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UNIQUE CONTINUATION AND AN INVERSE PROBLEM FOR HYPERBOLIC EQUATIONS ACROSS A GENERAL HYPERSURFACE

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Abstract. For a hyperbolic equation $p(x,t)\partial_t^2 u(x,t) = \Delta u(x,t) + \sum_{j=1}^n q_j(x,t)\partial_j u + q_{n+1}(x,t)\partial_t u + r(x,t)u$ in $\mathbb{R}^n \times \mathbb{R}$ with $p \in C^1$ and $q_1, ..., q_{n+1}, r \in L^\infty$, we consider the unique continuation and an inverse problem across a non-convex hypersurface Γ. Let Γ be a part of the boundary of a domain and let $\nu(x)$ be the inward unit normal vector to Γ at x. Then we prove the unique continuation near a point x_0 across Γ if $\nabla p(x_0, t) \cdot \nu(x_0) < 0$. Moreover we establish the conditional stability in the continuation. Next we prove the conditional stability in the inverse problem of determining a coefficient $r(x)$ from Cauchy data on Γ over a time interval. The key is a Carleman estimate in level sets of paraboloid shapes.

§1. Introduction and main result.

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We consider a hyperbolic equation:

$$(Au)(x,t) \equiv p(x,t)\partial_t^2 u(x,t) - \Delta u(x,t)$$
$$-\sum_{k=1}^{n} q_k(x,t)\partial_k u(x,t) - q_{n+1}(x,t)\partial_t u(x,t) - r(x,t)u(x,t),$$
$$x \in \mathbb{R}^n,\ t \in \mathbb{R}, \tag{1.1}$$

where $p \in C^1(\mathbb{R}^n_x \times \mathbb{R}_t), > 0$, $q_j, r \in L^\infty_{loc}(\mathbb{R}^n_x \times \mathbb{R}_t)$ for $1 \le j \le n+1$. We always set $x = (x_1, ..., x_n) \in \mathbb{R}^n$, $\partial_t = \frac{\partial}{\partial t}$, $\partial_t^2 = \frac{\partial^2}{\partial t^2}$, $\partial_j = \frac{\partial}{\partial x_j}$, $1 \le j \le n$, etc. and $\Delta = \sum_{j=1}^n \partial_j^2$, $B_\rho(x_0) = \{x \in \mathbb{R}^n;\ |x - x_0| < \rho\}$ with $x_0 \in \mathbb{R}^n$ and $\rho > 0$.

Let $\Gamma \subset \mathbb{R}^n_x$ be a hypersurface of class C^2. For small $\rho > 0$ and $x_0 \in \Gamma$, the hypersurface Γ divides the open ball $B_\rho(x_0)$ into D^+ and D^-. Let $\nu = \nu(x)$ be the unit normal vector to Γ at x which is oriented inward to D^+ and we set $\frac{\partial u}{\partial \nu} = \nabla u \cdot \nu$.

In this paper, we discuss

(1) unique continuation

(2) inverse problem

First we consider:

Unique continuation. Let $u = u(x,t)$ satisfy $Au = 0$ in $D^+ \times (-T,T)$ and $u = \frac{\partial u}{\partial \nu} = 0$ on $\Gamma \times (-T,T)$. Then can we find a neighbourhood $\mathcal{U}$ of x_0 where $u = 0$?

This is the classical unique continuation, and there are many results. In the case where the coefficients p, q_j, r, $1 \le j \le n+1$, are analytic, we can apply the Holmgren theorem or Fritz John's global Holmgren theorem (e.g., Rauch [28]), so that one can prove the unique continuation across Γ, provided that Γ is not the characteristics of the hyperbolic operator P. In the case where the coefficients are not analytic, for proving the unique continuation, one can apply Carleman estimates, and the unique continuation holds if D^+ is convex near Γ (e.g., Hörmander [11], Isakov [18], [19], Khaĭdarov [20]).

