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by

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Quasi-finite algebras graded by Hamiltonian and vertex operator algebras

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Abstract

A general notion of a quasi-finite algebra is introduced as an algebra graded by the set of all integers equipped with topologies on the homogeneous subspaces satisfying certain properties. An analogue of the regular bimodule is introduced and various module categories over quasi-finite algebras are described. When applied to the current algebras (universal enveloping algebras) of vertex operator algebras satisfying Zhu's C_2 -finiteness condition, our general consideration derives important consequences on representation theory of such vertex operator algebras. In particular, the category of modules over such a vertex operator algebra is shown to be equivalent to the category of modules over a finite-dimensional associative algebra.

Introduction

In order to construct conformal field theories on Riemann surfaces associated with a vertex operator algebra V and to obtain their properties such as the finite-dimensionality of the space of conformal blocks, factorization of the blocks along the boundaries of the moduli space of Riemann surfaces and the fusion functors or the tensor product of V -modules, we need first to impose an appropriate finiteness condition on V and second to study the structure of the abelian category of V -modules to some extent.

One of the candidates of such a finiteness condition is the one introduced by Y.-C. Zhu ([?]), usually called the C_2 -finiteness (or C_2 -cofiniteness), saying that a certain quotient space $V/C_2(V)$ is finite-dimensional. We will call this condition *Zhu's finiteness condition* in the rest of the paper. This condition was used in [?] in an essential way to the proof of the modular invariance of characters of V -modules, as well as the condition that the category of V -modules is semisimple. The modular invariance is a part of the characteristic properties of rational conformal field theories.

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Another important ingredient in Zhu's derivation of the modular invariance is the use of an associative algebra $A(V)$, called Zhu's algebra, and a functor from the category of V -modules to the category of $A(V)$ -modules. The fundamental result of Zhu is as follows: the functor sends an irreducible V -module to an irreducible $A(V)$ -module in such a way that the equivalence classes of irreducibles in the two categories are in one-to-one correspondence. In particular, the number of irreducible classes is finite if $A(V)$ is finite-dimensional. The last property actually follows from Zhu's finiteness condition mentioned above.

However, the above-mentioned functor need not give us an equivalence of categories. Specifically, if we include the cases when the category of V -modules is not semisimple, the algebra $A(V)$ is not enough to understand the properties of the category of V -modules in general. In this regard, it seems to the authors that detailed analysis of the structure of the abelian category of V -modules has not yet been done. (See [?] for some related results.)

The purpose of the present paper is to fill this gap by utilizing the universal enveloping algebra associated with the vertex operator algebra.

The universal enveloping algebra $\mathbf{U} = \mathbf{U}(V)$ associated with V is a certain topological algebra first considered by Frenkel and Zhu in [?]. The presence of topology is inevitable as the universal defining relations of vertex operator algebras contain infinite sums. We will call \mathbf{U} the *current algebra* for short in the rest of the paper. More precisely, the current algebra \mathbf{U} is an associative algebra with unity graded by the set of integers equipped with a separated linear topology defined by a certain sequence I_0, I_1, \dots of open left ideals such that each homogeneous subspace $\mathbf{U}(d)$ is complete with respect to the relative topology. Note that the quotient space $Q_n = \mathbf{U}/I_n$ is a discrete space, which inherits a grading from \mathbf{U} . We will denote the subspace of degree d by $Q_n(d)$. The algebra \mathbf{U} is important in that there is a one-to-one correspondence between V -modules of certain type and continuous discrete \mathbf{U} -modules. (See Section 6 for the precise statement.)

The authors noticed while trying to understand the category of V -modules by means of \mathbf{U} that the finiteness property should better be imposed on \mathbf{U} rather than on V as far as the properties of the category as an abelian category are concerned. Thus we arrived at the concept of *quasi-finiteness*, which is defined as follows: \mathbf{U} is quasi-finite if and only if the spaces $Q_n(d)$ are finite-dimensional for all $d \in \mathbb{Z}$ and $n \in \mathbb{N}$. As we will show in Theorem ??, Zhu's finiteness condition actually implies the quasi-finiteness of \mathbf{U} .

Let us assume that \mathbf{U} is quasi-finite. Then the spaces $Q_n(d)$ have generalized eigenspace decompositions with respect to the action of the Virasoro operator L_0 , which we will call the Hamiltonian of \mathbf{U} . By using this, we can construct a series of finite-dimensional algebras \mathbf{U}_n and functors from the category of continuous discrete left \mathbf{U} -modules to the category of left \mathbf{U}_n -modules which give rise to equivalences of categories when n is sufficiently large. Therefore, under the quasi-finiteness, the category of continuous discrete left \mathbf{U} -modules as an abelian category is completely described by the properties of the finite-dimensional algebra \mathbf{U}_n .

In application to conformal field theories on Riemann surfaces, we have to enlarge the algebra \mathbf{U} and to take into account the concept of duality of \mathbf{U} -modules.

Therefore we consider two different topologies on \mathbf{U} , one is the original left linear one and the other a right linear one, and take the filterwise completion \mathcal{U} and \mathcal{U}^\vee with respect to the two topologies. Then we may consider the left \mathcal{U} -modules, the right \mathcal{U} -modules, the left \mathcal{U}^\vee -modules, and the right \mathcal{U}^\vee -modules. We can now formulate the concept of quasi-finiteness for each of the four and can establish the equivalences or the dualities among them. We will call this type of results the *finiteness theorems*.

As the argument above deriving the equivalences and the dualities of categories only uses quite general properties of \mathbf{U} and L_0 , we now postulate them: we will call a topological algebra A with a distinguished element h a *quasi-finite algebra graded by Hamiltonian* if it shares the same properties with \mathbf{U} as mentioned above. (See Section 2 for the precise definition.)

The present paper is divided into two parts: Part I consists in explaining general theory of quasi-finite algebras graded by Hamiltonian and categories of modules over them and Part II in proving that Zhu's finiteness condition on V implies the quasi-finiteness of \mathbf{U} .

The plan of Part I is as follows. We will begin by describing in Section 1 the concept of compatible degreewise topological algebras and the associated topological filtered algebras which are modeled on the topological features of \mathbf{U} . In Section 2, we define the concept of quasi-finite algebras graded by Hamiltonian and give some consequences. In particular, we introduce an analogue of the regular bimodule and define certain finite-dimensional algebras A_n . We will show that the regular bimodule is a dense subspace of the algebra (Theorem ??). Section 3 is devoted to the equivalence of categories between the continuous discrete modules, which we will call *exhaustive modules*, and the category of A_n -modules (Theorem ??). In Section 4, we will introduce a notion of *coexhaustive modules* which is a *dual* notion of exhaustive modules and establish an equivalence between the category of exhaustive modules and the category of coexhaustive modules (Theorem ??). The duality of modules will be formulated in Section 5 as the duality between quasi-finite exhaustive left modules and quasi-finite coexhaustive right modules. We will then summarize various equivalences and the dualities for quasi-finite modules (Theorem ??).

The proof of quasi-finiteness of the current algebra \mathbf{U} in Part II under Zhu's finiteness condition will be done by considering a certain filtration \mathbf{G} on \mathbf{U} , which was introduced in [?], and a certain universal Poisson algebra $\tilde{\mathbf{S}} = \tilde{\mathbf{S}}(\mathfrak{p})$ associated with $\mathfrak{p} = V/C_2(V)$. We will construct a surjective homomorphism of Poisson algebra from $\tilde{\mathbf{S}}$ to the degreewise completion $\mathbf{S} = \mathbf{S}(V)$ of $\text{gr}^{\mathbf{G}}\mathbf{U}$. The quasi-finiteness of \mathbf{U} then follows from that of $\tilde{\mathbf{S}}$. We will call $\tilde{\mathbf{S}}$ the *Poisson current algebra*.

In Section 6, we will describe the construction and some properties of the current algebra \mathbf{U} associated with a vertex operator algebra V . In Section 7, we will consider the filtration \mathbf{G} on \mathbf{U} and show that the associated graded algebra has a structure of a Poisson algebra. In Section 8, we will construct the Poisson current algebra $\tilde{\mathbf{S}}$ associated with any Poisson algebra \mathfrak{p} and we will show that $\tilde{\mathbf{S}}$ is quasi-finite if \mathfrak{p} is finite-dimensional (Theorem ??). In the final section, we will consider the case when $\mathfrak{p} = V/C_2(V)$ and construct the surjective homomorphism $\Psi : \tilde{\mathbf{S}} \rightarrow \mathbf{S}$ of Poisson

algebras (Theorem ??). We will then combine these results to show that Zhu's finiteness condition on V implies quasi-finiteness of the current algebra \mathbf{U} (Theorem ??).

The finiteness theorems mentioned above will give us not only a nice conceptual understanding of the role of Zhu's finiteness condition in representation theory of vertex operator algebras but also a foundation in the strategy mentioned at the beginning of this introduction of constructing the spaces of conformal blocks on general Riemann surfaces and of showing their expected properties. This will be developed in our forthcoming paper, which will serve as the continuation of the previous paper [?] by two of the authors.

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Part I

Quasi-finite algebras graded by Hamiltonian

1 Linear topologies on graded algebras

In this section, we introduce compatible degreewise topological algebras and some related notions.

1.1 Preliminaries

We denote the set of integers by \mathbb{Z} and the set of nonnegative integers by \mathbb{N} .

Throughout the paper, we will work over an algebraically closed field \mathbf{k} of characteristic zero. A vector space always means a vector space over \mathbf{k} and the scalar multiplication of a vector space is denoted by juxtaposition.

An algebra means an associative algebra over \mathbf{k} with unity. The multiplication of an algebra A is denoted by the dot \cdot and the unity by 1_A or simply by 1 . For subsets S and T of A , we denote by $S \cdot T$ the linear span of the elements of the form $s \cdot t$ with $s \in S$ and $t \in T$. The action of A on an A -module M is denoted again by the dot. We always assume that the unity 1_A acts by the identity operator.

We endow the field \mathbf{k} with the discrete topology. Let

$$I_0, I_1, \dots, I_n, \dots \quad (1.1.1)$$

be a decreasing sequence of linear subspaces of a vector space V . A *linear topology defined by I_n* means a topology on V such that for each $v \in V$ the set $\{v + I_n \mid n \in \mathbb{N}\}$ forms a fundamental system of open neighborhoods of v .

Throughout the paper, a topology on a vector space always means a linear topology given in this way. Such a space is usually called a linearly topologized vector space in the literatures.

Let V be a vector space with the linear topology defined by I_n . Any subspace U which contains I_n for some n is open and closed, and the quotient topology on V/U is the discrete topology. The closure of a subspace U is given by $\bigcap_n (U + I_n)$ and U is dense in V if and only if the composite $U \rightarrow V \rightarrow V/I_n$ of canonical maps is surjective for any n .

The completion of V with respect to the linear topology defined by I_n is the projective limit

$$\hat{V} = \varprojlim_n V/I_n \quad (1.1.2)$$

endowed with the projective limit topology, namely the relative topology induced from $\prod_{n=0}^{\infty} V/I_n$ with the product topology of the discrete topologies on V/I_n . Let \hat{I}_n be the closure of the image of I_n under the canonical map $V \rightarrow \hat{V}$. Then \hat{I}_n agrees with the kernel of the canonical map $\hat{V} \rightarrow V/I_n$ and the topology on \hat{V} is the linear topology defined by \hat{I}_n .

The space V is said to be complete if the canonical map $V \rightarrow \hat{V}$ is a homeomorphism. Thus a complete space is separated (i.e. Hausdorff) in our convention. If V is complete then the closure of a subspace U is given by

$$\hat{U} = \varprojlim_n (U + I_n)/I_n. \quad (1.1.3)$$

and for a closed subspace F the quotient space V/F with the quotient topology is also complete.

We refer the reader to [?], [?] and [?] for linear topologies.

1.2 Compatible degreewise topological algebras

Let A be an algebra and suppose given a grading

$$A = \bigoplus_{d \in \mathbb{Z}} A(d) \quad (1.2.1)$$

indexed by integers such that $A(d) \cdot A(e) \subset A(d+e)$. We simply call such an A a *graded algebra*. Let us set

$$F_p A = \bigoplus_{d \leq p} A(d), \quad F_p^\vee A = \bigoplus_{d \geq -p} A(d) \quad (1.2.2)$$

where p is an integer. We call the filtration F the *associated filtration* and F^\vee the *opposite filtration*.

Let $A = \bigoplus_d A(d)$ be a graded algebra and suppose given a linear topology on each $A(d)$. In such a situation, we will say that A is endowed with a *degreewise topology*. We assume that the multiplication maps $A(d) \times A(e) \rightarrow A(d+e)$ are continuous. We will say that A is *degreewise complete* if each $A(d)$ is complete.

Now consider the subspace

$$A(d) \cap (A \cdot F_{-n-1} A) = \sum_{k \leq -n-1} A(d-k) \cdot A(k) \quad (1.2.3)$$

and let $I_n(A(d))$ be its closure in $A(d)$. We assume that the sequence $\{I_n(A(d))\}$ forms a fundamental system of open neighborhoods of zero in each $A(d)$.

Definition 1.2.1 A *compatible degreewise topological algebra* is a graded algebra A endowed with a degreewise topology such that all the conditions mentioned above are satisfied. A *compatible degreewise complete algebra* is a compatible degreewise topological algebra A such that it is degreewise complete.

Let A be a compatible degreewise topological algebra. Since the multiplication maps $A(d) \times A(e) \rightarrow A(d+e)$ are continuous, we have

$$A(d) \cdot I_n(A(e)) \subset I_n(A(d+e)), \quad I_n(A(d)) \cdot A(e) \subset I_{n-e}(A(d+e)). \quad (1.2.4)$$

Therefore, for $a \in A(d)$ and $b \in A(e)$, we have

$$(a + I_{n+e}(A(d))) \cdot (b + I_n(A(e))) \subset a \cdot b + I_n(A(d+e)). \quad (1.2.5)$$

Instead of $I_n(A(d))$, we may consider the closure $I_n^\vee(A(d))$ of $A(d) \cap (F_{-n-1}^\vee A \cdot A)$. Since we have

$$I_n^\vee(A(d)) = I_{n-d}(A(d)), \quad (1.2.6)$$

the topology defined by $I_n^\vee(A(d))$ agrees with the one defined by $I_n(A(d))$.

In the sequel, we will use the following terminologies: a graded subspace is said to be *degreewise dense* if each homogeneous subspace is dense; the sum of the closures of the homogeneous subspaces of a graded subspace U is called the *degreewise closure* of U .

