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Abstract. In this paper we consider an inverse boundary value problem of determining three dimensional unknown inclusions in an elliptic equation in a bounded domain $\Omega \subset \mathbb{R}^3$ from finite boundary measurements on $\partial\Omega$. We will show that polyhedral inclusions in Ω can be uniquely determined up to their convex edges from a single boundary measurement on $\partial\Omega$.

1. Introduction

Let $\Omega \subset \mathbb{R}^3$ be a simply connected bounded domain with smooth boundary, let D be a subdomain of Ω with Lipschitz boundary, and let us denote by χ_D the characteristic function of D, that is, $\chi_D(x) = 1$ if $x \in D$ and $\chi_D(x) = 0$ otherwise. In this paper, we consider an elliptic equation

$$-\Delta u(x) + q(x)\chi_D(x)u(x) = 0, \qquad x \in \Omega \tag{1.1}$$

with a Dirichlet boundary condition

$$u(x) = f(x), \qquad x \in \partial\Omega. \tag{1.2}$$

The function $q \in C^2(\Omega)$ is assumed to be a positive function: $q(x) > 0$ for all $x \in \overline{\Omega}$. For any given $f \in H^{\frac{1}{2}}(\partial\Omega)$ and a subdomain D of Ω the Dirichlet boundary value problem (1.1), (1.2) has a unique solution u in $H^1(\Omega)$. Hence its outward normal derivative $\frac{\partial u}{\partial \nu} = \nabla u \cdot \nu$ to the boundary $\partial\Omega$ belongs to $H^{-\frac{1}{2}}(\partial\Omega)$.

In this paper, we discuss an inverse boundary value problem where we are requested to determine the discontinuous boundary of the lower order term of equation (1.1) or the subdomain D of Ω from the boundary measurements $\frac{\partial u}{\partial \nu}$ on $\partial\Omega$. This inverse boundary value problem appears in many applications. For example, we can mention the reconstruction problem of the metal-to-semiconductor contact and its resistivity inside electric devices [2, 6, 7, 10], the support of a heat source [9], and absorbing inclusions in optical tomography [1]. We are interested in the uniqueness problem for this inverse boundary value problem in three dimensions with a single measurement. The purpose of this paper is to establish the uniqueness within three dimensional polyhedra D from a single boundary measurement $\frac{\partial u}{\partial \nu}$ on $\partial\Omega$. Kim and Yamamoto [9] proved the global uniqueness for convex hulls of polygons $D \subset \mathbb{R}^2$ from a single measurement of $\frac{\partial u}{\partial \nu}$ on $\partial\Omega$. For this uniqueness result in two dimensions, they make essential use of the following imbedding theorem: if D is a polygon in $\mathbb{R}^2$ and $p > 2$, then

$$W^{3,p}(D) \longrightarrow C^{2,\lambda}(\overline{D}), \qquad 0 < \lambda < 1 - \frac{2}{p}. \tag{1.3}$$

If the dimension is 3, however, then this imbedding theorem (1.3) does not hold, and therefore the proof in [9] does not mean any uniqueness result for polyhedra $D \subset \mathbb{R}^3$.

As a similar inverse problem, we can refer to the inverse conductivity problem [3, 5, 8, 13] of determining piecewise continuous $\gamma = \gamma(x) = 1 + k\chi_D(x)$ in an elliptic equation $\nabla \cdot (\gamma \nabla u) = 0$ in Ω. Here k is supposed to be a non-zero constant. In particular, the papers [5, 13] consider the uniqueness problem for two dimensional polygons D by one (or two) boundary measurement(s). They basically use the index theory, and so this idea cannot be applied to the three dimensional case too.

