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The absolute continuity of a measure induced by infinite dimensional stochastic differential equations

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0 Introduction

Let (B, H, μ) be an abstract Wiener space, that is, B is a separable real Banach space, H is a separable real Hilbert space imbedded in B densely and continuously, and μ is a Gaussian measure on B satisfying

$$\int_B e^{\sqrt{-1}\langle x,\phi\rangle}\mu(dx) = \exp\Big\{-\frac{1}{2}|\phi|_H^2\Big\}\ ,\ \phi \in B^*.$$

Here B^* denotes the dual space of B. We identify the dual space H^* of H with H itself. Then B^* can be regarded as a subset of $H^* = H$. Let μ_t be the distribution of $x \in B \mapsto \sqrt{t}x \in B$ under μ.

In this paper, we study the following type stochastic differential equations on (B, H, μ):

$$dX_t = dW_t + A(X_t)dW_t + b(X_t)dt \tag{0.1}$$

with $X_0 = 0$, where W_t is a *B-valued Wiener process* and $A : B \to H \otimes H$ and $b : B \to H$ are measurable maps with certain regularities. We assume that

$$E\Big[|(tI_H + \sigma(t))^{-1}|^p_{L(H;H)}\Big] < \infty, \quad \forall p \in (1, \infty) \tag{0.2}$$

where $\sigma(t)$ is the modified Malliavin covariance which will be defined in Section 3. Our main theorem is following:

Theorem 1 . *Let $\nu_T = P\circ X_T^{-1}$ be the distribution of X_T. Then, ν_T is absolutely continuous with respect to μ_T. Moreover, its Radon-Nikodým density $\rho_T(x)$ with respect to μ_T satisfies*

$$\int_B \rho_T(x)(\log \rho_T(x) \vee 1)^\alpha \mu_T(dx) < \infty$$

for $0 \le \alpha < 1/2$.

If $B = H = \mathbf{R}^d$, this is a well-known result. However, in the infinite dimensional setting, we cannot use some important items such as Sobolev inequalities that play essential roles there. Bell [1] studied about the quasi-invariance of the measures induced by certain stochastic differential equations with values in an infinite dimensional Hilbert space. It does not imply the absolute continuity as in finite dimensions. So we have to take a rather different strategy from the finite dimensional case.

Let us summarize the results in the present paper. In Section 1, we prepare some notations for Malliavin calculus and remind some known results. In Section 2, we prove the existence and uniqueness theorem for the SDE(0.1), and then we show that solutions to the SDE(0.1) are "smooth functions" in the sense of Malliavin calculus, that is, the H-valued part $Y_t = X_t - W_t$ is in $W^\infty(H)$ (cf. Section 1) for each $t \geq 0$. In Section 3, we define the modified Malliavin covariance $\sigma(t)$ and show that $\gamma(t) = (tI_H + \sigma(t))^{-1} - t^{-1}I_H \in W^\infty(H \otimes H)$ for each $t > 0$. Section 4 is devoted to show an integration in parts formula (Theorem 4.1) which is essential to show the main theorem in Section 5 (Theorem 1). Finally, in Section 6, we show that the condition (0.2) (therefore, Theorem 1) holds if the diffusion coefficient is uniformly elliptic. We will discuss the more general case in the forthcoming paper.

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1 Preliminary

1.1 Some known theorems

Let $\{e_i\}_{i=1}^\infty \subset B^*$ be a complete orthonormal system(abbreviated by CONS) in H. Let P_n be the orthogonal projection from H onto the finite dimensional subspace spanned by $\{e_1, \ldots, e_n\}$. Define the map $\tilde{P}_n : B \to B^*$ by

$$\tilde{P}_n x = \sum_{i=1}^n {}_B\langle x, e_i \rangle_{B^*} e_i \ , x \in B. \tag{1.1}$$

For each $h \in H$, it is easy to see that the sequence $\{\langle \tilde{P}_n \cdot, h \rangle_H\}_{n=1}^\infty$ is a Cauchy sequence in $L^2(B; \mathbf{R}, d\mu)$ and hence converges to an element in $L^2(B; \mathbf{R}, d\mu)$ which is denoted by $\langle \cdot, h \rangle_H$. It is easy to see that $\langle \cdot, h \rangle_H$ is determined independently of the choice of CONS and is normally distributed with mean 0 and variance $|h|_H^2$.

The following theorems are well known. (See e.g. Kuo [5] for the proofs.)

Theorem 1.1 (Landau-Shepp-Fernique). *Let p be a continuous semi-norm on B. Then there exists a constant $\beta = \beta_p > 0$ such that*

$$\int_B e^{\beta\, p(x)^2} \mu(dx) < \infty.$$

Theorem 1.2 (Cameron-Martin). *If $h \in H$ then $\mu(\,\cdot - h)$ is equivalent to μ and the Radon-Nikodým derivative is given by*

$$\frac{d\mu(\,\cdot - h)}{d\mu}(x) = \exp\Big\{ -\frac{1}{2}|h|_H^2 + \langle x, h\rangle_H \Big\},\ x \in B.$$

The following is due to Ito and Nisio [4]

Theorem 1.3. $\lim_{n\to\infty} |\tilde{P}_n x - x|_B = 0$ *for μ-a.e.$x \in B$.*

1.2 B-valued abstract Wiener space

Let $\mathbf{W} = \{w \in C([0,T] \to B); w_0 = 0\}$. Then $\mathbf{W}$ is a separable real Banach space with norm $\|w\|_{\mathbf{W}} = \sup_{0\le t\le T} |w_t|_B$. Let $\mathcal{B}(B)$ be the Borel σ-algebra on B. Let μ_t $(t \ge 0)$ be the image measure of μ induced by the map $x \mapsto \sqrt{t}x$, and set $P_t(x, A) := \mu_t(A - x)$ for $x \in B$, $A \in \mathcal{B}(B)$. Then, there exists a unique probability measure P on $\mathbf{W}$ such that

$$P(W_{t_0} \in dy_0, \dots, W_{t_n} \in dy_n) = \delta_0(dy_0) \prod_{i=1}^{n} P_{t_i - t_{i-1}}(y_{i-1}, dy_i),$$

for $0 = t_0 < t_1 < \cdots < t_n$ where $W_t(w) = w_t,\ w \in \mathbf{W}$. We call $W = \{W_t\}_{t\ge 0}$ *B-valued Wiener process* and P *Wiener measure on* $\mathbf{W}$. We often write as $W_t(h) = \langle W_t, h\rangle_H$ for $h \in H$. Then $W_t(h)/|h|_H$ is a 1-dimensional $\{\mathcal{F}_t\}$-Wiener process with $\mathcal{F}_t = \sigma\{w_s;\ 0 \le s \le t\}$.

Let $\mathbf{H}$ be a subspace of $\mathbf{W}$ consisting of $\mathbf{h}$ for which $\mathbf{h}(t)$ is absolutely continuous in $t \in [0,T]$ and $\int_0^T |\dot{\mathbf{h}}(t)|_H^2 dt < \infty$. Then $\mathbf{H}$ is a separable Hilbert space with norm $\|\mathbf{h}\|_{\mathbf{H}} = \{\int_0^T |\dot{\mathbf{h}}(t)|_H^2 dt\}^{1/2}$. The triple $(\mathbf{W}, \mathbf{H}, P)$ is also an abstract Wiener space and called *B-valued abstract Wiener space.*

1.3 Sobolev spaces

Let K be a separable real Hilbert space. Let $\{T_t\}_{t\ge 0}$ be the Ornstein-Uhlenbeck semigroup, i.e.,

$$T_t F(w) = \int_{\mathbf{W}} F(e^{-t}w + \sqrt{1 - e^{-2t}}z) P(dz)$$

for $t \ge 0$ and $F \in L^1(\mathbf{W}; K)$. Let L be the infinitesimal generator of $\{T_t\}_{t\ge 0}$. For $p \in (1, \infty)$ and $s \in \mathbf{R}$, $W^{s,p}(K)$ denote the completion of

$\mathcal{P}(K)$(the space of K-valued polynomials, see e.g. Watanabe [12]) by the norm $\|F\|_{s,p} = \|(I-L)^{s/2}F\|_p$ and is called the *Sobolev space*. Let us denote

$$W^{\infty}(K) = \bigcap_{p\in(1,\infty)} \bigcap_{s\in\mathbf{R}} W^{s,p}(K).$$

The $\mathbf{H}$-derivative D is defined as a linear operator from $\mathcal{P}(K)$ to $\mathcal{P}(\mathbf{H}\otimes K)$ such that

$$\frac{d}{dt}F(w+t\mathbf{h})\Big|_{t=0} = DF(w)[\mathbf{h}],\ \forall w \in \mathbf{W},\ \forall \mathbf{h}\in\mathbf{H}$$

for $F \in \mathcal{P}(K)$. More precisely, for $F(w) = f(\mathbf{h}_1(w), \dots, \mathbf{h}_n(w))k \in \mathcal{P}(K)$ where f is a polynomial, $k \in K$ and $\{\mathbf{h}_i\} \subset \mathbf{W}^*$ is ONS in $\mathbf{H}$, DF is given by

$$DF(w) = \sum_{i=1}^{n} \partial_i f(\mathbf{h}_1(w), \dots, \mathbf{h}_n(w))\mathbf{h}_i \otimes k$$

(cf. Shigekawa [9], Watanabe [12]). The following is due to Meyer [8].

