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An inverse spectral problem for a nonsymmetric differential operator: Uniqueness and reconstruction formula

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Abstract

We consider an eigenvalue problem for a system in $[0,1]$:

$$\begin{cases} \left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \dfrac{\mathrm{d}}{\mathrm{d}x} + \begin{pmatrix} p_{11}(x) & p_{12}(x) \\ p_{21}(x) & p_{22}(x) \end{pmatrix}\right] \begin{pmatrix} \varphi^{(1)}(x) \\ \varphi^{(2)}(x) \end{pmatrix} = \lambda \begin{pmatrix} \varphi^{(1)}(x) \\ \varphi^{(2)}(x) \end{pmatrix} \\ \varphi^{(2)}(0)\cosh\mu - \varphi^{(1)}(0)\sinh\mu = \varphi^{(2)}(1)\cosh\nu + \varphi^{(1)}(1)\sinh\nu = 0 \end{cases}$$

with constants $\mu, \nu \in \mathbb{C}$.
Under the assumption that p_{21}, p_{22} are known, we prove a uniqueness theorem and provide a reconstruction formula for p_{11} and p_{12} from the spectral characteristics consisting of one spectrum and the associated norming constants.

1 Introduction

In this paper, we consider an eigenvalue problem for a system:

$$\begin{cases} B\dfrac{\mathrm{d}\varphi}{\mathrm{d}x}(x) + P(x)\varphi(x) = \lambda\varphi(x), \qquad 0 < x < 1, \\ \varphi^{(2)}(0)\cosh\mu - \varphi^{(1)}(0)\sinh\mu = \varphi^{(2)}(1)\cosh\nu + \varphi^{(1)}(1)\sinh\nu = 0, \end{cases} \tag{1.1}$$

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where $B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\varphi(x) = \begin{pmatrix} \varphi^{(1)}(x) \\ \varphi^{(2)}(x) \end{pmatrix}$, $P = \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix} \in (C^1[0,1])^4$ is complex-valued, and the constants $\mu, \nu \in \mathbb{C}$.

The eigenvalue problem (1.1) can describe proper vibrations for various phenomena such as an electric oscillation in a transmission line (cf. Trooshin and Yamamoto [19], Cox and Knobel [1]), a vibration of a string with viscous drag (cf. Yamamoto [21]), etc. On the other hand, this eigenvalue problem can also generalize the Sturm-Liouville problem (cf. Yamamoto [20]). Besides, the time-independent Dirac equation with the external field (cf. Thaller[16]) for one spatial variable is actually described by our system, which will be shown as follows.

In the one dimensional Dirac equation with a 2×2 matrix-valued potential $V(x)$

$$\left(i\hbar \frac{\partial}{\partial t} - H \right) \psi = 0$$

where $H = -i\hbar c \sigma_1 \dfrac{\partial}{\partial x} + mc^2 \sigma_3 + V(x)$ and σ_1, σ_3 are Pauli matrices: $\sigma_1 = B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $\hbar$ is Planck's constant, c the velocity of light, m the mass of the particle, if we put $\psi = \exp(-iWt/\hbar)\phi$ where the constant W is the energy, then we have $(W - H)\phi = 0$, i.e.,

$$B \frac{\mathrm{d}\phi}{\mathrm{d}x} + \frac{i}{\hbar c} \left(mc^2 \sigma_3 + V(x) \right) \phi = \frac{iW}{\hbar c} \phi. \tag{1.2}$$

Hence if we consider (1.2) with suitable boundary conditions, then it is given by our system. Especially, if $V(x) = \mathcal{V}(x)E$ (cf. Kostin[7]), where $\mathcal{V}(x)$ is a scalar function, E the 2×2 unit matrix, then our main result (Theorem 2) implies that we can determine not only $\mathcal{V}(x)$ but also the mass m of the Dirac particle from the spectral characteristics (see the definition below).

For (1.1), we study an inverse spectral problem, in other words, determination of two components of $P(x)$ from the spectral characteristics when the other two ones of $P(x)$ are given a priori. Without loss of generality, we can assume that the second row p_{21}, p_{22} of $P(x)$ are given. In this paper, we shall prove a theorem of uniqueness and provide a formula of reconstruction through a modified Gel'fand-Levitan equation (cf. Gel'fand-Levitan [3] and Levitan & Sargsjan [12]). Without the assumption that two components of $P(x)$ are known a priori we prove also that the spectral characteristics determine all the four components of $P(x)$ uniquely under a condition on a certain eigenvector.

In 1909, for a differential equation, H. Weyl introduced a so-called spectral function related to the Parseval equality which generalizes the Plancherel theorem in the Fourier transform. For the derivation of the Parseval equality related with Sturm-Liouville problems we refer to Titchmarsh [17], Levinson [10, 11] and Yosida [23]. In Gel'fand and Levitan [3], they reconstructed a differential equation with a given spectral function. Since then, many authors such as V. A. Marchenko (cf. [8]), B. Simon (cf. [15], [4], [5], [14]) and M. G. Gasymov (cf. [2]) etc. have done much work in inverse spectral problem mainly concerning symmetric systems such as the Sturm-Liouville equation and the one dimensional Dirac system. As for the recent work for the Dirac system, we refer to Watson[18] and Lesch and Malamud [9]. As for more details about inverse spectral theory, we refer for example to Pöschel and Trubowitz [13], Yurko [25]. However, to our knowledge, most of researchers consider only the inverse spectral problems for self-adjoint operators. For the non-selfadjoint case, inverse problems of determining the matrix coefficient $P(x)$ are recently discussed by means of (i) two spectra (e.g., Yamamoto [20], Cox and Knobel [1]) (ii) the Weyl function (e.g., Yurko [24]). For the spectral characteristics which are related with the spectral function, M. Yamamoto proved the reconstruction and the uniqueness under the assumption that the eigenvectors of $A_{P,\mu,\nu}$ forms a Riesz basis in $(L^2(0,1))^2$ (cf. Yamamoto [22]). However, in general, only the eigenvectors are not enough for a Riesz basis (cf. Cox and Knobel [1], Trooshin and Yamamoto [19]), and so the results in [22] are not true for general $P(x)$ in (1.1).

The rest part of this paper is composed of four sections and one appendix. In Section 2, we show Theorem 1 and 2 as the main results. Section 3 and 4 are devoted to the proof of Theorem 1 and that of Theorem 2 respectively. In section 5, we give another reconstruction procedure and concluding remarks.

2 Auxiliary Propositions and Main Results

For the statement of the main results we need several propositions.
Let us introduce a nonsymmetric first-order differential operator in $(L^2(0,1))^2$:

$$(\mathcal{A}_P\varphi)(x) = B\frac{\mathrm{d}\varphi}{\mathrm{d}x}(x) + P(x)\varphi(x), \qquad 0 < x < 1, \tag{2.1}$$

where B and $P(x)$ are given in Section 1. We define an operator $A_{P,\mu,\nu}$ in $(L^2(0,1))^2$ by

$$(A_{P,\mu,\nu}\varphi)(x) = (\mathcal{A}_P\varphi)(x), \qquad \varphi \in D(A_{P,\mu,\nu}),\ 0 < x < 1, \tag{2.2}$$

where

$$\begin{aligned} D(A_{P,\mu,\nu}) = \Big\{\varphi \in \left(H^1(0,1)\right)^2 : \varphi^{(2)}(0)\cosh\mu - \varphi^{(1)}(0)\sinh\mu = 0, \\ \varphi^{(2)}(1)\cosh\nu + \varphi^{(1)}(1)\sinh\nu = 0\Big\}. \end{aligned} \tag{2.3}$$

Throughout this paper, $L^2(0,1)$ and $H^1(0,1)$ are the Lebesgue space and the Sobolev space of complex-valued functions respectively, and $(L^2(0,1))^2$,$(H^1(0,1))^2$

denote the product spaces. By $(\cdot,\cdot)$ we denote the scalar product in $(L^2(0,1))^2$:

$$(f,g)=\int_0^1 f^T(x)\overline{g(x)}\mathrm{d}x=\int_0^1\left(f^{(1)}(x)\overline{g^{(1)}(x)}+f^{(2)}(x)\overline{g^{(2)}(x)}\right)\mathrm{d}x \tag{2.4}$$

for $f=\begin{pmatrix} f^{(1)} \\ f^{(2)} \end{pmatrix}\in (L^2(0,1))^2$, $g=\begin{pmatrix} g^{(1)} \\ g^{(2)} \end{pmatrix}\in (L^2(0,1))^2$. Here and henceforth $\overline{c}$ denotes the complex conjugate of $c\in\mathbb{C}$ and $\cdot^T$ denotes the transpose of a vector or matrix under consideration. The quantity with the symbol $*$ denotes the adjoint one, and the variable x is in the interval $[0,1]$. Let

$$\xi=\begin{pmatrix} \cosh\mu \\ \sinh\mu \end{pmatrix},\quad \eta=\begin{pmatrix} \cosh\overline{\mu} \\ -\sinh\overline{\mu} \end{pmatrix}.$$

It is not hard to see that the adjoint operator $A^*_{P,\mu,\nu}$ of $A_{P,\mu,\nu}$ in $(L^2(0,1))^2$ is given by

$$\begin{cases} (A^*_{P,\mu,\nu}\varphi^*)(x)=-B\dfrac{\mathrm{d}\varphi^*}{\mathrm{d}x}(x)+\overline{P^T(x)}\varphi^*(x), \quad \varphi^*\in D(A^*_{P,\mu,\nu}),\ 0<x<1, \\ D(A^*_{P,\mu,\nu})=\Big\{\varphi^*\in (H^1(0,1))^2: \varphi^{*(2)}(0)\cosh\overline{\mu}+\varphi^{*(1)}(0)\sinh\overline{\mu}=0, \\ \qquad\qquad \varphi^{*(2)}(1)\cosh\overline{\nu}-\varphi^{*(1)}(1)\sinh\overline{\nu}=0\Big\} \end{cases} \tag{2.5}$$

and $\mathcal{A}^*_P=-\mathcal{A}_{-\overline{P^T}}$.

We call $w\neq 0$ a *root vector* of an operator A for λ if $(A-\lambda)^m w=0$ for some $m\in\mathbb{N}$. Moreover we call $\{w_n\}_{n\in\mathbb{Z}}$ a *Riesz basis* in $(L^2(0,1))^2$ if each $f\in(L^2(0,1))^2$ has a unique expansion

$$f=\sum_{n=-\infty}^{\infty} c_n w_n$$

with $c_n\in\mathbb{C}, n\in\mathbb{Z}$ and

$$J^{-1}\sum_{n=-\infty}^{\infty}|c_n|^2\le \| f\|^2_{(L^2(0,1))^2}\le J\sum_{n=-\infty}^{\infty}|c_n|^2,$$

where a constant $J>0$ is independent of f. We note that if $\{w_n\}_{n\in\mathbb{Z}}$ is a Riesz basis in $(L^2(0,1))^2$ and if, in the Hilbert space $(L^2(0,1))^2$, an element f_0 is orthogonal to each w_n for $n\in\mathbb{Z}$, then $f_0=0$.

For the spectrum $\sigma(A_{P,\mu,\nu})$ we have

Proposition 2.1.
(i) There exists $N_1\in\mathbb{N}$ and $\Sigma_1,\Sigma_2\subset\sigma(A_{P,\mu,\nu})$ such that $\sigma(A_{P,\mu,\nu})=\Sigma_1\bigcup\Sigma_2$, $\Sigma_1\bigcap\Sigma_2=\emptyset$ and the following properties hold:

(1) Σ_1 *consists of* $2N_1 - 1$ *eigenvalues including algebraic multiplicities in*

$$\left\{\lambda \in \mathbb{C} : \left|\mathrm{Im}\left(\lambda - \frac{1}{2}\int_0^1 (p_{11}+p_{22})(s)\mathrm{d}s + \mu + \nu\right)\right| \le (N_1 - \frac{1}{2})\pi\right\}.$$

(2) Σ_2 *consists of eigenvalues with algebraic multiplicity 1 in a neighborhood of*

$$\frac{1}{2}\int_0^1 (p_{11}+p_{22})(s)\mathrm{d}s - \mu - \nu + n\pi\sqrt{-1}$$

for every $|n| \ge N_1$.

Moreover with a suitable numbering $\{\lambda_n\}_{n\in\mathbb{Z}}$ *of* $\sigma(A_{P,\mu,\nu})$*, the eigenvalues have an asymptotic behavior*

$$\lambda_n = \frac{1}{2}\int_0^1 (p_{11}+p_{22})(s)\mathrm{d}s - \mu - \nu + n\pi\sqrt{-1} + O\left(\frac{1}{|n|}\right) \tag{2.6}$$

as $|n| \to \infty$.

(*ii*) *The set of all the root vectors of* $A_{P,\mu,\nu}$ *is a Riesz basis in* $(L^2(0,1))^2$.

For the proof, see Theorem 1.1 in [19].

Remark 2.1. We can prove that the geometric multiplicity of any eigenvalue is 1.

Here and Henceforth we say that an eigenvalue λ is *simple* if both the algebraic and geometric multiplicity of λ are 1. Henceforth, for the convenience of notations, we reset the spectrum $\sigma(A_{P,\mu,\nu}) = \Sigma_1 \bigcup \Sigma_2$ by a suitable renumbering as follows:

$$\begin{aligned} \Sigma_1 &= \left\{\lambda^i \in \sigma(A_{P,\mu,\nu}) : m_i \ge 2, 1 \le i \le N\right\}, \\ \Sigma_2 &= \{\lambda_n \in \sigma(A_{P,\mu,\nu}) : \lambda_n \text{ is simple}, n \in \mathbb{Z}\}, \end{aligned} \tag{2.7}$$

where m_i denotes the algebraic multiplicity of λ^i.

Remark 2.2. If $\sigma(A_{P,\mu,\nu})$ only consists of simple eigenvalues, then Σ_1 does not appear and the problem becomes much easier.

