mu(x) \right)^{\frac{1}{q}}$.

For this purpose put $u = q/t,\ v = r/t,\ w = p/t$, then we have $1 < u \le w < \infty$ and $1 < v < \infty$. We rewrite I as follows: $\mathrm{I} = \left\{ \mu(100Q)^{\frac{1}{w}-\frac{1}{u}} \left(\int_Q \left(\sum_{j=1}^{\infty} (Nf_j(x)^t)^v \right)^{\frac{u}{v}} d\mu(x) \right)^{\frac{1}{u}} \right\}^{\frac{1}{t}}$.

By using $L^v(l^u)$-$L^{v'}(l^{u'})$ duality, where $u' = u/(u-1)$ and $v' = v/(v-1)$, there exists a system of functions $\{g_j\}_{j=1}^{\infty}$ supported on Q such that

$$\mathrm{I} = \left\{ \mu(100Q)^{\frac{1}{w}-\frac{1}{u}} \left(\int_Q \sum_{j=1}^{\infty} Nf_j(x)^t g_j(x)\, d\mu(x) \right) \right\}^{\frac{1}{t}} \text{ and } \|g_j \,:\, L^{u'}(l^{v'})\| = 1.$$

Take an auxiliary δ so that $\max\left(\frac{1}{v'}, \frac{1}{w'}\right) < \delta < 1$. We have from Corollary 3.8 that

$$\begin{aligned}
&\mathrm{I}\\
&\le C \left\{ \mu(100Q)^{\frac{1}{w}-\frac{1}{u}} \left(\int_Q \sum_{j=1}^{\infty} \left(Nf_j(x)^t \{M_{36}(|g_j|^{\frac{1}{\delta}})(x)\}^{\delta} \right) d\mu(x) \right) \right\}^{\frac{1}{t}}\\
&\le \left\{ \mu(100Q)^{\frac{1}{w}-\frac{1}{u}} \left(\sum_{j=1}^{\infty} \int_{\mathbf{R}^d} (C_{\eta,\delta} M^{\sharp,\alpha} f_j(x)^t + C_2 \eta N f_j(x)^t) \{M_{\frac{3}{2}}(|g_j|^{\frac{1}{\delta}})(x)\}^{\delta}\, d\mu(x) \right) \right\}^{\frac{1}{t}}\\
&\le \left\{ C_{\eta,\delta}\, \mu(100Q)^{\frac{1}{w}-\frac{1}{u}} \left(\sum_{j=1}^{\infty} \int_{\mathbf{R}^d} M^{\sharp,\alpha} f_j(x)^t \{M_{\frac{3}{2}}(|g_j|^{\frac{1}{\delta}})(x)\}^{\delta}\, d\mu(x) \right) \right.\\
&\qquad \left. + C_2 \eta \mu(100Q)^{\frac{1}{w}-\frac{1}{u}} \left(\sum_{j=1}^{\infty} \int_{\mathbf{R}^d} N f_j(x)^t \{M_{\frac{3}{2}}(|g_j|^{\frac{1}{\delta}})(x)\}^{\delta}\, d\mu(x) \right) \right\}^{\frac{1}{t}}\\
&= \left\{ C_{\eta,\delta}\, \mu(100Q)^{\frac{1}{w}-\frac{1}{u}} \int_{\mathbf{R}^d} \left(\sum_{j=1}^{\infty} M^{\sharp,\alpha} f_j(x)^t \{M_{\frac{3}{2}}(|g_j|^{\frac{1}{\delta}})(x)\}^{\delta} \right) d\mu(x) \right.\\
&\qquad \left. + C_2 \eta \mu(100Q)^{\frac{1}{w}-\frac{1}{u}} \int_{\mathbf{R}^d} \left(\sum_{j=1}^{\infty} N f_j(x)^t \{M_{\frac{3}{2}}(|g_j|^{\frac{1}{\delta}})(x)\}^{\delta} \right) d\mu(x) \right\}^{\frac{1}{t}}.
\end{aligned}$$