In this work, roughly speaking, we will prove that we can determine uniquely polyhedrons D up to their convex edges from a single measurement of $\frac{\partial u}{\partial \nu}$ on $\partial\Omega$. Here a convex edge of D means a line segment where two adjacent faces of D meet and the interior angle of D is less than π. Before stating our main theorem, we will give some notations. Let x, P_1, P_2, $P_3 \in \mathbb{R}^3$, $r > 0$, and let A be a subset of $\mathbb{R}^3$. In the sequel we will use the following notations:

$$\bar{x} = (x_1, x_2) = (x_1, x_2, 0) \quad \text{ for } x = (x_1, x_2, x_3) \in \mathbb{R}^3,$$

$$\nabla = (\partial_{x_1}, \partial_{x_2}, \partial_{x_3}),$$

$$\Delta = \partial_{x_1}^2 + \partial_{x_2}^2 + \partial_{x_3}^2,$$

$$\nabla_{\bar{x}} = (\partial_{x_1}, \partial_{x_2}) = (\partial_{x_1}, \partial_{x_2}, 0),$$

$$\Delta_{\bar{x}} = \partial_{x_1}^2 + \partial_{x_2}^2,$$

$$B(P_1, r) = \{x \in \mathbb{R}^3 \ : \ |x - P_1| < r\},$$

$$d(x, A) = \inf\{|x - z| \ : \ z \in A\}.$$

Furthermore let us denote by $co(A)$ the convex hull of A, that is, the smallest convex set containing A and by $\triangle P_1 P_2 P_3$ the triangular domain which has three vertices P_1, P_2, P_3. We always assume that a polyhedron is a three dimensional solid which is composed of a finite number of polygonal faces, all the faces meet on their edges, and the edges meet at points, which are called vertices. In the sequel by a polyhedron, its faces, and its edges we mean relatively open sets respectively in $\mathbb{R}^3$, $\mathbb{R}^2$, and $\mathbb{R}$. In other words, we understand, for example, that the edge does not contain the both end points. Let D be a polyhedron, let ℓ be an edge of D, and let α_1, α_2 be adjacent faces to ℓ of D. Then by $\theta(\ell)$ we denote the interior angle of D between α_1 and α_2, and define ℓ as a convex edge if $\theta(\ell)$ is less than π.

Here is the first main theorem concerning the uniqueness up to convex edges of a polyhedron.

Theorem 1.1. *We assume that $f \in H^{\frac{1}{2}}(\partial\Omega)$ is non-negative, not identically zero, and that $q \in C^2(\Omega)$ is positive on $\overline{\Omega}$. Let D_j, $j = 1, 2$, be a polyhedron in Ω so that $\overline{D_j} \subset \Omega$ and $\Omega \setminus \overline{D_1 \cup D_2}$ is simply connected. Let u_j be the corresponding solution of the Dirichlet problem* (1.1) *and* (1.2) *with D_j and f. If $\frac{\partial u_1}{\partial \nu} = \frac{\partial u_2}{\partial \nu}$ on $\partial\Omega$, then $\overline{D_1} \setminus \overline{D_2}$ and $\overline{D_2} \setminus \overline{D_1}$ do not contain any point lying on a convex edge of D_1 and D_2, respectively.*

The next theorem is a uniqueness result within convex hulls of polyhedra $D \subset \Omega$. Theorem 1.2 can be proved in the same way as Theorem 1.1, and so we will skip its proof. The rest of this paper is devoted to the proof of Theorem 1.1.

Theorem 1.2. *Let us assume the same assumptions as in Theorem 1.1 except for that $\Omega \setminus \overline{D_1 \cup D_2}$ is simply connected. Then $\frac{\partial u_1}{\partial \nu} = \frac{\partial u_2}{\partial \nu}$ on $\partial\Omega$ implies $co(D_1) = co(D_2)$.*

The uniqueness for convex polyhedra is a corollary of Theorem 1.2.

Corollary 1.3. *Under the same assumptions as in Theorem 1.1, if D_1 and D_2 are convex polyhedra, then $\frac{\partial u_1}{\partial \nu} = \frac{\partial u_2}{\partial \nu}$ on $\partial\Omega$ implies $D_1 = D_2$.*