Note 1.2.2 Let A be a graded algebra endowed with a degreewise topology. In general, there is no canonical way of extending the topologies on the homogeneous subspaces to the whole space A which makes A into a topological algebra.

1.3 Degreewise completion

Let A be a compatible degreewise topological algebra. Set

$$\hat{A} = \bigoplus_d \hat{A}(d) \quad (1.3.1)$$

where $\hat{A}(d)$ is the completion of the space $A(d)$. We call \hat{A} the *degreewise completion* of A .

Since the multiplication $A(d) \times A(e) \rightarrow A(d+e)$ are continuous, they induce continuous bilinear maps $\hat{A}(d) \times \hat{A}(e) \rightarrow \hat{A}(d+e)$ which make \hat{A} into an algebra endowed with a degreewise topology.

Proposition 1.3.1 *The degreewise completion \hat{A} is a compatible degreewise complete algebra.*

For instance, let $A = \bigoplus_d A(d)$ be any graded algebra. We endow the space $A(d)$ with the linear topology defined by $A(d) \cap (A \cdot F_{-n-1} A)$. Then A becomes a compatible degreewise topological algebra and the degreewise completion \hat{A} is degreewise complete. We call this degreewise topology on A the *standard degreewise topology* and the algebra \hat{A} the *standard degreewise completion*.

Note 1.3.2 Giving topologies on the homogeneous subspaces of a graded algebra in the way described above was considered by some authors, see e.g. [?], [?], [?], [?], [?].

1.4 Associated filtered topological algebra

A topological algebra is said to be left linear if the topology is a linear topology defined by a sequence of left ideals. Right linearity is defined similarly.

Let A be a compatible degreewise topological algebra. Let us endow the algebra A with the associated filtration F . Consider the left ideal $I_n(A)$ of A defined by

$$I_n(A) = \bigoplus_d I_n(A(d)). \quad (1.4.1)$$

By the compatibility of A , the space $I_n(A)$ is the degreewise closure of $A \cdot F_{-n-1}A$. Let us endow the space A with the topology defined by $I_n(A)$. Then the relative topology on $A(d)$ induced from A agrees with the original topology on $A(d)$. For any $a \in F_p A$ and $b \in F_q A$ we have

$$(a + I_{n+q}(A)) \cdot (b + I_n(A)) \subset a \cdot b + I_n(A). \quad (1.4.2)$$

Hence the multiplication $A \times A \rightarrow A$ is continuous. In particular, as $I_n(A)$ are left ideals, A is a left linear topological algebra. We simply call this topology *the left linear topology* of A .

Now consider the opposite filtration F^\vee and set

$$I_n^\vee(A) = \bigoplus_d I_n^\vee(A(d)). \quad (1.4.3)$$

The space $I_n^\vee(A)$ is the degreewise closure of $F_{-n-1}^\vee A \cdot A$. Then the linear topology on A defined by $I_n^\vee(A)$ now makes A into a right linear topological algebra. We call this topology *the right linear topology* of A .

The left linear topology and the right linear topology on the same algebra A do *not* agree with each other in general although the restrictions to $A(d)$ are the same.

1.5 Filterwise completion

Let A be a compatible degreewise topological algebra. Let us endow A with the left linear topology and let $F_p \mathcal{A}$ be the completion

$$F_p \mathcal{A} = \varprojlim_n F_p A / F_p A \cap I_n(A) \quad (1.5.1)$$

with the projective limit topology. Then the multiplication $F_p A \times F_q A \rightarrow F_{p+q} A$ induces a multiplication $F_p \mathcal{A} \times F_q \mathcal{A} \rightarrow F_{p+q} \mathcal{A}$. Consider the space

$$\mathcal{A} = \varinjlim_p F_p \mathcal{A} \quad (1.5.2)$$

and give it the linear topology defined by

$$I_n(\mathcal{A}) = \text{Ker}(\mathcal{A} \rightarrow A/I_n(A)). \quad (1.5.3)$$

Then, for any $a \in F_p \mathcal{A}$ and $b \in F_q \mathcal{A}$, we have

$$(a + I_{n+q}(\mathcal{A})) \cdot (b + I_n(\mathcal{A})) \subset a \cdot b + I_n(\mathcal{A}). \quad (1.5.4)$$

Thus the space \mathcal{A} becomes a filtered left linear topological algebra such that the subspaces $F_p \mathcal{A}$ with the relative topologies are complete and that the image of A

under the canonical map $A \rightarrow \mathcal{A}$ is a dense subspace of \mathcal{A} . We call \mathcal{A} the *left linear filterwise completion* of A .

Now consider the right linear topology on A and the corresponding filterwise completion

$$\mathcal{A}^\vee = \varinjlim_p F_p^\vee \mathcal{A}^\vee, \quad F_p^\vee \mathcal{A}^\vee = \varprojlim_n F_p^\vee A / F_p^\vee A \cap I_n^\vee(A). \quad (1.5.5)$$

We will call \mathcal{A}^\vee the *right linear filterwise completion* of A .

If A is degreewise complete then the canonical maps $A \rightarrow \mathcal{A}$ and $A \rightarrow \mathcal{A}^\vee$ are injective. We will then identify A with its images. Thus we have two inclusions

$$\mathcal{A} \leftarrow A \rightarrow \mathcal{A}^\vee \quad (1.5.6)$$

such that the relative topologies on each $A(d)$ induced from \mathcal{A} and from \mathcal{A}^\vee agree with the original topology of a compatible degreewise topological algebra. Note that the filterwise completions \mathcal{A} and \mathcal{A}^\vee are *not* complete in general.

1.6 Hamiltonian of a graded algebra

Let A be a graded algebra. An element $h \in A$ is called a *Hamiltonian* of A if

$$A(d) = \{a \in A \mid [h, a] = da\} \quad (1.6.1)$$

holds for any d , where $[h, a] = h \cdot a - a \cdot h$ denotes the commutator. If this is the case then any central element as well as h itself belongs to $A(0)$ and an element h' is a Hamiltonian if and only if $h - h'$ is central.

An *algebra graded by Hamiltonian* is a pair (A, h) of a graded algebra A and a Hamiltonian $h \in A(0)$. We will denote by H the subalgebra of $A(0)$ generated by the Hamiltonian h .

Remark 1.6.1 Let (A, h) be a compatible degreewise complete algebra graded by Hamiltonian. Then the images of the canonical injections (??) are given respectively by $\sum_d \mathcal{A}(d)$ and $\sum_d \mathcal{A}^\vee(d)$, where

$$\mathcal{A}(d) = \{a \in \mathcal{A} \mid [h, a] = da\}, \quad \mathcal{A}^\vee(d) = \{a \in \mathcal{A}^\vee \mid [h, a] = da\}. \quad (1.6.2)$$

In the rest of Part I, we will be mainly concerned with a compatible degreewise complete algebra with Hamiltonian.

2 Quasi-finite algebras

In this section, we will formulate a finiteness condition on compatible degreewise complete algebras and describe its consequences. In particular, we will formulate an analogue of the regular bimodule as a degreewise dense subspace of the algebra.

2.1 Canonical quotient modules

Let A be a compatible degreewise topological algebra and recall the spaces $I_n(A)$. We set

$$Q_n = A/I_n(A). \quad (2.1.1)$$

Since $I_n(A)$ is a left ideal, the quotient Q_n is a left A -module. We will call this module the *left canonical quotient module*.

By (??), we have $I_n(A) \cdot F_0A \subset I_n(A)$. Hence the multiplication $A \times F_0A \rightarrow A$ induces an action of F_0A on Q_n for any n . Thus the space Q_n is an (A, F_0A) -bimodule. In particular, it is an $(A(0), A(0))$ -bimodule.

Since $I_n(A)$ is a graded subspace, the grading of A induces a grading on the quotient by setting $Q_n(d) = A(d)/I_n(A(d))$. If $d \leq -n - 1$ then since $A(d) \subset I_n(A)$ we have $Q_n(d) = 0$. Thus

$$Q_n = \bigoplus_{d=-n}^{\infty} Q_n(d). \quad (2.1.2)$$

Lemma 2.1.1 *For any $v \in Q_n$ there exists an m such that $I_m(A) \cdot v = 0$.*

Proof. By (??) we have $I_{n+d}(A) \cdot A(d) \subset I_n(A)$ and hence $I_m(A) \cdot Q_n(d) = 0$ for $m = n + d$. \square

Note that the action $A \times Q_n \rightarrow Q_n$ is continuous when A is endowed with the left linear topology and Q_n with the discrete topology.

Lemma 2.1.2 *Let v be an element of Q_n . Then $F_{-n-1}A \cdot v = 0$ implies $I_n(A) \cdot v = 0$.*

Proof. Suppose $F_{-n-1}A \cdot v = 0$ and choose an m such that $I_m(A) \cdot v = 0$. Since $I_n(A)$ is the closure of $A \cdot F_{-n-1}A$, we have $I_n(A) \subset A \cdot F_{-n-1}A + I_m(A)$. Hence $I_n(A) \cdot v \subset A \cdot F_{-n-1}A \cdot v + I_m(A) \cdot v = 0$. \square

We may likewise consider the right canonical quotient module $Q_n^\vee = A/I_n^\vee(A)$. We have analogous statements for the right canonical quotient modules as well. By the definitions of the filterwise completions \mathcal{A} and \mathcal{A}^\vee , the spaces Q_n and Q_n^\vee are canonically isomorphic to the spaces $\mathcal{A}/I_n(\mathcal{A})$ and $\mathcal{A}^\vee/I_n^\vee(\mathcal{A}^\vee)$, respectively.

2.2 Quasi-finite algebra graded by Hamiltonian

Let A be a compatible degreewise topological algebra. Let us consider the subspace $A(0)$ of degree zero, which is a subalgebra of A . Then $I_n(A(0))$ and $I_n^\vee(A(0))$ agree with each other and they give a two-sided ideal of $A(0)$. Therefore, the quotient $Q_n(0) = A(0)/I_n(A(0))$ is an algebra.

Lemma 2.2.1 *The space $Q_n(d)$ is a $(Q_{n+d}(0), Q_n(0))$ -bimodule.*

Proof. By (??), we have $I_{n+d}(A(0)) \cdot A(d) \subset I_n(A(d))$ and $A(d) \cdot I_n(A(0)) \subset I_n(A(d))$. The result follows. \square

Let h be a Hamiltonian and let h_n be the image of h in the quotient $Q_n(0)$. Let H_n be the subalgebra of $Q_n(0)$ generated by h_n . By Lemma ??, the space $Q_n(d)$ is in particular an (H_{n+d}, H_n) -bimodule.

Definition 2.2.2 A *weakly quasi-finite algebra* is a compatible degreewise complete algebra such that $Q_n(0)$ are finite-dimensional for all n . A *weakly quasi-finite algebra graded by Hamiltonian* is a pair (A, h) of a weakly quasi-finite algebra A and a Hamiltonian h of A .

Let (A, h) be a weakly quasi-finite algebra graded by Hamiltonian and let h_n and H_n be as above. Then H_n is a finite-dimensional commutative algebra for any $n \in \mathbb{N}$. This last property is what we will need in practice in considering a weakly quasi-finite algebra graded by Hamiltonian.

Now let us consider the following conditions for each d :

- (a) The spaces $Q_n(d)$ are finite-dimensional for all n .
- (b) The spaces $Q_n^\vee(d)$ are finite-dimensional for all n .

Thanks to the relation (??), these conditions are actually equivalent.

Definition 2.2.3 A *quasi-finite algebra* is a compatible degreewise complete algebra such that the equivalent conditions (a) and (b) above are satisfied for all integers d . A *quasi-finite algebra graded by Hamiltonian* is a pair (A, h) of a quasi-finite algebra A and a Hamiltonian h of A .

2.3 Spectrum of the Hamiltonian

Let (A, h) be a weakly quasi-finite algebra graded by Hamiltonian and recall the image h_n of h in $Q_n(0)$. We let φ_n be the minimal polynomial of h_n and let Ω_n be the set of roots of φ_n . Then the set Ω_n agrees with the eigenvalues of the left action of h on $Q_n(0)$.

Let us introduce a partial order on \mathbf{k} by letting $\lambda \succcurlyeq \mu$ when $\lambda - \mu$ is a nonnegative integer and let Γ_0 be the set of minimal elements of Ω_0 . We set

$$\Gamma_\infty = \{\gamma + k \mid \gamma \in \Gamma_0 \text{ and } k \in \mathbb{N}\}. \quad (2.3.1)$$

We will also use the following notation:

$$\Gamma_m = \{\gamma + k \mid \gamma \in \Gamma_0 \text{ and } k \in \mathbb{N} \text{ with } 0 \leq k \leq m\}. \quad (2.3.2)$$

Let g be the maximum of the integral differences among elements of Ω_0 :

$$g = \max \{\lambda - \mu \mid \lambda, \mu \in \Omega_0, \lambda - \mu \in \mathbb{N}\}. \quad (2.3.3)$$

Then we have $\Gamma_0 \subset \Omega_0 \subset \Gamma_g$.

Recall the subalgebras $H \subset A(0)$ and $H_n \subset Q_n(0)$. For any left H -module W , we denote by $W[\lambda]$ the generalized eigenspace of the Hamiltonian h acting on W :

$$W[\lambda] = \{v \in W \mid (h - \lambda)^r \cdot v = 0 \text{ for some } r \in \mathbb{N}\}. \quad (2.3.4)$$

If W is an H_n -module then $W[\lambda] \neq 0$ implies $\lambda \in \Omega_n$.

Now consider the space

$$K_m(Q_n) = \{v \in Q_n \mid I_m(A) \cdot v = 0\}. \quad (2.3.5)$$

Then this is a left $Q_m(0)$ -module and hence a left H_m -module. Thus the set of the eigenvalues of the left action of h on $K_m(Q_n)$ is contained in the set Ω_m . Note that $v \in K_m(Q_n)$ if and only if $F_{-m-1}A \cdot v = 0$ by Lemma ??.

