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UNIQUENESS IN DETERMINING POLYGONAL SOUND-HARD OBSTACLES

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Abstract. We consider the two dimensional inverse scattering problem of determining a sound-hard obstacle by the far field pattern. We establish the uniqueness within the class of polygonal domains by two incoming plane waves without further geometric constraints on the scatterers. This improves the uniqueness result by Cheng and Yamamoto [3].

§1. Introduction and the main result.

Let $D \subset \mathbb{R}^2$ be a polygonal bounded domain such that $\mathbb{R}^2 \setminus D$ is connected, and let $k \in \mathbb{R}$. By a polygonal domain, we mean that the boundary ∂D is composed of a finite number of segments. We note that polygonal domains under consideration are not necessarily convex.

For $x = (x_1, x_2) \in \mathbb{R}^2$, we set $r = |x|$. We consider the scattering problem with a sound-hard obstacle:

$$\Delta u + k^2 u = 0 \qquad in \quad \mathbb{R}^2 \setminus \overline{D}, \tag{1.1}$$

$$\frac{\partial u}{\partial \nu} = 0 \qquad on \quad \partial D \tag{1.2}$$

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and

$$\lim_{r\to\infty}\sqrt{r}\left(\frac{\partial}{\partial r}u^S(x)-iku^S(x)\right)=0. \tag{1.3}$$

Condition (1.2) corresponds to a sound-hard obstacle and (1.3) is the Sommerfeld radiation condition.

Throughout this paper, we always assume that D, D_1, D_2 are bounded polygonal open domains whose complements are connected. Moroever $\overline{D}$ denotes the closure of a domain D. We set $i=\sqrt{-1}$, $d\in S^1\equiv\{x\in\mathbb{R}^2\,;\,|x|=1\}$, and we call u and

$$u^S(x)\equiv u(x)-e^{ikx\cdot d},$$

respectively the scattered field and the total field. We consider $d\in S^1$ and $k\in\mathbb{R}$ respectively as the direction of the incoming plane wave (i.e., $e^{ikx\cdot d}$) and the wave number given by the medium in $\mathbb{R}^2\setminus\overline{D}$.

Then, for $k>0$ and $d\in S^1$, there exists a unique H^1-solution $u(x)=u(D;d)(x)\in H^1_{loc}(\mathbb{R}^2\setminus\overline{D})$ to (1.1) - (1.3) (e.g., Chapter 9 in McLean [16]). Furthermore $u(D;d)$ is smooth in any compact subset in $\mathbb{R}^2\setminus D$. We can define the far field pattern $u_\infty(D;d)\left(\frac{x}{r}\right)$:

$$u^S(D;d)(x)=\frac{e^{ikr}}{\sqrt{r}}\left\{u_\infty(D;d)\left(\frac{x}{r}\right)+O\left(\frac{1}{r}\right)\right\}\quad\text{as } r\longrightarrow\infty. \tag{1.4}$$

As for the scattering problem, we refer to Colton, Coyle and Monk [5], Colton and Kress [6], Ghosh Roy and Couchman [9], Kirsch [12], Potthast [17] for example.

Henceforth we fix $k>0$ arbitrarily. The main purpose of this article is to prove the uniqueness in

Inverse scattering problem. Determine D from the far field pattern $u_\infty(D;d)$ for given k and d (possibly by changing d).

By the uniqueness, we mean :

$$u_\infty(D_1;d)(x) = u_\infty(D_2;d)(x), \qquad |x| = 1 \tag{1.5}$$

(for possible several d) implies $D_1 = D_2$?

In the case of sound-hard obstacles, there are a few results on the uniqueness.

(1) For smooth domains D_1, D_2, if (1.5) holds for infinitely many $d \in S^1$, then $D_1 = D_2$ follows. The proof is based on Schiffer's idea (see Lax and Phillips [14]) and see Theorem 5.6 in Colton and Kress [6].

(2) Yun [20] proved the uniqueness of sound-hard balls with a single incident direction.

(3) Cheng and Yamamoto [3] proved the uniqueness within polygonal domains under an extra "non-trapping" assumption by two incident directions. See also Cheng and Yamamoto [4] for a similar uniqueness result for the impedance boundary condition.

In this paper, we get rid of the extra "non-trapping" assumption in [3] to establish the uniqueness in determining polygonal domains by two incident directions, which is stated as follows.

Theorem. *Let D_1 and D_2 be bounded polygonal domains whose complements are connected. Let $k > 0$ and let $d^1, d^2 \in S^1$ be arbitrarily fixed linearly independent vectors. If*

$$u_\infty(D_1;d^\alpha)(x) = u_\infty(D_2;d^\alpha)(x), \quad x \in S^1,\ \alpha = 1, 2,$$

then $D_1 = D_2$.

In the sound-soft case where the boundary condition (1.2) is replaced by $u = 0$ on ∂D, Alessandrini and Rondi [2] proved that the far field pattern $u_\infty(D;d)$ for a single incident direction $d \in S^1$ determines polygonal domains uniquely. As for the uniqueness with smooth domains in the sound-soft case, we can refer to Theorems 5.1 and 5.2 in [6], Colton and Sleeman [7], Kirsch and Kress [13], Liu [15], Sleeman [19]. Moreover see Chapter 6 in Isakov [11], and Isakov [10], Rondi [18].