2. Proof of Theorem 1.1

Since $\overline{D_1}, \overline{D_2} \subset \Omega$, we can take a subdomain Ω' of Ω so that $\overline{D_1} \cup \overline{D_2} \subset \Omega' \subset \overline{\Omega'} \subset \Omega$. Then it can be shown that

$$u_j \in C^1(\Omega') \cap H^2(\Omega') \tag{2.1}$$

and moreover

$$u_j \in C^2(\Omega' \setminus \partial D_j). \tag{2.2}$$

For regularity (2.1) and (2.2), see [4, 11]. Owing to non-negative and positive , respectively, a priori assumptions on f and q, the maximum principle applied to u_j shows that

$$u_j(x) > 0 \qquad \text{for all } x \in \overline{\Omega'}. \tag{2.3}$$

First we will show that $\overline{D_1} \setminus \overline{D_2}$ contains no point lying on a convex edge of D_1. Otherwise there exist a convex edge ℓ_0 of D_1 and a point $P_0 \in \ell_0$ such that $P_0 \in \overline{D_1} \setminus \overline{D_2}$. Since Laplace's operator is rotation and translation invariant, if necessary, we may assume that P_0 is the origin O of $\mathbb{R}^3$ and ℓ_0 lies on the x_3-axis. Let α_1, α_2 be adjacent faces to ℓ_0 of D_1 and let $\epsilon_0 := \frac{1}{4}\min\{d(P_0, D_2), d(P_0, \alpha) \mid \alpha \text{ is a face of } D_1 \text{ except for } \alpha_1, \alpha_2\}$. For any positive $\epsilon < \epsilon_0$ let $E(\epsilon) := \{x = (x_1, x_2, 0) \in D_1 \mid x_1^2 + x_2^2 < \epsilon^2\}$ and $E(\epsilon) \times (0, \epsilon_0) := \{(x_1, x_2, t) \in \mathbb{R}^3 \mid (x_1, x_2, 0) \in E(\epsilon) \text{ and } 0 < t < \epsilon_0\}$. Our choice of ϵ_0 implies that $E(\epsilon)$ is a sector with angle $\theta_0 := \theta(\ell_0)$ about the origin and

$$E(\epsilon) \times (0, \epsilon_0) \subset D_1 \setminus \overline{D_2}. \tag{2.4}$$

In fact, by using the potential argument [4, 12] we can prove that

$$\partial_{x_3} u_j \in C^1(\overline{E(\epsilon)} \times [0, \epsilon_0]), \qquad j = 1, 2. \tag{2.5}$$

For completeness, we will give a proof of (2.5) in Appendix.

Let Q_1^ϵ, Q_2^ϵ be two points on $\partial E(\epsilon)$ satisfying

$$|Q_1^\epsilon| = |Q_2^\epsilon| = \epsilon \quad \text{and} \quad \overline{OQ_1^\epsilon}, \overline{OQ_2^\epsilon} \subset \partial E(\epsilon). \tag{2.6}$$

Let us define two functions u on Ω and ω on $\overline{E(\epsilon)}$ as follows:

$$u(x) = u_1(x) - u_2(x), \qquad x \in \Omega, \tag{2.7}$$

and

$$\omega(x) = \omega(x_1, x_2, 0) = \int_0^{\epsilon_0} u(x_1, x_2, t)\, dt, \qquad x \in \overline{E(\epsilon)}. \tag{2.8}$$

From (2.1) and (2.2) it is easy to show that

$$\partial_{x_j}\omega(x) = \int_0^{\epsilon_0} \partial_{x_j} u(x_1, x_2, t)\, dt, \qquad x \in \overline{E(\epsilon)} \text{ and } j = 1, 2, \tag{2.9}$$

$$\partial_{x_i}\partial_{x_j}\omega(x) = \int_0^{\epsilon_0} \partial_{x_i}\partial_{x_j}u(x_1,x_2,t)\ dt, \qquad x \in E(\epsilon) \text{ and } i,j = 1,2, \tag{2.10}$$

and

$$||\partial_{x_i}\partial_{x_j}\omega||_{L^2(E(\epsilon))} \le \sqrt{\epsilon_0}||u||_{H^2(\Omega')}, \qquad i,j = 1,2. \tag{2.11}$$