Proposition 2.3.1 *The set Ω_n is a subset of Γ_{n+g} .*

Proof. Let λ be an element of Ω_n . Then there exists a generalized eigenvector v in $Q_n(0) \subset K_n(Q_n)$ with the eigenvalue λ . Since $v \neq 0$ and $F_{-n-1}A \cdot v = 0$, there exists a nonnegative integer k with $0 \leq k \leq n$ for which $v \notin K_{k-1}(Q_n)$ but $v \in K_k(Q_n)$. Then $A(-k) \cdot v$ is a nonzero subspace of $K_0(Q_n)$. Since $A(-k) \cdot v \subset Q_n[\lambda - k]$, we have $\lambda - k \in \Omega_0$ and hence $\lambda \in \Gamma_{k+g} \subset \Gamma_{n+g}$. \square

Now Lemma ?? implies $Q_n = \bigcup_{m=0}^{\infty} K_m(Q_n)$. Therefore, Proposition ?? implies that

$$Q_n = \bigoplus_{\lambda \in \Gamma_{\infty}} Q_n[\lambda]. \quad (2.3.6)$$

Lemma 2.3.2 *The space $Q_n[\lambda]$ with $\lambda \in \Gamma_m$ is a subspace of $K_m(Q_n)$.*

Proof. Since $\lambda - m - 1 \notin \Gamma_{\infty}$, we have $F_{-m-1}A \cdot Q_n[\lambda] = 0$. Hence $Q_n[\lambda] \subset K_m(Q_n)$. \square

Let ℓ be the maximum of the multiplicities of the roots of the minimal polynomial φ_g .

Proposition 2.3.3 *The multiplicities of the roots of φ_n are at most ℓ for any nonnegative integer n .*

Proof. Let v be an element of Q_n . We show that $(h - \lambda)^k \cdot v = 0$ for some k implies $(h - \lambda)^{\ell} \cdot v = 0$. Let λ be a minimal counterexample to this claim: there exists a nonzero vector v such that $(h - \lambda)^k \cdot v = 0$ for some k but $(h - \lambda)^{\ell} \cdot v \neq 0$. Then, by the minimality, we have $F_{-1}A \cdot (h_n - \lambda)^{\ell} \cdot v = 0$ and hence $(h_n - \lambda)^{\ell} \cdot v \in K_0(Q_n)$ by Lemma ??. Hence $\lambda \in \Omega_0 \subset \Gamma_g$ and so $v \in Q_n[\lambda] \subset K_g(Q_n)$ by Lemma ??. Hence $(h - \lambda)^{\ell} \cdot v = 0$, a contradiction. \square

2.4 The regular bimodule

Let (A, h) be a weakly quasi-finite algebra graded by Hamiltonian. We denote by $A[\lambda, \mu]$ the simultaneous generalized eigenspace of the left and the right actions of the Hamiltonian h on A :

$$A[\lambda, \mu] = \{a \in A \mid (h - \lambda)^r \cdot a = a \cdot (h - \mu)^r = 0 \text{ for some } r\}. \quad (2.4.1)$$

We will call this λ the *left eigenvalue* and μ the *right eigenvalue* of h .

By the definition, $A[\lambda, \mu] \neq 0$ implies that $d = \lambda - \mu$ is an integer and $A[\lambda, \mu] \subset A(d)$. We also note that

$$A(d) \cdot A[\lambda, \mu] \subset A[\lambda + d, \mu], \quad A[\lambda, \mu] \cdot A(e) \subset A[\lambda, \mu - e]. \quad (2.4.2)$$

It is easy to see that

$$A[\kappa, \lambda] \cdot A[\mu, \nu] = 0 \text{ if } \lambda \neq \mu. \quad (2.4.3)$$

Indeed, if $a \cdot (h - \lambda)^k = 0$ and $(h - \mu)^m \cdot b = 0$ with $\lambda \neq \mu$ then $a \cdot b = a \cdot 1 \cdot b = a \cdot (h - \lambda)^k f(h) \cdot b + a \cdot (h - \mu)^m g(h) \cdot b = 0$ for some polynomials $f(x)$ and $g(x)$. Also note that

$$A[\kappa, \lambda] \cdot A[\lambda, \mu] \subset A[\kappa, \mu]. \quad (2.4.4)$$

We now set

$$B = \sum_d B(d), \quad B(d) = \sum_{\lambda - \mu = d} A[\lambda, \mu]. \quad (2.4.5)$$

Then by (??) this is an (A, A) -bimodule. We call B the *regular bimodule* of A .

Remark 2.4.1 The structure of an (A, A) -bimodule on B actually prolongs to a structure of an $(\mathcal{A}, \mathcal{A}^\vee)$ -bimodule. (See Proposition ??.)

2.5 Denseness of the regular bimodule

To investigate the spaces $A[\lambda, \mu]$ we consider the left canonical quotient module Q_n , which is an $(A, F_0 A)$ -bimodule. Consider the simultaneous generalized eigenspaces of the left and the right actions of h on Q_n :

$$Q_n[\lambda, \mu] = \{v \in Q_n \mid (h - \lambda)^r \cdot v = v \cdot (h - \mu)^r = 0 \text{ for some } r\}. \quad (2.5.1)$$

Then $Q_n[\lambda, \mu] \neq 0$ implies that $d = \lambda - \mu$ is an integer and that $Q_n[\lambda, \mu] \subset Q_n(d)$.

Recall that the space $Q_n(d)$ is a $(Q_{n+d}(0), Q_n(0))$ -bimodule. In particular, it is an (H_{n+d}, H_n) -bimodule. Hence we have

$$Q_n(d) = \sum_{\mu \in \Omega_n} Q_n[\mu + d, \mu]. \quad (2.5.2)$$

Note that $A(d) \cdot Q_n[\lambda, \mu] \subset Q_n[\lambda + d, \mu]$ and that $Q_n[\lambda, \mu] \cdot A(e) \subset Q_n[\lambda, \mu - e]$ whenever $e \leq 0$.

Lemma 2.5.1 *If $A[\lambda, \mu] \neq 0$ then $\lambda, \mu \in \Gamma_\infty$.*

Proof. Let a be a nonzero element of $A[\lambda, \mu]$ with $d = \lambda - \mu$. Since we have assumed that $A(d)$ is complete, it is separated. Hence there exists an n such that the image v of a in $Q_n(d)$ is nonzero. Then since $v \in Q_n[\lambda, \mu]$, we have $\lambda \in \Gamma_\infty$ and $\mu \in \Gamma_n \subset \Gamma_\infty$. \square

Lemma 2.5.2 *If $\mu \in \Gamma_m$ and $n \geq m$ then the restriction $Q_n[\lambda, \mu] \rightarrow Q_m[\lambda, \mu]$ of the canonical surjection is an isomorphism for any λ .*

Proof. Set $d = \lambda - \mu$. Consider the kernel $I_m(A)/I_n(A)$ of the canonical surjection $Q_n \rightarrow Q_m$. Consider the map $A \times F_{-m-1}A \rightarrow A$ which induces

$$A \times F_{-m-1}A \rightarrow I_m(A)/I_n(A). \quad (2.5.3)$$

Since $I_m(A)$ is the closure of $A \cdot F_{-m-1}A$, this map is surjective. Hence the induced map

$$Q_n \times F_{-m-1}A \rightarrow I_m(A)/I_n(A) \quad (2.5.4)$$

is also surjective. The right eigenvalues of h on $Q_n \cdot F_{-m-1}A$ exceed those of Q_n by more than m . Hence it follows that the set of the right eigenvalues on the space $I_m(A)/I_n(A)$ does not intersect Γ_m . This implies the result. \square

By this lemma, we have the following result.

Proposition 2.5.3 *Let (A, h) be a weakly quasi-finite algebra graded by Hamiltonian and let m be a nonnegative integer. Then the canonical surjection $A[\lambda, \mu] \rightarrow Q_n[\lambda, \mu]$ is an isomorphism whenever $\mu \in \Gamma_m$ and $n \geq m$.*

Proof. By Lemma ??, we have a canonical splitting $Q_m[\lambda, \mu] \rightarrow \varprojlim_n Q_n[\lambda, \mu]$. Set $d = \lambda - \mu$. Since $A(d)$ is complete, the projective limit is isomorphic to the subspace $A[\lambda, \mu]$ of $A(d)$. \square

The following is the first main result of Part I.

Theorem 2.5.4 *Let (A, h) be a weakly quasi-finite algebra graded by Hamiltonian. Then the regular bimodule B is degreewise dense in A .*

Proof. By Proposition ??, the map $B(d) \rightarrow Q_n(d) = A(d)/I_n(A(d))$ is surjective for any n . This means that $B(d)$ is dense in $A(d)$. \square

Note that the whole space B is dense in A with respect to the left linear and the right linear topologies of A .

2.6 The associated finite algebras

Let (A, h) be a weakly quasi-finite algebra graded by Hamiltonian. We set

$$A_n = \sum_{\lambda, \mu \in \Gamma_n} A[\lambda, \mu]. \quad (2.6.1)$$

Then it follows from (??) that the space A_n is closed under the multiplication of A .

By Proposition ??, we may identify $A_n(0)$ with a subspace of $Q_n(0)$. Recall the elements 1_n and h_n of $Q_n(0)$, which are the images of 1 and h under the map $A(0) \rightarrow Q_n(0)$, respectively. Let

$$1_n = \sum_{\lambda \in \Omega_n} 1_n[\lambda, \lambda], \quad h_n = \sum_{\lambda \in \Omega_n} h_n[\lambda, \lambda] \quad (2.6.2)$$

be the decompositions to sums of simultaneous generalized eigenvectors. We set

$$1_{A_n} = \sum_{\lambda \in \Gamma_n} 1_n[\lambda, \lambda], \quad h_{A_n} = \sum_{\lambda \in \Gamma_n} h_n[\lambda, \lambda], \quad (2.6.3)$$

and regard them as elements of A_n .

Proposition 2.6.1 *The graded algebra structure on A induces a graded algebra structure on A_n for which 1_{A_n} is the unity and h_{A_n} is a Hamiltonian.*

We set

$$P_n = \sum_{\lambda \in \Gamma_\infty} \sum_{\mu \in \Gamma_n} A[\lambda, \mu], \quad P_m^\vee = \sum_{\lambda \in \Gamma_m} \sum_{\mu \in \Gamma_\infty} A[\lambda, \mu]. \quad (2.6.4)$$

Then P_n is an (A, A_n) -bimodule and P_m^\vee is an (A_m, A) -bimodule, which will play prominent roles in the next section.

Remark 2.6.2 The structure of an (A, A_n) -bimodule on P_n prolongs to a structure of an (\mathcal{A}, A_n) -bimodule and the structure of an (A_m, A) -bimodule on P_m^\vee to an (A_m, \mathcal{A}^\vee) -bimodule. (See Remark ?? and Proposition ??.)

Let us consider the case when A is quasi-finite.

Proposition 2.6.3 *Let (A, h) be a quasi-finite algebra graded by Hamiltonian. Then A_n are finite-dimensional for all n .*

Proof. For each λ and each μ , the space $A[\lambda, \mu]$ is isomorphic to $Q_m[\lambda, \mu]$ for sufficiently large m by Proposition ?.?. Since the range Γ_n of λ and μ is finite and $Q_m[\lambda, \mu] \subset Q_m(\lambda - \mu)$ is finite-dimensional, the space A_n is also finite-dimensional. \square

3 Exhaustive modules

We will define the notion of exhaustive modules and investigate the properties of the category of such modules over a weakly quasi-finite algebra A with Hamiltonian. In particular, we will show that the category of exhaustive A -modules and the category of A_n -modules are equivalent if $n \geq g$.

3.1 Exhaustive modules

Let (A, h) be a weakly quasi-finite algebra graded by Hamiltonian and endow it with the left linear topology.

Definition 3.1.1 A left A -module M is *exhaustive* if for any $v \in M$ there exists an m such that $I_m(A) \cdot v = 0$.

By this definition, it is easy to see that a left A -module M is exhaustive if and only if the action $A \times M \rightarrow M$ is continuous with respect to the left linear topology on A and the discrete topology on M . Note, however, that a topology on an exhaustive left A -module M need *not* be the discrete topology in order for the action $A \times M \rightarrow M$ to be continuous with respect to the left linear topology on A .

Lemma 3.1.2 *Let M be an exhaustive left A -module and let v be an element of M . Then $F_{-n-1}A \cdot v = 0$ implies $I_n(A) \cdot v = 0$.*

Proof. See the proof of Lemma ??.

□

Let us now consider the left linear filterwise completion \mathcal{A} . We may analogously define the notion of exhaustive left \mathcal{A} -modules as follows.

Definition 3.1.3 A left \mathcal{A} -module M is *exhaustive* if for any $v \in M$ there exists an m such that $I_m(\mathcal{A}) \cdot v = 0$.

It is fairly clear that results similar to those given above hold for exhaustive left \mathcal{A} -modules. The following proposition is a particular case of a general fact on topological algebras. (See e.g. [?].)

Proposition 3.1.4 *For any exhaustive left A -module M , the action $A \times M \rightarrow M$ induces an exhaustive left \mathcal{A} -module structure $\mathcal{A} \times M \rightarrow M$. Conversely, for any exhaustive left \mathcal{A} -module M , the action $\mathcal{A} \times M \rightarrow M$ restricts to an exhaustive left A -module structure.*

Thus the notion of exhaustive left A -modules and that of exhaustive left \mathcal{A} -modules have no essential differences.

Note 3.1.5 An exhaustive module is nothing else but a torsion module with respect to the left linear topology. (See [?].)

3.2 Generalized eigenspaces of exhaustive modules

Let A be a compatible degreewise topological algebra. For a left A -module M and a nonnegative integer n , we set

$$K_n(M) = \{v \in M \mid I_n(A) \cdot v = 0\}. \quad (3.2.1)$$

Note that M is exhaustive if and only if $M = \bigcup_n K_n(M)$ and if this is the case then $K_n(M) = \{v \in M \mid F_{-n-1}A \cdot v = 0\}$ by Lemma ??.

Consider the case when (A, h) is a weakly quasi-finite algebra graded by Hamiltonian. Since the space $K_n(M)$ has a structure of a left $Q_n(0)$ -module, it decomposes into the sum of generalized eigenspaces:

$$K_n(M) = \sum_{\lambda \in \Omega_n} K_n(M)[\lambda]. \quad (3.2.2)$$

Hence if M is exhaustive then the whole space M also decomposes into the sum of generalized eigenspaces.

Let us set

$$E_n(M) = \sum_{\lambda \in \Gamma_n} M[\lambda]. \quad (3.2.3)$$

The following is one of the key observations in the present paper.