Our proof of the main result is a combination of a modification of an idea in the proof of Lemma 3.7 in Alessandrini and Rondi [2] and an argument in Cheng and Yamamoto [3] which asserts the finiteness of the number of segments where normal derivatives of the total field vanish. Moreover if we apply our argument here to the sound-soft case, then we can obtain the uniqueness result in the sound-soft case within polygonal obstacles by a single incident direction which was already proved in [2].

The paper is composed of three sections. In Section 2, we show key lemmata and in Section 3, we complete the proof of the main theorem.

§2. Preliminaries.

In this section, we will show key lemmata. Let $k > 0$, and linearly independent vectors $d^1, d^2 \in S^1$ be arbitrarily fixed. Henceforth, for two distinct points $P, Q \in \mathbb{R}^2$, we understand that $\overline{PQ}$ is an open segment (not including the end points P and Q). Moreover for a polygonal domain D and $P \notin \overline{D}$, $Q \in \partial D$ such that $\overline{PQ} \in \mathbb{R}^2 \setminus \overline{D}$, by $\angle(\overline{PQ}, \partial D)$ we denote the least angle among the two angles in $\mathbb{R}^2 \setminus \overline{D}$ formed by $\overline{PQ}$ and ∂D. We note that the polygonal domains under consideration are always the complements of unbounded domains. Henceforth $\triangle ABC$ denotes the interior of the triangle ABC.

Lemma 1. *Let $\Omega \subset \mathbb{R}^2$ be a polygonal domain, and let $\overline{OA}$ be one of its sides such* *that Ω is located on one side of $\overline{OA}$. Let Π denote the symmetric transform in $\mathbb{R}^2$ with respect to the extended straight line OA. Let $v \in H^1(\Omega)$ satisfy*

$$\frac{\partial v}{\partial \nu} = 0 \qquad \text{on } \overline{OA}$$

and

$$\Delta v + k^2 v = 0 \qquad \text{in } \Omega.$$

We set

$$V(x_1, x_2) = \begin{cases} v(x_1, x_2), & (x_1, x_2) \in \Omega, \\ v(\Pi(x_1, x_2)), & (x_1, x_2) \in \Pi\Omega. \end{cases}$$

Then

$$V \in H^1(\Omega \cup \Pi\Omega \cup \overline{OA})$$

and

$$\Delta V + k^2 V = 0 \quad \text{in } \Omega \cup \Pi\Omega \cup \overline{OA}.$$

Moreover, if $\frac{\partial v}{\partial \nu} = 0$ on other side $\overline{BC}$ of $\partial\Omega$, then $\frac{\partial v}{\partial \nu} = 0$ on $\Pi\overline{BC}$.

The proof is directly done by the definition of H^1-solutions.

We can prove the following lemma exactly in the same way as Lemma 9 in Cheng and Yamamoto [3].

Lemma 2. *Let $d^1, d^2 \in S^1$ be linearly independent and let $u(D; d^\alpha) \in H^1_{loc}(\mathbb{R}^2 \setminus \overline{D})$, $\alpha = 1, 2$, satisfy (1.1) - (1.3). Then there does not exist an infinite straight half line $L \subset \mathbb{R}^2 \setminus \overline{D}$ where*

$$\frac{\partial u(D; d^\alpha)}{\partial \nu} = 0, \quad \alpha = 1, 2.$$

Furthermore we show two lemmata.

Lemma 3. *Let $E \subset \mathbb{R}^2$ be a domain and let $v \in H^1_{loc}(E)$ satisfy $\Delta v + k^2 v = 0$ in E. Let $L_0 \subset L \subset E$ be two segments. Then $\frac{\partial v}{\partial \nu} = 0$ on L_0 implies $\frac{\partial v}{\partial \nu} = 0$ on L.*

Proof. Since v satisfies the Helmholtz equation, the function v is real analytic in E (e.g., [6]). Therefore $\frac{\partial v}{\partial \nu}|_L$ is an analytic function in one variable and so the lemma follows.

Lemma 4. *Let $\widehat{E} \subset \mathbb{R}^2$ be a domain such that there exists $\rho_0 > 0$ with*

$$\widehat{E} \supset \{x \in \mathbb{R}^2; |x| > \rho_0\}. \tag{2.1}$$

Moreover let $(x_1, x_2) = (r\cos\mu, r\sin\mu)$ with $r \ge 0$ and $0 \le \mu < 2\pi$ be the polar coordinate. Let $v \in H^1(\widehat{E})$ satisfy

$$\Delta v + k^2 v = 0 \qquad \text{in } \widehat{E} \tag{2.2}$$

and

$$\text{there exists an open subset } U \subset \widehat{E} \text{ such that } \frac{\partial v}{\partial \mu} = 0 \text{ in } U. \tag{2.3}$$

Then for any $d \in S^1$, the function $v(x) - e^{ikx\cdot d}$ does not satisfy the Sommerfeld radiation condition (1.3).