Theorem 1.4. *For every $k \in \mathbf{N}$ and $p \in (1,\infty)$, there exist constants $0 < c_{k,p},\ C_{k,p} < \infty$ such that*

$$c_{k,p}\|D^kF\|_p \le \|F\|_{k,p} \le C_{k,p}\sum_{l=0}^{k}\|D^lF\|_p$$

for all $F \in \mathcal{P}(K)$.

By virtue of this theorem, the operator $D : \mathcal{P}(K) \to \mathcal{P}(\mathbf{H}\otimes K)$ is extended as a continuous operator from $W^{r+1,p}(K)$ to $W^{r,p}(\mathbf{H}\otimes K)$ for every $p \in (1,\infty)$ and $r \in \mathbf{R}$ (cf. Sugita [11]).

1.4 Stochastic integrals

Let E, F be Banach spaces. $L(E;F)$ denotes the Banach space consisting of bounded linear operators from E to F. For separable Hilbert spaces E, F, $L^k_{(2)}(E;F)$ denotes the Hilbert space consisting of Hilbert-Schmidt multilinear operators from $\underbrace{E\times\cdots\times E}_{k}$ to F. We denote $L^1_{(2)}(E;F)$ simply by $L_{(2)}(E;F)$ and often identify $L_{(2)}(E;F)$ with $E\otimes F$.

Let K be a real separable Hilbert space. $\mathcal{L}^p_2(H\otimes K)$, $p\in(1,\infty)$ denotes the collection of $(\mathcal{F}_t)$-adapted $H\otimes K$-valued processes Φ such that

$$E\Big[\Big\{\int_0^T |\Phi_t|^2_{H\otimes K}dt\Big\}^{p/2}\Big] < \infty.$$

For $\Phi \in \mathcal{L}_2^p(H \otimes K)$, we can define the stochastic integral $\int_0^t \Phi_s dW_s$ with respect to the Wiener process W_t as an element of $L^p(\mathbf{W}; K)$. In the case where $K = \mathbf{R}$, we often denote $\int_0^t \Phi_s dW_s$ by $\int_0^t \langle \Phi_s, dW_s \rangle_H$. $\mathcal{L}_1^p(K)$, $p \in (1, \infty)$, denotes the collection of $(\mathcal{F}_t)$-adapted K-valued processes ϕ such that

$$\int_0^T E[|\phi_t|_K^p]^{1/p} dt < \infty.$$

For $\phi \in \mathcal{L}_1^p(K)$ we can define $\int_0^T \phi_t dt$ as an element of $L^p(\mathbf{W}; K)$.

The following Burkholder's inequality for Hilbert space valued stochastic integrals will be used throughout the next section (see Kusuoka-Stroock [7] for the proof).

Theorem 1.5. *Let K be a separable real Hilbert space. Then for all $p \in (1, \infty)$, there exists a constant $c_p < \infty$ depending only on p such that*

$$E\Big[\sup_{0 \le s \le t} \Big| \int_0^s \Phi_r dW_r \Big|_K^p \Big]^{1/p} \le c_p E\Big[\Big\{ \int_0^t |\Phi_s|_{H \otimes K}^2 ds \Big\}^{p/2} \Big]^{1/p}$$

for all $\Phi \in \mathcal{L}_2^p(H \otimes K)$.

$\mathcal{L}_2^{n,p}(H \otimes K)$, $n \in \mathbf{N}$, $p \in (1, \infty)$, denotes the collection of $H \otimes K$-valued $(\mathcal{F}_t)$-adapted processes Φ such that $\Phi_t \in W^{n,p}(H \otimes K)$ for each $t \in [0, T]$ and $D^k \Phi_t \in \mathcal{L}_2^p(L_{(2)}^k(\mathbf{H}; H \otimes K))$ for each $k = 0, 1, \ldots, n$. $\mathcal{L}_1^{n,p}(K)$ denotes the collection of K-valued $(\mathcal{F}_t)$-adapted processes A such that $A_t \in W^{n,p}(K)$ for each $t \in [0, T]$ and $D^k A_t \in \mathcal{L}_1^p(L_{(2)}^k(\mathbf{H}; K))$ for each $k = 0, 1, \ldots, n$.

The following is well known in the finite dimensional setting (cf. Kusuoka-Stroock [6]). We can easily show that the same argument works in infinite dimensional cases.

Proposition 1.6. *Let $p \in (1, \infty)$ and $n \in \mathbf{Z}_+$. Let $\Phi \in \mathcal{L}_2^{n,p}(H \otimes K)$ and $\phi \in \mathcal{L}_1^{n,p}(K)$. Then*

$$\Psi_t = \int_0^t \Phi_s dW_s + \int_0^t \phi_s ds$$

is an element of $W^{n,p}(K)$ for each $t \in [0, T]$ and

$$E\Big[\sup_{0 \le t \le T} |D^k \Psi_t|_{L_{(2)}^k(\mathbf{H}; K)}^p\Big] < \infty$$

for $k = 0, 1, \ldots, n$. Moreover, $D\Psi_t$ is given by

$$D\Psi_t[\mathbf{h}] = \int_0^t D\Phi_s[\mathbf{h}] dW_s + \int_0^t \langle \Phi_s, \dot{\mathbf{h}}(s) \rangle_H ds + \int_0^t D\phi_s[\mathbf{h}] ds$$

for $\mathbf{h} \in \mathbf{H}$.

2 Stochastic differential equations

2.1 Existence and uniqueness theorem

Given two Borel functions $A : B \to H \otimes H$, $b : B \to H$. We consider the following stochastic differential equation for a B-valued process $\{X_t\}_{0 \le t \le T}$:

$$\begin{aligned} dX_t &= dW_t + A(X_t)dW_t + b(X_t)dt,\ 0 \le t \le T, \\ X_0 &= 0. \end{aligned} \tag{2.1}$$

Definition 2.1. We say that an $(\mathcal{F}_t)$-adapted B-valued stochastic process $\{X_t\}_{0 \le t \le T}$ is a solution of the SDE (2.1), if $A(X_t) \in \mathcal{L}_2^p(H \otimes H), b(X_t) \in \mathcal{L}_1^p(H)$ for each $t \in [0, T]$ and

$$X_t = W_t + \int_0^t A(X_s)dW_s + \int_0^t b(X_s)ds,\ 0 \le t \le T,\ \text{a.s.}$$

First we will show the existence and uniqueness of the solutions of (2.1).

Theorem 2.2. *Let $p \in [2, \infty)$. Assume that A and b are H-Lipschitz functions, that is, for some constant $K > 0$,*

$$|A(x+h) - A(x)|_{H \otimes H} + |b(x+h) - b(x)|_H \le K|h|_H$$

for every $x \in B$ and $h \in H$. Also assume that $A(W_.) \in \mathcal{L}_2^p(H \otimes H)$ and $b(W_.) \in \mathcal{L}_1^p(H)$. Then, there exists a unique H-valued process Y such that $E\Big[\sup_{0 \le t \le T} |Y_t|_H^p\Big] < \infty$ and $X := W + Y$ is a solution to the SDE (2.1).

Proof. As in finite dimensions, we apply the method of successive approximation. Set $Y_t^0 = 0$ and define Y_t^n inductively by

$$Y_t^n = \int_0^t A(W_s + Y_s^{n-1})dW_s + \int_0^t b(W_s + Y_s^{n-1})ds,$$

for $n = 1, 2, \cdots$. By Burkholder's inequality, we can see that Y^n ($n = 0, 1, \cdots$) are well-defined as elements of $L^p(\mathbf{W}; H)$ and that there exists a constant $C_1 > 0$ depending only on p, T and K such that

$$E\Big[\sup_{0 \le s \le t} |Y_s^{n+1} - Y_s^n|_H^p\Big]^{1/p} \le \Big\{C_1 \int_0^t E\big[|Y_s^n - Y_s^{n-1}|_H^p\big]ds\Big\}^{1/p}$$

for $n = 1, 2, \ldots$. Thus we have

$$E\Big[\sup_{0 \le s \le t} |Y_s^{n+1} - Y_s^n|_H^p\Big] \le C_2 \frac{(C_1 t)^n}{n!} \tag{2.2}$$

where $C_2 = E\Big[\sup_{0 \le s \le t} |Y_s^1 - Y_s^0|_H^p\Big] < \infty$. This concludes that $\{Y_\cdot^n\}$ is a Cauchy sequence in $L^p(\mathbf{W}; C([0, T]; H))$ and hence there exists an H-valued continuous process $Y_\cdot$ such that $\sup_{0\le t\le T}|Y_t - Y_t^n|_H \longrightarrow 0$ in L^p and it holds that

$$Y_t = \int_0^t A(W_s+Y_s)dW_s + \int_0^t b(W_s+Y_s)ds,\ 0\le t\le T \text{ a.s.}$$

Setting $X_t = W_t + Y_t$, we have

$$X_t = W_t + \int_0^t A(X_s)dW_s + \int_0^t b(X_s)ds,\ 0\le t\le T \text{ a.s.}$$

It remains to show the uniqueness. Let Y and Y' have the desired properties. Let

$$\tau_N = \inf\{t\ge 0; |Y_t|\vee|Y'_t|\ge N\}\ (\inf\emptyset=\infty),\ N\in\mathbf{N}.$$

Then using Burkholder's inequality again, we have

$$E[|Y_{t\wedge\tau_N} - Y'_{t\wedge\tau_N}|_H^p] \le \Big\{C_1\int_0^t E[|Y_{s\wedge\tau_N} - Y'_{s\wedge\tau_N}|_H^p]ds\Big\}.$$