We note that $\sigma(A_{P,\mu,\nu}) = \overline{\sigma(A^*_{P,\mu,\nu})}$ (cf. p.184 Remark 6.23 of Kato [6]). It means that if $\lambda \in \sigma(A_{P,\mu,\nu})$, then $\overline{\lambda} \in \sigma(A^*_{P,\mu,\nu})$ with the same algebraic and geometric multiplicity. Here and henceforth let $\varphi_n = \varphi_n(x)$ be the eigenvector of $A_{P,\mu,\nu}$ for λ_n such that $\varphi_n(0) = \xi$ and $\varphi^*_n = \varphi^*_n(x)$ be the eigenvector of $A^*_{P,\mu,\nu}$ for $\overline{\lambda_n}$ such that $\varphi^*_n(0) = \eta$ $(n \in \mathbb{Z})$. It is easy to see that

$$(\varphi_n, \varphi^*_m) = 0 \ \text{ if } n \ne m, \ n, m \in \mathbb{Z}. \tag{2.8}$$

Proposition 2.2.

There exist root vectors $\left\{\varphi^i_j\right\}_{1\le j\le m_i}$ *of* $A_{P,\mu,\nu}$ *for* λ^i *and* $\left\{\varphi^{i*}_j\right\}_{1\le j\le m_i}$ *of* $A^*_{P,\mu,\nu}$

for $\overline{\lambda^i}$ *($1 \le i \le N$) satisfying*
(i)

$$\begin{cases} (\mathcal{A}_P - \lambda^i)\varphi_1^i = 0,\ (\mathcal{A}_P - \lambda^i)\varphi_j^i = \varphi_{j-1}^i, \quad 2 \le j \le m_i, 1 \le i \le N, \\ \varphi_j^i(0) = \xi,\ \varphi_j^i \in D(A_{P,\mu,\nu}), \quad 1 \le j \le m_i, 1 \le i \le N \end{cases} \tag{2.9}$$

and

$$\begin{cases} (\mathcal{A}_P^* - \overline{\lambda^i})\varphi_{m_i}^{i*} = 0, (\mathcal{A}_P^* - \overline{\lambda^i})\varphi_j^{i*} = \varphi_{j+1}^{i*}, 1 \le j \le m_i - 1, 1 \le i \le N, \\ \varphi_{m_i}^{i*}(0) = \eta,\ \varphi_j^{i*}(0) = \overline{\alpha_j^i}\eta, \quad 1 \le j \le m_i - 1, 1 \le i \le N, \\ \varphi_j^{i*} \in D(A_{P,\mu,\nu}^*),\ 1 \le j \le m_i, 1 \le i \le N, \end{cases} \tag{2.10}$$

where the constants α_j^i *($1 \le j \le m_i - 1, 1 \le i \le N$) are defined through* $(2), (5), (10), (12), (13)$ *and* (16) *in the appendix.*
(ii)

$$\left(\varphi_j^i, \varphi_n^*\right) = 0, \quad \left(\varphi_n, \varphi_j^{i*}\right) = 0, \quad \textit{for } 1 \le j \le m_i,\ 1 \le i \le N,\ n \in \mathbb{Z}.$$

(iii)

$$\left(\varphi_j^i, \varphi_l^{k*}\right) = 0 \ \text{ if } i \ne k \text{ or } j \ne l,\ 1 \le j \le m_i,\ 1 \le l \le m_k, \quad 1 \le i, k \le N,$$

and

$$\left(\varphi_j^i, \varphi_j^{i*}\right) = \left(\varphi_{m_i}^i, \varphi_{m_i}^{i*}\right), \quad \textit{for } 1 \le j \le m_i,\ 1 \le i \le N. \tag{2.11}$$

In the appendix we will prove this proposition. The constants α_j^i are introduced for the sake of the orthogonality of the root vectors. We call $\left\{\varphi_j^{i*}\right\}_{1 \le j \le m_i}$ the *normalized root vectors* of $A_{P,\mu,\nu}^*$ for $\overline{\lambda^i}$ with respect to $\left\{\varphi_j^i\right\}_{1 \le j \le m_i}$ ($1 \le i \le N$). Noting Proposition 2.1 (ii), we see that

$$\text{both } \left\{\varphi_j^i\right\}_{1 \le j \le m_i, 1 \le i \le N} \bigcup \left\{\varphi_n\right\}_{n \in \mathbb{Z}} \text{ and } \left\{\varphi_j^{*i}\right\}_{1 \le j \le m_i, 1 \le i \le N} \bigcup \left\{\varphi_n^*\right\}_{n \in \mathbb{Z}} \text{ are Riesz bases in } (L^2(0,1))^2. \tag{2.12}$$

We set $\rho^i = \left(\varphi_{m_i}^i, \varphi_{m_i}^{i*}\right)$, $\boldsymbol{\alpha}^i = (\alpha_1^i, \cdots, \alpha_{m_i-1}^i)$, $1 \le i \le N$, and $\rho_n = (\varphi_n, \varphi_n^*)$, $n \in \mathbb{Z}$. Obviously, noting (2.11), we have

$$\left(\varphi_j^i, \varphi_j^{i*}\right) = \rho^i, \quad \forall 1 \le j \le m_i. \tag{2.13}$$

By (2.8), Proposition 2.1 (ii) and Proposition 2.2, it is not hard to see

$$\rho^i \ne 0,\ 1 \le i \le N; \quad \rho_n \ne 0,\ n \in \mathbb{Z}. \tag{2.14}$$

Definition. We call $S(P,\mu,\nu) := \{\lambda^i, m_i, \rho^i, \boldsymbol{\alpha}^i\}_{1\le i\le N} \bigcup \{\lambda_n, \rho_n\}_{n\in\mathbb{Z}}$ the *spectral characteristics* of $A_{P,\mu,\nu}$.

Proposition 2.3. *Let $f, g \in (L^2(0,1))^2$.*
(i) (the Parseval equality with respect to $A_{P,\mu,\nu}$)

$$(f,g) = \sum_{i=1}^{N}\sum_{j=1}^{m_i} \frac{\left(f,\varphi_j^{i*}\right)\left(\varphi_j^i, g\right)}{\rho^i} + \sum_{n\in\mathbb{Z}} \frac{\left(f,\varphi_n^*\right)\left(\varphi_n, g\right)}{\rho_n}. \tag{2.15}$$

(ii) (expansion)

$$f = \sum_{i=1}^{N}\sum_{j=1}^{m_i} \frac{\left(f,\varphi_j^{i*}\right)}{\rho^i}\varphi_j^i + \sum_{n\in\mathbb{Z}} \frac{\left(f,\varphi_n^*\right)}{\rho_n}\varphi_n, \tag{2.16}$$

$$g = \sum_{i=1}^{N}\sum_{j=1}^{m_i} \frac{\left(g,\varphi_j^{i}\right)}{\overline{\rho^i}}\varphi_j^{i*} + \sum_{n\in\mathbb{Z}} \frac{\left(g,\varphi_n\right)}{\overline{\rho_n}}\varphi_n^*, \tag{2.17}$$

where both series are convergent in $(L^2(0,1))^2$.

Proposition 2.3 can be proved by Proposition 2.1 (ii) and Proposition 2.2. Here we omit the details.

Remark 2.3. For f, g in $(L^2(0,1))^2$ or $(L^2(0,1))^4$ we denote still the product of f and g by

$$(f,g) = \int_0^1 f^T(x)\overline{g(x)}\mathrm{d}x.$$

Then

$$(F,G) = \sum_{i=1}^{N}\sum_{j=1}^{m_i} \frac{\left(F,\varphi_j^{i*}\right)\left(\varphi_j^i, G\right)}{\rho^i} + \sum_{n\in\mathbb{Z}} \frac{\left(F,\varphi_n^*\right)\left(\varphi_n, G\right)}{\rho_n} \tag{2.18}$$

holds for $F, G \in (L^2(0,1))^4$. In this case we call still (2.18) the Parseval equality.

For $\lambda \in \mathbb{C}$, let $S(x,\lambda)$ and $S^*(x,\overline{\lambda})$ satisfy the following initial value problems respectively:

$$\begin{cases} \left(\mathcal{A}_0 - \lambda\right) S = 0, \\ S(0,\lambda) = \xi \end{cases} \tag{2.19}$$

$$\begin{cases} \left(\mathcal{A}_0^* - \overline{\lambda}\right) S^* = 0, \\ S^*(0,\overline{\lambda}) = \eta. \end{cases} \tag{2.20}$$

Obviously, $S(x,\lambda) = \begin{pmatrix} \cosh(\lambda x+\mu) \\ \sinh(\lambda x+\mu) \end{pmatrix}$, $S^*(x,\overline{\lambda}) = \begin{pmatrix} \cosh(\overline{\lambda} x+\overline{\mu}) \\ -\sinh(\overline{\lambda} x+\overline{\mu}) \end{pmatrix}$ and $\big(S(\cdot,\lambda), S^*(\cdot,\overline{\lambda})\big) = 1$. For $n \in \mathbb{Z}$, let $\mu_n \in \sigma(A_{0,\mu,0})$ and let us denote $S_n(x) = S(x,\mu_n)$, $S_n^*(x) = S(x,\overline{\mu_n})$. Here a short calculation shows that $\mu_n = n\pi\sqrt{-1} - \mu$, $n \in \mathbb{Z}$.

Remark 2.4. Each μ_n ($n \in \mathbb{Z}$) is simple, and hence both $\{S_n\}_{n\in\mathbb{Z}}$ and $\{S_n^*\}_{n\in\mathbb{Z}}$ are Riesz bases in $(L^2(0,1))^2$.

Let $S_{(j)}(x,\lambda)$ and $S^*_{(j)}(x,\overline{\lambda})$ ($1 \le j \le m_i$) satisfy the following initial value problems respectively:

$$\begin{cases} (\mathcal{A}_0 - \lambda)\, S_{(1)} = 0,\ (\mathcal{A}_0 - \lambda)\, S_{(j)} = S_{(j-1)},\ 2 \le j \le m_i, \\ S_{(j)}(0,\lambda) = \xi,\ 1 \le j \le m_i, \end{cases} \tag{2.21}$$

$$\begin{cases} \big(\mathcal{A}_0^* - \overline{\lambda}\big)\, S^*_{(m_i)} = 0,\ \big(\mathcal{A}_0^* - \overline{\lambda}\big)\, S^*_{(j)} = S^*_{(j+1)},\ 1 \le j \le m_i - 1, \\ S^*_{(m_i)}(0,\overline{\lambda}) = \eta,\ S^*_{(j)}(0,\overline{\lambda}) = \overline{\alpha^i_j}\eta,\ 1 \le j \le m_i - 1. \end{cases} \tag{2.22}$$

Then, we can find the solutions of (2.21) and (2.22) possess the following forms:

$$S_{(j)}(x,\lambda) = \begin{pmatrix} \displaystyle\sum_{k=0}^{j-1} \frac{x^k}{k!}\gamma_k(x,\lambda,\mu) \\ \displaystyle\sum_{k=0}^{j-1} \frac{x^k}{k!}\delta_k(x,\lambda,\mu) \end{pmatrix},$$

$$S^*_{(j)}(x,\overline{\lambda}) = \begin{pmatrix} \displaystyle\sum_{k=j}^{m_i} \overline{\alpha^i_k}\frac{x^{k-j}}{(k-j)!}\gamma_{k-j}(x,\overline{\lambda},\overline{\mu}) \\ \displaystyle -\sum_{k=j}^{m_i} \overline{\alpha^i_k}\frac{x^{k-j}}{(k-j)!}\delta_{k-j}(x,\overline{\lambda},\overline{\mu}) \end{pmatrix},$$

where $\alpha^i_{m_i} = 1$,

$$\gamma_k(x,\lambda,\mu) = \begin{cases} \cosh(\lambda x+\mu),\ k \text{ even} \\ \sinh(\lambda x+\mu),\ k \text{ odd} \end{cases},\quad \delta_k(x,\lambda,\mu) = \begin{cases} \sinh(\lambda x+\mu),\ k \text{ even} \\ \cosh(\lambda x+\mu),\ k \text{ odd} \end{cases}.$$

Put

$$C^*(x,\overline{\lambda}) = \int_0^x S^*(t,\overline{\lambda})\mathrm{d}t,\quad C^*_{(j)}(x,\overline{\lambda}) = \int_0^x S^*_{(j)}(t,\overline{\lambda})\mathrm{d}t, \tag{2.23}$$

$$C(y,\lambda) = \int_0^y S(t,\lambda)\mathrm{d}t,\quad C_{(j)}(y,\lambda) = \int_0^y S_{(j)}(t,\lambda)\mathrm{d}t, \tag{2.24}$$

and

$$f(x,y) = \sum_{i=1}^{N}\sum_{j=1}^{m_i}\frac{\overline{C^*_{(j)}(x,\overline{\lambda^i})}C^T_{(j)}(y,\lambda^i)}{\rho^i} + \sum_{n\in\mathbb{Z}}\left\{\frac{\overline{C^*(x,\overline{\lambda_n})}C^T(y,\lambda_n)}{\rho_n} - \overline{C^*(x,\overline{\mu_n})}C^T(y,\mu_n)\right\}. \tag{2.25}$$

Proposition 2.4.
(i) *The series in (2.25) is convergent absolutely and uniformly in* $[0,1]^2$.
(ii) $f\in\left(C[0,1]^2\right)^4$ *and* $\frac{\partial f}{\partial x},\frac{\partial f}{\partial y},\frac{\partial^2 f}{\partial x\partial y}\in\left(C^1(\overline{\Omega})\right)^4,\in\left(C^1\left(\overline{(0,1)^2\backslash\Omega}\right)\right)^4$.

The proof of Proposition 2.4 is given in Section 4.
We further put

$$F(x,y)=\frac{\partial^2 f}{\partial x\partial y}(x,y) \tag{2.26}$$

and

$$\Omega=\left\{(x,y)\in[0,1]^2:\ 0<y<x<1\right\}.$$

We are ready to state our main results.

Theorem 1 (Uniqueness). *Let* $P=\begin{pmatrix} p_1 & p_2\\ u & v\end{pmatrix}$, $Q=\begin{pmatrix} q_1 & q_2\\ u & v\end{pmatrix}\in\left(C^1[0,1]\right)^4$. *If* $S(P,\mu,\nu)=S(Q,\mu,\nu)$, *then* $P\equiv Q$.