Proof. Without loss of generality, we may assume that

$$U \supset \{(r, \mu); 0 \le r < \varepsilon,\ 0 \le \mu < \theta\}$$

with some $\varepsilon > 0$ and $\theta \in (0, 2\pi)$. Then, by (2.3), we obtain

$$v(x_1, x_2) = v(r), \qquad (x_1, x_2) \in U.$$

Since

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \mu^2},$$

by (2.2), we have

$$\frac{\partial^2 v}{\partial r^2}(r) + \frac{1}{r}\frac{\partial v}{\partial r}(r) + k^2 v(r) = 0, \quad 0 < r < \varepsilon.$$

Therefore, in terms of the Bessel functions J_0 of the first kind and Y_0 of the second kind (e.g., Abramowitz and Stegun [1]), we represent v as

$$v(r) = C_1 J_0(kr) + C_2 Y_0(kr) \quad \text{in } U,$$

where C_1, C_2 are constants. By the singularity of $\frac{dY_0}{dr}$ at $r = 0$ (e.g., pp.360–361 in [1]) and $v \in H^1(E)$, noting that J_0 is smooth near $r = 0$, we see that $C_2 = 0$:

$$v(r) = C_1 J_0(kr) \quad \text{in } U. \tag{2.4}$$

Hence the classical unique continuation yields that (2.4) holds also in $\widehat{E}$. Direct calculations and the asymptotic behaviour of J_0 and $\frac{dJ_0}{dr}$ at $r = \infty$ (e.g., [1]), imply

$$\begin{aligned} &\sqrt{r}\left(\frac{\partial}{\partial r} - ik\right) v(x) \\ &= -\sqrt{\frac{2k}{\pi}} C_1 \left\{\cos\left(kr - \frac{\pi}{2} - \frac{\pi}{4}\right) + i\cos\left(kr - \frac{\pi}{4}\right)\right\} + O\left(\frac{1}{r}\right) \end{aligned} \tag{2.5}$$

and

$$\sqrt{r}\left(\frac{\partial}{\partial r} - ik\right) e^{ikx\cdot d} = ike^{ikx\cdot d}\sqrt{r}(d_1\cos\mu + d_2\sin\mu - 1) \tag{2.6}$$

as $r \longrightarrow \infty$, where we set $d = (d_1, d_2)$ and $x = (x_1, x_2) = (r\cos\mu, r\sin\mu)$.

By (2.1), we can choose some θ_1 and sufficiently large $\rho_1 > 0$ such that $d_1\cos\theta_1 + d_2\sin\theta_1 - 1 \neq 0$ and $L \equiv \{x = x_0 + \rho(\cos\theta_1, \sin\theta_1);\ \rho > \rho_1\} \subset \widehat{E}$. Therefore (2.6) implies

$$\lim_{r\to\infty, x\in L}\left|\sqrt{r}\left(\frac{\partial}{\partial r} - ik\right) e^{ikx\cdot d}\right| = \lim_{r\to\infty, x\in L} |k|\sqrt{r}|d_1\cos\theta_1 + d_2\sin\theta_1 - 1| = \infty,$$

which with (2.5) yields

$$\lim_{r\to\infty, x\in L} \left| \sqrt{r}\left(\frac{\partial}{\partial r} - ik\right)(v(x) - e^{ikx\cdot d})\right| = \infty.$$

Therefore the Sommerfeld radiation condition does not hold. Thus the proof of Lemma 4 is complete.

We will further state two lemmata, which are proved similarly to Lemmata 6 and 7 in Cheng and Yamamoto [3]. We omit the proofs.

Lemma 5. *Let* $A = (\varepsilon, 0)$, $O = (0,0)$, $B = (\varepsilon\cos\theta, \varepsilon\sin\theta)$, $E = \{x \in \mathbb{R}^2; 0 < \arg x < \theta,\ |x| < \varepsilon\}$ *for* $\varepsilon > 0$ *and* $0 < \theta < 2\pi$. *We take* $P \in E$ *and set* $\phi = \angle AOP \in (0, \theta)$. *We assume*

$$\frac{\phi}{\theta} \notin \mathbb{Q}.$$

Moreover let $\widehat{E} \subset \mathbb{R}^2$ *be an unbounded domain such that (2.1) holds and* $E \subset \widehat{E}$. *If* $v \in H^1_{loc}(\widehat{E})$ *satisfies*

$$\Delta v + k^2 v = 0 \qquad \text{in } \widehat{E} \tag{2.7}$$

$$\frac{\partial v}{\partial \nu} = 0 \quad \text{on } \overline{OA} \cup \overline{OB} \tag{2.8}$$

$$\frac{\partial v}{\partial \nu} = 0 \quad \text{on } \overline{OP}, \tag{2.9}$$

then $v(x) - e^{ikx\cdot d}$ *does not satisfy the Sommerfeld radiation condition (1.3).*

Lemma 6. *Let the sector* E *and the points* A, B, O *be defined as in Lemma 5, and let* $P \in E$ *and* $\phi = \angle AOP \in (0, \theta)$. *Let* $v \in H^1(E)$ *satisfy (2.7) - (2.9) and let us assume that*

$$\frac{\phi}{\theta} = \frac{n}{m} \in \mathbb{Q},$$

where $m, n \in \mathbb{N}$, $1 \le n \le m-1$ and the greatest common divisor of m and n is one. Then:

(i) There exist $m-1$ points $P^j \in E$, $1 \le j \le m-1$, such that $\angle AOP^j = \frac{j}{m}\theta$ and $\frac{\partial v}{\partial \nu} = 0$ on $\overline{OP^j}$.