By Gronwall's lemma we have $E[|Y_{t\wedge\tau_N} - Y'_{t\wedge\tau_N}|_H^p] = 0,\ t\in[0,T]$, and hence $P(Y_{t\wedge\tau_N} = Y'_{t\wedge\tau_N},\ 0\le t\le T) = 1$. Since $\lim_{N\to\infty}\tau_N = \infty$, we have $P(Y_t = Y'_t,\ 0\le t\le T) = 1$. $\square$

2.2 The smoothness of solutions

Definition 2.3. Given a separable real Hilbert space K, we say that a map $f: B\to K$ is *continuously H-Fréchet differentiable* if there exists a continuous map $f^{(1)}: B\to L_{(2)}(H;K)$ such that

$$\lim_{|h|_H\to 0}\frac{|f(x+h)-f(x)-f^{(1)}(x)h|_K}{|h|_H} = 0$$

for each $x\in B$. For $n=2,3,\dots$, we say that $f: B\to K$ is *n-times continuously H-Fréchet differentiable* if f is continuously H-Fréchet differentiable and $f^{(1)}: B\to L_{(2)}(H;K)$ is $(n-1)$-times continuously H-Fréchet differentiable. We write $f^{(n)} = (f^{(1)})^{(n-1)}$, $n=2,3,\dots$, and may regard $f^{(n)}$ as a map from B to $L^n_{(2)}(H;K)$. We denote by $\mathcal{CH}_b^\infty(K)$ the collection of infinitely many times continuously H-Fréchet differentiable function $f: B\to K$ such that $\sup_{x\in B}|f^{(n)}(x)|_{L^n_{(2)}(H;K)} < \infty$ for all $n\in\mathbf{Z}_+$.

Lemma 2.4. *Let $f\in\mathcal{CH}_b^\infty(K)$. Let $F\in W^\infty(H)$ and $G_t = W_t + F$. Then $f(G_t)\in W^\infty(K)$ and*

$$\begin{aligned}\|D^k f(G_t)\|_p \le \sum_{j=1}^k \frac{1}{j!}&\sum_{\substack{m_1+\cdots+m_j=k,\\ 1\le m_i\le k,\ i=1,\dots,j}}\frac{k!}{m_1!\cdots m_j!}\\ &\times\sup_{x\in B}|f^{(j)}(x)|_{L^j_{(2)}(H;K)}\prod_{i=1}^j(\sqrt{t}+\|D^{m_i}F\|_{jp})\end{aligned}\tag{2.3}$$

for each $t \in [0,T]$, $k \in \mathbf{Z}_+$ *and* $p \in (1,\infty)$.

Proof. It is easy to see that $f(\tilde{P}_N G_t) \in W^\infty(K)$ for all $N \in \mathbf{N}$. Given $k \in \mathbf{Z}_+$, $p \in (1,\infty)$, we have by the chain rule

$$D^k f(\tilde{P}_N G_t) = \sum_{j=1}^k \frac{1}{j!} \sum_{\substack{m_1+\cdots+m_j=k,\\ 1\le m_i\le k,\ i=1,\dots,j}} \frac{1}{m_1!\cdots m_j!} \times \sum_{\tau\in\mathcal{S}_k} \left[f^{(j)}(\tilde{P}_N G_t)\Big(D^{m_1}(\tilde{P}_N G_t),\dots,D^{m_j}(\tilde{P}_N G_t)\Big)_{H^j}\right]_\tau . \tag{2.4}$$

where

$$\begin{aligned} &f^{(j)}(\tilde{P}_N G_t)\Big(D^{m_1}(\tilde{P}_N G_t),\dots,D^{m_j}(\tilde{P}_N G_t)\Big)_{H^j}[\mathbf{h}_1,\dots,\mathbf{h}_k] \\ &\quad = f^{(j)}(\tilde{P}_N G_t)\Big(D^{m_1}(\tilde{P}_N G_t)[\mathbf{h}_1,\dots,\mathbf{h}_{m_1}],\dots \\ &\qquad\qquad ,D^{m_j}(\tilde{P}_N G_t)[\mathbf{h}_{m_1+\cdots+m_{j-1}+1},\dots,\mathbf{h}_k]\Big)_{H^j} \end{aligned}$$

for $\mathbf{h}_1,\dots,\mathbf{h}_k \in \mathbf{H}$, $\mathcal{S}_k$ denotes the symmetric group and $[\Psi]_\tau$ means that

$$[\Psi]_\tau(\mathbf{h}_1,\dots,\mathbf{h}_k) = \Psi(\mathbf{h}_{\tau(1)},\dots,\mathbf{h}_{\tau(k)})$$

for a map Ψ on $\mathbf{H}^k$ and $\tau \in \mathcal{S}_k$. But since $|D(\tilde{P}_N G_t)|_{L(\mathbf{H};H)} \le \sqrt{t} + |DF|_{L_{(2)}(\mathbf{H};H)}$ and $|D^m(\tilde{P}_N G_t)|_{L^m_{(2)}(\mathbf{H};H)} \le |D^m F|_{L^m_{(2)}(\mathbf{H};H)}$, $m \ge 2$ for all $N \in \mathbf{Z}_+$, we have

$$\sup_N \|D^k f(\tilde{P}_N G_t)\|_p \le \sum_{j=1}^k \frac{1}{j!} \sum_{\substack{m_1+\cdots+m_j=k,\\ 1\le m_i\le k,\ i=1,\dots,j}} \frac{k!}{m_1!\cdots m_j!} \times \sup_{x\in B} |f^{(j)}(x)|_{L^j_{(2)}(H;K)} \prod_{i=1}^j (\sqrt{t} + \|D^{m_i}F\|_{jp}) < \infty. \tag{2.5}$$

Since $W^{k,p}(K)$ is reflexive, (2.5) implies that there exist $\hat{f} \in W^{k,p}(K)$ and a subsequence N_j such that $f(\tilde{P}_{N_j} G_t)$ converges to $\hat{f}$ weakly in $W^{k,p}(K)$ as $j \to \infty$. But by the continuity of f, we have $\lim_{N\to\infty} \|f(G_t) - f(\tilde{P}_N G_t)\|_p = 0$ and hence $f(G_t) = \hat{f}$. (2.3) follows from (2.5). $\square$

Recall that in the proof of Theorem 2.2 we define $\{Y^n\}$ by

$$\begin{aligned} Y_t^0 &= 0, \\ Y_t^n &= \int_0^t A(W_s + Y_s^{n-1})dW_s + \int_0^t b(W_s + Y_s^{n-1})ds, \quad n = 1,2,\dots \end{aligned}$$

Lemma 2.5. *Let $A \in \mathcal{CH}_b^\infty(H \otimes H)$, $b \in \mathcal{CH}_b^\infty(H)$. Then $Y_t^n \in W^\infty(H)$ for all $n \in \mathbf{Z}_+$, $t \in [0,T]$ and*

$$E\Big[\sup_{0\le t\le T} |D^k Y_t^n|^p_{L^k_{(2)}(\mathbf{H};H)}\Big] < \infty$$

for all $k \in \mathbf{Z}_+, p \in (1,\infty)$.

Proof. We use induction in n. If $n = 0$, the claim is trivial. Assume that $Y_t^n \in W^\infty(H)$ for each $t \in [0,T]$ and

$$E\Big[\sup_{0\le t\le T} |D^k Y_t^n|^p_{L^k_{(2)}(\mathbf{H};H)}\Big] < \infty$$

for all $k \in \mathbf{Z}_+$ and $p \in (1,\infty)$. Then, by Lemma 2.4, we have $A(W_\cdot + Y_\cdot^n) \in \mathcal{L}_2^{k,p}(H \otimes H)$ and $b(W_\cdot + Y_\cdot^n) \in \mathcal{L}_1^{k,p}(dt;H)$. Hence, by Proposition 1.6 we have $Y_t^{n+1} \in W^\infty(H)$ for each $t \in [0,T]$ and

$$E\Big[\sup_{0\le t\le T} |D^k Y_t^{n+1}|^p_{L^k_{(2)}(\mathbf{H};H)}\Big] < \infty.$$

for all $k \in \mathbf{Z}_+, p \in (1,\infty)$. This completes the induction. □

Proposition 2.6. *Let $A \in \mathcal{CH}_b^\infty(H \otimes H)$, $b \in \mathcal{CH}_b^\infty(H)$. Then, for each $k \in \mathbf{Z}_+$ and $p \in [2,\infty)$, there exists a constant $M = M_{k,p} > 0$ such that*

$$\sup_n \Big\| \sup_{0\le s\le t} |D^k Y_s^n|_{L^k_{(2)}(\mathbf{H};H)} \Big\|_p \le e^{Mt}.$$

Proof. We use induction in k!%For $k = 0$, (2.2) implies our assertion. Assume that for each $l = 0, \dots, k$ and $p \in [2,\infty)$ there exists a constant $M_{l,p}$ such that

$$\sup_n \Big\| \sup_{0\le s\le t} |D^l Y_s^n|_{L^l_{(2)}(\mathbf{H};H)} \Big\|_p \le e^{M_{l,p}t}.$$