We use the lemma with $h_j(x) = M^{\sharp,\alpha} f_j(x)^t$ and $h_j(x) = Nf_j(x)^t$ respectively to obtain $\mathrm{I} \le C \left\| M^{\sharp,\alpha} f_j(x) \,:\, \mathcal{M}^p_q(l^r) \right\| + C_3 \eta^{\frac{1}{t}} \left\| Nf_j(x) \,:\, \mathcal{M}^p_q(l^r) \right\|$. Since $Q \in \mathcal{Q}(\mu)$ is arbitrary we have $\left\| Nf_j(x) \,:\, \mathcal{M}^p_q(l^r) \right\| \le C \left\| M^{\sharp,\alpha} f_j(x) \,:\, \mathcal{M}^p_q(l^r) \right\| + C_3 \eta^{\frac{1}{t}} \left\| Nf_j(x) \,:\, \mathcal{M}^p_q(l^r) \right\|$ for sufficiently small η. η is still at our disposal and every term of this formula is finite, we have

$$\left\| Nf_j(x) \,:\, \mathcal{M}^p_q(l^r) \right\| \le C \left\| M^{\sharp,\alpha} f_j(x) \,:\, \mathcal{M}^p_q(l^r) \right\|.$$

This is the desired result. □

5 An application to commutators

In this section we shall extend Propositions 2.8 and 2.9 to l^r-valued inequalities.

Theorem 5.1. *Suppose that $a \in$RBMO. Let $1 < q \le p < \infty$ and T be a singular integral operator with associated kernel K. Then*

$$\left\| [a,T] f_j \,:\, \mathcal{M}^p_q(l^r) \right\| \le C \left\| f_j \,:\, \mathcal{M}^p_q(l^r) \right\|.$$

Theorem 5.2. *Suppose that $a \in$RBMO. If the parameters satisfy* (15)*, then*

$$\| [a,I_\alpha] f_j \,:\, \mathcal{M}^s_t(l^r) \| \le C \left\| f_j \,:\, \mathcal{M}^p_q(l^r) \right\|.$$

In Appendix we consider another type of commutators. The proof of Theorem 5.1 will be somehow easier than that of Theorem 5.2. So we prove only Theorem 5.2 and the proof of Theorem 5.1 is omitted.

To prove the theorem we need the following pointwise estimates of commutators.

Lemma 5.3. *Let $f \in \mathcal{M}^p_q(\mu)$.*

(1) *Suppose that T is a singular integral operator and a is an* RBMO *function.*

$$(M^{\sharp,0}[a,T]f)(x) \le C\{(M_{(\frac{4}{3})}f)(x) + (M_{(\frac{4}{3})}(Tf))(x)\}.$$

For details we refer [15] (in Section 9)*.*

(2) *Let $0 < \alpha < n$. We have for almost μ-every $x \in \operatorname{supp}(\mu)$*

$$(M^{\sharp,\alpha}[a,I_\alpha]f)(x) \le C\,\|a\|_* \left(M^\alpha_{(\frac{9}{8})} f(x) + (M_{(\frac{3}{2})} I_\alpha f)(x) + I_\alpha|f|(x) \right).$$

For details we refer ([9] p1293)*.*

Proof of Theorems 5.1 and 5.2. The proofs of these theorems are similar to each other, so that we prove Theorem 5.2. Suppose firstly that $\mu(\mathbf{R}^d) = \infty$. In this case we can use Theorem 1.3, Propositions 2.9, 2.10 and Lemma 5.3 (2). Combining them, we can easily prove the theorem.

Suppose instead that $\mu(\mathbf{R}^d) < \infty$. Then the treatment of $\|M^{\sharp,\alpha}[a,I_\alpha]f_j \,:\, \mathcal{M}^s_t(l^r)\|$ is the same as in the case $\mu(\mathbf{R}^d) = \infty$.

As for the estimate of the second term $\text{III} := \left(\sum_{j=1}^\infty m_{\mathbf{R}^d}(|[a,I_\alpha]f_j|)^r \right)^{\frac{1}{r}}$, we have only to show that this is estimated from above by $C\|f_j \,:\, L^u(l^r)\|$, where $u = \dfrac{1+\min\left(q,r,\frac{n}{\alpha}\right)}{2}$, since $\|f_j \,:\, L^u(l^r)\| \le C\|f_j \,:\, \mathcal{M}^p_q(l^r)\|$. Define an auxiliary v by $\dfrac{1}{v} = \dfrac{1}{u} - \dfrac{\alpha}{n}$ for this purpose. By using the Minkowski's inequality and the boundedness of $[a,I_\alpha]$, III is estimated from above by

$$\left(\sum_{j=1}^\infty m_{\mathbf{R}^d}(|[a,I_\alpha]f_j|^v)^{\frac{r}{v}} \right)^{\frac{1}{r}} \le C \left(\sum_{j=1}^\infty (m_{\mathbf{R}^d}(|f_j|^u))^{\frac{r}{u}} \right)^{\frac{1}{r}} \le C\|f_j \,:\, L^u(l^r)\|.$$