Proposition 2.5. *Let* $P,Q\in\left(C^1[0,1]\right)^4$. *If* $S(P,\mu,\nu)=S(Q,\mu,\nu)$ *and there exist a sufficiently large* $|n|$ *and some eigenvector* ψ_n^* *of* $A^*_{Q,\mu,\nu}$ *such that for any* $m\neq n$ *and any* $1\le j\le m_i, 1\le i\le N$,

$$\rho_{mn}:=(\varphi_m,\psi_n^*)=0,\ \ \rho^i_{jn}:=(\varphi^i_j,\psi^*_n)=0,$$

then $P\equiv Q$.

Theorem 2 (Reconstruction). *Let* $P=\begin{pmatrix} p_1 & p_2\\ u & v\end{pmatrix}\in\left(C^1[0,1]\right)^4$, $S(P,\mu,\nu)=\left\{\lambda^i,m_i,\rho^i,\boldsymbol{\alpha}^i\right\}_{1\le i\le N}\bigcup\left\{\lambda_n,\rho_n\right\}_{n\in\mathbb{Z}}$ *be the spectral characteristics of* $A_{P,\mu,\nu}$ *and let* $F(x,y)$ *be given by* (2.25) *and* (2.26). *Then there exists* $M\in\left(C^1\left(\overline{\Omega}\right)\right)^4$ *such that*

$$F(x,y)+M(x,y)+\int_0^x M(x,\tau)F(\tau,y)\mathrm{d}\tau=0,\ \ (x,y)\in\overline{\Omega}. \tag{2.27}$$

Moreover, for $0 \le x \le 1$ *we have*

$$2(M_{12} - M_{21})(x,x) = (v(x) - p_1(x)) \cosh\left(\int_0^x (p_1 + v)(s)\mathrm{d}s\right) + (p_2(x) - u(x)) \sinh\left(\int_0^x (p_1 + v)(s)\mathrm{d}s\right), \quad (2.28)$$

$$2(M_{11} - M_{22})(x,x) = (v(x) - p_1(x)) \sinh\left(\int_0^x (p_1 + v)(s)\mathrm{d}s\right) + (p_2(x) - u(x)) \cosh\left(\int_0^x (p_1 + v)(s)\mathrm{d}s\right). \quad (2.29)$$

To our knowledge the existing results on inverse spectral problems for systems of differential equations do not give any simultaneous determination of all components of the unknown coefficient matrix, even for the Dirac system. Proposition 2.5 guarantees the uniqueness of all the components in some case. Theorem 2 gives a reconstruction procedure of $P(x)$ from $S(P,\mu,\nu)$. For fixed x, integral equation (2.27) is a Fredholm equation of the second kind with respect to $M(x,y)$ which corresponds to the Gel'fand-Levitan equation in the Sturm-Liouville equation. Thus we call (2.27) the *Gel'fand-Levitan* equation. If for given $S(P,\mu,\nu)$ and $F(x,y)$ determined by (2.25) and (2.26), the homogeneous equation with respect to 1×2 function $M(y)$

$$M(y) + \int_0^x M(\tau)F(\tau,y)\mathrm{d}\tau = 0 \quad (2.30)$$

has only the trivial solution, then (2.27) admits a unique solution by Fredholm's alternative theorem. Then we can solve (2.28) and (2.29) with respect to p_1 and p_2 since $M_{ij}(x,x), 1 \le i,j \le 2$ have been obtained by (2.27).

3 Proof of Theorem 1.

First we show the unique existence of solution to a boundary value problem for a hyperbolic system (Lemma 3.1) and a transformation formula (Lemma 3.2). For the proofs we refer to Yamamoto [20].

Lemma 3.1. *Suppose that* $Q = \begin{pmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{pmatrix}$, $P = \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix} \in \left(C^1[0,1]\right)^4$. *Let*

$$\theta_1(x) = \frac{1}{2}\int_0^x (p_{12} + p_{21} - q_{12} - q_{21})(s)\mathrm{d}s,$$

$$\theta_2(x) = \frac{1}{2}\int_0^x (p_{11} + p_{22} - q_{11} - q_{22})(s)\mathrm{d}s$$

and $\mu \in \mathbb{C}$. *Then there exists a unique* $K(Q,P,\mu) = (K_{kl}(Q,P,\mu)(x,y))_{1\le k,l\le 2}$ $\in \left(C^1\left(\overline{\Omega}\right)\right)^4$ *satisfying (3.1)-(3.4):*

$$\begin{aligned} &B\frac{\partial K(Q,P,\mu)}{\partial x}(x,y) + \frac{\partial K(Q,P,\mu)}{\partial y}(x,y)B \\ &\quad + P(x)K(Q,P,\mu)(x,y) - K(Q,P,\mu)(x,y)Q(y) = 0, \ (x,y)\in\Omega. \end{aligned} \tag{3.1}$$

$$\begin{cases} K_{12}(Q,P,\mu)(x,0) = -\tanh\mu \ K_{11}(Q,P,\mu)(x,0), \\ K_{22}(Q,P,\mu)(x,0) = -\tanh\mu \ K_{21}(Q,P,\mu)(x,0), \end{cases} \tag{3.2}$$

$$\begin{aligned} &K_{12}(Q,P,\mu)(x,x) - K_{21}(Q,P,\mu)(x,x) \\ &= \tfrac{1}{4}\exp(-\theta_1-\theta_2)(x)\times(p_{11}+p_{12}-p_{21}-p_{22}-q_{11}+q_{12}-q_{21}+q_{22})(x) \\ &+ \tfrac{1}{4}\exp(\theta_2-\theta_1)(x)\times(p_{11}-p_{12}+p_{21}-p_{22}-q_{11}-q_{12}+q_{21}+q_{22})(x), \end{aligned} \tag{3.3}$$

$$\begin{aligned} &K_{11}(Q,P,\mu)(x,x) - K_{22}(Q,P,\mu)(x,x) \\ &= \tfrac{1}{4}\exp(-\theta_1-\theta_2)(x)\times(p_{11}+p_{12}-p_{21}-p_{22}+q_{11}-q_{12}+q_{21}-q_{22})(x) \\ &+ \tfrac{1}{4}\exp(\theta_2-\theta_1)(x)\times(p_{12}-p_{11}-p_{21}+p_{22}-q_{11}-q_{12}+q_{21}+q_{22})(x). \end{aligned} \tag{3.4}$$

Set

$$R(Q,P)(x) = e^{-\theta_1(x)}\begin{pmatrix} \cosh\theta_2(x) & -\sinh\theta_2(x) \\ -\sinh\theta_2(x) & \cosh\theta_2(x) \end{pmatrix}. \tag{3.5}$$

We notice that $R(Q,P)(x)$ is continuously twice differentiable and $R^{-1}(Q,P)(x)=$ $R(P,Q)(x)$. Moreover it is easy to see that

$$R\left(-\overline{Q^T}, -\overline{P^T}\right)(x) = \overline{R\,(P,Q)\,(x)} = \overline{R^{-1}\,(Q,P)\,(x)}. \tag{3.6}$$

We note that (3.3) and (3.4) can be rewritten as follows:

$$\begin{aligned} &K(Q,P,\mu)(x,x)B - BK(Q,P,\mu)(x,x) \\ &= B\frac{\mathrm{d}R(Q,P)}{\mathrm{d}x}(x) + P(x)R(Q,P)(x) - R(Q,P)(x)Q(x). \end{aligned} \tag{3.7}$$

Now we define a *transformation operator* $X(Q,P,\mu)$ on $\left(H^1(0,1)\right)^2$ by

$$\left(X(Q,P,\mu)w\right)(x) = R(Q,P)(x)w(x) + \int_0^x K(Q,P,\mu)(x,y)w(y)\mathrm{d}y. \tag{3.8}$$

Lemma 3.2. *Let* $h = \begin{pmatrix} h^{(1)} \\ h^{(2)} \end{pmatrix} \in (C[0,1])^2$ *and* $\beta = \begin{pmatrix} \beta^{(1)} \\ \beta^{(2)} \end{pmatrix} \in \mathbb{C}^2$ *satisfy* $\beta^{(2)}\cosh\mu - \beta^{(1)}\sinh\mu = 0$. *For* $\lambda\in\mathbb{C}$, *if* $\psi = \psi(\cdot,\lambda) \in \left(C^1[0,1]\right)^2$ *satisfies*

$$\begin{cases} B\dfrac{\mathrm{d}\psi}{\mathrm{d}x}(x) + Q(x)\psi(x) = \lambda\psi(x) + h(x), \\ \psi(0) = \beta, \end{cases} \tag{3.9}$$

then $\varphi=\varphi(\cdot,\lambda)\in\left(C^1[0,1]\right)^2$ *defined by*

$$\varphi(x,\lambda)=R(Q,P)(x)\psi(x,\lambda)+\int_0^x K(Q,P,\mu)(x,y)\psi(y,\lambda)\mathrm{d}y \tag{3.10}$$

satisfies

$$\begin{cases} B\dfrac{\mathrm{d}\varphi}{\mathrm{d}x}(x)+P(x)\varphi(x)= & \lambda\varphi(x)+R(Q,P)(x)h(x) \\ & +\displaystyle\int_0^x K(Q,P,\mu)(x,y)h(y)\mathrm{d}y, \\ \varphi(0)=\beta. & \end{cases} \tag{3.11}$$

Obviously Lemma 3.2 can be rewritten as follows:
Let $\lambda\in\mathbb{C}$. *If* $h=\begin{pmatrix} h^{(1)} \\ h^{(2)} \end{pmatrix}\in(C[0,1])^2$, $(\mathcal{A}_Q-\lambda)\psi=h$, $\psi(0)=\beta$, *then* $\varphi=X(Q,P,\mu)\psi$ *satisfies*

$$(\mathcal{A}_P-\lambda)\varphi=X(Q,P,\mu)h,\quad \varphi(0)=\beta.$$

Now let $Q=\begin{pmatrix} q_1 & q_2 \\ u & v \end{pmatrix}\in\left(C^1[0,1]\right)^4$, $P=\begin{pmatrix} p_1 & p_2 \\ u & v \end{pmatrix}\in\left(C^1[0,1]\right)^4$. Assume that $S(P,\mu,\nu)=S(Q,\mu,\nu)=\left\{\lambda^i,m_i,\rho^i,\boldsymbol{\alpha}^i\right\}_{1\le i\le N}\bigcup\{\lambda_n,\rho_n\}_{n\in\mathbb{Z}}$. Since the solutions of (3.9) and (3.11) are unique, in terms of Lemma 3.2 we can obtain the following transformation formulae:

Lemma 3.3 (Transformation formulae). *Let* $\lambda\in\mathbb{C}$.
(i) If $(\mathcal{A}_Q-\lambda)\psi=0$, $\psi(0)=\xi$ *and* $(\mathcal{A}_P-\lambda)\varphi=0$, $\varphi(0)=\xi$, *then*

$$\varphi=X(Q,P,\mu)\psi \tag{3.12}$$

and

$$\psi=X(P,Q,\mu)\varphi. \tag{3.13}$$

(ii) If

$$\begin{cases} (\mathcal{A}_Q-\lambda^i)\psi_1^i=0,\ (\mathcal{A}_Q-\lambda^i)\psi_j^i=\psi_{j-1}^i, \quad 2\le j\le m_i, 1\le i\le N, \\ \psi_j^i(0)=\xi,\ \psi_j^i\in D(A_{Q,\mu,\nu}), \quad 1\le j\le m_i, 1\le i\le N, \end{cases}$$

and

$$\begin{cases} (\mathcal{A}_P-\lambda^i)\varphi_1^i=0,\ (\mathcal{A}_P-\lambda^i)\varphi_j^i=\varphi_{j-1}^i, \quad 2\le j\le m_i, 1\le i\le N, \\ \varphi_j^i(0)=\xi,\ \varphi_j^i\in D(A_{P,\mu,\nu}), \quad 1\le j\le m_i, 1\le i\le N, \end{cases}$$

then

$$\varphi_j^i = X(Q,P,\mu)\psi_j^i, \quad 1 \le j \le m_i, 1 \le i \le N, \tag{3.14}$$

and

$$\psi_j^i = X(P,Q,\mu)\varphi_j^i, \quad 1 \le j \le m_i, 1 \le i \le N. \tag{3.15}$$

(iii) If $(\mathcal{A}_Q^* - \overline{\lambda})\psi^* = 0$, $\psi^*(0,\overline{\lambda}) = \eta$ *and* $(\mathcal{A}_P^* - \overline{\lambda})\varphi^* = 0$, $\varphi^*(0,\overline{\lambda}) = \eta$, *then*

$$\varphi^*\left(x,\overline{\lambda}\right) = \left(X\left(-\overline{Q^T}, -\overline{P^T}, -\overline{\mu}\right)\psi^*\right)\left(x,\overline{\lambda}\right) \tag{3.16}$$

and

$$\psi^*\left(x,\overline{\lambda}\right) = \left(X\left(-\overline{P^T}, -\overline{Q^T}, -\overline{\mu}\right)\varphi^*\right)\left(x,\overline{\lambda}\right). \tag{3.17}$$

(iv) If

$$\begin{cases} (\mathcal{A}_Q^* - \overline{\lambda^i})\psi_{m_i}^{i*} = 0, \ (\mathcal{A}_Q^* - \overline{\lambda^i})\psi_j^{i*} = \psi_{j+1}^{i*}, \quad 1 \le j \le m_i - 1, 1 \le i \le N, \\ \psi_{m_i}^{i*}(0) = \eta, \ \psi_j^{i*}(0) = \alpha_j^i\eta, \quad 1 \le j \le m_i - 1, 1 \le i \le N, \\ \psi_j^{i*} \in D(A_{Q,\mu,\nu}^*), \quad 1 \le j \le m_i, 1 \le i \le N, \end{cases}$$

and

$$\begin{cases} (\mathcal{A}_P^* - \overline{\lambda^i})\varphi_{m_i}^{i*} = 0, \ (\mathcal{A}_P^* - \overline{\lambda^i})\varphi_j^{i*} = \varphi_{j+1}^{i*}, \quad 1 \le j \le m_i - 1, 1 \le i \le N, \\ \varphi_{m_i}^{i*}(0) = \eta, \ \varphi_j^{i*}(0) = \alpha_j^i\eta, \quad 1 \le j \le m_i - 1, 1 \le i \le N, \\ \varphi_j^{i*} \in D(A_{P,\mu,\nu}^*), \quad 1 \le j \le m_i, 1 \le i \le N, \end{cases}$$

then

$$\varphi_j^{i*} = X\left(-\overline{Q^T}, -\overline{P^T}, -\overline{\mu}\right)\psi_j^{i*}, \quad 1 \le j \le m_i, 1 \le i \le N, \tag{3.18}$$

and

$$\psi_j^{i*} = X\left(-\overline{P^T}, -\overline{Q^T}, -\overline{\mu}\right)\varphi_j^{i*}, \quad 1 \le j \le m_i, 1 \le i \le N. \tag{3.19}$$

Moveover, in order to prove Theorem 1, we need the following two lemmata.