(ii) There exists a point $Q \in E$ such that $\angle AOP = \angle BOQ$ and $\frac{\partial v}{\partial \nu} = 0$ on $\overline{OQ}$.

§3. Proof of Theorem.

First Step. Assume contrarily that $D_1 \ne D_2$. For simplicity, we set

$$u_j^\alpha = u(D_j; d^\alpha), \qquad \alpha = 1, 2,\ j = 1, 2.$$

By the Rellich theorem (e.g., Lemma 2.11 in [6]), we see from $u_\infty(D_1; d^\alpha) \equiv u_\infty(D_2; d^\alpha)$ that

$$u_1^\alpha = u_2^\alpha \quad \text{in unbounded connected components of } \mathbb{R}^2 \setminus \overline{(D_1 \cup D_2)} \tag{3.1}$$

(e.g., Theorem 2.13 in [6]).

Since $D_1 \ne D_2$, there exists a segment $\overline{A_1A_2}$ which is on $\partial D_1 \cap (\mathbb{R}^2 \setminus \overline{D_2})$ or on $\partial D_2 \cap (\mathbb{R}^2 \setminus \overline{D_1})$.

In fact, if there does not exist such a segment, then $\partial D_1 \subset \overline{D_2}$ and $\partial D_2 \subset \overline{D_1}$. Consequently $D_1 \subset \overline{D_2}$ and $D_2 \subset \overline{D_1}$ by the connectedness of the complements of D_1 and D_2. Indeed, let us assume contrarily that $D_1 \subset \overline{D_2}$ does not hold. Then there exists a point $P \in D_1$ such that $P \in \mathbb{R}^2 \setminus \overline{D_2}$. Let $Q \in (\mathbb{R}^2 \setminus \overline{D_1}) \cap (\mathbb{R}^2 \setminus \overline{D_2})$ be arbitrarily fixed. Since $\mathbb{R}^2 \setminus \overline{D_2}$ is connected by the assumption, there exists a continuous curve γ connecting P and Q, and $\gamma \subset \mathbb{R}^2 \setminus \overline{D_2}$. Moreover, by $P \in D_1$ and $Q \notin D_1$, noting that ∂D_1 is a simple closed curve, we see that γ intersects ∂D_1 at some point R. Hence $R \in \partial D_1 \cap \gamma \subset \partial D_1 \cap (\mathbb{R}^2 \setminus \overline{D_2})$, which contradicts that

$\partial D_1 \subset \overline{D_2}$. Thus $D_1 \subset \overline{D_2}$ follows. Similarly we can see that $D_2 \subset \overline{D_1}$. Therefore $\overline{D_1} = \overline{D_2}$, that is, $D_1 = D_2$. Hence such a segment $\overline{A_1A_2}$ exists.

Without loss of generality, we may assume that

$$\overline{A_1A_2} \subset \partial D_2 \cap (\mathbb{R}^2 \setminus \overline{D_1}).$$

By (3.1), we see that

$$\frac{\partial u_1^\alpha}{\partial \nu} = 0 \quad \text{on } \overline{A_1A_2}.$$

Hence, setting

$$\begin{aligned}\mathcal{G} = \Bigg\{ & S;\ S \text{ is a (finite or infinite) open segment such that} \\ & S \subset \mathbb{R}^2 \setminus \overline{D_1}, \text{ at least one end point of } S \text{ is on } \partial D_1 \\ & \text{and } \frac{\partial u_1^\alpha}{\partial \nu}|_S = 0, \quad \alpha = 1, 2 \Bigg\},\end{aligned} \tag{3.2}$$

we see that $\mathcal{G} \neq \emptyset$.

Next, in terms of Lemma 2, we can prove that

$$\mathcal{G} \text{ does not contain an infinite segment.} \tag{3.3}$$

Henceforth we extend each $S \in \mathcal{G}$ to maximal length in $\mathbb{R}^2 \setminus \overline{D_1}$. Then, in view of Lemma 3, we see that

$$\begin{aligned}\mathcal{G} = \Bigg\{ & S;\ S \text{ is a finite open segment such that} \\ & S \subset \mathbb{R}^2 \setminus \overline{D_1}, \text{ the both end points of } S \text{ are on } \partial D_1 \\ & \text{and } \frac{\partial u_1^\alpha}{\partial \nu}|_S = 0, \quad \alpha = 1, 2 \Bigg\}.\end{aligned} \tag{3.4}$$

Second Step. In this step, we will prove that

$$\mathcal{G} \text{ contains only a finite number of finite segments.} \tag{3.5}$$

Proof of (3.5). The proof is similar to [3]. Assume on the contrary that $\mathcal{G}$ contains infinitely many segments. Henceforth let α vary over $\{0,1\}$. Then we can choose sequences of points $\{P_j\}_{j\in\mathbb{N}}$ and $\{Q_j\}_{j\in\mathbb{N}}$ such that

$$P_j \neq P_{j'} \quad \text{if } j \neq j', \quad P_j, Q_j \in \partial D_1, \overline{P_jQ_j} \in \mathbb{R}^2 \setminus \overline{D_1} \tag{3.6}$$

and

$$\frac{\partial u_1^\alpha}{\partial \nu} = 0 \quad \text{on } \overline{P_jQ_j}, \quad j \in \mathbb{N}, \ \alpha = 1, 2. \tag{3.7}$$

Here we note that $\{Q_j\}_{j\in\mathbb{N}}$ may not be mutually distinct.