We will just give a brief proof of (2.11) here. Owing to (2.1), (2.2) and (2.10), we have

$$\begin{aligned} \int_{E(\epsilon)} |\partial_{x_i}\partial_{x_j}\omega(\bar{x})|^2\ d\bar{x} &\le \int_{E(\epsilon)} \left[\int_0^{\epsilon_0} |\partial_{x_i}\partial_{x_j}u(\bar{x},t)|\ dt\right]^2\ d\bar{x} \\ &\le \epsilon_0 \int_{E(\epsilon)}\int_0^{\epsilon_0} |\partial_{x_i}\partial_{x_j}u(\bar{x},t)|^2\ dt\ d\bar{x} \\ &\le \epsilon_0||u||^2_{H^2(\Omega')}. \end{aligned} \tag{2.12}$$

Since u_1, u_2 have the same Cauchy data on $\partial\Omega$ and $\Omega \setminus \overline{D_1 \cup D_2}$ is simply connected, the unique continuation means that

$$u(x) = 0 \qquad \text{for all } x \in \Omega \setminus \overline{D_1 \cup D_2}, \tag{2.13}$$

and, remembering the definition of ω, we now obtain

$$\omega(x) = |\nabla_{\bar{x}}\omega(x)| = 0 \qquad \text{for all } x \in \overline{OQ_1^\epsilon} \cup \overline{OQ_2^\epsilon} \cup \{O\}. \tag{2.14}$$

Next we will show that the function ω satisfies in weak sense the following elliptic equation in $E(\epsilon)$

$$\begin{aligned} \Delta_{\bar{x}}\omega(x) &= \Delta_{\bar{x}}\omega(\bar{x}) \\ &= \int_0^{\epsilon_0} q(\bar{x},t)u_1(\bar{x},t)\ dt - \partial_{x_3}u(\bar{x},\epsilon_0) + \partial_{x_3}u(\bar{x},0), \\ &\qquad x = (x_1,x_2,0) \in E(\epsilon). \end{aligned} \tag{2.15}$$

Fix any $\psi \in C_0^\infty(E(\epsilon))$, for any $\delta \in \left(0, \frac{\epsilon_0}{4}\right)$ take a cut-off function η_δ satisfying $0 \le \eta_\delta \le 1$, $\eta_\delta \in C^\infty(\mathbb{R})$, and

$$\eta_\delta(t) = \begin{cases} 1 & \text{if } x \in (\delta, \epsilon_0 - \delta), \\ 0 & \text{if } x \in (-\infty, 0) \cup (\epsilon_0, \infty), \end{cases}$$

and let $\psi_\delta(x_1,x_2,x_3) := \psi(x_1,x_2)\eta_\delta(x_3)$, $x = (x_1,x_2,x_3) \in \mathbb{R}^3$. It is clear that $\psi_\delta \in C_0^\infty(\Omega)$. Since u_1, u_2 are the weak solutions to (1.1) and $u = u_1 - u_2$, we have

$$\begin{aligned}
& \int_{E(\epsilon)} \int_0^{\epsilon_0} q(\bar{x}, t) u_1(\bar{x}, t)\ dt\ \psi(\bar{x})\ d\bar{x} \\
= & \int_{E(\epsilon)\times(0,\epsilon_0)} q(\bar{x}, t) u_1(\bar{x}, t) \psi(\bar{x})\ dt d\bar{x} \\
= & \lim_{\delta\downarrow 0} \int_\Omega q(\bar{x}, t) u_1(x) \psi_\delta(x)\ dx \\
= & -\lim_{\delta\downarrow 0} \int_\Omega \nabla u(x) \cdot \nabla \psi_\delta(x)\ dx \qquad (2.16) \\
= & -\lim_{\delta\downarrow 0} \int_\Omega \nabla u(x) \cdot [(\nabla \psi(\bar{x})) \eta_\delta(x_3) + \psi(\bar{x})(\nabla \eta_\delta(x_3))]\ dx \\
= & -\lim_{\delta\downarrow 0} \int_\Omega [\nabla_{\bar{x}} u(x) \cdot \nabla_{\bar{x}} \psi(\bar{x})] \eta_\delta(x_3)\ dx \\
& \qquad -\lim_{\delta\downarrow 0} \int_\Omega \partial_{x_3} u(x) \partial_{x_3} \eta_\delta(x_3) \psi(\bar{x})\ dx \\
:= & I + J.
\end{aligned}$$