Lemma 3.4. *For* $0 < b < y < a < x < 1$, *we have*

$$\begin{aligned} I \ &:= \ \sum_{i=1}^{N}\sum_{j=1}^{m_i}\frac{1}{\rho^i}\int_a^x R\left(Q,P\right)(t)\overline{\varphi_j^{i*}(t)}\mathrm{d}t\int_b^y \left(\psi_j^i(t)\right)^T \mathrm{d}t \\ &\quad + \sum_{n\in\mathbb{Z}}\frac{1}{\rho_n}\int_a^x R\left(Q,P\right)(t)\overline{\varphi_n^*(t)}\mathrm{d}t\int_b^y \psi_n^T(t)\mathrm{d}t \\ &= \ 0, \end{aligned} \tag{3.20}$$

and

$$
\begin{aligned}
I_0 \quad := \quad & \sum_{i=1}^{N}\sum_{j=1}^{m_i}\frac{1}{\rho^i}\int_a^x \overline{\psi_j^{i*}(t)}\mathrm{d}t\int_b^y \left(\psi_j^i(t)\right)^T \mathrm{d}t \\
& +\sum_{n\in\mathbb{Z}}\frac{1}{\rho_n}\int_a^x \overline{\psi_n^{*}(t)}\mathrm{d}t\int_b^y \psi_n^T(t)\mathrm{d}t \\
= \quad & 0.
\end{aligned}
\tag{3.21}
$$

Proof. By Lemma 3.3 (i) and (ii), it is true that

$$\psi_n(t) = \left(X(P,Q,\mu)\varphi_n\right)(t), \quad n\in\mathbb{Z}$$

and

$$\psi_j^i(t) = \left(X(P,Q,\mu)\varphi_j^i\right)(t), \quad 1\le j\le m_i, 1\le i\le N.$$

By the symmetry of $R(P,Q)(x)$, changing the order of integrals, we obtain

$$
\begin{aligned}
\int_b^y \psi_n^T(t)\mathrm{d}t \quad = \quad & \int_b^y \left(R\left(P,Q\right)(t)\varphi_n(t) + \int_0^t K\left(P,Q,\mu\right)(t,\tau)\varphi_n(\tau)\mathrm{d}\tau\right)^T \mathrm{d}t \\
= \quad & \int_b^y \varphi_n^T(t)R\left(P,Q\right)(t)\mathrm{d}t + \int_0^b \varphi_n^T(t)\mathrm{d}t\int_b^y K^T\left(P,Q,\mu\right)(\tau,t)\mathrm{d}\tau \\
& + \int_b^y \varphi_n^T(t)\mathrm{d}t\int_t^y K^T\left(P,Q,\mu\right)(\tau,t)\mathrm{d}\tau \\
= \quad & \int_0^1 \varphi_n^T(t)\left(\chi_{(b,y)}(t)G_1(t) + \chi_{(0,b)}(t)G_2(t)\right)\mathrm{d}t \\
= \quad & \left(\varphi_n(\cdot),\ \chi_{(b,y)}(\cdot)\overline{G_1(\cdot)} + \chi_{(0,b)}(\cdot)\overline{G_2(\cdot)}\right),
\end{aligned}
$$

where

$$G_1(t) = R\left(P,Q\right)(t) + \int_t^y K^T\left(P,Q,\mu\right)(\tau,t)\mathrm{d}\tau,$$

$$G_2(t) = \int_b^y K^T\left(P,Q,\mu\right)(\tau,t)\mathrm{d}\tau.$$

Similarly,

$$\int_b^y \left(\psi_j^i(t)\right)^T \mathrm{d}t = \left(\varphi_j^i(\cdot), \chi_{(b,y)}(\cdot)\overline{G_1(\cdot)} + \chi_{(0,b)}(\cdot)\overline{G_2(\cdot)}\right).$$

Therefore

$$
\begin{aligned}
I = & \sum_{i=1}^{N}\sum_{j=1}^{m_i}\frac{1}{\rho^i}\left(\chi_{(a,x)}(\cdot)R(Q,P)(\cdot), \varphi_j^{i*}(\cdot)\right)\left(\varphi_j^i(\cdot), \chi_{(b,y)}(\cdot)\overline{G_1(\cdot)} + \chi_{(0,b)}(\cdot)\overline{G_2(\cdot)}\right) \\
& +\sum_{n\in\mathbb{Z}}\frac{1}{\rho_n}\left(\chi_{(a,x)}(\cdot)R(Q,P)(\cdot), \varphi_n^{*}(\cdot)\right)\left(\varphi_n(\cdot), \chi_{(b,y)}(\cdot)\overline{G_1(\cdot)} + \chi_{(0,b)}(\cdot)\overline{G_2(\cdot)}\right).
\end{aligned}
$$

It is obvious that $R(Q,P), G_1, G_2 \in (L^2(0,1))^4$. By the Parseval equality with respect to $A_{P,\mu,\nu}$, we obtain

$$I = \left(\chi_{(a,x)}(\cdot)R(Q,P)(\cdot), \chi_{(b,y)}(\cdot)\overline{G_1(\cdot)} + \chi_{(0,b)}(\cdot)\overline{G_2(\cdot)}\right) = 0$$

since $(a,x) \cap \{(0,b) \cup (b,y)\} = \emptyset$.

Similarly, by the Parseval equality with respect to $A_{Q,\mu,\nu}$,

$$I_0 = \left(\chi_{(a,x)}(\cdot)E, \chi_{(b,y)}(\cdot)E\right) = 0,$$

where E denotes the unit matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. $\square$

Lemma 3.5. *For $0 < b < y < a < x < 1$, we have*

$$\begin{aligned} I &= \sum_{i=1}^{N}\sum_{j=1}^{m_i}\frac{1}{\rho^i}\left\{\int_a^x \overline{\psi_j^{i*}(t)}\mathrm{d}t\right. \\ &+ \left.\int_a^x R(Q,P)(t)\mathrm{d}t\int_0^t \overline{K\left(-\overline{Q^T}, -\overline{P^T}, -\overline{\mu}\right)(t,\tau)}\overline{\psi_j^{i*}(\tau)}\mathrm{d}\tau\right\}\int_b^y \left(\psi_j^i(t)\right)^T\mathrm{d}t \\ &\quad +\sum_{n\in\mathbb{Z}}\frac{1}{\rho_n}\left\{\int_a^x \overline{\psi_n^*(t)}\mathrm{d}t\right. \\ &\quad +\left.\int_a^x R(Q,P)(t)\mathrm{d}t\int_0^t \overline{K\left(-\overline{Q^T}, -\overline{P^T}, -\overline{\mu}\right)(t,\tau)}\overline{\psi_n^*(\tau)}\mathrm{d}\tau\right\}\int_b^y \psi_n^T(t)\mathrm{d}t \\ &= 0. \end{aligned}$$

Proof. If one notices (3.16),(3.18) and (3.6), then the proof of Lemma 3.5 is complete.

Proof of Theorem 1. First by Lemma 3.4 and 3.5, we have

$$\begin{aligned} 0 = &\sum_{i=1}^{N}\sum_{j=1}^{m_i}\frac{1}{\rho^i}\left\{\int_a^x R(Q,P)(t)\mathrm{d}t\int_0^t \overline{K\left(-\overline{Q^T}, -\overline{P^T}, -\overline{\mu}\right)(t,\tau)}\overline{\psi_j^{i*}(\tau)}\mathrm{d}\tau\right\} \\ &\qquad\times\int_b^y \left(\psi_j^i(t)\right)^T\mathrm{d}t \\ &+\sum_{n\in\mathbb{Z}}\frac{1}{\rho_n}\left\{\int_a^x R(Q,P)(t)\mathrm{d}t\int_0^t \overline{K\left(-\overline{Q^T}, -\overline{P^T}, -\overline{\mu}\right)(t,\tau)}\overline{\psi_n^*(\tau)}\mathrm{d}\tau\right\}\int_b^y \psi_n^T(t)\mathrm{d}t \end{aligned}$$

$$
\begin{aligned}
&=\sum_{i=1}^{N}\sum_{j=1}^{m_i}\frac{1}{\rho^i}\left\{\int_0^a\left[\int_a^x R(Q,P)(t)\overline{K\left(-\overline{Q^T},-\overline{P^T},-\overline{\mu}\right)(t,\tau)}\mathrm{d}t\right]\overline{\psi_j^{i*}(\tau)}\mathrm{d}\tau\right.\\
&\left.+\int_a^x\left[\int_y^x R(Q,P)(t)\overline{K\left(-\overline{Q^T},-\overline{P^T},-\overline{\mu}\right)(t,\tau)}\mathrm{d}t\right]\overline{\psi_j^{i*}(\tau)}\mathrm{d}\tau\right\}\int_b^y\left(\psi_j^i(t)\right)^T\mathrm{d}t\\
&+\sum_{n\in\mathbb{Z}}\frac{1}{\rho_n}\left\{\int_0^a\left[\int_a^x R(Q,P)(t)\overline{K\left(-\overline{Q^T},-\overline{P^T},-\overline{\mu}\right)(t,\tau)}\mathrm{d}t\right]\overline{\psi_n^*(\tau)}\mathrm{d}\tau\right.\\
&\left.+\int_a^x\left[\int_y^x R(Q,P)(t)\overline{K\left(-\overline{Q^T},-\overline{P^T},-\overline{\mu}\right)(t,\tau)}\mathrm{d}t\right]\overline{\psi_n^*(\tau)}\mathrm{d}\tau\right\}\int_b^y\psi_n^T(t)\mathrm{d}t\\
&\qquad\text{(by changing the order of integrals)}\\
&=\left(\chi_{(0,a)}(\cdot)\left(\int_a^x R(Q,P)(t)\overline{K\left(-\overline{Q^T},-\overline{P^T},-\overline{\mu}\right)(t,\cdot)}\mathrm{d}t\right)^T,\chi_{(b,y)}(\cdot)E\right)\\
&+\left(\chi_{(a,x)}(\cdot)\left(\int_y^x R(Q,P)(t)\overline{K\left(-\overline{Q^T},-\overline{P^T},-\overline{\mu}\right)(t,\cdot)}\mathrm{d}t\right)^T,\chi_{(b,y)}(\cdot)E\right)\\
&\qquad\text{(by the Parseval equality with respect to } A_{Q,\mu,\nu}\text{).}
\end{aligned}
$$

Note that $(0,a)\bigcap(b,y)=(b,y)$ and $(a,x)\bigcap(b,y)=\emptyset$. It follows that

$$
\int_b^y\int_a^x R(Q,P)(t)\overline{K\left(-\overline{Q^T},-\overline{P^T},-\overline{\mu}\right)(t,\tau)}\mathrm{d}t\mathrm{d}\tau=0
$$

for $0\le b\le y\le a\le x\le 1$. It implies that

$$
R(Q,P)(x)\overline{K\left(-\overline{Q^T},-\overline{P^T},-\overline{\mu}\right)(x,y)}=0,\quad (x,y)\in\overline{\Omega}. \tag{3.22}
$$

Since $R(Q,P)(x)$ is invertible, we see that $K\left(-\overline{Q^T},-\overline{P^T},-\overline{\mu}\right)(x,x)=0$.

By means of (3.3) and (3.4) in Lemma 3.1, we have

$$
\begin{aligned}
&\exp\left(\frac{1}{2}\int_0^x(p_1+p_2-q_1-q_2)(s)\mathrm{d}s\right)\times(-p_1+p_2+q_1+q_2-2u)(x)\\
&+\exp\left(\frac{1}{2}\int_0^x(-p_1+p_2+q_1-q_2)(s)\mathrm{d}s\right)\times(-p_1-p_2+q_1-q_2+2u)(x)=0
\end{aligned}
$$

and

$$
\begin{aligned}
&\exp\left(\frac{1}{2}\int_0^x(p_1+p_2-q_1-q_2)(s)\mathrm{d}s\right)\times(-p_1+p_2-q_1-q_2+2v)(x)\\
&+\exp\left(\frac{1}{2}\int_0^x(-p_1+p_2+q_1-q_2)(s)\mathrm{d}s\right)\times(p_1+p_2+q_1-q_2-2v)(x)=0,
\end{aligned}
$$

that is,

$$\exp\left(\int_0^x (p_1-q_1)(s)\mathrm{d}s\right)\times(-p_1+p_2+q_1+q_2-2u)(x)$$
$$+(-p_1-p_2+q_1-q_2+2u)(x)=0 \tag{3.23}$$

and

$$\exp\left(\int_0^x (p_1-q_1)(s)\mathrm{d}s\right)\times(-p_1+p_2-q_1-q_2+2v)(x)$$
$$+(p_1+p_2+q_1-q_2-2v)(x)=0. \tag{3.24}$$

Setting $r_1(x)=p_1(x)-q_1(x)$, $r_2(x)=p_2(x)+q_2(x)$, we rewrite (3.23) as

$$\exp\left(\int_0^x r_1(s)\mathrm{d}s\right)\times(-r_1+r_2-2u)(x)+(-r_1-r_2+2u)(x)=0,$$

which is equivalent to

$$r_1(x)\left(1+\exp\left(\int_0^x r_1(s)\mathrm{d}s\right)\right)=a(x)\left(1-\exp\left(\int_0^x r_1(s)\mathrm{d}s\right)\right), \tag{3.25}$$

where $a=2u-r_2\in C^1[0,1]$.