Since the length $|\partial D_1|$ of the curve ∂D_1 is finite and $P_j \neq P_{j'}$ if $j \neq j'$, we can choose subsequences $\{P_j\}_{j\in\mathbb{N}}$ and $\{Q_j\}_{j\in\mathbb{N}}$, which are denoted by the same letters, such that

$$\lim_{j\to\infty} P_j = P_\infty, \quad \lim_{j\to\infty} Q_j = Q_\infty. \tag{3.8}$$

Without loss of generality, by further taking subsequences of $\{P_j\}_{j\in\mathbb{N}}$ and $\{Q_j\}_{j\in\mathbb{N}}$, we may assume that

$$\begin{aligned}&P_j,\ Q_j,\ j \in \mathbb{N}, \text{ are located at one side of } P_\infty,\ Q_\infty \text{ respectively}\\ &\text{and } P_j,\ j \in \mathbb{N} \text{ are not vertices of } D_1.\end{aligned} \tag{3.9}$$

Then we note that

$$\overline{P_jP_{j+1}}, \quad \overline{Q_jQ_{j+1}} \subset \partial D_1, \quad j \in \mathbb{N}. \tag{3.10}$$

Moreover we see that

$$\frac{\angle(\overline{Q_jP_j}, \partial D_1)}{\pi} \neq \frac{1}{2}, \in \mathbb{Q}, \quad j \in \mathbb{N}, \tag{3.11}$$

provided that we extract subsequences if necessary.

In fact, let $\frac{\angle(\overline{Q_jP_j},\partial D_1)}{\pi} \notin \mathbb{Q}$ for some $j \in \mathbb{N}$. Then, by Lemma 5, $u_1^\alpha(x) - e^{ikx\cdot d^\alpha}$ cannot satisfy (1.3), which is a contradiction. Next let us assume that $\frac{\angle(\overline{Q_{j_k}P_{j_k}},\partial D_1)}{\pi} = \frac{\pi}{2}$ for infinitely many j_k, $k \in \mathbb{N}$. Then, since

$$\frac{\partial u_1^\alpha}{\partial \nu} = 0 \qquad \text{on } \overline{P_{j_k}Q_{j_k}},\ k \in \mathbb{N},$$

and $\lim_{k\to\infty} |\overline{P_{j_{k+1}}P_{j_k}}| = 0$, we repeat applications of Lemma 1 with respect to the symmetric axes $\overline{P_{j_k}Q_{j_k}}$, $k \in \mathbb{N}$, so that there exists $\delta_0 > 0$ such that for any $\varepsilon > 0$ and any point P with $\operatorname{dist}(P, \partial D_1) < \delta_0$, we can choose points P' and Q' such that $\overline{P'Q'} \perp \partial D_1$, $Q' \in \mathbb{R}^2 \setminus \overline{D_1}$, $|\overline{P'Q'}| = \delta_0$, $\operatorname{dist}(P, \overline{P'Q'}) < \varepsilon$ and $\frac{\partial u_1^\alpha}{\partial \nu} = 0$ on $\overline{P'Q'}$. Since Δ is invariant with respect to a rotation, without loss of generality, we may take ∂D_1 on the x_1-axis. Then $\left|\frac{\partial u_1^\alpha}{\partial \nu}\right| = \left|\frac{\partial u_1^\alpha}{\partial x_1}\right|$ on any segment which is perpendicular to ∂D_1. Hence, since $\frac{\partial u_1^\alpha}{\partial x_1}$ is continuous in $\mathbb{R}^2 \setminus \overline{D_1}$ and $\varepsilon > 0$ is arbitrary, we see that $\frac{\partial u_1^\alpha}{\partial x_1} = 0$, $\alpha = 1, 2$, in an open set in $\mathbb{R}^2 \setminus \overline{D_1}$. Therefore, by the classical unique continuation, we see that $u_1^\alpha(x_1, x_2) = v(x_2)$ for $(x_1, x_2) \in \mathbb{R}^2 \setminus \overline{D_1}$. By (1.2), we have $\frac{\partial v}{\partial x_2}(x_2) = 0$ on ∂D_1. Consequently u_1^α is a constant function, which yields $u_1^\alpha = 0$ in $\mathbb{R}^2 \setminus \overline{D_1}$ by (1.1) and $k \neq 0$. Hence Lemma 4 yields a contradiction because $u_1^\alpha - e^{ikx\cdot d^\alpha}$ satisfies (1.3). Thus the proof of (3.11) is complete.

By [3], under condition (3.11), we can construct $\triangle P_jP_{j+1}R_j \subset \mathbb{R}^2 \setminus \overline{D_1}$, $j \in \mathbb{N}$, which satisfy

$$\Delta u_1^\alpha + k^2 u_1^\alpha = 0 \quad \text{in} \quad \triangle P_jP_{j+1}R_j, \tag{3.12}$$

$$\frac{\partial u_1^\alpha}{\partial \nu} = 0 \quad \text{on } \partial(\triangle P_jP_{j+1}R_j) \tag{3.13}$$

and

$$\text{(3.14)} \quad \begin{aligned} &\triangle P_j P_{j+1} R_j \text{ is included in a rectangle } \Omega_j \text{ whose side lengths are} \\ &|\overline{P_j P_{j+1}}| \text{ and } \rho_j \text{ such that } \lim_{j\to\infty} \rho_j = 0. \end{aligned}$$

For completeness, we will give the construction of the triangles in the appendix.