Proposition 3.2.1 *If M is exhaustive then $E_n(M) \subset K_n(M) \subset E_{n+g}(M)$.*

Proof. The containment $K_n(M) \subset E_{n+g}(M)$ follows from $\Omega_n \subset \Gamma_{n+g}$ because $K_n(M)$ is a left $Q_n(0)$ -module. The rest is similar to Lemma ???. Consider the space $M[\lambda]$ with $\lambda \in \Gamma_n$. Then we have $A(d) \cdot M[\lambda] \subset M[\lambda + d]$. Therefore, since $\lambda - n - 1 \notin \Gamma_\infty$, we have $F_{-n-1}A \cdot M[\lambda] = 0$. This implies that $E_n(M) \subset K_n(M)$. \square

3.3 Equivalence of categories

We will mean by the category of exhaustive left A -modules the full subcategory of the category of left A -modules consisting of exhaustive left A -modules.

Let M be an exhaustive left A -module. Then the space $E_n(M)$ is a left A_n -module. A homomorphism $\phi : M' \rightarrow M''$ of left A -modules induces a map

$$E_n(\phi) : E_n(M') \rightarrow E_n(M''). \quad (3.3.1)$$

Thus we have a functor $E_n(-)$ from the category of exhaustive left A -module to the category of left A_n -modules.

Lemma 3.3.1 *If M is exhaustive then $K_0(M) = 0$ implies $M = 0$.*

Proof. Suppose $K_0(M) = 0$ and $M \neq 0$. Let v be a nonzero element of M . Since M is exhaustive, $I_n(A) \cdot v = 0$ and hence $F_{-n-1}A \cdot v = 0$ for sufficiently large n . Take the maximal integer n for which $F_{-n-1}A \cdot v \neq 0$. Then by the maximality $F_{-1}A \cdot (F_{-n-1}A \cdot v) = 0$ and hence $I_0(A) \cdot (F_{-n}A \cdot v) = 0$ by Lemma ???. Hence $F_{-n}A \cdot v \subset K_0(M) = 0$ which is a contradiction. \square

Lemma 3.3.2 *A homomorphism $\phi : M' \rightarrow M''$ of exhaustive left A -modules is an isomorphism if and only if $E_g(\phi)$ is an isomorphism of vector spaces.*

Proof. Assume that $E_g(\phi)$ is an isomorphism of vector spaces. Then, by Lemma ??, we see $\text{Ker } \phi = 0$ and $\text{Coker } \phi = 0$ since $K_0(\text{Ker } \phi) \subset E_g(\text{Ker } \phi) = \text{Ker}(E_g(\phi)) = 0$ and $K_0(\text{Coker } \phi) \subset E_g(\text{Coker } \phi) = \text{Coker}(E_g(\phi)) = 0$. \square

Recall the (A, A_n) -bimodule P_n defined in Subsection ??, which is an exhaustive left A -module. Therefore, for a left A_n -module X , the space $P_n \otimes_{A_n} X$ is an exhaustive left A -module. We note that $A_n = E_n(P_n)$.

Lemma 3.3.3 *For a left A_n -module X , the map $A_n \otimes_{A_n} X \rightarrow P_n \otimes_{A_n} X$ induced by the inclusion $A_n \rightarrow P_n$ is injective.*

Proof. We set $E_n^\perp(P_n) = \sum_{\lambda \in \Gamma_\infty \setminus \Gamma_n} P_n[\lambda]$. Then the decomposition $P_n = E_n(P_n) \oplus E_n^\perp(P_n) = A_n \oplus E_n^\perp(P_n)$ is a direct sum decomposition of a right A_n -module. Hence the map $E_n(P_n) \otimes_{A_n} X \rightarrow P_n \otimes_{A_n} X$ is injective. \square

Now we come to the second main result of Part I. Recall the number g defined by (??).

Theorem 3.3.4 *Let (A, h) be a weakly quasi-finite algebra graded by Hamiltonian and let n be an integer such that $n \geq g$. Then the functors $E_n(-)$ and $P_n \otimes_{A_n} -$ are mutually inverse equivalences of categories between the category of exhaustive left A -modules and the category of left A_n -modules.*

Proof. Let X be a left A_n -module. By Lemma ??, the map $A_n \otimes_{A_n} X \rightarrow P_n \otimes_{A_n} X$ is injective. Therefore

$$X \cong A_n \otimes_{A_n} X \cong E_n(P_n \otimes_{A_n} X). \quad (3.3.2)$$

Let M be an exhaustive left A -module. Then by letting $X = E_n(M)$ in (??), we have

$$E_n(M) \cong E_n(P_n \otimes_{A_n} E_n(M)). \quad (3.3.3)$$

Since $n \geq g$, we have $P_n \otimes_{A_n} E_n(M) \cong M$ by Lemma ??. \square

Corollary 3.3.5 *Let (A, h) be a weakly quasi-finite algebra graded by Hamiltonian and let n be an integer such that $n \geq g$. Then the module P_n is a progenerator of the category of exhaustive left A -modules.*

Let us consider the case when A is quasi-finite. Then the algebra A_n is finite-dimensional for all n by Proposition ??. Thus we have the following corollary.

Corollary 3.3.6 *If (A, h) is a quasi-finite algebra graded by Hamiltonian then the category of exhaustive left A -modules is equivalent to the category of left modules over a finite-dimensional algebra.*

We may likewise define the notion of exhaustive right A -modules and exhaustive right \mathcal{A}^\vee -modules by using the spaces $I_n^\vee(A)$ and $I_n^\vee(\mathcal{A}^\vee)$, respectively. We have analogous results for the right modules as well.

Note 3.3.7 Theorem ?? may be understood to be a particular case of a topological variant of Morita equivalences. See [?] for general theory of equivalences between abelian categories and module categories and [?] and [?] for results closely related to Theorem ??.

4 Coexhaustive modules

We will now consider the notion of coexhaustive modules, which is *dual* to that of exhaustive modules. In this section, we will give the definition of coexhaustive

modules and describe their topologies by means of the generalized eigenspaces. The precise duality will be considered in the next section under necessary finiteness assumptions.

4.1 Coexhaustive modules

Let (A, h) be a weakly quasi-finite algebra graded by Hamiltonian and let \mathcal{A}^\vee be the right filterwise completion.

Let \mathcal{M} be a left \mathcal{A}^\vee -module endowed with a linear topology such that the action $\mathcal{A}^\vee \times \mathcal{M} \rightarrow \mathcal{M}$ is continuous. Let us denote by $I_n^\vee(\mathcal{M})$ the closure of the space $F_{-n-1}^\vee \mathcal{A}^\vee \cdot \mathcal{M}$. We assume that the sequence $\{I_n^\vee(\mathcal{M})\}$ forms a fundamental system of open neighborhoods of zero.

Definition 4.1.1 A compatible topological left \mathcal{A}^\vee -module is a left \mathcal{A}^\vee -module endowed with a linear topology which satisfies the conditions mentioned above.

Let \mathcal{M} be a compatible topological left \mathcal{A}^\vee -module. Then we have

$$F_p^\vee \mathcal{A}^\vee \cdot I_n^\vee(\mathcal{M}) \subset I_{n-p}^\vee(\mathcal{M}), \quad I_n^\vee(\mathcal{A}^\vee) \cdot \mathcal{M} \subset I_n^\vee(\mathcal{M}) \quad (4.1.1)$$

because the action $\mathcal{A}^\vee \times \mathcal{M} \rightarrow \mathcal{M}$ is continuous.

Definition 4.1.2 A compatible topological left \mathcal{A}^\vee -module \mathcal{M} is *coexhaustive* if \mathcal{M} is complete as a topological vector space.

Obviously, the completion of a compatible topological left \mathcal{A}^\vee -module has a canonical structure of a coexhaustive left \mathcal{A}^\vee -module.

4.2 Generalized eigenspaces of coexhaustive modules

Let \mathcal{M} be a coexhaustive left \mathcal{A}^\vee -module. Consider the space

$$Q_n^\vee(\mathcal{M}) = \mathcal{M}/I_n^\vee(\mathcal{M}). \quad (4.2.1)$$

Then this is a left $Q_n^\vee(0)$ -module and hence has a generalized eigenspace decomposition of the form

$$Q_n^\vee(\mathcal{M}) = \sum_{\mu \in \Omega_n} Q_n^\vee(\mathcal{M})[\mu]. \quad (4.2.2)$$

Lemma 4.2.1 *If $\lambda \in \Gamma_m$ and $n \geq m$ then the restriction $Q_n^\vee[\lambda] \rightarrow Q_m^\vee[\lambda]$ of the canonical surjection is an isomorphism.*

Proof. See the proof of Lemma ??.

□

This lemma implies that the subspace $\sum_{\lambda \in \Gamma_\infty} \mathcal{M}[\lambda]$ is dense in \mathcal{M} . It also implies that the space

$$E_n(\mathcal{M}) = \sum_{\lambda \in \Gamma_n} \mathcal{M}[\lambda] \quad (4.2.3)$$

is a discrete subspace for each n . Consider the space $E_n^\perp(\mathcal{M}) = \sum_{\lambda \in \Gamma_\infty \setminus \Gamma_n} \mathcal{M}[\lambda]$ and let $\widehat{E_n^\perp(\mathcal{M})}$ be its closure in \mathcal{M} . Then the quotient

$$P_n(\mathcal{M}) = \mathcal{M} / \widehat{E_n^\perp(\mathcal{M})} \quad (4.2.4)$$

is canonically isomorphic to $E_n(\mathcal{M})$.

Lemma 4.2.2 *If \mathcal{M} is a coexhaustive left \mathcal{A}^\vee -module then the canonical map $E_m(\mathcal{M}) \rightarrow Q_m^\vee(\mathcal{M})$ is injective and the canonical map $Q_m^\vee(\mathcal{M}) \leftarrow E_{m+g}(\mathcal{M})$ is surjective.*

Proof. By Lemma ??, we know that the map $\mathcal{M}[\lambda] \rightarrow Q_n^\vee(\mathcal{M})[\lambda]$ is injective if $\lambda \in \Gamma_n$. Hence the map $E_n(\mathcal{M}) \rightarrow Q_n^\vee(\mathcal{M})$ is injective. Now recall that $Q_n^\vee(\mathcal{M}) = \sum_{\lambda \in \Gamma_{n+g}} Q_n^\vee(\mathcal{M})[\lambda]$. Hence the map $Q_m^\vee(\mathcal{M}) \leftarrow E_{m+g}(\mathcal{M})$ is surjective. \square

4.3 Exhaustion and coexhaustion

We will mean by the category of coexhaustive left \mathcal{A}^\vee -modules the category for which the objects are the coexhaustive left \mathcal{A}^\vee -modules and the morphisms are the continuous homomorphisms of left \mathcal{A}^\vee -modules.

Let M be an exhaustive left \mathcal{A} -module and regard it as a left A -module by identifying A with a subalgebra of \mathcal{A} via the canonical map $A \rightarrow \mathcal{A}$. We give M the linear topology defined by $I_n^\vee(M) = F_{-n-1}^\vee A \cdot M$. Then M becomes a left A -module such that the action $A \times M \rightarrow M$ is continuous. We let $Q_\infty^\vee(M)$ be the completion:

$$Q_\infty^\vee(M) = \varprojlim_n Q_n^\vee(M), \quad Q_n^\vee(M) = M / I_n^\vee(M). \quad (4.3.1)$$

Then $Q_\infty^\vee(M)$ is a coexhaustive left \mathcal{A}^\vee -module.

Let M' and M'' be exhaustive left \mathcal{A} -modules and let $\phi : M' \rightarrow M''$ be a homomorphism of left \mathcal{A} -modules. Then ϕ gives rise to a continuous homomorphism of left A -modules. Hence it induces a continuous homomorphism $Q_\infty^\vee(\phi) : Q_\infty^\vee(M') \rightarrow Q_\infty^\vee(M'')$ of coexhaustive left \mathcal{A}^\vee -modules. We call the functor $Q_\infty^\vee(-)$ the *coexhaustion functor*.

Conversely, let \mathcal{M} be a coexhaustive left \mathcal{A}^\vee -module and regard it as a left A -module via the canonical map $A \rightarrow \mathcal{A}^\vee$. Consider the space

$$K_\infty(\mathcal{M}) = \bigcup_n K_n(\mathcal{M}), \quad K_n(\mathcal{M}) = \{v \in \mathcal{M} \mid I_n(A) \cdot v = 0\}. \quad (4.3.2)$$

Then this is an exhaustive left A -module and hence an exhaustive left \mathcal{A} -module.

Let $\varphi : \mathcal{M}' \rightarrow \mathcal{M}''$ be a continuous homomorphism of coexhaustive left \mathcal{A}^\vee -modules. We regard φ as a homomorphism of left A -modules via the canonical map $A \rightarrow \mathcal{A}^\vee$. If $I_n(A) \cdot v = 0$ then $I_n(A) \cdot \varphi(v) = \varphi(I_n(A) \cdot v) = \varphi(0) = 0$. Therefore, φ restricts to a homomorphism $K_\infty(\varphi) : K_\infty(\mathcal{M}') \rightarrow K_\infty(\mathcal{M}'')$ of left A -modules and hence a homomorphism of left \mathcal{A} -modules. We call the functor $K_\infty(-)$ the *exhaustion functor*.

Let us set

$$\mathcal{P}_\infty(M) = \varprojlim_n \mathcal{P}_n(M), \quad \mathcal{E}_\infty(\mathcal{M}) = \bigcup_n \mathcal{E}_n(\mathcal{M}). \quad (4.3.3)$$

Lemma 4.3.1 *For an exhaustive left \mathcal{A} -module M its coexhaustion $\mathcal{Q}_\infty^\vee(M)$ is canonically isomorphic to $\mathcal{P}_\infty(M)$ as topological vector spaces. For a coexhaustive left \mathcal{A}^\vee -module \mathcal{M} its exhaustion $\mathcal{K}_\infty(\mathcal{M})$ is canonically isomorphic to $\mathcal{E}_\infty(\mathcal{M})$ as vector spaces.*

Proof. Conclusions are clear by the arguments in Subsections ?? and ??. \square

Now the following theorem is an immediate consequence of this lemma.

Theorem 4.3.2 *Let (A, h) be a weakly quasi-finite algebra graded by Hamiltonian and let \mathcal{A} and \mathcal{A}^\vee be the left and the right filterwise completions, respectively. Then the functors $\mathcal{Q}_\infty^\vee(-)$ and $\mathcal{K}_\infty(-)$ are mutually inverse equivalences of categories between the category of exhaustive left \mathcal{A} -modules and the category of coexhaustive left \mathcal{A}^\vee -modules.*

In particular, the category of coexhaustive left \mathcal{A}^\vee -modules is an abelian category.