Next we are going to prove that $r_1(x)\equiv 0$. First since $r_1, a\in C^1[0,1]$, we can choose a positive integer N_0 such that

$$\|r_1(\cdot)\|_{C^0[0,1]}\le N_0,\ \exp\left(\int_0^1 |r_1(s)|\mathrm{d}s\right)\le N_0,\ \|a(\cdot)\|_{C^0[0,1]}\le N_0.$$

Denote $\delta_0:=1/N_0$. Then for any $x\in[0,\delta_0]$, we have

$$\left|\int_0^x r_1(s)\mathrm{d}s\right|\le\delta_0\|r_1(\cdot)\|_{C^0[0,1]}\le 1.$$

On the other hand, if $z=z_1+\sqrt{-1}z_2,\ z_1,z_2\in\mathbb{R}$ satisfies $|z|\le 1$, then

$$|1+\exp(z)|=\sqrt{(1+\exp(z_1)\cos z_2)^2+(\exp(z_1)\sin z_2)^2}\ge 1+\exp(z_1)\cos z_2\ge 1$$

since $-1\le z_1,z_2\le 1$. This yields that for any $x\in[0,\delta_0]$,

$$\left|1+\exp\left(\int_0^x r_1(s)\mathrm{d}s\right)\right|\ge 1.$$

Therefore, applying the mean value theorem to the function $\exp\left(\left|\int_0^x r_1(s)\mathrm{d}s\right|\right)-1$ which is obviously not less than $\left|1-\exp\left(\int_0^x r_1(s)\mathrm{d}s\right)\right|$, we obtain from (3.25) that for any $x\in[0,\delta_0]$,

$$|r_1(x)|\le|a(x)|\times\left|\int_0^x r_1(s)\mathrm{d}s\right|\times\exp\left(\int_0^1|r_1(s)|\mathrm{d}s\right)\le N_0^2\int_0^x|r_1(s)|\mathrm{d}s.$$

The Gronwall inequality implies that $r_1(x) \equiv 0$ in $[0, \delta_0]$. Similarly, we can apply the same argument to the subinterval $[\delta_0, 2\delta_0]$, in which we obtain $r_1(x) \equiv 0$. Repeat the same argument in each subinterval $[(k-1)\delta_0, k\delta_0], 1 \le k \le N_0$. Consequently, it follows that $r_1(x) \equiv 0$ in $[0,1]$, that is, $p_1(x) = q_1(x)$. Substituting $p_1 = q_1$ into (3.24), we have $p_2(x) = q_2(x)$. Thus $P(x) = Q(x)$ follows and the proof is complete. $\square$

4 Proofs of Proposition 2.5 and Theorem 2.

Let $P = \begin{pmatrix} p_1 & p_2 \\ u & v \end{pmatrix} \in (C^1[0,1])^4$, $S(P,\mu,\nu) = \{\lambda^i, m_i, \rho^i, \boldsymbol{\alpha}^i\}_{1\le i\le N} \bigcup \{\lambda_n, \rho_n\}_{n\in\mathbb{Z}}$ be the spectral characteristics of $A_{P,\mu,\nu}$. We should note that $D(A_{P,\mu,\nu}) = D(A_{0,\mu,\nu})$.

We divide the proofs into three steps.

First step. In this step, we prove Proposition 2.5 and Proposition 2.4.

Similarly to Lemma 3.3, we have the following transformation formulae.

Lemma 4.1 (Transformation formulae). *Let $\lambda \in \mathbb{C}$ and $1 \le i \le N$.*
(i) If $(\mathcal{A}_0 - \lambda)S = 0$, $S(0,\lambda) = \xi$ and $(\mathcal{A}_P - \lambda)\varphi = 0$, $\varphi(0,\lambda) = \xi$, then

$$\varphi(x,\lambda) = (X(0,P,\mu)S)(x,\lambda)$$

and

$$S(x,\lambda) = (X(P,0,\mu)\varphi)(x,\lambda).$$

(ii) If

$$\begin{cases} (\mathcal{A}_0 - \lambda^i)S_{(1)}(x,\lambda^i) = 0, \ (\mathcal{A}_0 - \lambda^i)S_{(j)}(x,\lambda^i) = S_{(j-1)}(x,\lambda^i), \quad 2 \le j \le m_i, \\ S_{(j)}(0,\lambda^i) = \xi, \quad 1 \le j \le m_i, \end{cases}$$

and

$$\begin{cases} (\mathcal{A}_P - \lambda^i)\varphi_1^i = 0, \ (\mathcal{A}_P - \lambda^i)\varphi_j^i = \varphi_{j-1}^i, \quad 2 \le j \le m_i, \\ \varphi_j^i(0) = \xi, \ \varphi_j^i \in D(A_{P,\mu,\nu}), \quad 1 \le j \le m_i, \end{cases}$$

then

$$\varphi_j^i(x) = \left(X(0,P,\mu)S_{(j)}\right)(x,\lambda^i), \quad 1 \le j \le m_i,$$

and

$$S_{(j)}(x,\lambda^i) = \left(X(P,0,\mu)\varphi_j^i\right)(x), \quad 1 \le j \le m_i.$$

(iii) If $(\mathcal{A}_0^* - \overline{\lambda})S^* = 0$, $S^*(0,\overline{\lambda}) = \eta$ and $(\mathcal{A}_P^* - \overline{\lambda})\varphi^* = 0$, $\varphi^*(0,\overline{\lambda}) = \eta$, then

$$\varphi^*\left(x,\overline{\lambda}\right) = \left(X\left(0, -\overline{P^T}, -\overline{\mu}\right)S^*\right)\left(x,\overline{\lambda}\right)$$

and

$$S^*\left(x,\overline{\lambda}\right)=\left(X\left(-\overline{P^T},0,-\overline{\mu}\right)\varphi^*\right)\left(x,\overline{\lambda}\right).$$

(iv) If

$$\begin{cases}(\mathcal{A}_0^*-\overline{\lambda^i})S^*_{(m_i)}(x,\overline{\lambda^i})=0,\ (\mathcal{A}_0^*-\overline{\lambda^i})S^*_{(j)}(x,\overline{\lambda^i})=S^*_{(j+1)}(x,\overline{\lambda^i}),1\le j\le m_i-1,\\ S^*_{(m_i)}(0,\overline{\lambda^i})=\eta,\ S^*_{(j)}(0,\overline{\lambda^i})=\overline{\alpha^i_j}\eta,\quad 1\le j\le m_i-1,\end{cases}$$

and

$$\begin{cases}(\mathcal{A}_P^*-\overline{\lambda^i})\varphi^{i*}_{m_i}=0,\ (\mathcal{A}_P^*-\overline{\lambda^i})\varphi^{i*}_j=\varphi^{i*}_{j+1},\quad 1\le j\le m_i-1,\\ \varphi^{i*}_{m_i}(0)=\eta,\ \varphi^{i*}_j(0)=\overline{\alpha^i_j}\eta,\quad 1\le j\le m_i-1,\varphi^{i*}_j\in D(A^*_{P,\mu,\nu}),\quad 1\le j\le m_i,\end{cases}$$

then

$$\varphi^{i*}_j(x)=\left(X\left(0,-\overline{P^T},-\overline{\mu}\right)S^*_{(j)}\right)(x,\overline{\lambda^i}),\quad 1\le j\le m_i,$$

and

$$S^*_{(j)}(x,\overline{\lambda^i})=\left(X\left(-\overline{P^T},0,-\overline{\mu}\right)\varphi^{i*}_j\right)(x),\quad 1\le j\le m_i.$$

Lemma 4.2. *There exists a constant $\delta=\delta(P,\mu,\nu)>0$ such that $|\rho_n|\ge\delta$, $n\in\mathbb{Z}$.*

Proof. First we see that there exists a constant $c(\mu)>0$ such that $|S(x,\lambda)|\le c(\mu)$ provided that $|\mathrm{Re}\lambda|<\infty$.

Moreover, by (2.19) and integrating by parts, we have

$$\begin{aligned}&\left|\int_0^x K(0,P,\mu)(x,y)S(y,\lambda)\mathrm{d}y\right|\\ &=\left|\int_0^x K(0,P,\mu)(x,y)\frac{1}{\lambda}B\partial_yS(y,\lambda)\mathrm{d}y\right|\\ &=\frac{1}{|\lambda|}\left|K(0,P,\mu)(x,x)BS(x,\lambda)-K(0,P,\mu)(x,0)BS(0,\lambda)\right.\\ &\quad\left.-\int_0^x\partial_yK(0,P,\mu)(x,y)BS(y,\lambda)\mathrm{d}y\right| \qquad (4.1)\\ &\le\frac{1}{|\lambda|}\left(2\|K(0,P,\mu)\|_\infty+\max_{0\le x\le 1}\int_0^x|\partial_yK(0,P,\mu)(x,y)|\,\mathrm{d}y\right)\|S(\cdot,\lambda)\|_\infty\\ &\equiv\frac{c(P,\mu)}{|\lambda|}.\end{aligned}$$

By means of Lemma 4.1 (i), from (4.1), for large $|n|$ there exist $D_n \in (L^\infty(0,1))^2$ such that

$$\sup_{n\in\mathbb{Z},0\le x\le 1} |D_n(x)| \le c(P,\mu)$$

and

$$\varphi_n(x) = \varphi(x,\lambda_n) = R(0,P)(x)S(x,\lambda_n) + \frac{D_n(x)}{\lambda_n}. \tag{4.2}$$

Similarly, in view of Lemma 4.1 (iii), there exist $D_n^* \in (L^\infty(0,1))^2$ such that

$$\varphi_n^*(x) = \varphi^*(x,\overline{\lambda_n}) = \overline{R^{-1}(0,P)(x)}S^*(x,\overline{\lambda_n}) + \frac{D_n^*(x)}{\overline{\lambda_n}}. \tag{4.3}$$

By (2.6), we have

$$\begin{aligned}\rho_n &= (\varphi_n, \varphi_n^*)\\ &= \left(R(0,P)(\cdot)S(\cdot,\lambda_n), \overline{R^{-1}(0,P)(\cdot)}S^*(\cdot,\overline{\lambda_n})\right) + O\left(\frac{1}{|n|}\right)\\ &= 1 + O\left(\frac{1}{|n|}\right),\end{aligned} \tag{4.4}$$

which implies $|\rho_n| \ge \frac{1}{2}$ for sufficiently large $|n|$. Note that $\rho_n \ne 0$, $n \in \mathbb{Z}$. Therefore we can take $\delta = \delta(P,\mu,\nu) > 0$ such that $|\rho_n| \ge \delta > 0$. □

Proof of Proposition 2.5. From $S(P,\mu,\nu) = S(Q,\mu,\nu)$ we can prove that the transformation kernel $K\left(-\overline{Q^T}, -\overline{P^T}, -\overline{\mu}\right)(x,y) = 0$ in $\overline{\Omega}$ as in the proof of Theorem 1 (see (3.22)). Then by Lemma 3.3 (iii) we see that

$$\varphi_n^*(x) = R(-\overline{Q^T}, -\overline{P^T})(x)\psi_n^*(x). \tag{4.5}$$

On the other hand, from $\rho_{mn} = 0$, $\rho_{jn}^i = 0$ and Proposition 2.2 it follows that $\varphi_n^* - \psi_n^*$ is orthogonal to the Riesz basis $\left\{\varphi_j^i\right\}_{1\le j\le m_i, 1\le i\le N} \bigcup \{\varphi_m\}_{m\in\mathbb{Z}}$ and, consequently, $\varphi_n^* - \psi_n^* \equiv 0$. Substituting this into (4.5), we obtain

$$\varphi_n^*(x) = R(-\overline{Q^T}, -\overline{P^T})(x)\varphi_n^*(x). \tag{4.6}$$

Moreover (4.3) yields by (2.6) that

$$\varphi_n^*(x) = R(0, -\overline{P^T})S^*(x,\overline{\lambda_n}) + O(\frac{1}{n}),$$

which implies that

$$\left(\varphi_n^{*(1)}(x)\right)^2 - \left(\varphi_n^{*(2)}(x)\right)^2 \ne 0$$

for sufficiently large $|n|$.
Therefore, from (4.6) it follows that $R(-\overline{Q^T}, -\overline{P^T})(x) \equiv E$. Consequently, in view of (3.7), replacing Q, P by $-\overline{Q^T}, -\overline{P^T}$ respectively, we obtain from

$$K\left(-\overline{Q^T}, -\overline{P^T}, -\overline{\mu}\right)(x,x) = 0, \quad R(-\overline{Q^T}, -\overline{P^T})(x) \equiv E$$

that $P(x) \equiv Q(x)$. The proof is complete. $\square$

Proof of Proposition 2.4 (i). It is sufficient to prove

$$\left| \frac{\overline{C^*(x, \overline{\lambda_n})} C^T(y, \lambda_n)}{\rho_n} \right| \le \frac{c_1}{n^2}, \quad n \in \mathbb{Z}, \quad (x,y) \in [0,1]^2 \tag{4.7}$$

and

$$\left| \overline{C^*(x, \overline{\mu_n})} C^T(y, \mu_n) \right| \le \frac{c_2}{n^2}, \quad n \in \mathbb{Z}, \quad (x,y) \in [0,1]^2. \tag{4.8}$$

Here the constants c_1, $c_2 > 0$ are independent of $(x,y) \in [0,1]^2$ and $n \in \mathbb{Z}$.
By the definitions of $C(\cdot, \lambda)$ and $C^*(x, \overline{\lambda})$, we see that

$$|C(y,\lambda)| \le \frac{c(\mu)}{|\lambda|}, \quad \left|C^*(x, \overline{\lambda})\right| \le \frac{c(\mu)}{|\lambda|}.$$

Moreover, by Lemma 4.2 we have

$$\left| \frac{\overline{C^*(x, \overline{\lambda_n})} C^T(y, \lambda_n)}{\rho_n} \right| \le \frac{c^2(\mu)}{\delta |\lambda_n|^2}. \tag{4.9}$$

Then Proposition 2.1 (i) completes the proof. $\square$
Lemma 4.3.

$$\begin{aligned} I(P, \mu, \nu) \quad &:= \quad \sum_{i=1}^{N} \sum_{j=1}^{m_i} \frac{\overline{C^*_{(j)}(x, \overline{\lambda^i})} C^T_{(j)}(y, \lambda^i)}{\rho^i} + \sum_{n \in \mathbb{Z}} \frac{\overline{C^*(x, \overline{\lambda_n})} C^T(y, \lambda_n)}{\rho_n} \\ &= \quad \int_0^{\min(x,y)} \overline{\Psi(x,t)} \Phi^T(y,t) \mathrm{d}t, \end{aligned}$$

where

$$\Psi(x,t) = \begin{cases} \overline{R(0,P)(t)} + \displaystyle\int_t^x K\left(-\overline{P^T}, 0, -\overline{\mu}\right)(\tau, t)\mathrm{d}\tau, & 0 \le t \le x \le 1, \\ 0, & 0 \le x < t \le 1, \end{cases} \tag{4.10}$$

and

$$\Phi(y,t)=\begin{cases} R^{-1}(0,P)(t)+\displaystyle\int_t^y K\left(P,0,\mu\right)(\tau,t)\mathrm{d}\tau, & 0\le t\le y\le 1,\\ 0, & 0\le y<t\le 1. \end{cases} \tag{4.11}$$