Then we can arrive at a contradiction as follows, which completes the proof of (3.5). If u_1^α identically vanishes in $\triangle P_j P_{j+1} R_j$ for some $j \in \mathbb{N}$, then Lemma 4 immediately yields a contradiction. Next we assume that u_1^α does not vanish identically in $\triangle P_j P_{j+1} R_j$ for any $j \in \mathbb{N}$. Then k^2 is an eigenvalue of $-\Delta$ in $\triangle P_j P_{j+1} R_j$ with the homogeneous Neumann boundary condition.

By $\lambda_2(\Omega) > 0$ we denote the second smallest eigenvalue of $-\Delta$ in Ω with the homogeneous Neumann boundary condition. Note that the smallest eigenvalue in the Neumann case is always 0. By a comparison property of eigenvalues (e.g., Courant and Hilbert [8]), we have

$$\lambda_2(\triangle P_j P_{j+1} R_j) \ge \lambda_2(\Omega_j), \quad j \in \mathbb{N}.$$

Here Ω_j is a rectangle whose side lengths are $|\overline{P_j P_{j+1}}|$ and ρ_j. We can directly see that

$$\lambda_2(\Omega_j) = \pi^2 \min\left\{ \frac{1}{|\overline{P_j P_{j+1}}|^2}, \frac{1}{\rho_j^2} \right\}.$$

Since $k \ne 0$, properties (3.12) and (3.13) yield $k^2 \ge \lambda_2(\triangle P_j P_{j+1} R_j)$. Consequently

$$k^2 \ge \pi^2 \min\left\{ \frac{1}{|\overline{P_j P_{j+1}}|^2}, \frac{1}{\rho_j^2} \right\}, \quad j \in \mathbb{N}. \tag{3.15}$$

By (3.8) and (3.14), the right hand side tends to ∞ as $j \longrightarrow \infty$, so that (3.15) is impossible. Thus the proof of (3.5) is complete.

Third Step. In this step, we will complete the proof of the main theorem, by an idea in the proof of Lemma 3.7 in Alessandrini and Rondi [2]. By the second step, we can set $\mathcal{G} = \{S_1, ..., S_N\}$, where S_j, $1 \le j \le N$, are finite segments. We note that

$$
\begin{aligned}
&S_j \subset \mathbb{R}^2 \setminus \overline{D_1}, \text{ the both end points are on } \partial D_1 \text{ and} \\
&\frac{\partial u_1^\alpha}{\partial \nu} = 0 \quad \text{on } S_j,\ 1 \le j \le N,\ \alpha = 1, 2.
\end{aligned} \tag{3.16}
$$

Since S_j connects two points on ∂D_1,

$$
\begin{aligned}
&\text{every } S_j,\ 1 \le j \le N, \text{ divides } \mathbb{R}^2 \setminus \overline{D_1} \text{ into} \\
&\text{an unbounded domain and a bounded polygonal domain} \\
&\text{which is bounded by } S_j \text{ and } \partial D_1.
\end{aligned} \tag{3.17}
$$

In fact, let $S_j = \overline{A_j B_j}$ with $A_j, B_j \in \partial D_1$. Then, by noting that ∂D_1 is a Jordan closed curve because D_1 is connected, the points A_j, B_j divide ∂D_1 into two parts Λ_1 and Λ_2, and either Λ_1 or Λ_2, say Λ_1 and $\overline{A_j B_j}$ form a Jordan closed curve. That is, Λ_1 and $\overline{A_j B_j}$ limit a bounded polygonal domain. Thus (3.17) is seen.

Hence ∂D_1 and $S_1, ..., S_N$ divide $\mathbb{R}^2 \setminus \overline{D_1}$ into a finite number of parts Ω_∞, Ω_1, ..., Ω_M such that Ω_∞ is an unbounded domain and $\Omega_1, ..., \Omega_M$ are bounded polygonal domains. Let us choose one bounded polygonal domain, say Ω_1, among $\Omega_1, ..., \Omega_M$ whose boundary shares one open segment S^* with $\partial\Omega_\infty$.

By (3.16) and the definition of $\Omega_1, ..., \Omega_M$, we see that

$$
\Omega_\infty \bigcap \left(\bigcup_{j=1}^N S_j \right) = \emptyset. \tag{3.18}
$$

By Π, we denote the reflection transform with respect to the extended straight line of S^*. We set

$$\Omega^+ = \Omega_\infty \cap \Pi\Omega_1, \qquad \Omega^- = \Omega_1 \cap \Pi\Omega_\infty.$$

Then we have

$$\Omega^- = \Pi\Omega^+.$$

In fact, we have $\Pi\Omega^+ = \Pi(\Omega_\infty \cap \Pi\Omega_1) = \Pi\Omega_\infty \cap \Pi^2(\Omega_1) = \Omega_1 \cap \Pi\Omega_\infty$ because Π is injective and Π^2 is the identity transform.