We may likewise define the notion of coexhaustive right A -modules and coexhaustive right \mathcal{A} -modules by using the spaces $\mathcal{I}_n(A)$ and $\mathcal{I}_n(\mathcal{A})$, respectively. We have analogous results for the right modules as well.

5 Duality for quasi-finite modules

We now consider finiteness conditions for exhaustive modules and for coexhaustive modules and discuss the duality between the categories of such modules.

5.1 Quasi-finiteness of exhaustive modules

Let (A, h) be a weakly quasi-finite algebra graded by Hamiltonian and let \mathcal{A} and \mathcal{A}^\vee be the left and the right filterwise completions, respectively.

Definition 5.1.1 A *quasi-finite left \mathcal{A} -module* is an exhaustive left \mathcal{A} -module M such that the spaces $\mathcal{K}_n(M)$ are finite-dimensional for all n .

For an exhaustive right \mathcal{A}^\vee -module N , we define the space $\mathcal{K}_n^\vee(N)$ in the same way as $\mathcal{K}_n(M)$ for an exhaustive left \mathcal{A} -module M :

$$\mathcal{K}_n^\vee(N) = \{v \in N \mid v \cdot \mathcal{I}_n^\vee(\mathcal{A}^\vee) = 0\}. \quad (5.1.1)$$

Definition 5.1.2 A *quasi-finite right \mathcal{A}^\vee -module* is an exhaustive right \mathcal{A}^\vee -module N such that the spaces $\mathcal{K}_n^\vee(N)$ are finite-dimensional for all n .

The following proposition characterizes the quasi-finiteness of an algebra by means of the quasi-finiteness of the canonical quotient modules.

Proposition 5.1.3 *Let (A, h) be a weakly quasi-finite algebra with Hamiltonian. Then the following conditions are equivalent:*

- (a) A is a quasi-finite algebra.
- (b) The left canonical quotient modules Q_n are quasi-finite for all n .
- (c) The right canonical quotient modules Q_n^\vee are quasi-finite for all n .

Proof. We will show the equivalence of (a) and (b). Since $K_m(Q_n)$ is an (H_m, H_n) -bimodule, the pair of the left and the right eigenvalues of h on $K_m(Q_n)$ are in the finite set $\Omega_m \times \Omega_n$. Then if $K_m(Q_n)(d) \neq 0$ then $d = \lambda - \mu$ for some $\lambda \in \Omega_m$ and $\mu \in \Omega_n$. Therefore, we have

$$K_m(Q_n) \subset \bigoplus_{-n \leq d \leq m+n} Q_n(d). \quad (5.1.2)$$

On the other hand, it is easy to see that $Q_n(d) \subset K_{n+d+1}(Q_n)$. Therefore we immediately see that $K_m(Q_n)$ are finite-dimensional for all m and n if and only if $Q_n(d)$ are finite-dimensional for all d and n . The proof for the equivalence of (a) and (c) is similar. □

Now consider the case when (A, h) is a quasi-finite algebra graded by Hamiltonian. Since $E_n(M)$ is a left A_n -module and A_n is finite-dimensional, $E_n(M)$ is finite-dimensional if and only if it is finitely generated as a left A_n -module.

Proposition 5.1.4 *Let (A, h) be a quasi-finite algebra graded by Hamiltonian. Then the following conditions for an exhaustive left \mathcal{A} -module M are equivalent:*

- (a) M is a quasi-finite left \mathcal{A} -module.
- (b) M is finitely generated as a left \mathcal{A} -module.
- (c) $E_n(M)$ are finite-dimensional for all n .
- (d) $E_g(M)$ is finite-dimensional.

Proof. We will show (a) \Rightarrow (c) \Rightarrow (d) \Rightarrow (b) \Rightarrow (a). Assume that M is quasi-finite. Then $E_n(M)$ are finite-dimensional since $E_n(M) \subset K_n(M)$ by Proposition ???. In particular, $E_g(M)$ is finite-dimensional. Next assume that $E_g(M)$ is finite-dimensional and let M' be the left \mathcal{A} -submodule of M generated by $E_g(M)$. Then $K_0(M/M') \subset E_g(M/M') = 0$. By Lemma ???, we have $M = M'$ and hence M is finitely generated. Now assume that M is finitely generated and let v_1, \dots, v_k be a set of generators. Since M is exhaustive, there exists n_1, \dots, n_k such that $I_{n_i}(\mathcal{A}) \cdot v_i = 0$ for $i = 1, \dots, k$. This implies the existence of a surjective homomorphism $Q_{n_1} \times \dots \times Q_{n_k} \rightarrow M$ of left \mathcal{A} -modules. Since A is quasi-finite and hence the modules Q_{n_i} are quasi-finite, so is the image M . □

5.2 Quasi-finiteness of coexhaustive modules

Let us now turn to the quasi-finiteness of coexhaustive modules.

Definition 5.2.1 A *quasi-finite left \mathcal{A}^\vee -module* is a coexhaustive left \mathcal{A}^\vee -module \mathcal{M} such that the spaces $Q_n^\vee(\mathcal{M})$ are finite-dimensional for all n .

For a coexhaustive right \mathcal{A} -module \mathcal{N} , we set

$$Q_n(\mathcal{N}) = \mathcal{N}/I_n(\mathcal{N}). \quad (5.2.1)$$

Definition 5.2.2 A *quasi-finite right \mathcal{A} -module* is a coexhaustive right \mathcal{A} -module \mathcal{N} such that the spaces $Q_n(\mathcal{N})$ are finite-dimensional for all n .

For an exhaustive left \mathcal{A} -module M , the canonical map $E_n(M) \rightarrow P_n^\vee(M)$ is an isomorphism. Therefore, Lemma ?? implies that $E_n(M)$ are finite-dimensional for all n if and only if $P_n^\vee(M)$ are finite-dimensional for all n . Therefore, the consideration in Subsection ?? implies the following result.

Proposition 5.2.3 *Let (A, h) be a quasi-finite algebra graded by Hamiltonian. Let M be an exhaustive left \mathcal{A} -module and let \mathcal{M} be its coexhaustion. Then the exhaustive module M is quasi-finite if and only if the coexhaustive module \mathcal{M} is quasi-finite.*

We have analogous results for quasi-finite right \mathcal{A}^\vee -modules and for quasi-finite right \mathcal{A} -modules.

5.3 Duality

Let (A, h) be a weakly quasi-finite algebra graded by Hamiltonian. Let M be an exhaustive left \mathcal{A} -module and consider its full dual space \mathcal{N} :

$$\mathcal{N} = M^* = \text{Hom}_{\mathbf{k}}(M, \mathbf{k}). \quad (5.3.1)$$

Then this space becomes a right \mathcal{A} -module.

Since M is exhaustive, it has the generalized eigenspace decomposition $M = \bigoplus_{\lambda \in \Gamma_\infty} M[\lambda]$, which gives rise to

$$\mathcal{N} = \prod_{\lambda \in \Gamma_\infty} M[\lambda]^* \quad (5.3.2)$$

where $M[\lambda]^*$ is the set of linear functions $f : M \rightarrow \mathbf{k}$ such that $f(M[\mu]) = 0$ holds for any $\mu \neq \lambda$. Therefore, by setting

$$J_n(\mathcal{N}) = \{f \in \mathcal{N} \mid f(E_n(M)) = 0\}, \quad (5.3.3)$$

we have

$$\mathcal{N}/J_n(\mathcal{N}) = \prod_{\lambda \in \Gamma_n} M[\lambda]^*. \quad (5.3.4)$$

We give \mathcal{N} the linear topology defined by $J_n(\mathcal{N})$.

Lemma 5.3.1 *The topological right \mathcal{A} -module \mathcal{N} is a coexhaustive right \mathcal{A} -module.*

Proof. Completeness of \mathcal{N} as a topological vector space is clear from the definition. Since $(f \cdot a)(E_n(M)) = f(a \cdot E_n(M)) = 0$ holds for any $f \in \mathcal{N}$ and any $a \in I_n(\mathcal{A})$, we have $\mathcal{N} \cdot I_n(\mathcal{A}) \subset J_n(\mathcal{N})$. Hence the action $\mathcal{N} \times \mathcal{A} \rightarrow \mathcal{N}$ is continuous. Let $I_n(\mathcal{N})$ be the closure of $\mathcal{N} \cdot I_n(\mathcal{A})$. It remains to show that $\{I_n(\mathcal{N})\}$ forms a fundamental system of open neighborhoods of zero. Since $J_n(\mathcal{N})$ is closed, we have $I_n(\mathcal{N}) \subset J_n(\mathcal{N})$. Consider the quotient $\mathcal{N}/I_n(\mathcal{N})$. Since we have $I_n(\mathcal{N}) \cdot I_n(\mathcal{A}) \subset I_n(\mathcal{N})$, the quotient space $\mathcal{N}/I_n(\mathcal{N})$ is a right $Q_n(0)$ -module. In particular, it is a right H_n -module. Hence it has a generalized eigenspace decomposition with respect to the right action of h with the eigenvalues belonging to $\Omega_n \subset \Gamma_{n+g}$. Since $J_{n+g}(\mathcal{N})$ is the closure of $E_{n+g}^\perp(\mathcal{N})$, we have $J_{n+g}(\mathcal{N}) \subset I_n(\mathcal{N})$. Hence $I_n(\mathcal{N})$ with $n \in \mathbb{N}$ form a basis of open neighborhoods of zero. \square

Thus we have shown that for any exhaustive left \mathcal{A} -module, its full dual naturally becomes a coexhaustive right \mathcal{A} -module.

Conversely, let \mathcal{N} be a coexhaustive right \mathcal{A} -module and consider the continuous dual space M :

$$M = \text{Hom}_{\mathbf{k}}^{\text{cont}}(\mathcal{N}, \mathbf{k}). \quad (5.3.5)$$

Recall that the base field \mathbf{k} is given the discrete topology.

Take any element of M , which is given by a continuous map $f : \mathcal{N} \rightarrow \mathbf{k}$. Then, for any $a \in \mathcal{A}$, the map $a \cdot f : \mathcal{N} \rightarrow \mathbf{k}$ is also continuous since, for $a \in F_p \mathcal{A}$, we have

$$(a \cdot f)(v + I_n(\mathcal{N})) \subset f(v \cdot a) + f(I_{n-p}(\mathcal{N})) = f(v \cdot a) \quad (5.3.6)$$

for sufficiently large n . We then have

$$(I_n(\mathcal{A}) \cdot f)(v) = f(v \cdot I_n(\mathcal{A})) \subset f(I_n(\mathcal{N})) = 0 \quad (5.3.7)$$

for all v . Therefore, the module M is exhaustive.

Thus we have defined contravariant functors $\text{Hom}_{\mathbf{k}}(-, \mathbf{k})$ and $\text{Hom}_{\mathbf{k}}^{\text{cont}}(-, \mathbf{k})$ between the category of exhaustive left \mathcal{A} -modules and the category of coexhaustive right \mathcal{A} -modules.

Proposition 5.3.2 *Let (A, h) be a weakly quasi-finite algebra graded by Hamiltonian. Then the functors $\text{Hom}_{\mathbf{k}}(-, \mathbf{k})$ and $\text{Hom}_{\mathbf{k}}^{\text{cont}}(-, \mathbf{k})$ give rise to mutually inverse duality of categories between the category of quasi-finite left \mathcal{A} -modules and the category of quasi-finite right \mathcal{A} -modules.*

Note 5.3.3 A quasi-finite right \mathcal{A} -module is nothing else but a linearly compact right \mathcal{A} -module. We may understand the above-mentioned duality as a version of the Lefschetz duality [?] between discrete modules and linearly compact modules. See also [?].

5.4 Involution

Let (A, h) be a quasi-finite algebra graded by Hamiltonian. Suppose given a linear involution $\theta : A \rightarrow A$ satisfying the following conditions:

- (i) $\theta(a \cdot b) = \theta(b) \cdot \theta(a)$ for any $a, b \in A$ and $\theta(h) = h$.
- (ii) $\theta : A(d) \rightarrow A(-d)$ is continuous.

Note that by (i) we have $[h, \theta(a)] = [\theta(h), \theta(a)] = -\theta([h, a])$. Hence $\theta(A(d)) = A(-d)$ and the condition (ii) makes sense. We then have that $\theta(I_n(A)) = I_n^\vee(A)$. Hence θ extends to anti-isomorphisms

$$\theta_\infty : \mathcal{A} \rightarrow \mathcal{A}^\vee, \quad \theta_\infty^\vee : \mathcal{A}^\vee \rightarrow \mathcal{A} \quad (5.4.1)$$

of filtered topological algebras such that $\theta_\infty \circ \theta_\infty^\vee = 1$ and $\theta_\infty^\vee \circ \theta_\infty = 1$. We will denote the maps θ_∞ and θ_∞^\vee by the same symbol θ by abuse of notation.

Let M be a left \mathcal{A} -module and let $\vartheta(M)$ be the same space M as a vector space. We give $\vartheta(M)$ a structure of a right \mathcal{A}^\vee -module by letting $v \cdot a = \theta(a) \cdot v$ for $a \in \mathcal{A}^\vee$ and $v \in M$. Similarly, for a right \mathcal{A}^\vee -module N , we define a left \mathcal{A} -module $\vartheta^\vee(N)$ in a similar way.

Proposition 5.4.1 *The functors ϑ and ϑ^\vee are mutually inverse equivalences of categories between the category of left \mathcal{A} -modules and the category of right \mathcal{A}^\vee -modules.*

Let us compose the involution, the duality and the exhaustion. Then we get an auto-duality

$$D : M \mapsto \vartheta(K_\infty(\text{Hom}_{\mathbf{k}}(M, \mathbf{k}))) \quad (5.4.2)$$

of the category of quasi-finite left \mathcal{A} -modules. Note that $K_\infty(\text{Hom}_{\mathbf{k}}(M, \mathbf{k}))$ is the restricted dual space:

$$K_\infty(\text{Hom}_{\mathbf{k}}(M, \mathbf{k})) = \{f \mid f(E_n^\perp(M)) = 0 \text{ for some } n\}. \quad (5.4.3)$$

5.5 Finiteness theorems

Let (A, h) be a quasi-finite algebra and consider the following categories.