Proof. By Lemma 4.1 (ii),

$$\begin{aligned} C_{(j)}(y,\lambda^i) &= \int_0^y S_{(j)}(t,\lambda^i)\mathrm{d}t \\ &= \int_0^y \left\{R(P,0)(t)\varphi_j^i(t)+\int_0^t K\left(P,0,\mu\right)(t,\tau)\varphi_j^i(\tau)\mathrm{d}\tau\right\}\mathrm{d}t \\ &= \int_0^y \left\{R^{-1}(0,P)(t)+\int_t^y K\left(P,0,\mu\right)(\tau,t)\mathrm{d}\tau\right\}\varphi_j^i(t)\mathrm{d}t \\ &= \int_0^1 \Phi(y,t)\varphi_j^i(t)\mathrm{d}t. \end{aligned}$$

Therefore

$$C_{(j)}^T(y,\lambda^i)=\left(\varphi_j^i(\cdot),\overline{\Phi^T(y,\cdot)}\right). \tag{4.12}$$

Similarly,

$$C^T(y,\lambda_n)=\left(\varphi_n(\cdot),\overline{\Phi^T(y,\cdot)}\right). \tag{4.13}$$

By Lemma 4.1 (iv), noting (3.6), we have

$$\begin{aligned} & C_{(j)}^*\left(x,\overline{\lambda^i}\right) \\ =\ & \int_0^x S_{(j)}^*(t,\overline{\lambda^i})\mathrm{d}t \\ =\ & \int_0^x \left\{R(-\overline{P^T},0)(t)\varphi_j^{i*}(t)+\int_0^t K\left(-\overline{P^T},0,-\overline{\mu}\right)(t,\tau)\varphi_j^{i*}(\tau)\mathrm{d}\tau\right\}\mathrm{d}t \\ =\ & \int_0^x \left\{\overline{R(0,P)(t)}+\int_t^x K\left(-\overline{P^T},0,-\overline{\mu}\right)(\tau,t)\mathrm{d}\tau\right\}\varphi_j^{i*}(t)\mathrm{d}t \\ =\ & \int_0^1 \Psi(x,t)\varphi_j^{i*}(t)\mathrm{d}t. \end{aligned}$$

Therefore

$$\overline{C^*_{(j)}(x,\overline{\lambda^i})} = \left(\overline{\Psi^T(x,\cdot)}, \varphi_j^{i*}(\cdot)\right). \tag{4.14}$$

Similarly,

$$\overline{C^*(x,\overline{\lambda_n})} = \left(\overline{\Psi^T(x,\cdot)}, \varphi_n^{*}(\cdot)\right). \tag{4.15}$$

By (4.12), (4.13), (4.14), (4.15) and the Parseval equality with respect to $A_{P,\mu,\nu}$, we obtain

$$\begin{aligned} I(P,\mu,\nu) &= \sum_{i=1}^{N}\sum_{j=1}^{m_i} \frac{\left(\overline{\Psi^T(x,\cdot)}, \varphi_j^{i*}(\cdot)\right)\left(\varphi_j^i(\cdot), \overline{\Phi^T(y,\cdot)}\right)}{\rho^i} \\ &\quad + \sum_{n\in\mathbb{Z}} \frac{\left(\overline{\Psi^T(x,\cdot)}, \varphi_n^{*}(\cdot)\right)\left(\varphi_n(\cdot), \overline{\Phi^T(y,\cdot)}\right)}{\rho_n} \\ &= \left(\overline{\Psi^T(x,\cdot)}, \overline{\Phi^T(y,\cdot)}\right). \end{aligned}$$

Thus the proof of Lemma 4.3 is complete. □

Lemma 4.4.

$$I(0,\mu,\nu) := \sum_{n\in\mathbb{Z}} \overline{C^*(x,\overline{\mu_n})} C^T(y,\mu_n) = \min(x,y)E.$$

Proof. By the definition of $C^*(x,\overline{\mu_n})$, $C(y,\mu_n)$ and the Parseval equality with respect to $A_{0,\mu,0}$, the proof is complete. □

Proof of Proposition 2.4 (ii). By Lemmata 4.3 and 4.4, we see that

$$f(x,y) = \int_0^{\min(x,y)} \left(\overline{\Psi(x,t)}\Phi^T(y,t) - E\right) \mathrm{d}t.$$

In view of the definitions of Ψ and Φ a direct calculation yields

$$F(x,y) \equiv \frac{\partial^2 f}{\partial x \partial y}(x,y)$$

$$= \begin{cases} \overline{K\left(-\overline{P^T}, 0, -\overline{\mu}\right)(x,y)} R^{-1}(0,P)(y) \\ \qquad + \int_0^y \overline{K\left(-\overline{P^T}, 0, -\overline{\mu}\right)(x,t)} K^T(P,0,\mu)(y,t)\mathrm{d}t, \quad (x,y) \in \overline{\Omega}, \\ R(0,P)(x)K^T(P,0,\mu)(y,x) \\ \qquad + \int_0^x \overline{K\left(-\overline{P^T}, 0, -\overline{\mu}\right)(x,t)} K^T(P,0,\mu)(y,t)\mathrm{d}t, \quad (x,y) \in \overline{(0,1)^2 \backslash \Omega}. \end{cases}$$

Then Lemma 3.1 completes the proof of Proposition 2.4. □

Remark 4.1. The continuity of $F(x,y)$ at the diagonal implies that

$$\overline{K\left(-\overline{P^T}, 0, -\overline{\mu}\right)(x,x)} R^{-1}(0,P)(x) = R(0,P)(x)K^T(P,0,\mu)(x,x).$$

Second step. Similarly to Lemma 3.4, we apply Lemma 4.1 and the Parseval equality with respect to $A_{P,\mu,\nu}$ to obtain:
Lemma 4.5. *For $0 < b < y < a < x < 1$,*

$$\begin{aligned} \widetilde{I} \quad &:= \quad \sum_{i=1}^{N}\sum_{j=1}^{m_i}\frac{1}{\rho^i}\int_a^x R(0,P)(t)\overline{\varphi_j^{i*}(t)}\mathrm{d}t\int_b^y S_{(j)}^T(t,\lambda^i)\mathrm{d}t \\ &\quad + \sum_{n\in\mathbb{Z}}\frac{1}{\rho_n}\int_a^x R(0,P)(t)\overline{\varphi_n^{*}(t)}\mathrm{d}t\int_b^y S^T(t,\lambda_n)\mathrm{d}t \\ &= \quad 0. \end{aligned}$$

Now set

$$M(x,y) = R(0,P)(x)\overline{K\left(0, -\overline{P^T}, -\overline{\mu}\right)(x,y)} \in \left(C^1(\overline{\Omega})\right)^4 \tag{4.16}$$

and

$$H(x,\tau)=\begin{cases}\displaystyle\int_a^x M(t,\tau)\mathrm{d}t, & 0\le\tau\le a,\\[2ex] \displaystyle\int_\tau^x M(t,\tau)\mathrm{d}t, & a<\tau\le x,\\[2ex] 0, & x<\tau.\end{cases}\tag{4.17}$$

We establish

Lemma 4.6. *For $0<b<y<a<x<1$,*

$$\begin{aligned}\widetilde{I} &= \sum_{i=1}^{N}\sum_{j=1}^{m_i}\frac{1}{\rho^i}\int_a^x\overline{S_{(j)}^*(t,\overline{\lambda^i})}\mathrm{d}t\int_b^y S_{(j)}^T(t,\lambda^i)\mathrm{d}t\\ &\quad+\sum_{n\in\mathbb{Z}}\frac{1}{\rho_n}\int_a^x\overline{S^*(t,\overline{\lambda_n})}\mathrm{d}t\int_b^y S^T(t,\lambda_n)\mathrm{d}t\\ &\quad+\sum_{i=1}^{N}\sum_{j=1}^{m_i}\frac{1}{\rho^i}\int_0^1 H(x,\tau)\overline{S_{(j)}^*(\tau,\overline{\lambda^i})}\mathrm{d}\tau\int_b^y S_{(j)}^T(t,\lambda^i)\mathrm{d}t\\ &\quad+\sum_{n\in\mathbb{Z}}\frac{1}{\rho_n}\int_0^1 H(x,\tau)\overline{S^*(\tau,\overline{\lambda_n})}\mathrm{d}\tau\int_b^y S^T(t,\lambda_n)\mathrm{d}t\\ &= 0.\end{aligned}$$

Proof. By Lemma 4.1,

$$\varphi_j^{i*}(t)=\left(X\left(0,-\overline{P^T},-\overline{\mu}\right)S_{(j)}^*\right)(t,\overline{\lambda^i}),\quad 1\le j\le m_i, 1\le i\le N$$

and

$$\varphi_n^*(t)=\left(X\left(0,-\overline{P^T},-\overline{\mu}\right)S^*\right)(t,\overline{\lambda_n}),\quad n\in\mathbb{Z}.$$

Recalling the definition of the transformation operator and changing the order of integrals, by (3.6) we complete the proof of Lemma 4.6 directly by Lemma 4.5. □

The Parseval equality with respect to $A_{0,\mu,0}$ shows

Lemma 4.7. *For $0<b<y<a<x<1$,*

$$\sum_{n\in\mathbb{Z}}\int_a^x\overline{S_n^*(t)}\mathrm{d}t\int_b^y S_n^T(t)\mathrm{d}t=0.$$

Third step.

Proof of Theorem 2. Lemmata 4.6 and 4.7 show that

$$
\begin{aligned}
&0=\widetilde{I}\\
&=\Bigg\{\sum_{i=1}^{N}\sum_{j=1}^{m_i}\frac{1}{\rho^i}\int_a^x \overline{S^*_{(j)}(t,\overline{\lambda^i})}\mathrm{d}t\int_b^y S^T_{(j)}(t,\lambda^i)\mathrm{d}t\\
&\qquad+\sum_{n\in\mathbb{Z}}\left(\frac{1}{\rho_n}\int_a^x \overline{S^*(t,\overline{\lambda_n})}\mathrm{d}t\int_b^y S^T(t,\lambda_n)\mathrm{d}t-\int_a^x \overline{S^*_n(t)}\mathrm{d}t\int_b^y S^T_n(t)\mathrm{d}t\right)\Bigg\}\\
&+\sum_{n\in\mathbb{Z}}\int_0^1 H(x,\tau)\overline{S^*_n(\tau)}\mathrm{d}\tau\int_b^y S^T_n(t)\mathrm{d}t\\
&+\Bigg\{\sum_{i=1}^{N}\sum_{j=1}^{m_i}\frac{1}{\rho^i}\int_0^1 H(x,\tau)\overline{S^*_{(j)}(\tau,\overline{\lambda^i})}\mathrm{d}\tau\int_b^y S^T_{(j)}(t,\lambda^i)\mathrm{d}t\\
&\qquad\qquad+\sum_{n\in\mathbb{Z}}\frac{1}{\rho_n}\int_0^1 H(x,\tau)\overline{S^*(\tau,\overline{\lambda_n})}\mathrm{d}\tau\int_b^y S^T(t,\lambda_n)\mathrm{d}t\\
&\qquad\qquad-\sum_{n\in\mathbb{Z}}\int_0^1 H(x,\tau)\overline{S^*_n(\tau)}\mathrm{d}\tau\int_b^y S^T_n(t)\mathrm{d}t\Bigg\}\\
&\equiv I_1+I_2+I_3.
\end{aligned}
\tag{4.18}
$$

Next we will transform I_1, I_2 and I_3. First let us recall definitions (2.23)-(2.25) of $C^*_{(j)}(\cdot,\overline{\lambda^i})$, $C_{(j)}(\cdot,\lambda^i)$, $C^*(\cdot,\lambda)$, $C(\cdot,\lambda)$ and $f(\cdot,\cdot)$. Then

$$
\begin{aligned}
I_1 &= \sum_{i=1}^{N}\sum_{j=1}^{m_i}\frac{1}{\rho^i}\left(\overline{C^*_{(j)}(x,\overline{\lambda^i})}-\overline{C^*_{(j)}(a,\overline{\lambda^i})}\right)\left(C^T_{(j)}(y,\lambda^i)-C^T_{(j)}(b,\lambda^i)\right)\\
&\quad+\sum_{n\in\mathbb{Z}}\Bigg\{\frac{1}{\rho_n}\left(\overline{C^*(x,\overline{\lambda_n})}-\overline{C^*(a,\overline{\lambda_n})}\right)\left(C^T(y,\lambda_n)-C^T(b,\lambda_n)\right)\\
&\qquad\quad-\left(\overline{C^*(x,\overline{\mu_n})}-\overline{C^*(a,\overline{\mu_n})}\right)\left(C^T(y,\mu_n)-C^T(b,\mu_n)\right)\Bigg\}\\
&= f(x,y)-f(x,b)-f(a,y)+f(a,b).
\end{aligned}
\tag{4.19}
$$

By the Parseval equality with respect to $A_{0,\mu,0}$ and (4.17), we have

$$
\begin{aligned}
I_2 &= \sum_{n\in\mathbb{Z}}\int_0^1 H(x,\tau)\overline{S^*_n(\tau)}\mathrm{d}\tau\int_b^y S^T_n(t)\mathrm{d}t\\
&= \int_0^1 H(x,\tau)\chi_{(b,y)}(\tau)\mathrm{d}\tau\\
&= \int_b^y\int_a^x M(t,\tau)\mathrm{d}t\mathrm{d}\tau.
\end{aligned}
\tag{4.20}
$$