We shall show that for given $p \in [2,\infty)$, there exists a constant $M = M_{k+1,p}$ such that

$$\Big\| \sup_{0\le s\le t} |D^{k+1} Y_s^n|_{L^{k+1}_{(2)}(\mathbf{H};H)} \Big\|_p \le e^{Mt}$$

for every $n \in \mathbf{Z}_+$ by using induction in n. It is clear for $n = 0$. Assume that the claim holds for n and we show that for $n+1$. By Proposition 1.6 and Burkholder's inequality, we have

$$\begin{aligned}
&\Big\| \sup_{0\le s\le t} |D^{k+1} Y_s^{n+1}|_{L^{k+1}_{(2)}(\mathbf{H};H)} \Big\|_p \\
&\qquad \le c_p \Big\{ \int_0^t \|D^{k+1} A(X_s^n)\|_p^2 ds \Big\}^{1/2} + (k+1) \Big\{ \int_0^t \|D^k A(X_s^n)\|_p^2 ds \Big\}^{1/2} \\
&\qquad\qquad + \int_0^t \|D^{k+1} b(X_s^n)\|_p ds.
\end{aligned} \tag{2.6}$$

But Lemma 2.4 and the induction hypothesis imply that

$$
\begin{aligned}
&\|D^{k+1}A(X_s^n)\|_p \\
\leq\ & \sum_{j=1}^{k+1}\frac{1}{j!}\sum_{\substack{m_1+\cdots+m_j=k+1,\\ 1\leq m_i\leq k+1,\ i=1,\dots,j}}\frac{(k+1)!}{m_1!\cdots m_j!}\|A^{(j)}\|_\infty\prod_{i=1}^{j}(\sqrt{s}+\|D^{m_i}Y_s^n\|_{jp}) \\
\leq\ & \sum_{j=2}^{k+1}\frac{1}{j!}\sum_{\substack{m_1+\cdots+m_j=k+1,\\ 1\leq m_i\leq k+1,\ i=1,\dots,j}}\frac{(k+1)!}{m_1!\cdots m_j!}\|A^{(j)}\|_\infty\prod_{i=1}^{j}(e^{s/4}+e^{M_{m_i,(k+1)p}s}) \\
&\qquad +2\|A^{(1)}\|_\infty e^{Ms} \\
\leq\ & (C_k+2\|A^{(1)}\|_\infty)e^{Ms}
\end{aligned}
$$

where $\|A^{(j)}\|_\infty=\sup_{x\in B}|A^{(j)}(x)|_{L^j_{(2)}(H;H\otimes H)}$,

$$
C_k=\sum_{j=1}^{k+1}\frac{1}{j!}\sum_{\substack{m_1+\cdots+m_j=k+1,\\ 1\leq m_i\leq k+1,\ i=1,\dots,j}}\frac{(k+1)!}{m_1!\cdots m_j!}\|A^{(j)}\|_\infty 2^j
$$

and $M=M_{k+1,p}$ is taken such that $M>(k+1)\Big(\frac{1}{4}\vee\max_{1\leq l\leq k}M_{l,(k+1)p}\Big)$. Hence we have

$$
c_p\Big\{\int_0^t\|D^{k+1}A(X_s^n)\|_p^2ds\Big\}^{1/2}\leq\frac{1}{3}e^{Mt}. \tag{2.7}
$$

replacing M by larger one if necessary. By a similar way, we can show that

$$
(k+1)\Big\{\int_0^t\|D^kA(X_s^n)\|_p^2ds\Big\}^{1/2}\leq\frac{1}{3}e^{Mt} \tag{2.8}
$$

and that

$$
\int_0^t\|D^{k+1}b(X_s^n)\|_pds\leq\frac{1}{3}e^{Mt} \tag{2.9}
$$

replacing M by larger one if necessary.

Combining (2.6), (2.7), (2.8) and (2.9), we have

$$
\Big\|\sup_{0\leq s\leq t}|D^{k+1}Y_s^{n+1}|_{L^{k+1}_{(2)}(\mathbf{H};H)}\Big\|_p\leq e^{Mt}.
$$

This completes the proof. $\square$

Let X be a solution to SDE (2.1) and $Y_t=X_t-W_t$. Proposition 2.6 implies immediately the following:

Theorem 2.7. *Let $A \in \mathcal{CH}_b^\infty(H \otimes H)$, $b \in \mathcal{CH}_b^\infty(H)$. For every $k \in \mathbf{Z}_+$ and $p \in (1, \infty)$, $Y_\cdot \in \mathcal{L}_2^{k,p}(H)$ and there exists a constant $M = M_{k,p} > 0$ such that*

$$\Big\| \sup_{0 \le s \le t} |D^k Y_s|_{L^k_{(2)}(\mathbf{H};H)} \Big\|_p \le e^{Mt},$$

for any $t \in [0, T]$.

3 Modified Malliavin covariance operators

We denote by $GL(H)$ the collection of bounded linear operators on H with their inverse in $L(H;H)$.

Theorem 3.1. *Assume that $I_H + \sigma(w) \in GL(H)$ for P-a.e.w and*

$$E[|(I_H + \sigma)^{-1}|^p_{L(H;H)}] < \infty$$

for all $p \in (1, \infty)$. If $\sigma \in W^\infty(H \otimes H)$, then $\gamma := (I_H + \sigma)^{-1} - I_H \in W^\infty(H \otimes H)$.

To prove this theorem, we make some preparations. Let $\tilde{\phi}$ and $\tilde{\psi}$ be smooth functions $\mathbf{R} \to [0, 1]$ which satisfy

$$\tilde{\phi}(x) = \begin{cases} 1 & \text{if } |x| \le 1 \\ 0 & \text{if } |x| \ge 2 \end{cases}, \quad \tilde{\psi}(x) = \begin{cases} 1 & \text{if } |x| \ge 1 \\ 0 & \text{if } |x| \le \frac{1}{2} \end{cases}$$

Then, we define smooth functions $\phi_n : H \otimes H \to \mathbf{R}$, $\psi_m : H \otimes H \to \mathbf{R}$ by

$$\phi_n(A) = \tilde{\phi}\Big(\frac{1}{n}|A|^2_{H \otimes H}\Big), \ \psi_m(A) = \tilde{\psi}\Big(m\Delta_2(A)\Big),$$

where $\Delta_2(A) = \det_2(I_H + A)$ (see Dunford-Schwartz [2] Chapter XI.9 for the definition). Using these functions, we define $F_{n,m} : H \otimes H \to H \otimes H$ by

$$F_{n,m}(A) = \begin{cases} \phi_n(A)\psi_m(A)((I_H + A)^{-1} - I_H), & \text{if } I_H + A \in GL(H) \\ 0, & \text{otherwise} \end{cases}.$$

We first show that this $F_{n,m}$ is smooth and its Fréchet derivatives are bounded.

Lemma 3.2. *If $|\Delta_2(A)| > 0$, then $I_H + A \in GL(H)$. Moreover, there is a constant $C > 0$ such that*

$$|(I_H + A)^{-1}|_{L(H;H)} \le \exp(1 + C|A|^2_{H \otimes H})|\Delta_2(A)|^{-1}$$

for any $A \in H \otimes H$ with $|\Delta_2(A)| > 0$.

Proof. Let $A \in H \otimes H$ and $\{\lambda_i\}_{i\in\mathbf{N}}$ be its eigenvalues. We note that there exists a constant $0 < C' < \infty$ such that $|(1+z)e^{-z}| \le e^{C'|z|^2}$ for every $z \in \mathbf{C}$. Then, for any i,

$$\begin{aligned}
|\Delta_2(A)| &= \prod_{j=1}^{\infty} |(1+\lambda_j)e^{-\lambda_j}| \\
&= |(1+\lambda_i)e^{-\lambda_i} \prod_{j\neq i} |(1+\lambda_j)e^{-\lambda_j}| \\
&\le |1+\lambda_i|e^{|\lambda_i|} \prod_{j\neq i} e^{C'|\lambda_j|^2} \\
&\le |1+\lambda_i|e^{1+|\lambda_i|^2} e^{C'|A|^2} \\
&\le |1+\lambda_i|e^{1+(1+C')|A|^2}.
\end{aligned}$$

Hence

$$|(I_H + A)^{-1}|_{L(H;H)} \le \sup_i |1+\lambda_i|^{-1} \le e^{1+(1+C')|A|^2} |\Delta_2(A)|^{-1}.$$

□

Lemma 3.3. *For every $m \in \mathbf{N}$, ψ_m is Fréchet differentiable and*

$$|\psi_m'(A)|_{L(H\otimes H;\mathbf{R})} \le \sup_{x\in\mathbf{R}} |\tilde{\psi}'(x)||(I_H + A)^{-1}|_{L(H;H)}|A|_{H\otimes H},$$

if $I_H + A \in GL(H)$.