We will deal with the terms I and J:

$$\begin{aligned}
I &= -\lim_{\delta\downarrow 0} \int_{E(\epsilon)\times(0,\epsilon_0)} [\nabla_{\bar{x}} u(x) \cdot \nabla_{\bar{x}} \psi(\bar{x})] \eta_\delta(x_3)\ dx \\
&= -\lim_{\delta\downarrow 0} \int_{E(\epsilon)} \int_0^{\epsilon_0} [\nabla_{\bar{x}} u(\bar{x}, t) \cdot \nabla_{\bar{x}} \psi(\bar{x})] \eta_\delta(t)\ dt d\bar{x} \\
&= -\int_{E(\epsilon)} \int_0^{\epsilon_0} \nabla_{\bar{x}} u(\bar{x}, t) \cdot \nabla_{\bar{x}} \psi(\bar{x})\ dt d\bar{x} \qquad (2.17) \\
&= -\int_{E(\epsilon)} \nabla_{\bar{x}} \int_0^{\epsilon_0} u(\bar{x}, t)\ dt \cdot \nabla_{\bar{x}} \psi(\bar{x})\ d\bar{x} \\
&= -\int_{E(\epsilon)} \nabla_{\bar{x}} \omega(\bar{x}) \cdot \nabla_{\bar{x}} \psi(\bar{x})\ d\bar{x},
\end{aligned}$$

and

$$\begin{aligned} J &= -\lim_{\delta\downarrow 0}\int_{E(\epsilon)\times(0,\epsilon)} \partial_{x_3}u(\bar{x},t)\partial_{x_3}\eta_\delta(t)\ \psi(\bar{x})\ dx \\ &= -\lim_{\delta\downarrow 0}\int_{E(\epsilon)}\int_0^{\epsilon_0} \partial_{x_3}u(\bar{x},t)\partial_{x_3}\eta_\delta(t)\ dt\ \psi(\bar{x})\ d\bar{x} \\ &= \lim_{\delta\downarrow 0}\int_{E(\epsilon)}\int_0^{\epsilon_0} \partial_{x_3}^2u(\bar{x},t)\eta_\delta(t)\ dt\ \psi(\bar{x})\ d\bar{x} \qquad (2.18) \\ &= \int_{E(\epsilon)}\int_0^{\epsilon_0} \partial_{x_3}^2u(\bar{x},t)\ dt\ \psi(\bar{x})\ d\bar{x} \\ &= \int_{E(\epsilon)} [\partial_{x_3}u(\bar{x},\epsilon_0)-\partial_{x_3}u(\bar{x},0)]\psi(\bar{x})\ d\bar{x}. \end{aligned}$$

Equations (2.16), (2.17), (2.18) imply

$$\begin{aligned} &-\int_{E(\epsilon)} \nabla_{\bar{x}}\omega(\bar{x})\cdot\nabla_{\bar{x}}\psi(\bar{x})\ d\bar{x} \qquad (2.19) \\ &= \int_{E(\epsilon)}\left[\int_0^{\epsilon_0} q(\bar{x},t)u_1(\bar{x},t)\ dt-\partial_{x_3}u(\bar{x},\epsilon_0)+\partial_{x_3}u(\bar{x},0)\right]\psi(\bar{x})\ d\bar{x}, \end{aligned}$$

which completes our claim (2.15).

From (2.1) and (2.5) it follows that

$$\int_0^{\epsilon_0} q(\cdot,t)u_1(\cdot,t)\ dt-\partial_{x_3}u(\cdot,\epsilon_0)+\partial_{x_3}u(\cdot,0)\in C^1(\overline{E(\epsilon)}) \qquad (2.20)$$

and by (2.3), (2.13) and (2.20) there exists a positive $\epsilon_1\le\epsilon_0$ such that

$$\int_0^{\epsilon_0} u_1(\bar{x},t)\ dt-\partial_{x_3}u(\bar{x},\epsilon_0)+\partial_{x_3}u(\bar{x},0)>0, \quad x=(x_1,x_2,0)\in\overline{E(\epsilon_1)}. \qquad (2.21)$$