- L1. The category of quasi-finite left \mathcal{A} -modules.
- L2. The category of quasi-finite left \mathcal{A}^\vee -modules.
- L3. The category of finitely generated left A_n -modules.
- R1. The category of quasi-finite right \mathcal{A}^\vee -modules.
- R2. The category of quasi-finite right \mathcal{A} -modules.
- R3. The category of finitely generated right A_n -modules.

Recall that quasi-finite left \mathcal{A} -modules and right \mathcal{A}^\vee -modules are exhaustive whereas quasi-finite left \mathcal{A}^\vee -modules and right \mathcal{A} -modules are coexhaustive.

The following theorem summarizes the results obtained so far regarding the quasi-finite modules over quasi-finite algebras.

Theorem 5.5.1 *Let (A, h) be a quasi-finite algebra with Hamiltonian and let n be an integer such that $n \geq g$.*

- (1) *The categories L1, L2 and L3 are equivalent to each other.*
- (2) *The categories R1, R2 and R3 are equivalent to each other.*
- (3) *The categories L1, L2 and L3 and the categories R1, R2 and R3 are dual to each other.*
- (4) *If A has an involution θ then the categories L1, L2 and L3 and the categories R1, R2 and R3 are equivalent to each other.*

Part II

Quasi-finiteness and Zhu's finiteness condition

6 Vertex operator algebras and current algebras

We now turn our attention to vertex operator algebras. In this section, we will describe in detail the construction and properties of the universal enveloping algebra associated with a vertex operator algebra, which we will simply call the *current algebra*, in order to give precise statements which seem to have been overlooked in the literatures.

6.1 Vertex operator algebra

Recall that a vertex operator algebra is a graded vector space V with the grading being indexed by integers equipped with countably many bilinear maps indexed by integers and two distinguished elements, called the vacuum vector and the conformal vector (or the Virasoro element), satisfying a number of axioms ([?], [?], [?]), which we will describe briefly below. See [?] for an account.

Let us denote the homogeneous subspaces of the grading of V by $V[r]$ with $r \in \mathbb{Z}$. It is assumed that there exists an $m \in \mathbb{N}$ such that $V[r] = 0$ for $r < -m$. Therefore, the grading of V is of the following form

$$V = \bigoplus_{r=-m}^{\infty} V[r]. \quad (6.1.1)$$

We will write $\Delta(u) = r$ when u belongs to the subspace $V[r]$ and call $\Delta(u)$ the *weight* of the element u .

Let us denote the countably many bilinear maps by

$$\mu_n : V \times V \rightarrow V, \quad (u, v) \mapsto u_{(n)}v. \quad (6.1.2)$$

It is assumed that they satisfy $V[r]_{(n)}V[s] \subset V[r + s - n - 1]$. In other words, for homogeneous u and v , we have

$$\Delta(u_{(n)}v) = \Delta(u) + \Delta(v) - n - 1. \quad (6.1.3)$$

Then the sums in the following expression are finite:

$$\begin{aligned} & \sum_{i=0}^{\infty} \binom{p}{i} (u_{(r+i)}v)_{(p+q-i)}w \\ &= \sum_{i=0}^{\infty} (-1)^i \binom{r}{i} (u_{(p+r-i)}(v_{(q+i)}w) - (-1)^r v_{(q+r-i)}(u_{(p+i)}w)). \end{aligned} \quad (6.1.4)$$

The bilinear maps (??) are assumed to satisfy this identity for any $u, v, w \in V$ and any $p, q, r \in \mathbb{Z}$. This set of identities is the main identity of vertex operator algebras called the *Borcherds identity* or the *Cauchy-Jacobi identity*.

The vacuum vector $\mathbf{1}$ is an element of weight 0. It enjoys, for any $u \in V$, the relation $u_{(-1)}\mathbf{1} = u$ and $u_{(n)}\mathbf{1} = 0$ for $n \geq 0$. We set

$$Tu = u_{(-2)}\mathbf{1}. \quad (6.1.5)$$

Then it follows that the operator $T : V \rightarrow V$ is a derivation with respect to the operations $u_{(n)}$ for every n .

The conformal vector ω is an element of weight 2. It satisfies

$$\omega_{(n)}\omega = 0, \quad (n \geq 4 \text{ or } n = 2), \quad \omega_{(1)}\omega = 2\omega, \quad \omega_{(3)}\omega \in \mathbf{k}\mathbf{1}. \quad (6.1.6)$$

Then it follows that the operators $L_n : V \rightarrow V$ defined by $L_n u = \omega_{(n+1)}u$ satisfy the Virasoro commutation relation with the central charge c given by $2\omega_{(3)}\omega = c\mathbf{1}$. Among the Virasoro operators L_n , the L_0 and L_{-1} have special roles: the weights of V are given by the eigenvalues of L_0 and the derivation T agrees with L_{-1} . In other words,

$$V[r] = \{v \in V \mid L_0 v = rv\}, \quad Tu = L_{-1}u. \quad (6.1.7)$$

The weight subspaces are usually assumed to be finite-dimensional.

Among the conditions regarding the Virasoro operators, only the one concerning the eigenvalues of L_0 will be used in the rest of the paper.

6.2 The current Lie algebras

Let us consider the space

$$V[t, t^{-1}] = V \otimes_{\mathbf{k}} \mathbf{k}[t, t^{-1}]. \quad (6.2.1)$$

We define a bilinear map $V[t, t^{-1}] \times V[t, t^{-1}] \rightarrow V[t, t^{-1}]$ by setting

$$[u \otimes t^m, v \otimes t^n] = \sum_{i=0}^{\infty} \binom{m}{n} (u_{(i)}v) \otimes t^{m+n-i-1}. \quad (6.2.2)$$

Consider the quotient space

$$\mathfrak{g} = V[t, t^{-1}] / \partial V[t, t^{-1}] \quad (6.2.3)$$

where $\partial : V[t, t^{-1}] \rightarrow V[t, t^{-1}]$ is defined by

$$\partial(u \otimes t^n) = Tu \otimes t^n + nu \otimes t^{n-1}. \quad (6.2.4)$$

Then the bracket operation on $V[t, t^{-1}]$ induces a bilinear operation on \mathfrak{g} denoted by the same symbol which gives a structure of a Lie algebra on \mathfrak{g} ([?]). We will call the Lie algebra \mathfrak{g} the associated *current Lie algebra*. We will denote the image of an element of $V[t, t^{-1}]$ in \mathfrak{g} by the same symbol.

Since $n\mathbf{1} \otimes t^{n+1} = \partial(\mathbf{1} \otimes t^n)$, we know that $\mathbf{1} \otimes t^n = 0$ in \mathfrak{g} is zero unless $n = -1$, when $\mathbf{1} \otimes t^{-1}$ is central.

It will be useful to introduce the following notation:

$$J_n(u) = u \otimes t^{n+\Delta(u)-1} \quad (6.2.5)$$

for a homogeneous u and extend it linearly. We denote its image in \mathfrak{g} by the same symbol and assign the degree $-n$ to $J_n(u)$. Then the associated current Lie algebra is graded by the degree:

$$\mathfrak{g} = \bigoplus_{d \in \mathbb{Z}} \mathfrak{g}(d). \quad (6.2.6)$$

Note the relation

$$[L_0, J_n(u)] = -nJ_n(u) \quad (6.2.7)$$

for the element $L_0 = J_0(\omega)$, which follows from the axioms for vertex operator algebras.

Let \mathbb{U} be the quotient algebra of the universal enveloping algebra of the Lie algebra \mathfrak{g} by the two sided ideal generated by $J_0(\mathbf{1}) - 1$ and let us denote the image of $J_n(u)$ by the same symbol. We give the degree $d_1 + \dots + d_k$ to the element of the form $J_{-d_1}(u_1) \cdots J_{-d_k}(u_k)$ with $u_1, \dots, u_k \in V$. Let $\mathbb{U}(d)$ be the span of these vectors of degree d . Then we have

$$\mathbb{U} = \bigoplus_{d \in \mathbb{Z}} \mathbb{U}(d). \quad (6.2.8)$$

by which the algebra \mathbb{U} becomes a graded algebra. Note that the relation (??) says that the image of L_0 is a Hamiltonian of \mathbb{U} .

Consider the standard degreewise topology on \mathbb{U} and let $\hat{\mathbb{U}}$ denote the degreewise completion. (See Subsection ??.)

6.3 The current algebras

For $u, v \in V$, consider the following expressions in $\hat{\mathbb{U}}$ for k, m, n :

$$\begin{aligned} B_{k,m,n}(u, v) = & \sum_{i=0}^{\infty} \binom{k + \Delta(u) - 1}{i} J_{k+m+n}(u_{(n+i)}v) \\ & - \sum_{i=0}^{\infty} (-1)^i \binom{n}{i} J_{k+n-i}(u) \cdot J_{m+i}(v) \\ & + (-1)^n \sum_{i=0}^{\infty} (-1)^i \binom{n}{i} J_{m+n-i}(v) \cdot J_{k+i}(u). \end{aligned} \quad (6.3.1)$$

Then the first sum in the right-hand side is actually a finite sum whereas the second and the last are infinite sums which converge in the linear topology of $\hat{\mathbb{U}}(-k-m-n)$. The relations $B_{k,m,n}(u, v) = 0$ are nothing else but the Borcherds identities for the actions on a module where the sums in the right-hand side become finite sums when they act on a fixed element of a module. (See the next subsection for the definition of modules.)

Let \mathbb{B} be the ideal of $\hat{\mathbb{U}}$ generated by the elements of the form $B_{k,m,n}(u, v)$ with $u, v \in V$ and integers k, m, n , and let $\hat{\mathbb{B}}$ be the degreewise closure of \mathbb{B} . Then $\hat{\mathbb{B}}$ is also an ideal of $\hat{\mathbb{U}}$.

Remark 6.3.1 The ideal \mathbb{B} is in fact generated by the elements of the form $B_{k,m,n}(u, v)$ with $k = -\Delta(u) + 1$ and $m, n \in \mathbb{Z}$. Alternatively, it is also generated by the elements of the form $B_{k,m,n}(u, v)$ with $k, m \in \mathbb{Z}$ and $n \in \mathbb{Z} \setminus \mathbb{N}$.

We now define the *current algebra* \mathbf{U} associated with V to be the quotient algebra of $\hat{\mathbf{U}}$ by the ideal $\hat{\mathbb{B}}$:

$$\mathbf{U} = \hat{\mathbf{U}} / \hat{\mathbb{B}}. \quad (6.3.2)$$

Then \mathbf{U} is a graded algebra, since $\hat{\mathbb{B}}$ is a graded ideal, and the image of L_0 is a Hamiltonian.

Proposition 6.3.2 *The algebra \mathbf{U} is a compatible degreewise complete algebra with Hamiltonian.*

Note 6.3.3 This construction is essentially due to Frenkel and Zhu [?]. The left linear filterwise completion of \mathbf{U} as in Subsection ?? is isomorphic to the current algebra $\mathcal{U}(V)$ considered in [?].

6.4 Denseness of the current Lie algebra

Let \mathbf{U} be the current algebra associated with a vertex operator algebra V . Let us regard the current Lie algebra \mathfrak{g} as a subspace of \mathbf{U} and let ϕ denote the composition of the canonical maps $\mathbf{U} \rightarrow \hat{\mathbf{U}} \rightarrow \mathbf{U}$. By construction, $\mathbf{U}(d)$ is a dense subspace of $\hat{\mathbf{U}}(d)$.

The following observation is insightful.

Proposition 6.4.1 *The image $\phi(\mathfrak{g}(d))$ is a dense subspace of $\mathbf{U}(d)$ for each integer d .*

Proof. It suffices to show that $\phi(\mathfrak{g})$ is dense in \mathbf{U} with respect to the left linear topology on \mathbf{U} . Let us denote by $\phi_n : \mathbf{U} \rightarrow \mathbb{Q}_n$ the composite of ϕ with the canonical surjection $\mathbf{U} \rightarrow \mathbb{Q}_n$. By the Borcherds relations, we have

$$\begin{aligned} & J_s(u) \cdot J_t(v) \cdot 1_n \\ &= \sum_{m=0}^n \sum_{j=0}^{\Delta(u)+n} (-1)^m \binom{n-s+m}{n-s} \binom{\Delta(u)+n}{j} J_{s+t}(u_{(s+\Delta(u)-m-j-1)}v) \cdot 1_n \end{aligned} \quad (6.4.1)$$

in the quotient \mathbb{Q}_n for any integers s and t provided $s \leq n$. Hence by induction we have $\phi_n(\mathbf{U}) = \phi_n(\mathfrak{g})$ for any nonnegative integer n . Therefore, since $\phi(\mathbf{U}(d))$ is dense, $\phi(\mathfrak{g}(d))$ is also dense. \square

6.5 Exhaustive modules

Let M be a vector space and suppose given a series of maps $V \times M \rightarrow M$ indexed by integers which we denote by $(u, v) \mapsto \pi_n^M(u)v$. Such an M is said to be a *weak V -module* if it satisfies the conditions listed below.

Set $J_n^M(u) = \pi_{n+\Delta(u)-1}^M(u)$ and let $B_{k,m,n}^M(u, v)$ be the expression (??) with $\{J_n\}$ being replaced by $\{J_n^M\}$. Then the conditions are as follows:

- (i) For any $u \in V$ and $v \in M$ there exists an m such that $J_n^M(u)v = 0$ for all $n \geq m$.
- (ii) The operator $J_n^M(\mathbf{1})$ is the identity if $n = 0$ and is zero otherwise.
- (iii) The identity $B_{k,m,n}^M(u, v) = 0$ holds for any k, m, n and $u, v \in V$.

We will say that a weak V -module is an *exhaustive V -module* if instead of the condition (i) the following stronger condition is satisfied:

- (i)' For any $v \in M$ there exists an m such that $J_{n_1}(u_1) \cdots J_{n_k}(u_k)v = 0$ for all $u_1, \dots, u_k \in V$ whenever $n_1 + \cdots + n_k \geq m$.

Remark 6.5.1 Thanks to the condition (iii), the condition (i)' follows from the apparently weaker condition that for any $v \in M$ there exists an m such that $J_n(u)v = 0$ for all $u \in V$ and $n \geq m$ by successive use of the relation (??).

Let us consider the map $J_n : V \rightarrow \mathbf{U}$ which sends $u \in V$ to the image of $J_n(u) = u \otimes t^{n+\Delta(u)-1}$ in \mathbf{U} . Then any exhaustive left \mathbf{U} -module M becomes an exhaustive V -module by letting $J_n^M(u)$ be the action of $J_n(u)$ on M . We will call this V -module structure on M the *associated V -module structure*.