By (1.2) and the construction of Ω^+ and Ω^-, we apply Lemma 1 to see that

$$\frac{\partial u_1^\alpha}{\partial \nu} = 0 \quad \text{on } \partial\Omega^+ \cup \partial\Omega^-,\ \alpha = 1, 2. \tag{3.19}$$

Moreover $\partial\Omega^+ \setminus S^*$ contains a segment which is not on ∂D_1.

In fact, since $\Omega^+ \subset \Omega_\infty$, we can choose a continuous curve $\gamma = \gamma(t) \subset \Omega_\infty \subset \mathbb{R}^2 \setminus \overline{D_1}$ such that $\{\gamma(t); t > 0\} \cap \Omega^+ \ne \emptyset$, $\lim_{t\to\infty} \gamma(t) = \infty$ and $\gamma(0) \in S^*$. Furthermore, Ω_1 is bounded and so Ω^+ is bounded. Therefore, γ must intersect $\partial\Omega^+$ at some point which is not in $S^* \cup \partial D_1$.

Let us choose a segment S^{**} from $(\partial\Omega^+ \setminus S^*) \setminus \partial D_1$. We extend S^{**} in $\mathbb{R}^2 \setminus \overline{D_1}$, so that we can assume that its both end points are on ∂D_1. In fact, otherwise we can extend S^{**} to ∞ in $\mathbb{R}^2 \setminus \overline{D_1}$. We denote the extension by S^{**} again. Then (3.19) and Lemma 3 imply that

$$\frac{\partial u_1^\alpha}{\partial \nu} = 0 \quad \text{on } S^{**},\ \alpha = 1, 2.$$

This is impossible by Lemma 2. Thus after necessary extension, S^{**} is a segment connecting two points on ∂D_1. Therefore

$$S^{**} \in \{S_1, ..., S_N\}. \tag{3.20}$$

Since $\Omega^+ \subset \Omega_\infty$, we have

$$S^{**} \subset \Omega_\infty. \tag{3.21}$$

By (3.18), we see that (3.20) and (3.21) are not compatible. This contradiction leads us to the completion of the proof of the theorem.

Appendix. Construction of $\triangle P_j P_{j+1} R_j$ satisfying (3.12) - (3.14).

We consider the following two cases separately.

Case a. $P_\infty = Q_\infty$.

Case b. $P_\infty \neq Q_\infty$.

Case a. Then, by extracting a subsequence if necessary, we can assume that $Q_j \neq Q_{j'}$ if $j \neq j'$. Otherwise $Q_j = Q_\infty$ for $j \in \mathbb{N}$, which is impossible because $\overline{P_j P_\infty} = \overline{P_j Q_j} \subset \mathbb{R}^2 \subset \overline{D_1}$.

By (3.9) and (3.10), we have $\overline{P_j P_\infty}$, $\overline{Q_j Q_\infty} \subset \partial D_1$. Hence, since $\overline{P_j Q_j} \subset \mathbb{R}^2 \setminus \overline{D_1}$ by (3.6), we see that the three points P_j, Q_j, P_∞ are not on one line, that is, they form a triangle. Moreover $\triangle P_j Q_j P_\infty \subset \mathbb{R}^2 \setminus \overline{D_1}$. Therefore, setting $R_j = P_\infty$ for $j \in \mathbb{N}$, we see that $\triangle P_j Q_j P_\infty$ satisfies (3.12) - (3.14). In fact, (3.12) and (3.13) are straightforward from (3.6) - (3.8). Since $\lim_{j\to\infty} |\overline{P_j P_\infty}| = \lim_{j\to\infty} |\overline{Q_j P_\infty}| = 0$ by (3.8), the lengths of all the sides of $\triangle P_j Q_j P_\infty$ tend to 0 as $j \longrightarrow \infty$, so that (3.14) follows.

Case b. Let L be the side of D_1 including $\overline{P_\infty P_j}$, $j \in \mathbb{N}$. With (3.8) and (3.9), by further taking subsequences, we can assume that

$$|\overline{P_j P_\infty}| \text{ and } |\overline{Q_j Q_\infty}| \text{ are monotonically decreasing in } j \in \mathbb{N}. \tag{1}$$

In terms of (3.8), if we choose the minor angle or the major angle suitably, then

$$\lim_{j\to\infty} \angle(\overline{Q_j P_j}, L) = \angle(\overline{Q_\infty P_\infty}, L). \tag{2}$$

By (3.11), there exist $m_j, n_j \in \mathbb{N}$ such that the greatest common divisor of m_j and n_j is one, $1 \le n_j \le m_j - 1$ and

$$\angle(\overline{Q_jP_j}, L) = \frac{n_j}{m_j}\pi, \quad j \in \mathbb{N}. \tag{3}$$

In view of (2), the sequence $\frac{n_j}{m_j}$, $j \in \mathbb{N}$, converges. We have the two cases:

Case b-(i). $\sup_{j\in\mathbb{N}} m_j = \infty$.

Case b-(ii). $\sup_{j\in\mathbb{N}} m_j < \infty$.

Case b-(i). We choose a subsequence if necessary, so that $\lim_{j\to\infty} m_j = \infty$. Since D_1 is a polygon, we can choose $\triangle P_\infty AP_1$ such that $\triangle P_\infty AP_1 \subset \mathbb{R}^2 \setminus \overline{D_1}$.