Proof. By virtue of Dunford-Schwartz [2], $\Delta_2 : H \otimes H \to \mathbf{R}$ is Fréchet differentiable and

$$\Delta_2'(A)[B] = \Delta_2(A)\mathrm{tr}\{-(I_H + A)^{-1}AB\}$$

for A, $B \in H \otimes H$. Hence

$$|\Delta_2'(A)|_{L(H\otimes H;\mathbf{R})} \le |\Delta_2(A)||(I_H + A)^{-1}|_{L(H;H)}|A|_{H\otimes H}.$$

Noting that $\psi_m'(A)$ vanishes if $\Delta_2(A) > 1/m$, we have

$$\begin{aligned}
|\psi_m'(A)|_{L(H\otimes H;\mathbf{R})} &\le m \sup_{x\in\mathbf{R}} |\tilde{\psi}'(x)||\Delta_2'(A)|_{L(H\otimes H;\mathbf{R})} \\
&\le \sup_{x\in\mathbf{R}} |\tilde{\psi}'(x)||(I_H + A)^{-1}|_{L(H;H)}|A|_{H\otimes H}.
\end{aligned}$$

□

Lemma 3.4. *For every m, $n \in \mathbf{N}$, $F_{m,n}$ is Fréchet differentiable and*

$$\sup_{A\in H\otimes H} |F_{m,n}'(A)|_{L(H\otimes H;H\otimes H)} < \infty.$$

Proof. Note that $F_{m,n}(A)$ vanishes if $|\Delta_2(A)| < 1/2m$ or $|A|^2_{H\otimes H} > 2n$. By Lemma 3.2, if $|\Delta_2(A)| \geq 1/2m$ and $|A|^2_{H\otimes H} \leq 2n$, then

$$|(I_H + A)^{-1}|_{L(H;H)} \leq e^{1+C|A|^2}|\Delta_2(A)|^{-1} \leq 2me^{1+2Cn}. \tag{3.1}$$

Moreover, it is easy to see that

$$|\phi_n'(A)|_{L(H\otimes H;\mathbf{R})} \leq \frac{2}{\sqrt{n}} \sup_{x\in\mathbf{R}} |\tilde{\phi}'(x)|. \tag{3.2}$$

Combining (3.1), (3.2) and Lemma 3.3, we have the conclusion. □

Now, let us prove Theorem 3.1.

Proof of Theorem 3.1. By Lemma 3.4, we can see $F_{m,n}(\sigma) \in W^{1,p}(H\otimes H)$ for all $p \in (1,\infty)$ and

$$\begin{aligned} DF_{m,n}(\sigma) = \psi_m'(\sigma)[D\sigma]\phi_n(\sigma)\gamma + \psi_m(\sigma)\phi_n'(\sigma)[D\sigma]\gamma \\ + \psi_m(\sigma)\phi_n(\sigma)\{(I_H+\gamma)D\sigma(I_H+\gamma)\}. \end{aligned} \tag{3.3}$$

By Lemma 3.3, we have

$$\begin{aligned} &E\Big[|\psi_m'(\sigma)[D\sigma]\phi_n(\sigma)\gamma|^p_{L_{(2)}(\mathbf{H};H\otimes H)}\Big] \\ &\leq E\Big[|\psi_m'(\sigma)|^p|D\sigma|^p_{L_{(2)}(\mathbf{H};H\otimes H)}|(I_H+\sigma)^{-1}|^p_{L(H;H)}|\sigma|^p_{H\otimes H}\Big] \\ &\leq \sup_{x\in\mathbf{R}}|\tilde{\psi}'(x)|^p E\Big[|D\sigma|^p_{L_{(2)}(\mathbf{H};H\otimes H)}|(I_H+\sigma)^{-1}|^{2p}_{L(H;H)}|\sigma|^{2p}_{H\otimes H}\,,\ |\Delta_2(\sigma)| \leq \frac{1}{m}\Big] \end{aligned} \tag{3.4}$$

Since $|\Delta_2(\sigma)| > 0$, a.s., the right-hand side in (3.4) converges to 0 as $m \to \infty$. Thus

$$DF_{m,n}(\sigma) \to \phi_n'(\sigma)[D\sigma]\gamma + \phi_n(\sigma)\{(I_H+\gamma)D\sigma(I_H+\gamma)\}$$

as $m \to \infty$ in $L^p(\mathbf{W}; L_{(2)}(\mathbf{H}; H\otimes H))$. Hence, noting that $F_{m,n}(\sigma) \to \phi_n(\sigma)\gamma$ as $m \to \infty$ in $L^p(\mathbf{W}; H\otimes H)$, we conclude that $\phi_n(\sigma)\gamma \in W^{1,p}(H\otimes H)$ and

$$D\{\phi_n(\sigma)\gamma\} = \phi_n'(\sigma)[D\sigma]\gamma + \phi_n(\sigma)\{(I_H+\gamma)D\sigma(I_H+\gamma)\}. \tag{3.5}$$

Moreover, (3.2) implies that the right-hand side in (3.5) converges to $(I_H + \gamma)D\sigma(I_H+\gamma)$ as $n \to \infty$ in $L^p(\mathbf{W}; L_{(2)}(\mathbf{H}; H\otimes H))$. Hence, noting that $\phi_n(\sigma)\gamma \to \gamma$ as $n \to \infty$ in $L^p(\mathbf{W}; H\otimes H)$, we have $\gamma \in W^{1,p}(H\otimes H)$ and

$$D\gamma = (I_H+\gamma)D\sigma(I_H+\gamma). \tag{3.6}$$

Therefore, if we assume $\gamma \in W^{k,p}(H\otimes H)$ for all $p \in (1,\infty)$, (3.6) implies that $D\gamma \in W^{k,p}(L_{(2)}(\mathbf{H}; H\otimes H))$ for all $p \in (1,\infty)$. We have our theorem by induction. □

Let X be a solution to SDE (2.1) with $A \in \mathcal{CH}_b^\infty(H \otimes H)$, $b \in \mathcal{CH}_b^\infty(H)$. Let $Y_t^h = \langle Y_t, h\rangle_H$ and $X_t^h = \langle W_t, h\rangle_H + Y_t^h$ for $h \in H$. Then we have

$$\begin{aligned}
&\Big(DX_t^h, DX_t^g\Big)_{\mathbf{H}} \\
&\qquad = \quad t\langle h, g\rangle_H + \Big(\mathbf{h}_t^h, DY_t^g\Big)_{\mathbf{H}} + \Big(\mathbf{h}_t^g, DY_t^h\Big)_{\mathbf{H}} + \Big(DY_t^h, DY_t^g\Big)_{\mathbf{H}},
\end{aligned}$$

where $\mathbf{h}_t^h = \int_0^{\cdot\wedge t} h\, ds$. It is easy to see that the maps $(h, g) \mapsto (\mathbf{h}_t^h, DY_t^g)_{\mathbf{H}}$ and $(h, g) \mapsto (DY_t^h, DY_t^g)_{\mathbf{H}}$ are in $L^2_{(2)}(H; \mathbf{R})$. Hence we can define an $H \otimes H$-valued process $\sigma(t)$ such that

$$\langle (tI_H + \sigma(t))h, g\rangle_H = \Big(DX_t^h, DX_t^g\Big)_{\mathbf{H}}$$

for $h, g \in H$.

Let J and $\tilde{J}_t$ be the solutions to the following SDE's:

$$J_t h = \int_0^t A'(X_s)\Big[(I_H + J_s)h\Big]_H dW_s + \int_0^t b'(X_s)\Big[(I_H + J_s)h\Big]_H ds, \quad h \in H, \tag{3.7}$$

$$\begin{aligned}
\tilde{J}_t h = &- \int_0^t (I_H + \tilde{J}_s)A'(X_s)[h]_H dW_s - \int_0^t (I_H + \tilde{J}_s)b'(X_s)[h]_H ds \\
&+ \int_0^t \sum_{i=1}^\infty (I_H + \tilde{J}_s)A'(X_s)\Big[A'(X_s)[h]_H e_i\Big]_H e_i ds, \quad h \in H
\end{aligned} \tag{3.8}$$

where $\{e_i\}_{i=1}^\infty$ is a CONS in H. By a similar way as in Section 2, we can see that (3.7) and (3.8) determine the unique solutions, respectively. Moreover, $J_t, \tilde{J}_t \in W^\infty(H \otimes H)$ for each $t \in [0, T]$ and for any $k \in \mathbf{Z}_+$ and $p \in (1, \infty)$, there exist constants $\tilde{L} = \tilde{L}_{k,p}, L = L_{k,p}$ such that

$$E\Big[\sup_{0\le t\le T} |D^k J_t|^p_{H\otimes H}\Big]^{1/p} \le e^{Lt},$$

$$E\Big[\sup_{0\le t\le T} |D^k \tilde{J}_t|^p_{H\otimes H}\Big]^{1/p} \le e^{\tilde{L}t}.$$

Proposition 3.5. *For each $t \in [0, T]$, $(I_H + J_t)(I_H + \tilde{J}_t) = (I_H + \tilde{J}_t)(I_H + J_t) = I_H$ and*

$$\begin{aligned}
&(I_H + \tilde{J}_t)(tI_H + \sigma(t))(I_H + \tilde{J}_t^*) \\
&\qquad = \int_0^t (I_H + \tilde{J}_s)(I_H + A(X_s))(I_H + A(X_s)^*)(I_H + \tilde{J}_s{}^*)ds.
\end{aligned} \tag{3.9}$$

Proof. Using Itô formula, we have $(I_H + J_t)(I_H + \tilde{J}_t) = (I_H + \tilde{J}_t)(I_H + J_t) = I_H$ and

$$\mathbf{h}(t) + DY_t[\mathbf{h}] = (I_H + J_t)\int_0^t (I_H + \tilde{J}_s)\{I_H + A(X_s)\}\dot{\mathbf{h}}(s)ds, \quad \mathbf{h} \in \mathbf{H},$$

which implies (3.9). □

The following theorem is immediately derived from Theorem 3.1 and Proposition 3.9.

Theorem 3.6. *Let $A \in \mathcal{CH}_b^\infty(H \otimes H)$ and $b \in \mathcal{CH}_b^\infty(H)$. If $tI_H + \sigma(t) \in GL(H)$, a.s. and $E[|(tI_H + \sigma(t))^{-1}|^p_{L(H;H)}] < \infty$ for all $p \in (1, \infty)$ and $t \in (0, T]$, then*

$$\gamma(t) = (tI_H + \sigma(t))^{-1} - t^{-1}I_H \in W^\infty(H \otimes H)$$

for all $t \in (0.T]$.