So the proof is finished. □

6 Appendix

Another boundedness of the commutator on Morrey space Finally we consider another commutator with a Lipschitz function and a singular integral operator T or with a Lipschitz function and the fractional integral operator. Shirai [14] considered a commutator with $b \in \Lambda_\gamma$ and T and proved the boundedness of $[b, T]$ with the Lebesgue measure. The same proof also holds in our nonhomogeneous space. The proof is similar to the usual case with the aid of Proposition 2.10. For the proof we refer [14].

Proposition 6.1. *Assume that the parameters satisfy that*

$$1 < q \le p < \infty,\ 1 < t \le s < \infty,\ \frac{p}{q} = \frac{s}{t},\ \frac{1}{s} = \frac{1}{p} - \frac{\alpha + \gamma}{n},\ 0 < \alpha < n,\ 0 < \gamma \le 1$$

Suppose that a continuous function b satisfies

$$|b(x) - b(y)| \le C|x - y|^\gamma \tag{24}$$

for $C > 0$. Then we have

$$\|[b, I_\alpha] f_j\ :\ \mathcal{M}^s_t(l^r)\| \le \left\| f_j\ :\ \mathcal{M}^p_q(l^r) \right\|.$$

Proposition 6.2. *Assume that the parameters satisfy that*

$$1 < q \le p < \infty,\ 1 < t \le s < \infty,\ \frac{p}{q} = \frac{s}{t},\ \frac{1}{s} = \frac{1}{p} - \frac{\gamma}{n},\ 0 < \gamma \le 1.$$

Suppose that b is the same function as in the previous theorem. Then

$$\|[b, T] f_j\ :\ \mathcal{M}^s_t(l^r)\| \le C \left\| f_j\ :\ \mathcal{M}^p_q(l^r) \right\|.$$

A note on the weighted norm inequality of Stein-type

In considering the weighted norm inequalities it could not be better if it held that

$$\int_{\{x \in \mathbf{R}^d\,:\,M_\kappa f(x) > \lambda\}} |g(x)| d\mu(x) \le C_\kappa \int_{\mathbf{R}^d} |f(x)| M_\kappa g(x)\, d\mu(x), \tag{25}$$

but this does not hold without doubling assumption. Here we disprove (25) with $\kappa = 3$, as is announced in Section 2.

Counterexample of (25) . We consider the case $d = 2$. We define a measure μ by posing a weight function f given below:

$$f(x) = \begin{cases} 1 & (|x| \ge 1) \\ 1/m! & (2^{-m} < |x| < 2^{-m+1} \text{ for some } m \in \mathbf{N}) \\ 0 & (\text{otherwise}) \end{cases}.$$

Let $\mu := f(x)dx$.

We disprove (25) by reduction to the absurdity. Suppose we have the inequality (25) with $\kappa = 3$. First of all fix an integer α. We are going to let α tend to infinity later.

Claim 6.3. *Set $R_m = \mu(B((2^{-m}, 0), 3 \cdot 2^{-m}))^{-1}$. Then we have*

$$B(O, 2 \cdot 2^{-m}) \subset \{(x, y) \in \mathbf{R}^2\ :\ M_3 \delta_O(x, y) > R_m\},$$

where δ_O is a dirac measure massed on $O = (0, 0)$.