Therefore (2.11), (2.14), (2.15), (2.20) and (2.21) imply that $\int_0^{\epsilon_0} q(\cdot,t)$ $u_1(\cdot,t)\ dt\ -\partial_{x_3}u(\cdot,\epsilon_0)+\partial_{x_3}u(\cdot,0)\in C^1(\overline{E(\epsilon)})$ is strictly positive in the triangle $\triangle OQ_1^{\epsilon_1}Q_2^{\epsilon_1}$ and ω is an H^2-solution of the following Poisson problem

$$\begin{cases} \Delta\omega=\int_0^{\epsilon_0} u_1(\cdot,t)\ dt-\partial_{x_3}u(\cdot,\epsilon_0)+\partial_{x_3}u(\cdot,0) & \text{in } \triangle OQ_1^{\epsilon_1}Q_2^{\epsilon_1}, \\ \omega=|\nabla_{\bar{x}}\omega|=0 \qquad \text{on } \overline{OQ_1^{\epsilon_1}}\cup\overline{OQ_2^{\epsilon_1}}\cup\{O\}, \end{cases} \qquad (2.22)$$

which is a contradiction to Proposition 2.2 in [9], a non-existence proposition about an H^2-solution of a Cauchy problem for Poisson's equation.

Proposition 2.2 *Let $\triangle P_1P_2P_3$ be the interior of a triangle which has three vertices $P_j\in\mathbb{R}^2$, $j=1,\ 2,\ 3$. Let $G\in W^{1,\infty}(\triangle P_1P_2P_3)$*

be strictly positive in $\triangle P_1P_2P_3$. *Then there exists no solution* $v \in H^2(\triangle P_1P_2P_3)$ *to*

$$\begin{cases} \Delta v = G & \text{in } \triangle P_1P_2P_3 \\ v = |\nabla v| = 0 & \text{on } \overline{P_1P_2} \cup \overline{P_1P_3}. \end{cases}$$

Notice that $\triangle OQ_1^{\epsilon_1}Q_2^{\epsilon_1} \subset E(\epsilon_1)$, because $0 < \theta_0 = \theta(\ell_0) < \pi$ or ℓ_0 is convex. Therefore we can conclude that $\overline{D_1}\setminus\overline{D_2}$ contains no point lying on a convex edge of D_1. In the same way we can prove that $\overline{D_2} \setminus \overline{D_1}$ contains no point lying on a convex edge of D_2, and so the proof of Theorem 1.1 is completed.

Appendix. Proof of (2.5)

Since $E(2\epsilon) \times (-\epsilon_0, 2\epsilon_0)$ is still contained in $D_1 \setminus \overline{D_2}$, we deduce from (2.2)

$$u_2 \in C^2(E(2\epsilon) \times (-\epsilon_0, 2\epsilon_0)), \tag{1}$$

which enables us to focus on u_1. Let $\Phi(\cdot,\cdot)$ be the fundamental solution of Laplace's operator in $\mathbb{R}^3$:

$$\Phi(x,y) = -\frac{1}{4\pi}\frac{1}{|x-y|}, \qquad x, y \in \mathbb{R}^3, \tag{2}$$

and let us define a function H on Ω as follows

$$H(x) := \int_{\partial\Omega} \left[f(y)\frac{\partial \Phi}{\partial \nu}(x,y) - \Phi(x,y)\frac{\partial u_1}{\partial \nu}(y) \right] \, d\sigma_y, \qquad x \in \Omega. \tag{3}$$

It follows from Green's representation formula that the solution u_1 to the Dirichlet problem (1.1), (1.2) can be represented as

$$u_1(x) = H(x) + \int_{D_1} \Phi(x,y)q(y)u_1(y)\, dy, \qquad x \in \Omega. \tag{4}$$

Since the function H is smooth in Ω, it is sufficient to show that

$$\partial_{x_3} \int_{D_1} \frac{1}{|\cdot - y|} q(y)u_1(y)\, dy \in C^1(\overline{E(2\epsilon)} \times (-\epsilon_0, 2\epsilon_0)). \tag{5}$$