4 Integration by parts formula

Let K be a separable Hilbert space. We denote by $\mathcal{F}C_b^\infty(K)$ the collection of K-valued functions f on B expressed as

$$f(x) = \sum_{l=1}^m \tilde{f}_l(\,{}_B\langle x, e_1\rangle_{B^*}, \dots ,{}_B\langle x, e_n\rangle_{B^*})\,k_l,\ x \in B.$$

where $\{e_1, \dots , e_n\} \subset B^*$ is an ONS in H, $\{k_1, \dots , k_m\} \subset K$ is an ONS in K and $\tilde{f} \in C_b^\infty(\mathbf{R}^n)$. For $f \in \mathcal{F}C_b^\infty(K)$ expressed as above, define

$$\nabla f(x) = \sum_{i=1}^n \sum_{l=1}^m \partial_i \tilde{f}_l(\,{}_B\langle x, e_1\rangle_{B^*}, \dots ,{}_B\langle x, e_n\rangle_{B^*})\,e_i \otimes k_l,\ x \in B.$$

Theorem 4.1. *Let $A \in \mathcal{CH}_b^\infty(H \otimes H)$ and $b \in \mathcal{CH}_b^\infty(H)$. Assume that $tI_H + \sigma(t) \in GL(H)$, a.s. and $E[|(tI_H + \sigma(t))^{-1}|^p_{L(H;H)}] < \infty$ for all $p \in (1, \infty)$ and $t \in (0, T]$. Then, for every $G \in W^\infty(H \otimes K)$ and $t \in (0, T]$, there exists $\rho_t \in W^\infty(K)$ such that*

$$E[\langle \nabla u(X_t), G\rangle_{H\otimes K}] = E[\langle u(X_t), \rho_t\rangle_K]$$

for every $u \in \mathcal{F}C_b^\infty(K)$. In fact, ρ_t is given by

$$\rho_t = D^*\Big\{G(t^{-1}I_H + \gamma(t))(Z_t + DY_t)\Big\}$$

where Z_t is a bounded linear operator $\mathbf{H} \to H$ defined by $Z_t(\mathbf{h}) = \mathbf{h}(t)$, we identify $H \otimes K$ with $L_{(2)}(H; K)$ and regard $G(t^{-1}I_H + \gamma(t))(Z_t + DY_t) \in L_{(2)}(\mathbf{H}; K)$ as an element of $\mathbf{H} \otimes K$.

Moreover, for each $k \in \mathbf{Z}_+$ and $p \in (1,\infty)$, there exists a constant $C < \infty$ depending only k, p and T such that

$$\|\rho_t\|_{k,p} \le Ct^{-1}\|G\|_{k+1,3p}(1 + \|\gamma(t)\|_{k+1,3p})$$

for all $t \in (0,T]$.

Proof. Let $u \in \mathcal{F}C_b^\infty(K)$ be given by

$$u(x) = \sum_{l=1}^{m} \tilde{u}_l(\,_B\langle x, e_1\rangle_{B^*}\,, \ldots\,,\,_B\langle x, e_n\rangle_{B^*})k_l$$

where $\{e_i\}_{i=1}^n \subset B^*$ and $\{k_l\}_{l=1}^m$ are ONS's in H and K, respectively. Let $W_t^j = \langle W_t, e_j\rangle_H$ and $Y_t^j = \langle Y_t, e_j\rangle_H$. Then

$$\begin{aligned}&\Big(D\langle u(X_t), k_l\rangle_K, DW_t^j + DY_t^j\Big)_{\mathbf{H}}\\ &\qquad = \sum_{i=1}^{n} \partial_i\tilde{u}_l(X_t^1, \ldots, X_t^n)\langle(tI_H + \sigma(t))e_i, e_j\rangle_H\end{aligned}$$

and hence

$$\Big(D\langle u(X_t), k\rangle_K, \langle Z_t + DY_t, h\rangle_H\Big)_{\mathbf{H}} = \Big\langle (tI_H + \sigma(t))\nabla\langle u(X_t), k\rangle_K, h\Big\rangle_H$$

for all $h \in H$ and $k \in K$. This implies

$$\langle \nabla u(X_t), G\rangle_{H\otimes K} = \Big(D\{u(X_t)\}, G(t^{-1}I_H + \gamma(t))(Z_t + DY_t)\Big)_{\mathbf{H}\otimes K}.$$

Hence

$$\begin{aligned}&E\Big[\langle \nabla u(X_t), G\rangle_{H\otimes K}\Big]\\ &\qquad = E\Big[\Big(Du(X_t), G(t^{-1}I_H + \gamma(t))(Z_t + DY_t)\Big)_{\mathbf{H}\otimes K}\Big]\\ &\qquad = E\Big[\Big\langle u(X_t), D^*\Big\{G(t^{-1}I_H + \gamma(t))(Z_t + DY_t)\Big\}\Big\rangle_K\Big].\end{aligned}$$

The last inequality is derived from Theorem 2.7 and Theorem 3.6. □

5 The absolute continuity of a measure induced by SDE

Throughout this section, we assume that $A \in \mathcal{CH}_b^\infty(H\otimes H)$, $b \in \mathcal{CH}_b^\infty(H)$ and that $tI_H + \sigma(t) \in GL(H)$ a.s., $E[|(tI_H + \sigma(t))^{-1}|_{L(H;H)}^p] < \infty$ for all $p \in (1,\infty)$ and $t \in (0,T]$.

Let $P_tf(x) = E[f(x + W_t)]$ for $f \in \mathcal{F}C_b^\infty \equiv \mathcal{F}C_b^\infty(\mathbf{R}), x \in B$.

Lemma 5.1. *Let $1 < p' < p < \infty$ and $0 \le t < T$. Then there exist constants $c_1, c_2 > 0$ such that*

$$E[P_{T-t}(|f|^{p'})^{1/p'}(X_t)] \quad \le \quad \exp\Big(c_1 \frac{r}{T-t}\Big) P_T(|f|^p)^{1/p}(0) + c_2 \frac{\|f\|_\infty}{r^2}$$

for all $r > 0$ and $f \in \mathcal{F}C_b^\infty$. Here $c_1 = \dfrac{1}{2(p-p')}$ and $c_2 = E[\sup_{0\le t\le T} |Y_t|^4]$.

Proof. By Theorem 1.2, we have

$$\begin{aligned} P_t f(x+h) &= E[f(x+h+W_t)] \\ &= E\Big[f(x+W_t)\exp\Big\{\frac{\langle W_t, h\rangle_H - \frac{1}{2}|h|_H^2}{t}\Big\}\Big]. \end{aligned}$$

for all $h \in H$, $x \in B$ and $f \in \mathcal{F}C_b^\infty$. Let $p \in (1,\infty)$, $1/p + 1/q = 1$. Using Hölder's inequality, we have

$$\begin{aligned} |P_t f(x+h)| &\le e^{-\frac{1}{2t}|h|_H^2} E\Big[|f(x+W_t)|^p\Big]^{1/p} E\Big[\exp\Big\{\frac{\langle W_t, h\rangle_H}{t} q\Big\}\Big]^{1/q} \\ &= e^{-\frac{1}{2t}|h|_H^2} P_t(|f|^p)^{1/p}(x) E\Big[\exp\Big\{\frac{\langle W_1, h\rangle_H}{\sqrt{t}} q\Big\}\Big]^{1/q} \\ &= e^{\frac{1}{2t(p-1)}|h|_H^2} P_t(|f|^p)^{1/p}(x). \end{aligned}$$

Hence we have

$$\begin{aligned} & E\Big[P_{T-t}(|f|^{p'})^{1/p'}(X_t)\Big] \\ &\quad \le E\Big[P_{T-t}(|f|^{p'})^{1/p'}(W_t+Y_t); |Y_t|_H \le \sqrt{r}\Big] \\ &\qquad\qquad + \|f\|_\infty P\Big(|Y_t|_H \ge \sqrt{r}\Big) \\ &\quad \le E\Big[\sup_{|h|_H \le \sqrt{r}} P_{T-t}(|f|^{p'})^{1/p'}(W_t+h)\Big] \\ &\qquad\qquad + \frac{\|f\|_\infty}{r^2} E\Big[\sup_{0\le t\le T} |Y_t|_H^4\Big] \\ &\quad \le \exp\Big(\frac{r}{2(T-t)(p-p')}\Big) E\Big[P_{T-t}(|f|^p)^{1/p}(W_t)\Big] + c_2 \frac{\|f\|_\infty}{r^2} \\ &\quad = \exp\Big(c_1 \frac{r}{T-t}\Big) P_T(|f|^p)^{1/p}(0) + c_2 \frac{\|f\|_\infty}{r^2}. \end{aligned}$$

□

Lemma 5.2. *Let $p \in (1,\infty)$ and $a_{ij} \in \mathbf{R}$, $i,j = 1,\dots,n$, with $a_{ij} = a_{ji}$. Then*

$$\begin{aligned} &\Big\{\int_{\mathbf{R}^n} \Big(\sum_{i,j=1}^n a_{ij}(\eta_i\eta_j - \delta_{ij})\Big)^p \frac{1}{(2\pi)^{n/2}} e^{-|\eta|^2/2} d\eta\Big\}^{1/p} \\ &\qquad\qquad \le \sqrt{2}(p-1)\Big(\sum_{i,j=1}^n a_{ij}^2\Big)^{1/2} \end{aligned} \tag{5.1}$$