In particular, in the case where the coefficients are independent of t, Robbiano [29] proved the unique continuation for not necessarily convex D^+. Also see Lerner [27]. The result by Robbiano was generalized by Hörmander [12] and Tataru [30] where the analyticity of the coefficients in some components of (x, t) is essential. See Eller, Isakov, Nakamura and Tataru [10] for applications to the Maxwell's system and the Lamé system.

The Carleman estimates used in Hörmander [12] and Tataru [30], are very difficult to be applied to inverse problems which we are going to consider in this paper. On the other hand, even for the analytic coefficient case, the unique continuation breaks for general domain D^+ (i.e., in the case where Γ is across the characteristics of P). Moreover, in the case where D^+ is not convex near Γ, there are very few trials by classical Carleman estimates, which are applicable also to the inverse problems. In the case where Γ is flat and A is a ultrahyperbolic operator, Amirov [2] - [4] proved a Carleman estimate to apply it to an inverse problem of determining a source term by lateral Cauchy data. Isakov [19] established a Carleman estimate for a hyperbolic operator A and proved a unique continuation result across flat Γ. In Amirov [2] - [4], Isakov [19], we note that the principal coefficient p cannot be constant. On the other hand, in the case of $p \equiv 1$, Khaĭdarov [20] showed a counterexample of the nonuniqueness in the continuation: there exists $u \in C^\infty(\mathbb{R}^n \times \mathbb{R})$

and $q \in C^\infty(\mathbb{R}^n \times \mathbb{R})$ such that

$$\begin{cases} \partial_t^2 u = \Delta u - q(x,t)\partial_t u \quad \text{in } \mathbb{R}^n \times \mathbb{R}, \\ u = 0 \quad \text{in } x_1 \ge 0, \\ u \ne 0 \quad \text{in } x_1 < 0. \end{cases}$$

Note that q depends on t also. As for other counterexmples, see Alinhac [1], Kumano-go [25]. If q is t-independent or analytic for some component of (x,t), then we can know that if

$$\begin{cases} \partial_t^2 u = \Delta u - q(x,t)\partial_t u \quad \text{in } \mathbb{R}^n \times \mathbb{R}, \\ u = 0 \quad \text{in } x_1 \ge 0, \end{cases}$$

then for any $\widetilde{x} = (0, x_2, ..., x_n)$, there exist a neighbourhood $\mathcal{U}$ of $\widetilde{x}$ and $t_0 > 0$ such that $u = 0$ in $\mathcal{U} \times (-t_0, t_0)$.

In this paper, in contrast with those existing papers, we will discuss a sufficient condition on the principal coefficient p and the boundary Γ for the unique continuation, under that

(1) the coefficients p, q_j, r, $1 \le j \le n+1$, are not analytic in any components of (x,t).

(2) D^+ is not necessarily convex near Γ.

As is seen by the counterexample by Alinhac [1], Khaĭdarov [20] and Kumano-go [25] and by Amirov [2] - [4] and Isakov [19], we cannot expect the unique continuation if p is constant. Furthermore for any Γ, we cannot have the unique continuation across Γ. For this, we will assume that the normal derivative of p at $x_0 \in \Gamma$ is negative. For specifying the condition on Γ, we introduce

Definition. Let $x_0 \in \Gamma$ and $\kappa > 0$. We say that Γ satisfies *the outer paraboloid condition* with κ at x_0 if there exist a neighbourhood $\mathcal{V}$ of x_0 and a paraboloid

$\mathcal{P}$ which is tangential to Γ at x_0 and that $\mathcal{P} \cap \mathcal{V} \subset D^-$ and $\mathcal{P}$ is congruent to $x_1 = \kappa \sum_{j=2}^n |x_j|^2$ (after rotations, translation and symmetric transforms).

Now we are ready to state our first main result.