In fact, it is well known that for all $x \in \mathbb{R}^3$ and $k = 1, 2, 3$

$$\begin{aligned} \partial_{x_k} \int_{D_1} \frac{1}{|x-y|} q(y)u_1(y)\, dy &= \int_{D_1} \partial_{x_k} \frac{1}{|x-y|} q(y)u_1(y)\, dy \\ &= -\int_{D_1} \partial_{y_k} \frac{1}{|x-y|} q(y)u_1(y)\, dy. \end{aligned} \tag{6}$$

Hence by the integration by parts we have

$$\begin{aligned}
&\partial_{x_3} \int_{D_1} \frac{1}{|x-y|} q(y)u_1(y)\ dy \\
&= - \int_{\partial D_1} \frac{1}{|x-y|} q(y)u_1(y)\nu_3(y)\ d\sigma_y \\
&\qquad + \int_{D_1} \frac{1}{|x-y|} \partial_{y_3}[q(y)u_1(y)]\ dy, \quad x \in \mathbb{R}^3,
\end{aligned} \tag{7}$$

where $\nu_3(y)$ is the third component of the outward normal vector to ∂D_1. Here remember that q and u_1 are C^1 functions in Ω'. Since $\nu_3(y) = 0$ for all $y \in (\overline{OQ_1^{2\epsilon}} \cup \overline{OQ_2^{2\epsilon}}) \times (-\epsilon_0, 2\epsilon_0)$, we get for all $x \in \mathbb{R}^3$

$$\begin{aligned}
&\partial_{x_3} \int_{D_1} \frac{1}{|x-y|} q(y)u_1(y)\ dy = \\
&- \int_{\partial D_1 \setminus \left[(\overline{OQ_1^{2\epsilon}} \cup \overline{OQ_2^{2\epsilon}}) \times (-\epsilon_0, 2\epsilon_0)\right]} \frac{q(y)u_1(y)\nu_3(y)}{|x-y|}\ d\sigma_y + \int_{D_1} \frac{\partial_{y_3}[q(y)u_1(y)]}{|x-y|}\ dy,
\end{aligned} \tag{8}$$

which implies our claim (5).

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References

[1] G. Bal, Optical tomography for small volume absorbing inclusions, Inverse Problems, **19**(2003), 371-386

[2] W. Fang and E. Cumberbatch, Inverse problems for metal oxide semiconductor field-effect transistor contact resistivity, SIAM J. Appl. Math., **52**(1992), 699-709

[3] A. Friedman and V. Isakov, On the uniqueness in the inverse conductivity problem with one measurement, Indiana Univ. Math. J., **38**(1989), 563-579

[4] D. Gilbarg and N.S. Trudinger, Elliptic Partial Differential Equations of Second Order, Springer-Verlag Berlin Heidelberg, 1977

[5] V. Isakov , On uniqueness of recovery of a discontinuous conductivity coefficient, Commun. Pure Appl. Math., **41**(1988), 865-877

[6] S. Kim, Recovery of an unknown support of a source term in an elliptic equation, Inverse Problems, **20**(2004), 565-574

[7] H. Kang, K. Kwon and K. Yun, Recovery of an inhomogeneity in an elliptic equation, Inverse Problems, **17**(2001), 25-44

[8] H. Kang and J.K. Seo, The layer potential technich for the inverse conductivity problem, Inverse Problems, **12**(1996), 267-278

[9] S. Kim and M. Yamamoto, Uniqueness in identification of the support of a source term in an elliptic equation, SIAM J. Math. Anal., **35**(2003), 148-159

[10] W. H. Loh, S. E. Swirhun, T. A. Schreyer, R.M. Swanson, and K. C. Saraswat, Modelling and measurement of contact resistances, IEEE Trans. Electron Devices, **34**(1987), 512-524

[11] O.A. Ladyzhenskaya and N.N. Ural'tseva, Linear and Quasilinear Elliptic Equations, Academic Press, New York, 1968

[12] C. Miranda, Partial Differential Equations of Elliptic Type, Springer-Verlag Berlin · Heidelberg · New York, 1970

[13] J.K. Seo, On the uniqueness in the inverse conductivity problem, J. Fourier Anal. Appl., **2**(1996), 227-235

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