Proof. Let $\{e_1, e_2, \dots, e_n\} \subset B^*$ be an ONS in H and define $F_{ij}(x) = \langle x, e_i\rangle\langle x, e_j\rangle - \delta_{ij},\ x \in B$. Then F_{ij}'s are Fourier-Hermitian functionals. Hence using hyper-contractivity property of the Ornstein-Uhlenbeck semi-group, we have

$$\begin{aligned}
&\Big\{ \int_{\mathbf{R}^n} \Big(\sum_{i,j=1}^n a_{ij}(\eta_i\eta_j - \delta_{ij}) \Big)^p \frac{1}{(2\pi)^{n/2}} e^{-|\eta|^2/2} d\eta \Big\}^{1/p} \\
&\qquad = \Big\| \sum_{i,j=1}^n a_{ij} F_{ij} \Big\|_{L^p(B;\mathbf{R},d\mu)} \\
&\qquad \le (p-1) \Big\| \sum_{i,j=1}^n a_{ij} F_{ij} \Big\|_{L^2(B;\mathbf{R},d\mu)} \\
&\qquad = (p-1) \Big(2\sum_{i=1}^n a_{ii}^2 + 4 \sum_{i<j} a_{ij}^2 \Big)^{1/2} \\
&\qquad = \sqrt{2}(p-1) \Big(\sum_{i,j=1}^n a_{ij}^2 \Big)^{1/2}.
\end{aligned}$$

□

Theorem 5.3. *Let $\nu_T = P \circ X_T^{-1}$ be the distribution of X_T. For $p \in (1, \infty)$, there exists a constant $C > 0$ depending only p and T such that*

$$\Big| \int_B f(x) \nu_T(dx) \Big| \le C e^{C/\varepsilon^2} \Big\{ \int_B |f(x)|^p \mu_T(dx) \Big\}^{1/p} + \varepsilon \|f\|_\infty$$

for every bounded Borel function $f : B \to \mathbf{R}$ and $\varepsilon \in (0, 1)$.

Proof. It suffices to show for $f \in \mathcal{F}C_b^\infty$. In fact, for every $n \in \mathbf{N}$ and bounded Borel function f, we can find a compact set $K \subset B$ and $f_n \in \mathcal{F}C_b^\infty$ such that $\nu_T(B\backslash K) + \mu_T(B\backslash K) \le 1/n$, $\max_{x\in K} |f(x) - f_n(x)| \le 1/n$ and $\|f_n\|_\infty \le \|f\|_\infty + 1/n$. Then, we have

$$\int_B |f - f_n| d\nu_T \longrightarrow 0, \quad \int_B |f - f_n|^p d\mu_T \longrightarrow 0$$

as $n \to \infty$. Hence if the assertion for f_n holds, then so does for f.

Given

$$f(x) = \tilde{f}({}_B\langle x, e_1\rangle_{B^*}, \dots, {}_B\langle x, e_n\rangle_{B^*}),\ x \in B$$

for some $\tilde{f} \in C_b^\infty$ and an ONS $\{e_1, e_2, \dots e_n\}$ in H. Let $u(x, t) = P_t f(x)$. Then, since $\frac{\partial}{\partial t} u(t, x) = \frac{1}{2} \triangle u(t, x)$, Ito's formula implies

$$u(T - t, X_t) = \int_0^t \Big\langle \{I_H + A(X_s)^*\} \nabla u(T - s, X_t), dW_s \Big\rangle_H$$

$$+\int_0^t \Big\langle \nabla u(T-s,X_s), b(X_s)\Big\rangle_H ds$$
$$+\int_0^t \Big\langle \nabla^2 u(T-s,X_s), \tilde{A}(X_s)\Big\rangle_{H\otimes H} ds,$$

where $\tilde{A}(x) = \dfrac{1}{2}\{A(x)+A(x)^* + A(x)A(x)^*\}$. So we have

$$\begin{aligned} E[f(X_T)] &= E[u(0,X_T)] \\ &= u(T,0) \\ &\quad + E\Big[\int_0^T \Big\langle \nabla u(T-t,X_t), b(X_t)\Big\rangle_H dt\Big] \\ &\quad + E\Big[\int_0^T \Big\langle \nabla^2 u(T-t,X_t), \tilde{A}(X_t)\Big\rangle_{H\otimes H} dt\Big]. \end{aligned} \tag{5.2}$$

First, we show that there exists a constant $0 < c_3 < \infty$ depending only on p and T such that

$$\begin{aligned} &\Big| E\Big[\int_0^T \Big\langle \nabla^2 u(T-t,X_t), \tilde{A}(X_t)\Big\rangle_{H\otimes H} dt\Big]\Big| \\ &\qquad \le c_3 e^{c_3 r^2} P_T(|f|^p)^{1/p}(0) + \frac{c_3}{r}\|f\|_\infty \end{aligned} \tag{5.3}$$

for all $r \ge \frac{2}{T} \vee 1$. Let

$$\gamma_t^n(d\xi) = \frac{1}{(2\pi t)^{n/2}} e^{\frac{|\xi|^2}{2t}} d\xi,\ \ \xi \in \mathbf{R}^n,$$

and $X_t^{(n)} = ({}_B\langle X_t, e_1\rangle_{B^*}, \dots, {}_B\langle X_t, e_n\rangle_{B^*})$. Then we have

$$\begin{aligned} &\Big\langle \nabla^2 u(T-t,X_t), \tilde{A}(X_t)\Big\rangle_{H\otimes H} \\ &\quad = \int_{\mathbf{R}^n} \tilde{f}(X_t^{(n)}+\eta) \sum_{i,j=1}^n \langle \tilde{A}(X_t)e_i, e_j\rangle_H \frac{\eta_i\eta_j - \delta_{ij}(T-t)}{(T-t)^2} \gamma_{T-t}^n(d\eta). \end{aligned} \tag{5.4}$$

Let $p' = (1+p)/2$ and $\dfrac{1}{p'} + \dfrac{1}{q'} = 1$. By (5.4), Hölder's inequality and Lemma 5.2,

$$\begin{aligned} &\Big|\Big\langle \nabla^2 u(T-t,X_t), \tilde{A}(X_t)\Big\rangle_{H\otimes H}\Big| \\ &\quad \le \Big\{\int_{\mathbf{R}^n} \tilde{f}(X_t^{(n)}+\eta)^{p'} \gamma_{T-t}^n(d\eta)\Big\}^{1/p'} \\ &\qquad\quad \times \Big\{\int_{\mathbf{R}^n} \Big(\sum_{i,j=1}^n \langle \tilde{A}(X_t)e_i, e_j\rangle_H \frac{\eta_i\eta_j - \delta_{ij}(T-t)}{(T-t)^2}\Big)^{q'} \gamma_{T-t}^n(d\eta)\Big\}^{1/q'} \\ &\quad \le C_p P_{T-t}(|f|^{p'})^{1/p'}(X_t) \frac{1}{T-t}\|\tilde{A}\|_\infty \end{aligned}$$

where $C_p = \sqrt{2}(q'-1) > 0$ and $\|\tilde{A}\|_\infty = \sup_{x\in B} |\tilde{A}|_{H\otimes H}$. Hence, by Lemma 5.1, we have

$$
\begin{aligned}
&\int_0^{T-\frac{1}{r}} E\Big[\Big|\Big\langle \nabla^2 u(T-t,X_t), \tilde{A}(X_t)\Big\rangle_{H\otimes H}\Big|\Big] dt \\
&\quad\le C_p \int_0^{T-\frac{1}{r}} E\Big[P_{T-t}(|f|^{p'})^{1/p'}(X_t)\Big] \frac{1}{T-t}\|\tilde{A}\|_\infty dt \\
&\quad\le C_p \int_0^{T-\frac{1}{r}} \Big\{ e^{c_1 \frac{r}{T-t}} P_T(|f|^p)^{1/p}(0) + \frac{c_2}{r^2}\|f\|_\infty \Big\} \frac{1}{T-t}\|\tilde{A}\|_\infty dt \\
&\quad\le C_p\|\tilde{A}\|_\infty P_T(|f|^p)^{1/p}(0) \int_0^{T-\frac{1}{r}} \frac{1}{T-t} e^{c_1\frac{r}{T-t}} dt \\
&\qquad\qquad\qquad + C_p\|\tilde{A}\|_\infty \frac{c_2}{r^2}\int_0^{T-\frac{1}{r}} \frac{1}{T-t} dt \|f\|_\infty \qquad (5.5)\\
&\quad\le C_p\|\tilde{A}\|_\infty \frac{T}{c_1} e^{c_1 r^2} P_T(|f|^p)^{1/p}(0) \\
&\qquad\qquad\qquad + C_p\|\tilde{A}\|_\infty \frac{c_2}{r^2}(\log T + \log r)\|f\|_\infty \\
&\quad\le c_3 e^{c_1 r^2} P_T(|f|^p)^{1/p}(0) + \frac{\tilde{c}_3}{r}\|f\|_\infty
\end{aligned}
$$

On the other hand, by Theorem 4.1, there exist $\rho_t \in L^2(\mathbf{W};\mathbf{R},dP)$ such that

$$
\left|E\Big[\Big\langle \nabla^2 u(T-t,X_t), \tilde{A}(X_t)\Big\rangle_H\Big]\right| = \left|E\Big[\rho_t u(T-t,X_t)\Big]\right| \le \|\rho_t\|_{L^2(\mathbf{W};\mathbf{R},dP)}\|f\|_\infty.
$$

Hence we have

$$
\left|\int_{T-\frac{1}{r}}^T E\Big[\Big\langle \nabla^2 u(T-t,X_t), \tilde{A}(X_t)\Big\rangle_H\Big] dt\right| \le \frac{1}{r}\sup_{T/2\le t\le T}\|\rho_t\|_{L^2(\mathbf{W};\mathbf{R},dP)}\|f\|_\infty, \tag{5.6}
$$

if $r \ge 2/T$. Thus (5.5) and (5.6) imply (5.3).