Proposition 6.5.2 *Let M be an exhaustive V -module. Then there exists a unique structure of an exhaustive \mathbf{U} -module on M such that the associated V -module structure agrees with the given V -module structure on M .*

Proof. Let $J_n^M : V \times M \rightarrow M$ be the given V -module structure on M . Then they induce a map $\mathfrak{g} \times M \rightarrow M$ which gives a \mathfrak{g} -module structure on M by the relation $B_{k,m,n}^M(u, v) = 0$ with $n \geq 0$. By the universal property of the universal enveloping algebra of \mathfrak{g} , this lifts to a \mathbf{U} -module structure on M because of the axiom (ii). Since M is an exhaustive V -module, the map $\mathbf{U}(d) \times M \rightarrow M$ is continuous for each d when M is endowed with the discrete topology. Hence this map prolongs to the action of the degreewise completion $\hat{\mathbf{U}}$. Now the axiom (iii) is nothing else but the defining relations of the algebra \mathbf{U} . Hence the $\hat{\mathbf{U}}$ -module structure induces a \mathbf{U} -module structure on M , which is exhaustive by Lemma ???. The uniqueness is clear on each step. \square

Thus we have obtained the following result.

Theorem 6.5.3 *The category of exhaustive V -modules is canonically equivalent to the category of exhaustive \mathbf{U} -modules.*

7 Associated Poisson algebras

We will consider the associated graded algebra with respect to a filtration on \mathbf{U} and show that it has a structure of a degreewise complete Poisson algebra.

7.1 Zhu's Poisson algebra

Recall that a (commutative) *Poisson algebra* is a vector space \mathfrak{p} equipped with two bilinear maps $\cdot : \mathfrak{p} \times \mathfrak{p} \rightarrow \mathfrak{p}$ and $\{ , \} : \mathfrak{p} \times \mathfrak{p} \rightarrow \mathfrak{p}$ called the multiplication and the Poisson bracket, respectively, such that \mathfrak{p} is a commutative associative algebra with unity with respect to the multiplication, \mathfrak{p} is a Lie algebra with respect to the Poisson bracket and the Leibniz identity holds:

$$\{x \cdot y, z\} = x \cdot \{y, z\} + y \cdot \{x, z\}. \quad (7.1.1)$$

We denote the unity of \mathfrak{p} by $1_{\mathfrak{p}}$.

Let V be a vertex operator algebra. We let $C_2(V)$ be the span of the elements of the form $u_{(n)}v$ with $u, v \in V$ and $n \leq -2$. We set

$$u \cdot v = u_{(-1)}v \quad \text{and} \quad \{u, v\} = u_{(0)}v. \quad (7.1.2)$$

The following result is obtained in [?].

Proposition 7.1.1 (Zhu) *The operations \cdot and $\{ , \}$ induces a Poisson algebra structure on the quotient space $V/C_2(V)$.*

Let us denote by \mathfrak{p} the Poisson algebra $V/C_2(V)$ obtained as above, which we will call *Zhu's Poisson algebra*.

7.2 Poisson filtrations and the associated graded algebras

Let V be a vertex operator algebra and let \mathfrak{g} , \mathbb{U} , $\hat{\mathbb{U}}$ and \mathbf{U} as in the preceding section.

Let $G_p \mathfrak{g}$ be the image of $\bigoplus_{r \leq p} V[r] \otimes_{\mathbf{k}} \mathbf{k}[t, t^{-1}]$ in \mathfrak{g} and let $G_p \mathbb{U}$ be the sum of subspaces $G_{p_1} \mathfrak{g} \cdots G_{p_k} \mathfrak{g}$ with $k = 0, 1, 2, \dots$ and $p_1 + \cdots + p_k = p$ in \mathbb{U} . Then G is a separated filtration on \mathbb{U} satisfying

$$G_p \mathbb{U} \cdot G_q \mathbb{U} \subset G_{p+q} \mathbb{U} \quad (7.2.1)$$

for any integers p and q .

Let $G_p \hat{\mathbb{U}}(d)$ be the closure of the image of $G_p \mathbb{U}(d) = G_p \mathbb{U} \cap \mathbb{U}(d)$ in $\hat{\mathbb{U}}(d)$. Then the associated graded algebra is given by

$$\text{gr}^G \hat{\mathbb{U}} = \bigoplus_d \bigoplus_p \text{gr}_p^G \hat{\mathbb{U}}(d), \quad \text{gr}_p^G \hat{\mathbb{U}}(d) = G_p \hat{\mathbb{U}}(d) / G_{p-1} \hat{\mathbb{U}}(d). \quad (7.2.2)$$

Considering the quotients by $\hat{\mathbb{B}}$, we obtain the induced filtration $G_p \mathbf{U}$ of the current algebra \mathbf{U} and the associated graded algebra:

$$\text{gr}^G \mathbf{U} = \bigoplus_{d=-\infty}^{\infty} \text{gr}^G \mathbf{U}(d), \quad \text{gr}_p^G \mathbf{U}(d) = \bigoplus_p \text{gr}_p^G \mathbf{U}(d). \quad (7.2.3)$$

Let us give the space $\text{gr}^G \mathbf{U}$ the induced degreewise topology and let \mathbf{S} be the degreewise completion of the algebra $\text{gr}^G \mathbf{U}$:

$$\mathbf{S} = \bigoplus_d \mathbf{S}(d), \quad \mathbf{S}(d) = \varprojlim_n \text{gr}^G \mathbf{U}(d) / I_n(\text{gr}^G \mathbf{U}(d)). \quad (7.2.4)$$

Then this is a compatible degreewise complete algebra. We will denote the image of $J_n(u)$ in \mathbf{S} by $\Psi_n(u)$.

Proposition 7.2.1 *The algebra \mathbf{U} is quasi-finite if and only if \mathbf{S} is so.*

Proof. By the construction, we have

$$Q_n(\mathbf{S}(d)) = \text{gr}^G Q_n(\mathbf{U}(d)), \quad Q_n(\mathbf{U}(d)) = \bigcup_p G_p Q_n(\mathbf{U}(d)), \quad (7.2.5)$$

where $G_p Q_n(\mathbf{U}(d))$ denotes the induced filtration. Hence $Q_n(\mathbf{U}(d))$ is finite-dimensional if and only if $Q_n(\mathbf{S}(d))$ is so. \square

7.3 Associated Poisson structure

Consider the operation of taking commutator of elements of \mathbf{U} :

$$\mathbf{U} \times \mathbf{U} \rightarrow \mathbf{U}, \quad (a, b) \mapsto [a, b] = a \cdot b - b \cdot a. \quad (7.3.1)$$

The particular case of the identities $B_{k,m,n}(u, v) = 0$ with $k = 0$ is given by

$$[J_m(u), J_n(v)] = \sum_{i=0}^{\infty} \binom{m + \Delta(u) - 1}{i} J_{m+n}(u_{(i)}v). \quad (7.3.2)$$

Lemma 7.3.1 $[G_p \mathbf{U}, G_q \mathbf{U}] \subset G_{p+q-1} \mathbf{U}$.

Proof. The left-hand side of (??) belongs to $G_{\Delta(u)+\Delta(v)} \mathbf{U}$ whereas the element $J_{m+n}(u_{(i)}v)$ in the right-hand side belongs to $G_{\Delta(u)+\Delta(v)-i-1} \mathbf{U}$ for $i \geq 0$. Hence an element of $[G_p \mathbf{U}, G_q \mathbf{U}]$ is written as a sum of elements of $G_{p+q-1} \mathbf{U}$. \square

Thanks to this lemma, the multiplication of $\text{gr}^G \mathbf{U}$ is commutative and the operation $[\ , \] : G_p \mathbf{U} \times G_q \mathbf{U} \rightarrow G_{p+q-1} \mathbf{U}$ induces an operation

$$\text{gr}_p^G \mathbf{U} \times \text{gr}_q^G \mathbf{U} \rightarrow \text{gr}_{p+q-1}^G \mathbf{U}, \quad (\alpha, \beta) \mapsto \{\alpha, \beta\} \quad (7.3.3)$$

by letting $\{\alpha, \beta\}$ be the image of $[a, b]$ in $\text{gr}_{p+q-1}^G \mathbf{U}$, where a and b are representatives of α and β , respectively.

In general, we will call a compatible degreewise topological algebra with a Poisson algebra structure for which the Poisson bracket is continuous a *compatible degreewise topological Poisson algebra*. In case the degreewise topology is complete then we will say that the Poisson algebra is a *compatible degreewise complete Poisson algebra*.

Proposition 7.3.2 *The multiplication and the bracket operation defined as above endow the space \mathbf{S} with a structure of a compatible degreewise complete Poisson algebra.*

7.4 Relation to Zhu's Poisson algebra

Let us look more carefully at the relations $B_{k,m,n}(u,v) = 0$. When $k = -\Delta(u) + 1$ and $m + n = p + \Delta(u) - 1$, we have

$$\begin{aligned} J_p(u_{(n)}v) &= \sum_{i=0}^{\infty} (-1)^i \binom{n}{i} J_{k+n-i}(u) \cdot J_{m+i}(v) \\ &\quad - (-1)^n \sum_{i=0}^{\infty} (-1)^i \binom{n}{i} J_{m+n-i}(v) \cdot J_{k+i}(u). \end{aligned} \quad (7.4.1)$$

Then the left-hand side belongs to $G_{\Delta(u)+\Delta(v)-n-1}\mathbf{U}$ whereas the right-hand side to $G_{\Delta(u)+\Delta(v)}\mathbf{U}$. Therefore,

$$J_p(u_{(n)}v) \equiv 0 \quad \text{if } n \leq -2. \quad (7.4.2)$$

This implies that the map $J_p : V \rightarrow \text{gr}^G\mathbf{U}$ factors a map

$$\Psi_p : \mathfrak{p} \rightarrow \text{gr}^G\mathbf{U} \quad (7.4.3)$$

where \mathfrak{p} is Zhu's Poisson algebra $V/C_2(V)$.

Now substitute $n = -1$ in (??). Then we have

$$J_p(u_{(-1)}v) = \sum_{i=0}^{\infty} J_{k-i-1}(u) \cdot J_{m+i}(v) + \sum_{i=0}^{\infty} J_{m-i-1}(v) \cdot J_{k+i}(u). \quad (7.4.4)$$

Projecting this relation to the associated graded algebra, we immediately have

$$\Psi_p(u_{(-1)}v) = \sum_{j=-\infty}^{\infty} \Psi_{p-j}(u) \cdot \Psi_j(v). \quad (7.4.5)$$

Note 7.4.1 The results in this section and the next are reformulations of the arguments in Subsection 3.2 of [?].

8 Poisson current algebras

In this section, we will construct a universal Poisson algebra satisfying the relation (??), which we will call a Poisson current algebra, and will investigate its properties.

8.1 Symmetric algebras on the loop Lie algebras

Let \mathfrak{p} be a Poisson algebra. A *Poisson ideal* of \mathfrak{p} means a subspace \mathfrak{a} of \mathfrak{p} such that both $\mathfrak{p} \cdot \mathfrak{a} \subset \mathfrak{a}$ and $\{\mathfrak{p}, \mathfrak{a}\} \subset \mathfrak{a}$ hold.

Consider the case when \mathfrak{p} is given a grading $\mathfrak{p} = \bigoplus_r \mathfrak{p}[r]$ indexed by integers satisfying

$$\{\mathfrak{p}[r], \mathfrak{p}[s]\} \subset \mathfrak{p}[r+s-1], \quad \mathfrak{p}[r] \cdot \mathfrak{p}[s] \subset \mathfrak{p}[r+s]. \quad (8.1.1)$$

Then the unity $1_{\mathfrak{p}}$ must belong to $\mathfrak{p}[0]$. We will call a Poisson algebra endowed with such a grading a *graded Poisson algebra*. We denote $\Delta(x) = r$ when $x \in \mathfrak{p}[r]$ and call it the weight of x .

Recall the well-known fact that the symmetric algebra on a Lie algebra has a canonical structure of a Poisson algebra induced from the Lie bracket operation on the Lie algebra.

Let $\mathfrak{p} = \bigoplus_r \mathfrak{p}[r]$ be a graded Poisson algebra and let $\mathfrak{p}[t, t^{-1}]$ be the loop Lie algebra

$$\mathfrak{p}[t, t^{-1}] = \mathfrak{p} \otimes_{\mathbf{k}} \mathbf{k}[t, t^{-1}] \quad (8.1.2)$$

with the Lie bracket defined by

$$[x \otimes t^m, y \otimes t^n] = \{x, y\} \otimes t^{m+n}. \quad (8.1.3)$$

For each homogeneous x , we set

$$\tilde{\Psi}_n(x) = x \otimes t^{n+\Delta(x)-1} \quad (8.1.4)$$

and extend it linearly to all x . Then the Lie bracket operation takes the following form:

$$[\tilde{\Psi}_m(x), \tilde{\Psi}_n(y)] = \tilde{\Psi}_{m+n}(\{x, y\}). \quad (8.1.5)$$

Let \mathfrak{q} be the quotient of $\mathfrak{p}[t, t^{-1}]$ by the span of the elements $\tilde{\Psi}_n(1_{\mathfrak{p}})$ with $n \neq 0$. We will denote the image of $\tilde{\Psi}_n(x)$ in \mathfrak{q} by the same symbol. Since $[\tilde{\Psi}_m(1_{\mathfrak{p}}), \tilde{\Psi}_n(x)] = 0$, the space \mathfrak{q} becomes a Lie algebra.

Now let \mathbb{S} be the quotient algebra of the symmetric algebra on \mathfrak{q} by the ideal generated by $\tilde{\Psi}_0(1_{\mathfrak{p}}) - 1$ and let us denote the image of $\tilde{\Psi}_n(x)$ in \mathbb{S} by the same symbol. We give the degree $d_1 + \dots + d_k$ to the element of the form $\tilde{\Psi}_{-d_1}(x_1) \cdots \tilde{\Psi}_{-d_k}(x_k)$ with $x_1, \dots, x_k \in \mathfrak{p}$. Let $\mathbb{S}(d)$ be the span of these vectors of degree d . Then we have

$$\mathbb{S} = \bigoplus_d \mathbb{S}(d) \quad (8.1.6)$$

and the algebra \mathbb{S} becomes a Poisson algebra. Note that we have

$$\mathbb{S}(d) \cdot \mathbb{S}(e) \subset \mathbb{S}(d+e), \quad \{\mathbb{S}(d), \mathbb{S}(e)\} \subset \mathbb{S}(d+e). \quad (8.1.7)$$

Recall the standard degreewise topology on \mathbb{S} defined by

$$I_n(\mathbb{S}(d)) = \sum_{k \leq -n-1} \mathbb{S}(d-k) \cdot \mathbb{S}(k), \quad (8.1.8)$$

which is separated. Then the multiplication maps $\mathbb{S}(d) \times \mathbb{S}(e) \rightarrow \mathbb{S}(d+e)$ is continuous. Moreover we have the following.