Proof of the claim. Let $(x,y) \in B(O, 2\cdot 2^{-m})$. By the rotation invariance of the sets $B(O, 2\cdot 2^{-m})$ and $\{(x,y) \in \mathbf{R}^2 \,:\, M_3\delta_O(x,y) > \lambda\}$, we may assume that $0 \le x < 2^{-k+1}$ and $y = 0$. Since $O, (x,0) \in B((x/2,0),(1+s)x/2)$ for all $s > 0$, we have $M_3\delta_O(x,0) > \mu(B((x/2,0),(1+s)x/2))^{-1}$. If $s > 0$ is sufficiently small, we have $M_3\delta_O(x,0) > \mu(B((x/2,0),(1+s)x/2))^{-1} > R_k$. □

It follows from the claim that we have

$$\int_{B(O,2\cdot 2^{-m})} |g(x)|\, d\mu(x) \le C\mu(B((2^{-m},0), 3\cdot 2^{-m}))M_3 g(O).$$

Let ϕ be a function such that $\int_{\mathbf{R}^d} \phi(x)\,dx = 1$ supported on a small ball whose center is O. For $r \ll 1/2$ we take a function g_r of the form $g_r = \sum_{j=1}^{\alpha} \frac{1}{r^2}\phi(r^2 \cdot - x_j^r)$, where x_j^r satisfies

$$\lim_{r\to 0} x_j^r = \left(2^{-m+1}\cos\frac{2\pi j}{\alpha}, 2^{-m+1}\sin\frac{2\pi j}{\alpha}\right)$$

and that

$$\operatorname{supp}(g_r) \subset B(O, 2^{-m+1}) \bigcap \left(\bigcup_{j=1}^{\alpha} B\left(\left(2^{-m+1}\cos\frac{2\pi j}{\alpha}, 2^{-m+1}\sin\frac{2\pi j}{\alpha}\right), r\right)\right).$$

Tending $r \to 0$, we have $\alpha \le C\mu(B((2^{-m},0),3\cdot 2^{-m}))M_3\mu_{m,\alpha}(O)$, where denoting δ_x as the dirac measure supported on x, we set $\mu_{m,p} := \sum_{j=1}^{\alpha} \delta_{\left(2^{-m+1}\cos\frac{2\pi j}{\alpha}, 2^{-m+1}\sin\frac{2\pi j}{\alpha}\right)}$.

By the definition of $M_3\mu_{m,\alpha}(O)$ we have

$$M_3\mu_{m,\alpha}(O) = \sup_{\substack{(y,r)\in\mathbf{R}^n\times(0,\infty) \\ O\in B(y,r)}} \frac{\sharp\left\{1 \le j \le \alpha \,:\, \left(2^{-k+1}\cos\left(\frac{2\pi j}{\alpha}\right), 2^{-k+1}\sin\left(\frac{2\pi j}{\alpha}\right)\right) \in B(y,r)\right\}}{\mu(B(y,3r))}.$$

For a finite set $J = \{j_1, \ldots, j_m\}$ with $1 \le j_1 < j_2 < \ldots < j_m \le \alpha$ we set

$$S_J := \inf\left\{\mu(B(y,3r)) \,:\, O, \left(2^{-k+1}\cos\left(\frac{2\pi j_1}{\alpha}\right), 2^{-k+1}\sin\left(\frac{2\pi j_1}{\alpha}\right)\right), \ldots, \right.$$
$$\left. \left(2^{-k+1}\cos\left(\frac{2\pi j_m}{\alpha}\right), 2^{-k+1}\sin\left(\frac{2\pi j_m}{\alpha}\right)\right) \in B(y,r)\right\}.$$

Then $M_3\mu_{k,\alpha}(O)$ can be written as $M_3\mu_{k,p}(O) = \max_{J\subset\{1,\ldots,\alpha\}} \frac{\sharp J}{S_J}$.

Fixing α, if $\sharp J \ge 2$, we have by geometric observation that $\mu(S_J) \ge \frac{C_\alpha}{(m-1)!}$. Notice also that $\mu(B((2^{-k},0),3\cdot 2^{-k})) = O\left(\frac{1}{m!}\right)$. Thus we have $\lim_{k\to\infty} \frac{\mu(B((2^{-k},0),3\cdot 2^{-k}))}{S_J} = 0$. And $S_{\{j\}} \sim \mu(B((2^{-k},0),3\cdot 2^{-k})$, where $\sim$ does not depend on α and k. Thus keeping α fixed, we have

$$\alpha \le C \overline{\lim_{k\to\infty}} M_3\mu_{k,p}(O) \le C,$$

where C is independent on α. Since α is arbitrary, we obtained a desired contradiction and (25) is disproved. □

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