Similarly, we can show that

$$
\begin{aligned}
&\left|E\Big[\int_0^T \Big\langle \nabla u(T-t,X_t), b(X_t)\Big\rangle_H dt\Big]\right| \\
&\qquad \le c_4 e^{c_4 r^2} P_T(|f|^p)^{1/p}(0) + \frac{c_4}{r}\|f\|_\infty
\end{aligned} \tag{5.7}
$$

for all $r \ge \frac{2}{T} \vee 1$ for some constants $c_4 > 0$ depending only on p and T.

Therefore, by (5.2), (5.3) and (5.7), we conclude that there exists a constant $\tilde{C} > 0$ depending only on p and T such that

$$
\Big|E[f(X_T)]\Big| \le \tilde{C} e^{\tilde{C} r^2} P_T(|f|^p)^{1/p}(0) + \frac{\tilde{C}}{r}\|f\|_\infty
$$

for all $r \geq \frac{2}{T} \vee 1$. For given $\varepsilon \in (0,1)$, letting $r = \dfrac{1}{\varepsilon}\Big(\dfrac{2}{T} \vee 1\Big)$, we have the theorem. $\square$

Theorem 1 . *Let $\nu_T = P \circ X_T^{-1}$ be the distribution of X_T. Then, ν_T is absolutely continuous with respect to μ_T. Moreover, its Radon-Nikodým density $\rho_T(x)$ with respect to μ_T satisfies*

$$\int_B \rho_T(x)(\log \rho_T(x) \vee 1)^\alpha \mu_T(dx) < \infty$$

for $0 \leq \alpha < 1/2$.

Proof. Let $A \in \mathcal{B}(B)$. By Theorem 5.3, for every $\varepsilon > 0$ there exists a constant $C = C_{p,\varepsilon} < \infty$ such that

$$\nu_T(A) \leq C\mu_T(A)^{1/p} + \varepsilon.$$

Hence $\mu_T(A) = 0$ implies $\nu_T(A) = 0$.

Moreover, applying Theorem 5.3 to $\dfrac{\rho_T \wedge \xi}{\xi}$, we have

$$\begin{aligned}
\nu_T(\rho_T \geq \xi) &\leq \int_B \Big\{\frac{\rho_T(x) \wedge \xi}{\xi}\Big\}^{1/2} \nu_T(dx) \\
&\leq Ce^{C/\varepsilon^2}\Big\{\int_B \frac{\rho_T(x) \wedge \xi}{\xi} \mu_T(dx)\Big\}^{1/2} + \varepsilon \\
&\leq Ce^{C/\varepsilon^2}\xi^{-1/2} + \varepsilon
\end{aligned}$$

for all $\xi > 0$ and $\varepsilon > 0$. Hence, letting $\varepsilon = 4\sqrt{\dfrac{C}{\log \xi}}$, we have

$$\nu_T(\rho_T \geq \xi) \leq 4\sqrt{\frac{C}{\log \xi}}, \quad \forall \xi > 1. \tag{5.8}$$

Let $F_T(\xi) = \nu_T(\rho_T \leq \xi)$. Then, (5.8) implies

$$1 - F_T(\xi) \leq \frac{C'}{\sqrt{\log \xi}}, \quad \forall \xi > 1.$$

for some constant C'. Hence we have

$$\begin{aligned}
&\int_B \rho_T(x)(\log \rho_T(x) \vee 1)^\alpha \mu_T(dx) \\
&\qquad \leq 1 + \int_{\rho_T \geq e} (\log \rho_T)^\alpha d\nu_T \\
&\qquad = 1 + \int_e^\infty (\log \xi)^\alpha dF_T(\xi)
\end{aligned}$$

$$
\begin{aligned}
&= 1 - \Big[(\log\xi)^\alpha(1 - F_T(\xi))\Big]_e^\infty + \int_e^\infty \frac{\alpha(\log\xi)^{\alpha-1}}{\xi}(1-F_T(\xi))d\xi \\
&\leq 2 - F(e) + \alpha C' \int_e^\infty \frac{1}{\xi(\log\xi)^{3/2-\alpha}} d\xi < \infty.
\end{aligned}
$$

□

6 The uniformly elliptic case

The following is well known (cf. Kusuoka-Stroock [7]).

Lemma 6.1. *Let $\Phi \in \mathcal{L}_2^p(H \otimes K)$ and denote $I_t = \int_0^t \Phi_s dW_s$. If $C \equiv \sup_{0\leq t\leq T} \sup_{w\in\mathbf{W}} |\Phi_t(w)|_{H\otimes K} < \infty$ then*

$$
E\Big[\exp\Big\{\frac{\alpha}{2C^2 t}\sup_{0\leq s\leq t}|I_s|_K^2\Big\}\Big] \leq \frac{e}{(1-\alpha)^{1/2}}
$$

for every $\alpha \in (0,1)$ and $t \in (0,\infty)$.

Define a stopping time by

$$
\zeta(r) = \inf\{t > 0;\ |X_t|_B \geq r \text{ or } |J_t|_{H\otimes H} \geq r\}.
$$

Lemma 6.2. *There exist constants $C, M < \infty$ such that*

$$
P(\zeta(r) \leq t) \leq C e^{-Mr^2/t}
$$

for every $t > 0$ and $0 < r < 1$.

Proof. We may assume that $|h|_B \leq |h|_H$, $h \in H$. Let $\zeta_1(r) \inf\{t > 0; |X_t|_B \geq r\}$ and $\zeta_2(r) = \inf\{t > 0; |J_t|_{H\otimes H} \geq r\}$.

Set $I_t = \int_0^t A(X_s)dW_s$. By Theorem 1.1 and Lemma 6.1, we have

$$
\begin{aligned}
P(\zeta_1(r) \leq t) &\leq P\Big(\sup_{0\leq s\leq t}|W_s|_B \geq r/3\Big) \\
&\quad + P\Big(\sup_{0\leq s\leq t}|I_s|_H + \|b\|_\infty t \geq 2r/3\Big) \\
&\leq P\Big(\sqrt{t/T}\sup_{0\leq s\leq T}|W_s|_B \geq r/3\Big) \\
&\quad + P\Big(\sup_{0\leq s\leq t}|I_s|_H \geq r/3\Big) \\
&\leq \exp\Big\{-\beta\Big(\frac{r}{3}\Big)^2\Big(\frac{T}{t}\Big)\Big\} E\Big[\exp\Big\{\beta\sup_{0\leq s\leq T}|W_s|_B^2\Big\}\Big] \\
&\quad + \exp\Big\{-\frac{\alpha}{2\|A\|_\infty^2 t}\Big(\frac{r}{3}\Big)^2\Big\}\frac{e}{(1-\alpha)^{1/2}},
\end{aligned}
$$

if $0 < t \leq 1/3\|b\|_\infty$. Letting $M = (\beta T/9) \wedge (\alpha/18\|A\|^2_{H\otimes H})$ and

$$C = E\left[\exp\left\{\beta \sup_{0\leq s\leq T} |W_s|^2_B\right\}\right] \vee \frac{e}{(1-\alpha)^{1/2}} \vee 3\|b\|_\infty M,$$

we have $P(\zeta_1(r) \leq t) \leq Ce^{-Mr^2/t}$. By a similar way we can prove that $P(\zeta_2(r) \leq t) \leq Ce^{-Mr^2/t}$. □

Theorem 6.3. *Let $A \in \mathcal{CH}^\infty_b(H\otimes H)$ and $b \in \mathcal{CH}^\infty_b(H)$. Suppose there exists a constant $0 < \varepsilon_0 < 1$ such that*

$$\left|(I_H + A(0))h\right|_H \geq \varepsilon_0 |h|_H$$

for all $h \in H$. Then, for each $t \in (0,T]$, $I_H + \sigma(t) \in GL(H)$ a.e. and

$$E[|(tI_H + \sigma(t))^{-1}|^p_{L(H;H)}] < \infty$$

for all $p \in (1,\infty)$.

Proof. We choose $r > 0$ such that $|X_t|_B \leq r$ and $|J_t|_{H\otimes H} \leq r$ implies

$$\left|\{I_H + A(X_t)^*\}(I_H + \tilde{J_t}^*)h\right|_H \geq \varepsilon_0|h|_H$$

for all $h \in H$. Then we have

$$\begin{aligned} &\int_0^t \left|\{I_H + A(X_s)^*\}(I_H + \tilde{J_s}^*)h\right|^2_H ds \\ &\quad\geq \int_0^{t\wedge\zeta_r} \left|\{I_H + A(X_s)^*\}(I_H + \tilde{J_s}^*)h\right|^2_H ds \\ &\quad\geq (t\wedge\zeta(r))\varepsilon_0^2|h|^2_H \end{aligned}$$

for all $h \in H$. Hence Proposition 3.9 and Lemma 6.2 imply our assertion. □

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