Since

$$
\frac{\partial C^*_{(j)}}{\partial\tau}(\tau,\overline{\lambda^i})=S^*_{(j)}(\tau,\overline{\lambda^i}),\ \ H(x,x)=0,\quad\forall x\in[0,1]
$$

and

$$C^*_{(j)}(0,\overline{\lambda^i})=0,$$

integration by parts yields

$$\begin{aligned}
& \int_0^1 H(x,\tau)\overline{S^*_{(j)}(\tau,\overline{\lambda^i})}\mathrm{d}\tau \\
= \quad & \left(\int_0^a+\int_a^x\right) H(x,\tau)\overline{S^*_{(j)}(\tau,\overline{\lambda^i})}\mathrm{d}\tau \\
= \quad & \left[H(x,\tau)C^*_{(j)}(\tau,\overline{\lambda^i})\right]\Big|_{\tau=0}^{\tau=a}+\left[H(x,\tau)C^*_{(j)}(\tau,\overline{\lambda^i})\right]\Big|_{\tau=a}^{\tau=x} \\
& -\left(\int_0^a+\int_a^x\right)\frac{\partial H}{\partial\tau}(x,\tau)\overline{C^*_{(j)}(\tau,\overline{\lambda^i})}\mathrm{d}\tau \\
= \quad & -\int_0^x\frac{\partial H}{\partial\tau}(x,\tau)\overline{C^*_{(j)}(\tau,\overline{\lambda^i})}\mathrm{d}\tau.
\end{aligned} \tag{4.21}$$

Similarly,

$$\int_0^1 H(x,\tau)\overline{S^{i*}_j(\tau)}\mathrm{d}\tau=-\int_0^x\frac{\partial H}{\partial\tau}(x,\tau)\overline{C^*_j(\tau,\overline{\mu^i})}\mathrm{d}\tau, \tag{4.22}$$

$$\int_0^1 H(x,\tau)\overline{S^*(\tau,\overline{\lambda_n})}\mathrm{d}\tau=-\int_0^x\frac{\partial H}{\partial\tau}(x,\tau)\overline{C^*(\tau,\overline{\lambda_n})}\mathrm{d}\tau, \tag{4.23}$$

and

$$\int_0^1 H(x,\tau)\overline{S^*_n(\tau)}\mathrm{d}\tau=-\int_0^x\frac{\partial H}{\partial\tau}(x,\tau)\overline{C^*(\tau,\overline{\mu_n})}\mathrm{d}\tau. \tag{4.24}$$

Therefore, by (4.21)-(4.24) and Proposition 2.4 (i), we have

$$\begin{aligned}
I_3= \quad & \sum_{i=1}^N\sum_{j=1}^{m_i}\int_0^x\frac{\partial H}{\partial\tau}(x,\tau)\left[-\frac{1}{\rho^i}\overline{C^*_{(j)}(\tau,\overline{\lambda^i})}\left(C^T_{(j)}(y,\lambda^i)-C^T_{(j)}(b,\lambda^i)\right)\right]\mathrm{d}\tau \\
& +\sum_{n\in\mathbb{Z}}\int_0^x\frac{\partial H}{\partial\tau}(x,\tau)\left[-\frac{1}{\rho_n}\overline{C^*(\tau,\overline{\lambda_n})}\left(C^T(y,\lambda_n)-C^T(b,\lambda_n)\right)\right. \\
& \left.+\overline{C^*(\tau,\overline{\mu_n})}\left(C^T(y,\mu_n)-C^T(b,\mu_n)\right)\right]\mathrm{d}\tau \\
= \quad & \int_0^x\frac{\partial H}{\partial\tau}(x,\tau)\left(f(\tau,b)-f(\tau,y)\right)\mathrm{d}\tau
\end{aligned}$$

(exchange the order of sums and integrals).

Integrating by parts and noting that $f(0,\cdot)=0$, we obtain

$$\begin{aligned} I_3 = & \int_0^x H(x,\tau)\left(\frac{\partial f}{\partial\tau}(\tau,y)-\frac{\partial f}{\partial\tau}(\tau,b)\right)\mathrm{d}\tau \\ = & \int_a^x\int_0^t M(t,\tau)\left(\frac{\partial f}{\partial\tau}(\tau,y)-\frac{\partial f}{\partial\tau}(\tau,b)\right)\mathrm{d}\tau\mathrm{d}t. \end{aligned} \tag{4.25}$$

The last identity follows from the definition of $H(x,\tau)$ and change of the order of integrals.

Consequently, by (4.18), (4.19), (4.20) and (4.25), we obtain

$$\begin{aligned} 0 = f(x,y)-f(x,b)-f(a,y)+f(a,b)+\int_b^y\int_a^x M(t,\tau)\mathrm{d}t\mathrm{d}\tau \\ +\int_a^x\int_0^t M(t,\tau)\left(\frac{\partial f}{\partial\tau}(\tau,y)-\frac{\partial f}{\partial\tau}(\tau,b)\right)\mathrm{d}\tau\mathrm{d}t. \end{aligned} \tag{4.26}$$

Differentiating the both sides once with respect to x and then once with respect to y, we obtain (2.27).

For completing the proof of Theorem 2, we have to derive (2.28) and (2.29).

Since

$$M(x,x)=R(0,P)(x)\overline{K\left(0,-\overline{P^T},-\overline{\mu}\right)(x,x)}$$

by (4.16) and $K\left(0,-\overline{P^T},-\overline{\mu}\right)(x,x)$ satisfies (3.3) and (3.4), by the definition of $R(0,P)(x)$, we can directly verify (2.28) and (2.29). $\square$

5 Another Reconstruction Procedure and Remarks.

We assume that $P_0=P_0(x)=\begin{pmatrix} p_{11}^0(x) & p_{12}^0(x) \\ p_{21}^0(x) & p_{22}^0(x)\end{pmatrix}$ exists such that $\sigma(A_{P_0,\mu,\nu})$ has the same structure as $\sigma(A_{P,\mu,\nu})$, namely, $\sigma(A_{P_0,\mu,\nu})=\Sigma'\bigcup\Sigma''$, where $\Sigma'\bigcap\Sigma''=\emptyset$, $\Sigma'=\{\mu^i\in\sigma(A_{P_0,\mu,\nu}):$
$m(\mu^i)=m_i\ge 2, 1\le i\le N\}$, $\Sigma''=\{\mu_n\in\sigma(A_{P_0,\mu,\nu}):\mu_n \text{ is simple}, n\in\mathbb{Z}\}$, $m(\mu^i)$ denotes the algebraic multiplicity of μ^i. For convenience here we use the same symbols as before when the zero matrix is replaced by P_0. Furthermore, we set $S_j^i=S_j^i(x)=S_{(j)}(x,\mu^i)$, $S_j^{i*}=S_j^{i*}(x)=S_{(j)}^*(x,\overline{\mu^i})$ and

$$\sigma^i=\left(S^i_{(m_i)}(\cdot,\mu^i),S^{i*}_{(m_i)}(\cdot,\overline{\mu^i})\right),\ \sigma_n=(S_n,S_n^*),\quad 1\le i\le N,\ n\in\mathbb{Z}.$$

We define $f(x,y)$ by:

$$\begin{aligned} f(x,y) &= \sum_{i=1}^{N}\sum_{j=1}^{m_i}\left\{\frac{\overline{C^*_{(j)}(x,\overline{\lambda^i})}C^T_{(j)}(y,\lambda^i)}{\rho^i} - \frac{\overline{C^*_{(j)}(x,\overline{\mu^i})}C^T_{(j)}(y,\mu^i)}{\sigma^i}\right\} \\ & +\sum_{n\in\mathbb{Z}}\left\{\frac{\overline{C^*(x,\overline{\lambda_n})}C^T(y,\lambda_n)}{\rho_n} - \frac{\overline{C^*(x,\overline{\mu_n})}C^T(y,\mu_n)}{\sigma_n}\right\}, \end{aligned} \tag{5.1}$$

and set

$$F(x,y) = \frac{\partial^2 f}{\partial x\partial y}(x,y). \tag{5.2}$$

Replacing the zero matrix by P_0 and arguing similarly to Section 4, one can show

Theorem 3 (Reconstruction). *Let* $P = \begin{pmatrix} p_1 & p_2 \\ u & v \end{pmatrix} \in \left(C^1[0,1]\right)^4$, $S(P,\mu,\nu) = \left\{\lambda^i, m_i, \rho^i, \boldsymbol{\alpha}^i\right\}_{1\le i\le N}\bigcup\left\{\lambda_n,\rho_n\right\}_{n\in\mathbb{Z}}$ *be the spectral characteristics of* $A_{P,\mu,\nu}$. *Then there exists* $M\in\left(C^1\left(\overline{\Omega}\right)\right)^4$ *such that*

$$\widetilde{F}(x,y) + M(x,y) + \int_0^x M(x,\tau)\widetilde{F}(\tau,y)\mathrm{d}\tau = 0, \tag{5.3}$$

where

$$\widetilde{F}(x,y) = F(x,y) + \sum_{i=1}^{N}\sum_{j=1}^{m_i}\frac{1}{\sigma^i}\left(\overline{S^{i*}_j(x)} - \overline{\widetilde{S}^{i*}_j(x)}\right)\left(S^i_j(y)\right)^T \tag{5.4}$$

and $\left\{\widetilde{S}^{i*}_j\right\}_{1\le j\le m_i}$ *are the normalized root vectors of* $A^*_{P_0,\mu,\nu}$ *for* $\overline{\mu^i}$ *with respect to*
$\left\{S^i_j\right\}_{1\le j\le m_i}$ *(*$1\le i\le N$*).*

Moreover, for $0\le x\le 1$ *we have*

$$\begin{aligned} 2(M_{12}-M_{21})(x,x) &= (v(x)-p_1(x))\cosh\left(\int_0^x (p_1+v-p^0_{11}-p^0_{22})(s)\mathrm{d}s\right) \\ &+ (p_2(x)-u(x))\sinh\left(\int_0^x (p_1+v-p^0_{11}-p^0_{22})(s)\mathrm{d}s\right) + p^0_{11}(x) - p^0_{22}(x), \end{aligned} \tag{5.5}$$

$$\begin{aligned} 2(M_{11}-M_{22})(x,x) &= (v(x)-p_1(x))\sinh\left(\int_0^x (p_1+v-p^0_{11}-p^0_{22})(s)\mathrm{d}s\right) \\ &+ (p_2(x)-u(x))\cosh\left(\int_0^x (p_1+v-p^0_{11}-p^0_{22})(s)\mathrm{d}s\right) + p^0_{21}(x) - p^0_{12}(x) \end{aligned} \tag{5.6}$$

Although this paper extends the work of M. Yamamoto in [22] in some sense, some further research on this subject need be done. First we should specify conditions on spectral characteristics in order that the Gel'fand-Levitan equation admits a unique solution. Second, for the problem of stability, can we estimate deviation in p_1 and p_2 in a suitable norm when the spectral characteristics perturbs? In a forthcoming paper, we will discuss them.

Appendix. Proof of Proposition 2.2.

Let $\left\{\varphi_j^i\right\}_{1\le j\le m_i}$ and $\left\{\widetilde{\varphi}_j^i\right\}_{1\le j\le m_i}$ be the unique solutions to the initial value problems:

$$\begin{cases} (\mathcal{A}_P-\lambda^i)\varphi_1^i=0,\ (\mathcal{A}_P-\lambda^i)\varphi_j^i=\varphi_{j-1}^i, \quad 2\le j\le m_i, 1\le i\le N, \\ \varphi_j^i(0)=\xi,\ \varphi_j^i\in D(A_{P,\mu,\nu}), \quad 1\le j\le m_i, 1\le i\le N \end{cases} \tag{1}$$

and

$$\begin{cases} (\mathcal{A}_P^*-\overline{\lambda^i})\widetilde{\varphi}_{m_i}^i=0,\ (\mathcal{A}_P^*-\overline{\lambda^i})\widetilde{\varphi}_j^i=\widetilde{\varphi}_{j+1}^i, \quad 1\le j\le m_i-1, 1\le i\le N, \\ \widetilde{\varphi}_j^i(0)=\eta,\ \widetilde{\varphi}_j^i\in D(A_{P,\mu,\nu}^*), \quad 1\le j\le m_i, 1\le i\le N. \end{cases} \tag{2}$$

It is easy to see that $(\mathcal{A}_P-\lambda^i)^{m_i}\varphi_j^i=(\mathcal{A}_P^*-\overline{\lambda^i})^{m_i}\widetilde{\varphi}_j^i=0$ $(1\le j\le m_i, 1\le i\le N)$, so that $\left\{\varphi_j^i\right\}_{1\le j\le m_i}$ and $\left\{\widetilde{\varphi}_j^i\right\}_{1\le j\le m_i}$ are root vectors for $\lambda^i\in\sigma(A_{P,\mu,\nu})$ and $\overline{\lambda^i}\in\sigma(A_{P,\mu,\nu}^*)$ $(1\le i\le N)$ respectively. Then by Proposition2.1(ii), both $\left\{\varphi_j^i\right\}_{1\le j\le m_i, 1\le i\le N}\bigcup\{\varphi_n\}_{n\in\mathbb{Z}}$ and $\left\{\widetilde{\varphi}_j^i\right\}_{1\le j\le m_i, 1\le i\le N}\bigcup\{\varphi_n^*\}_{n\in\mathbb{Z}}$ are Riesz bases in $(L^2(0,1))^2$. Henceforth we set $\varphi_0^i=\widetilde{\varphi}_0^i=\varphi_{m_i+1}^i=\widetilde{\varphi}_{m_i+1}^i=0$ $(1\le i\le N)$.

Lemma 1. *For* $1\le l\le m_i, 1\le k\le m_j, 1\le i,j\le N, i\ne j$,

$$\left(\varphi_l^i,\widetilde{\varphi}_k^j\right)=0. \tag{3}$$

Proof. We divide the proof into five steps.

(i) Since $\lambda^i\neq\lambda^j$ and $\lambda^i\left(\varphi_1^i,\widetilde{\varphi}_{m_j}^j\right)=\left(\mathcal{A}_P\varphi_1^i,\widetilde{\varphi}_{m_j}^j\right)=\left(\varphi_1^i,\mathcal{A}_P^*\widetilde{\varphi}_{m_j}^j\right)=\left(\varphi_1^i,\overline{\lambda^j}\widetilde{\varphi}_{m_j}^j\right)=\lambda^j\left(\varphi_1^i,\widetilde{\varphi}_{m_j}^j\right)$, it follows that $\left(\varphi_1^i,\widetilde{\varphi}_{m_j}^j\right)=0$.