Lemma 8.1.1 *The Poisson bracket operation $\mathbb{S}(d) \times \mathbb{S}(e) \rightarrow \mathbb{S}(d+e)$ is continuous with respect to the standard degreewise topology.*

Proof. Let i be any integer with $i \leq -n-1$. Then we have

$$\begin{aligned} & \{\mathbb{S}(d), \mathbb{S}(e-i) \cdot \mathbb{S}(i)\} \\ & \subset \{\mathbb{S}(d), \mathbb{S}(e-i)\} \cdot \mathbb{S}(i) + \{\mathbb{S}(d), \mathbb{S}(i)\} \cdot \mathbb{S}(e-i) \\ & = \mathbb{S}(d+e-i) \cdot \mathbb{S}(i) + \mathbb{S}(e-i) \cdot \mathbb{S}(d+i). \end{aligned}$$

Therefore, $\{u + I_k(\mathbb{S}(d)), v + I_m(\mathbb{S}(e))\} \subset \{u, v\} + I_n(\mathbb{S}(d + e))$ if k and m satisfy $k, k - e, m, m - d \geq n$. \square

Let $\hat{\mathbb{S}}$ be the degreewise completion of \mathbb{S} with respect to the standard degreewise topology. Then by Lemma ?? the Poisson algebra structure on \mathbb{S} extends to $\hat{\mathbb{S}}$. Let us denote the image of $\tilde{\Psi}_n(x)$ under the canonical map $\mathbb{S} \rightarrow \hat{\mathbb{S}}$ again by the same symbol.

The algebra $\hat{\mathbb{S}}$ is a compatible degreewise complete Poisson algebra.

8.2 Poisson current algebras

For each $x, y \in \mathfrak{p}$ and an integer m , we set

$$D_m(x, y) = \tilde{\Psi}_m(x \cdot y) - \sum_{j \in \mathbb{Z}} \tilde{\Psi}_{m-j}(x) \cdot \tilde{\Psi}_j(y). \quad (8.2.1)$$

and let $\hat{\mathbb{D}}$ denote the degreewise closure of the ideal of $\hat{\mathbb{S}}$ generated by the elements $D_m(x, y)$ with $x, y \in \mathfrak{p}$ in $\hat{\mathbb{S}}$ and $m \in \mathbb{Z}$.

Lemma 8.2.1 *The ideal $\hat{\mathbb{D}}$ is a Poisson ideal of $\hat{\mathbb{S}}$.*

Proof. Recall that the Poisson bracket operation is continuous. Hence we have

$$\begin{aligned} & \left\{ D_m(x, y), \tilde{\Psi}_n(z) \right\} \\ &= \tilde{\Psi}_{m+n}(\{x \cdot y, z\}) - \sum_j \{ \tilde{\Psi}_{m-j}(x) \cdot \tilde{\Psi}_j(y), \tilde{\Psi}_n(z) \} \\ &= \tilde{\Psi}_{m+n}(x \cdot \{y, z\}) - \sum_j \tilde{\Psi}_{m-j}(x) \cdot \tilde{\Psi}_{j+n}(\{y, z\}) \\ & \quad + \tilde{\Psi}_{m+n}(y \cdot \{x, z\}) - \sum_j \tilde{\Psi}_j(y) \cdot \tilde{\Psi}_{m+n-j}(\{x, z\}) \\ &= D_{m+n}(x, \{y, z\}) + D_{m+n}(y, \{x, z\}), \end{aligned}$$

so the conclusion follows. \square

We let $\tilde{\mathbb{S}} = \tilde{\mathbb{S}}(\mathfrak{p})$ be the quotient of $\hat{\mathbb{S}}$ by the Poisson ideal $\hat{\mathbb{D}}$:

$$\tilde{\mathbb{S}} = \hat{\mathbb{S}} / \hat{\mathbb{D}}. \quad (8.2.2)$$

We will call the Poisson algebra $\tilde{\mathbb{S}}$ the *Poisson current algebra* associated with the Poisson algebra \mathfrak{p} .

Proposition 8.2.2 *The algebra $\tilde{\mathbb{S}}$ is a compatible degreewise complete Poisson algebra.*

8.3 Quasi-finiteness

Now assume that our Poisson algebra \mathfrak{p} is finite-dimensional and let x_1, \dots, x_r be a basis of a linear complement of $\mathbf{k}1_{\mathfrak{p}}$ in \mathfrak{p} . Then the algebra \mathbb{S} is spanned by the elements of the form $\tilde{\Psi}_{-d_1}(x_{i_1}) \cdots \tilde{\Psi}_{-d_k}(x_{i_k})$ with $k \geq 0$ and $d_1 \geq \dots \geq d_k$.

Consider the left canonical quotient module Q_n in the case when the algebra A is $\tilde{\mathbb{S}}$. The module Q_n is generated as a left $\tilde{\mathbb{S}}$ -module by the image 1_n of the unit of $\tilde{\mathbb{S}}$.

For each $k \geq 0$ and $d \geq -n$, consider the set

$$\Pi_k(d) = \{(d_1, \dots, d_k) \mid d_1 \geq \dots \geq d_k \geq -n \text{ and } d_1 + \dots + d_k = d\}. \quad (8.3.1)$$

We will call a vector of the form $\tilde{\Psi}_{-d_1}(x_{i_1}) \cdots \tilde{\Psi}_{-d_k}(x_{i_k}) \cdot 1_n$ a *vector with index* (d_1, \dots, d_k) . We put

$$\mathring{\Pi}_k(d) = \{(d_1, \dots, d_k) \mid d_1 > \dots > d_k \geq -n \text{ and } d_1 + \dots + d_k = d\} \quad (8.3.2)$$

and set

$$\Pi(d) = \bigcup_{k=0}^{\infty} \Pi_k(d), \quad \mathring{\Pi}(d) = \bigcup_{k=0}^{\infty} \mathring{\Pi}_k(d). \quad (8.3.3)$$

Lemma 8.3.1 *The space $Q_n(d)$ is spanned by vectors with indices in $\mathring{\Pi}(d)$.*

Proof. Since the module Q_n is exhaustive and the image of each $\mathbb{S}(d)$ is dense in $\tilde{\mathbb{S}}(d)$, the space $Q_n(d)$ is spanned by vectors with indices in $\Pi(d)$. Let us show that any vector with index in $\Pi_k(d)$ is a linear combination of vectors with indices in $\mathring{\Pi}(d)$.

Introduce the lexicographic order on the set $\Pi_k(d)$: we define $(d_1, \dots, d_k) < (e_1, \dots, e_k)$ by $d_1 < e_1$ and in case $d_1 = e_1$ recursively by $(d_2, \dots, d_k) < (e_2, \dots, e_k)$. Then as this is a total order on a finite set, there exists a maximum element: that is $(d + (k-1)n, -n, \dots, -n)$.

We now proceed by induction on the length k . The case $k = 1$ is trivial. Assume that the claim is true for any vector with index shorter than k and suppose given a vector

$$\tilde{\Psi}_{-d_1}(x_{i_1}) \cdots \tilde{\Psi}_{-d_k}(x_{i_k}) \cdot 1_n \quad (8.3.4)$$

with $d_1 \geq \dots \geq d_k > -n - 1$. If $d_1 > \dots > d_k > -n - 1$ then we have nothing to prove so we consider the case when $d_i = d_{i+1}$ at some position i . Recall the relation $\tilde{\Psi}_{2k}(x \cdot y) = \sum_j \tilde{\Psi}_{k-j}(x) \cdot \tilde{\Psi}_{k+j}(y)$ which implies

$$\tilde{\Psi}_{-d_i}(x_i) \cdot \tilde{\Psi}_{-d_i}(x_{i+1}) = \tilde{\Psi}_{-2d_i}(x_i \cdot x_{i+1}) - \sum_{j \neq 0} \tilde{\Psi}_{-d_i-j}(x_i) \cdot \tilde{\Psi}_{-d_i+j}(x_{i+1}). \quad (8.3.5)$$

Hence the vector (??) is rewritten as the sum of a shorter vector and a finite number of vectors greater in the lexicographic order. By the inductive hypothesis, the shorter vector is written by the vectors with indices in $\mathring{\Pi}(d)$. Then apply the same argument to the vectors with greater indices in the rest of the sum. The recursion stops within a finite number of steps, at most at the maximum. \square

Note 8.3.2 The argument described above is a refined variation of the proof of Theorem 3.2.7 in [?]. The idea of utilizing the relation (??) goes back to [?]. See [?] and [?] for related results.

We will say that a compatible degreewise complete Poisson algebra is *quasi-finite* if the conditions of quasi-finiteness for a compatible degreewise complete algebras are satisfied except the existence of a Hamiltonian. Now the following result is an immediate consequence of Lemma ??.

Theorem 8.3.3 *If the Poisson algebra \mathfrak{p} is finite-dimensional then the algebra $\tilde{\mathfrak{S}}$ is quasi-finite.*

Proof. By Lemma ??, the space $Q_n(d)$ is spanned by the vectors associated with $\mathring{\Pi}(d) = \bigcup_{k=0}^{\infty} \mathring{\Pi}_k(d)$, which is a finite set. \square

9 Current algebras and Poisson current algebras

In this section, we will show that Zhu's finiteness condition on a vertex operator algebra implies the quasi-finiteness of the associated current algebra. This will be done by relating the results of the preceding section to the Poisson algebra \mathfrak{S} associated with \mathfrak{U} .

9.1 Relation to the current algebras

A *homomorphism of degreewise topological Poisson algebras* is a map from a degreewise topological Poisson algebra to another such that it is a homomorphism of Poisson algebras that preserves the gradings for which the restriction to each homogeneous subspace is continuous.

Let V be a vertex operator algebra and let \mathfrak{p} be Zhu's Poisson algebra $V/C_2(V)$. Recall the notations in the previous sections of Part II.

By the definition of Zhu's Poisson algebra, (??) implies that the maps $V \rightarrow \text{gr}^G \mathfrak{U}$ which sends $u \in V[r]$ to the image $\Psi_p(u)$ of $J_p(u)$ in $\text{gr}_r^G \mathfrak{U}$ factors a map

$$\Psi_p : \mathfrak{p} \rightarrow \text{gr}^G \mathfrak{U}. \quad (9.1.1)$$

Then the set of the maps $\Psi_p : \mathfrak{p} \rightarrow \text{gr}^G \mathfrak{U}$ gives rise to a single map

$$\Psi : \mathfrak{q} \rightarrow \text{gr}^G \mathfrak{U} \quad (9.1.2)$$

which sends $\tilde{\Psi}_n(x)$ to $\Psi_n(u)$, where $x = \bar{u}$ is the class of $u \in V$ in \mathfrak{p} .

Now the relations (??) and (??) imply that the map Ψ is a homomorphism of Lie algebras. Since $\text{gr}^G \mathfrak{U}$ is a Poisson algebra, this map induces a unique homomorphism

$$\mathfrak{S} \rightarrow \text{gr}^G \mathfrak{U} \quad (9.1.3)$$

of Poisson algebras by the universal property of the symmetric algebra. We denote this map by the same symbol Ψ .

Lemma 9.1.1 *The map Ψ prolongs to a surjective homomorphism $\hat{\mathbb{S}} \rightarrow \mathbf{S}$ of degreewise topological Poisson algebras.*

Proof. The assertion follows immediately from the construction by noting the relation $\Psi(I_n(\mathbb{S})) = I_n(\text{gr}^G \mathbf{U})$, where the latter space is the one induced from $I_n(\mathbf{U})$. \square

Now let $\tilde{\mathbb{S}}$ be the Poisson current algebra of \mathfrak{p} as defined in Subsection ???. The relation (??) implies that the ideal $\hat{\mathbb{D}}$ is mapped to the closure of $\text{gr}^G \hat{\mathbb{B}}$. Therefore, the map $\hat{\mathbb{S}} \rightarrow \mathbf{S}$ induces a homomorphism $\tilde{\mathbb{S}} \rightarrow \mathbf{S}$ of degreewise topological Poisson algebras.

Thus we have verified the following result.

Theorem 9.1.2 *Let V be a vertex operator algebra and let \mathbf{U} be the associated current algebra. Let \mathbf{S} be the degreewise completion of $\text{gr}^G \mathbf{U}$ and let $\tilde{\mathbb{S}}$ be the Poisson current algebra associated with Zhu's Poisson algebra $\mathfrak{p} = V/C_2(V)$. Then there exists a surjective homomorphism $\tilde{\mathbb{S}} \rightarrow \mathbf{S}$ of degreewise topological Poisson algebras.*

9.2 Consequences of Zhu's finiteness condition

A vertex operator algebra V is said to satisfy *Zhu's finiteness condition* or said to be *C_2 -finite* if Zhu's Poisson algebra $\mathfrak{p} = V/C_2(V)$ is finite-dimensional.

By combining Proposition ??, Theorem ?? and Theorem ??, we immediately see that Zhu's finiteness condition implies quasi-finiteness. Namely, we have the following theorem which is the main result of Part II.

Theorem 9.2.1 *If a vertex operator algebra V satisfies Zhu's finiteness condition then the associated current algebra \mathbf{U} is quasi-finite.*

Let V be a vertex operator algebra satisfying Zhu's finiteness condition and let \mathbf{U} be the associated current algebra. Let \mathbf{U}_n be the finite-dimensional algebra associated with \mathbf{U} as defined in Subsection ??? and let g be the number defined by (??). Then Theorem ?? allows us to apply the results of Part I to the algebra $A = \mathbf{U}$. For instance, we have the following.

Corollary 9.2.2 *Let V be a C_2 -finite vertex operator algebra and let n be an integer such that $n \geq g$. Then the category of exhaustive V -modules is canonically equivalent to the category of left \mathbf{U}_n -modules.*

Moreover, the finiteness theorems in Subsection ??? hold for the various categories of \mathbf{U} -modules.

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