Henceforth $j \in \mathbb{N}$ are arbitrary but sufficiently large. We can apply Lemma 6 twice, setting $O = P_j$, $P = Q_j$ and $O = P_{j+1}, P = Q_{j+1}$ respectively. Then there exist points $R_j \in \mathbb{R}^2 \setminus \overline{D_1}$ such that $\angle R_jP_{j+1}P_j = \frac{1}{m_{j+1}}\pi$, $\angle R_jP_jP_{j+1} = \frac{1}{m_j}\pi$ and $\frac{\partial u_1^\alpha}{\partial \nu} = 0$ on $\overline{R_jP_{j+1}} \cup \overline{R_jP_j}$. Since $\overline{P_jP_{j+1}} \subset \overline{P_\infty P_1}$ and $\angle R_jP_{j+1}P_j \longrightarrow 0$, $\angle R_jP_jP_{j+1} \longrightarrow 0$ as $j \longrightarrow \infty$, we see that $\triangle P_jP_{j+1}R_j \subset \triangle P_\infty AP_1 \subset \mathbb{R}^2 \setminus \overline{D_1}$ for large $j \in \mathbb{N}$. Therefore (3.12) and (3.13) follow. Let ρ_j be the length of the perpendicular to $\overline{P_jP_{j+1}}$ from R_j, that is, $\rho_j = |\overline{H_jR_j}|$ where $H_j \in \overline{P_jP_{j+1}}$ and $\overline{R_jH_j} \perp \overline{P_jP_{j+1}}$. Then

$$\rho_j = |\overline{P_jH_j}| \tan\frac{1}{m_j}\pi \le |\overline{P_jP_{j+1}}| \tan\frac{1}{m_j}\pi.$$

Since $\lim_{j\to\infty} |\overline{P_jP_{j+1}}| = 0$ by (3.8) and $\lim_{j\to\infty} m_j = \infty$, condition (3.14) follows.

Case b - (ii). If necessary, we can again choose subsequences, so that we can assume that for some $m, n \in \mathbb{N}$,

$$\angle(\overline{Q_jP_j}, L) = \frac{n}{m}\pi, \quad j \in \mathbb{N} \tag{4}$$

in terms of (2).

In this case, $P_jQ_jQ_{j+1}P_{j+1}$ forms a quadrilateral, because $\overline{P_jQ_j} \parallel \overline{P_{j+1}Q_{j+1}}$. Henceforth $P_jQ_jQ_{j+1}P_{j+1}$ means the interior of the quadrilateral. Then we can prove that

$$P_jQ_jQ_{j+1}P_{j+1} \subset \mathbb{R}^2 \setminus \overline{D_1}. \tag{5}$$

In fact, if not, then there exists a point $P \in P_jQ_jQ_{j+1}P_{j+1}$ such that $P \in \overline{D_1}$. Since D_1 is connected, we can choose a continuous curve γ such that γ connects P and P_{j+1}, and $\gamma \setminus \{P, P_{j+1}\} \subset D_1$. Since $\overline{P_jP_{j+1}}$, $\overline{Q_jQ_{j+1}}$ are sides of D_1 by (3.10) and $\overline{P_jQ_j}$, $\overline{P_{j+1}Q_{j+1}} \subset \mathbb{R}^2 \setminus \overline{D_1}$ by (3.6), the existence of such a curve γ is a contradiction. Thus (5) follows.

Let L_j be the infinite straight line passing P_j such that L_j is not parallel to $\overline{P_jQ_j}$ and the angle between L_j and L is $\frac{n}{m}\pi$. Since $\angle(\overline{Q_jP_j}, \partial D_1) = \frac{n}{m}\pi, \ne \frac{\pi}{2}$ by (3.11), such a straight line L_j exists. Let R_j be the intersection point of L_{j+1} and the infinite straight line passing P_j and Q_j. By (3.8) and $P_\infty \ne Q_\infty$, we have

$$\inf_{j\in\mathbb{N}} |\overline{P_jQ_j}| > 0. \tag{6}$$

Moreover, we see that $\angle R_jP_{j+1}P_j = \angle R_jP_jP_{j+1} = \frac{n}{m}\pi$, so that

$$\lim_{j\to\infty} |\overline{P_jR_j}| = \lim_{j\to\infty} \frac{|\overline{P_jP_{j+1}}|}{2} \left(\cos \frac{n}{m}\pi\right)^{-1} = 0 \tag{7}$$

by $\lim_{j\to\infty} |\overline{P_jP_{j+1}}| = 0$.

It follows from (6) and (7) that R_j is on the segment $\overline{P_jQ_j}$. Therefore (5) implies that $\triangle P_jP_{j+1}R_j \subset \mathbb{R}^2 \setminus \overline{D_1}$, $j \in \mathbb{N}$. Then Lemma 6 yields $\frac{\partial u_1^\alpha}{\partial\nu} = 0$ on $\overline{P_{j+1}R_j}$, and so (3.12) and (3.13) follow. Finally, by (3.8) and (7), condition (3.14) is seen. Thus the construction of $\triangle P_jP_{j+1}R_j$ satisfying (3.12) - (3.14) is complete.

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