(ii) For given k in $2\le k\le m_j$, if $\left(\varphi_1^i,\widetilde{\varphi}_k^j\right)=0$,then

$$\begin{aligned}\lambda^i\left(\varphi_1^i,\widetilde{\varphi}_{k-1}^j\right)&=\left(\mathcal{A}_P\varphi_1^i,\widetilde{\varphi}_{k-1}^j\right)=\left(\varphi_1^i,\mathcal{A}_P^*\widetilde{\varphi}_{k-1}^j\right)\\&=\left(\varphi_1^i,\overline{\lambda^j}\widetilde{\varphi}_{k-1}^j+\widetilde{\varphi}_k^j\right)=\lambda^j\left(\varphi_1^i,\widetilde{\varphi}_{k-1}^j\right).\end{aligned}$$

By $\lambda^i\neq\lambda^j$, it follows that $\left(\varphi_1^i,\widetilde{\varphi}_{k-1}^j\right)=0$.

(iii) From (i) and (ii), by induction we have $\left(\varphi_1^i, \widetilde{\varphi}_k^j\right) = 0$ for $1 \le k \le m_j$.

(iv) For given l in $1 \le l \le m_i$, if $\left(\varphi_l^i, \widetilde{\varphi}_k^j\right) = 0$, $1 \le k \le m_j$, we claim that $\left(\varphi_{l+1}^i, \widetilde{\varphi}_k^j\right) = 0$, $1 \le k \le m_j$.

First by the assumption we have $\lambda^i \left(\varphi_{l+1}^i, \widetilde{\varphi}_{m_j}^j\right) = \left(\mathcal{A}_P \varphi_{l+1}^i - \varphi_l^i, \widetilde{\varphi}_{m_j}^j\right) = \left(\varphi_{l+1}^i, \mathcal{A}_P^* \widetilde{\varphi}_{m_j}^j\right) = \lambda^j \left(\varphi_{l+1}^i, \widetilde{\varphi}_{m_j}^j\right)$, then $\left(\varphi_{l+1}^i, \widetilde{\varphi}_{m_j}^j\right) = 0$.

Now suppose that for given s with $1 \le s \le m_i - 1$, $\left(\varphi_{l+1}^i, \widetilde{\varphi}_{s+1}^j\right) = 0$. Then

$$\begin{aligned}\lambda^i \left(\varphi_{l+1}^i, \widetilde{\varphi}_s^j\right) &= \left(\mathcal{A}_P \varphi_{l+1}^i - \varphi_l^i, \widetilde{\varphi}_s^j\right) = \left(\varphi_{l+1}^i, \mathcal{A}_P^* \widetilde{\varphi}_s^j\right) \\ &= \left(\varphi_{l+1}^i, \overline{\lambda^j} \widetilde{\varphi}_s^j + \widetilde{\varphi}_{s+1}^j\right) = \lambda^j \left(\varphi_{l+1}^i, \widetilde{\varphi}_s^j\right),\end{aligned}$$

and $\left(\varphi_{l+1}^i, \widetilde{\varphi}_s^j\right) = 0$. By induction we have $\left(\varphi_{l+1}^i, \widetilde{\varphi}_k^j\right) = 0$ for $1 \le k \le m_j$.

(v) From (iii) (iv) and by induction we obtain $\left(\varphi_l^i, \widetilde{\varphi}_k^j\right) = 0$ for $1 \le l \le m_i, 1 \le k \le m_j$. □

Lemma 2. *For* $1 \le j \le m_i, 1 \le i \le N, n \in \mathbb{Z}$,

$$\left(\varphi_j^i, \varphi_n^*\right) = \left(\varphi_n, \widetilde{\varphi}_j^i\right) = 0. \tag{4}$$

Proof. Since $\left(\varphi_1^i, \varphi_n^*\right) = 0$ and

$$\begin{aligned}\lambda^i \left(\varphi_j^i, \varphi_n^*\right) &= \left(\mathcal{A}_P \varphi_j^i - \varphi_{j-1}^i, \varphi_n^*\right) \\ = \left(\varphi_j^i, \mathcal{A}_P^* \varphi_n^*\right) - \left(\varphi_{j-1}^i, \varphi_n^*\right) &= \lambda_n \left(\varphi_j^i, \varphi_n^*\right) - \left(\varphi_{j-1}^i, \varphi_n^*\right),\end{aligned}$$

by induction it follows that $\left(\varphi_j^i, \varphi_n^*\right) = 0$. Similarly, $\left(\varphi_n, \widetilde{\varphi}_j^i\right) = 0$. □

Lemma 3. *For* $1 \le j \le m_i, 1 \le i \le N$,

$$\left(\varphi_j^i, \widetilde{\varphi}_j^i\right) = \left(\varphi_{m_i}^i, \widetilde{\varphi}_{m_i}^i\right) \equiv \rho^i \ne 0, \tag{5}$$

and for $1 \le k < l \le m_i$,

$$\left(\varphi_k^i, \widetilde{\varphi}_l^i\right) = 0. \tag{6}$$

Proof. First we see that

$$\left(\varphi_j^i, \widetilde{\varphi}_j^i\right) = \left((\mathcal{A}_{\mathcal{P}} - \lambda^i)\varphi_{j+1}^i, \widetilde{\varphi}_j^i\right) = \left(\varphi_{j+1}^i, (\mathcal{A}_P^* - \overline{\lambda^i})\widetilde{\varphi}_j^i\right) = \left(\varphi_{j+1}^i, \widetilde{\varphi}_{j+1}^i\right).$$

By induction it follows that $\left(\varphi_j^i, \widetilde{\varphi}_j^i\right) = \left(\varphi_{m_i}^i, \widetilde{\varphi}_{m_i}^i\right) \equiv \rho^i$ for $1 \le j \le m_i, 1 \le i \le N$.

On the other hand, for $1 \le k \le m_i - 1$,

$$\left(\varphi_k^i, \widetilde{\varphi}_{m_i}^i\right) = \left((\mathcal{A}_{\mathcal{P}} - \lambda^i)\varphi_{k+1}^i, \widetilde{\varphi}_{m_i}^i\right) = \left(\varphi_{k+1}^i, (\mathcal{A}_P^* - \overline{\lambda^i})\widetilde{\varphi}_{m_i}^i\right) = 0. \tag{7}$$

Then by Lemmata 1 and 2, $\left(\varphi^i_{m_i}, \widetilde{\varphi}^i_{m_i}\right) \neq 0$ since $\left\{\varphi^i_j\right\}_{1\leq j\leq m_i, 1\leq i\leq N} \bigcup \{\varphi_n\}_{n\in\mathbb{Z}}$ forms a Riesz basis in $(L^2(0,1))^2$ and $\widetilde{\varphi}^i_{m_i} \neq 0$.

Moreover, since

$$\left(\varphi^i_k, \widetilde{\varphi}^i_l\right) = \left((\mathcal{A}_{\mathcal{P}} - \lambda^i)\varphi^i_{k+1}, \widetilde{\varphi}^i_l\right) = \left(\varphi^i_{k+1}, (\mathcal{A}^*_P - \overline{\lambda^i})\widetilde{\varphi}^i_l\right) = \left(\varphi^i_{k+1}, \widetilde{\varphi}^i_{l+1}\right), \tag{8}$$

(6) follows from (7). □

Proof of Proposition 2.2. We set

$$\begin{cases} \varphi^{i*}_{m_i} = \widetilde{\varphi}^i_{m_i} \in D(A^*_{P,\mu,\nu}), \\ \varphi^{i*}_k = \widetilde{\varphi}^i_k - \displaystyle\sum_{j=k+1}^{m_i} a^i_{j,k}\varphi^{i*}_j, \quad 1 \leq k \leq m_i - 1, \end{cases} \tag{9}$$

where

$$a^i_{j,k} = \overline{(\varphi^i_j, \widetilde{\varphi}^i_k)/\rho^i} \quad \text{for} \quad k+1 \leq j \leq m_i. \tag{10}$$

For given k in $1 \leq k \leq m_i - 1$ and $1 \leq i \leq N$, suppose that φ^{i*}_j $(k+1 \leq j \leq m_i)$ satisfies:

$$\begin{cases} \left(\varphi^i_l, \varphi^{i*}_j\right) = 0, \quad 1 \leq l \leq m_i, l \neq j, \\ \left(\varphi^i_j, \varphi^{i*}_j\right) = \left(\varphi^i_{m_i}, \varphi^{i*}_{m_i}\right) = \rho^i, \quad k+1 \leq j \leq m_i, \\ (\mathcal{A}^*_P - \overline{\lambda^i})\varphi^{i*}_j = \varphi^{i*}_{j+1}, \quad k+1 \leq j \leq m_i - 1, (\mathcal{A}^*_P - \overline{\lambda^i})\varphi^{i*}_{m_i} = 0. \end{cases} \tag{11}$$

Then we claim that the equalities in (11) still hold when the index j is replaced by k.

First, for $1 \leq l \leq k-1$, by (6) and the assumption,

$$\left(\varphi^i_l, \varphi^{i*}_k\right) = \left(\varphi^i_l, \widetilde{\varphi}^i_k\right) - \sum_{j=k+1}^{m_i} \overline{a^i_{j,k}}\left(\varphi^i_l, \varphi^{i*}_j\right) = 0.$$

For $k+1 \leq l \leq m_i$, by the assumption we have

$$\left(\varphi^i_l, \varphi^{i*}_k\right) = \left(\varphi^i_l, \widetilde{\varphi}^i_k\right) - \sum_{j=k+1}^{m_i} \overline{a^i_{j,k}}\left(\varphi^i_l, \varphi^{i*}_j\right)$$

$$= \left(\varphi^i_l, \widetilde{\varphi}^i_k\right) - \overline{a^i_{l,k}}\left(\varphi^i_l, \varphi^{i*}_l\right) = \left(\varphi^i_l, \widetilde{\varphi}^i_k\right) - \frac{\left(\varphi^i_l, \widetilde{\varphi}^i_k\right)}{\rho^i}\rho^i = 0.$$

Therefore, $\left(\varphi^i_l, \varphi^{i*}_k\right) = 0$ for $1 \leq l \leq m_i, l \neq k$.

Second, by (5) and the assumption

$$\left(\varphi_k^i, \varphi_k^{i*}\right) = \left(\varphi_k^i, \widetilde{\varphi}_k^i\right) - \sum_{j=k+1}^{m_i} \overline{a_{j,k}^i} \left(\varphi_k^i, \varphi_j^{i*}\right) = \rho^i.$$

Finally, by the assumption

$$(\mathcal{A}_P^* - \overline{\lambda^i})\varphi_k^{i*} = (\mathcal{A}_P^* - \overline{\lambda^i})\widetilde{\varphi}_k^i - \sum_{j=k+1}^{m_i} a_{j,k}^i (\mathcal{A}_P^* - \overline{\lambda^i})\varphi_j^{i*} = \widetilde{\varphi}_{k+1}^i - \sum_{j=k+1}^{m_i} a_{j,k}^i \varphi_{j+1}^{i*}.$$

Moreover by (8) we see that $a_{j,k}^i = a_{j+1,k+1}^i$, and

$$(\mathcal{A}_P^* - \overline{\lambda^i})\varphi_k^{i*} = \widetilde{\varphi}_{k+1}^i - \sum_{j=k+2}^{m_i} a_{j,k+1}^i \varphi_j^{i*} = \varphi_{k+1}^{i*}.$$

Here we note that if $k = m_i - 1$ then the last equality still holds by the assumption.

Now by (7) and induction, (11) holds for $1 \le j \le m_i$. Therefore, since each $\varphi_j^{i*}(1 \le j \le m_i)$ is a linear combination of $\left\{\widetilde{\varphi}_k^i\right\}_{j \le k \le m_i}$ (see (9)), by Lemma 1 and Lemma 2 we can derive Proposition 2.2 (ii) and (iii).

Now it remains to prove Proposition 2.2 (i).
Let $\overrightarrow{\varphi^{i*}} := \left(\varphi_1^{i*}, \varphi_2^{i*}, \ldots, \varphi_{m_i-1}^{i*}, \varphi_{m_i}^{i*}\right)^T$ and $\overrightarrow{\widetilde{\varphi}^i} := \left(\widetilde{\varphi}_1^i, \widetilde{\varphi}_2^i, \ldots, \widetilde{\varphi}_{m_i-1}^i, \widetilde{\varphi}_{m_i}^i\right)^T$. Put

$$U^i := \begin{pmatrix} 1 & a_{2,1}^i & a_{3,1}^i & \ldots & a_{m_i,1}^i \\ 0 & 1 & a_{3,2}^i & \ldots & a_{m_i,2}^i \\ \multicolumn{5}{c}{\dotfill} \\ 0 & \ldots & 0 & 1 & a_{m_i,m_i-1}^i \\ 0 & \ldots & 0 & 0 & 1 \end{pmatrix} \tag{12}$$

where $a_{j,k}^i$ $(1 \le k \le m_i - 1, k+1 \le j \le m_i)$ are defined by (10). It is easy to see that U^i is invertible. Hence, setting

$$V^i = (V_{jk}^i)_{1 \le j,k \le m_i} := \left(U^i\right)^{-1}, \tag{13}$$

from (9) we have

$$U^i \overrightarrow{\varphi^{i*}} = \overrightarrow{\widetilde{\varphi}^i} \quad \text{or} \quad \overrightarrow{\varphi^{i*}} = V^i \overrightarrow{\widetilde{\varphi}^i}. \tag{14}$$

The last equality yields for $1 \le j \le m_i$,

$$\varphi_j^{i*} = \sum_{k=j}^{m_i} V_{jk}^i \widetilde{\varphi}_k^i \tag{15}$$

since $V^i_{jk} = 0$ for $k < j$. Moreover, since $\widetilde{\varphi}^i_j(0) = \eta$ $(1 \le j \le m_i)$, if we set

$$\overline{\alpha^i_j} := \sum_{k=j}^{m_i} V^i_{jk} \quad \text{for} \quad 1 \le j \le m_i - 1, \tag{16}$$

then it follows from (15) that $\varphi^{i*}_j(0) = \overline{\alpha^i_j}\eta$ $\;(1 \le j \le m_i - 1)$. This completes the proof. □

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