Theorem 1. *Let $x_0 \in \Gamma \setminus \partial\Gamma$. In (1.1), let us assume that*

$$\begin{cases} p \in C^1(\mathbb{R}_x^n \times \mathbb{R}_t), \quad p > 0 \, in \, \mathbb{R}_x^n \times \mathbb{R}_t, \\ \\ q_j, r \in L^\infty_{loc}(\mathbb{R}_x^n \times \mathbb{R}_t), \quad 1 \le j \le n+1, \end{cases} \tag{1.2}$$

$$\frac{\partial p}{\partial \nu}(x_0, 0) < 0. \tag{1.3}$$

Moreover Γ is assumed to satisfy the outer paraboloid condition with

$$\kappa < \frac{-\frac{\partial p}{\partial \nu}(x_0, 0)}{4(\|p\|_{L^\infty(B_\rho(x_0,0))} + 1)}. \tag{1.4}$$

Let $u \in H^2(D^+ \times (-T, T))$ satisfy

$$Au = 0 \quad in \ D^+ \times (-T, T) \tag{1.5}$$

and

$$u = \frac{\partial u}{\partial \nu} = 0 \quad on \ \Gamma \times (-T, T). \tag{1.6}$$

Then there exist a neighbourhood $\mathcal{V}$ of x_0 and $T_1 \in (0, T)$ such that

$$u = 0 \quad in \ (\mathcal{V} \cap D^+) \times (-T_1, T_1). \tag{1.7}$$

Remark 1. Physically, $V(x,t) = \frac{1}{\sqrt{p(x,t)}}$ corresponds to the wave speed, and so assumption (1.3) means that $\frac{\partial V}{\partial \nu}(x_0, 0) > 0$, that is, the wave speed increases near x_0 along a transverse direction.

Notice that assumption (1.3) excludes constant principal coefficients, so that our result is compatible with the counterexamples by [1], [20], [25].

By the definition, we see that a hyperplane Γ always satisfies condition (1.4), because we can take $\kappa = 0$. Therefore Theorem 1 yields

Corollary. *We assume (1.2), (1.3), (1.5), (1.6) and that Γ is a hyperplane. Then the conclusion of Theorem 1 is true.*

The corollary corresponds with Isakov's unique continuation [19].

Remark 2. As is seen from the proof, we can further specify $\mathcal{V}$ and T_1 in conclusion (1.7).

Thus we can sum up the unique continuation across Γ for the equation $p(x,t)\partial_t^2 u = \Delta u + q(x,t)\partial_t u$ as follows:

(1) Let $p(x,t)$ and $q(x,t)$ be t-independent. Then we can prove the unique continuation across Γ which is flat or satisfies some geometric constraint ([12], [29], [30]).

(2) Let $p \equiv 1$ and $q(x,t)$ be t-dependent without any analyticity. Then the unique continuation across the flat Γ is not true in general (e.g., [20], [25]).

(3) Let $\frac{\partial p}{\partial \nu} < 0$ and $q \in L^\infty_{loc}(\mathbb{R}^n_x \times \mathbb{R}_t)$. Then the unique continuation across Γ is true under assumption (1.4).

Furthermore we can prove the conditional stability in the continuation.

Theorem 2. *Under the same assumptions as in Theorem 1, let $u \in H^2(D^+ \times (-T,T))$ satisfy*

$$Au = f \quad \text{in } D^+ \times (-T,T) \tag{1.8}$$

and

$$u = g, \quad \frac{\partial u}{\partial \nu} = h \quad \text{on } \Gamma \times (-T,T). \tag{1.9}$$

Then there exist a neighbourhood $\mathcal{V}$ of x_0, $T_1 \in (0,T)$ and constants $C > 0$,

$\theta \in (0,1)$ *such that*

$$\|u\|_{H^1((\mathcal{V}\cap D^+)\times(-T_1,T_1))} \le C\mathcal{E}^\theta(\mathcal{E}^{1-\theta} + \|u\|^{1-\theta}_{H^1(D^+\times(-T,T))}). \tag{1.10}$$

Here we set

$$\mathcal{E} = \|f\|_{L^2(D^+\times(-T,T))} + \|g\|_{H^{\frac{3}{2}}(\Gamma\times(-T,T))} + \|g\|_{H^2(-T,T;L^2(\Gamma))}$$

$$+\|h\|_{H^2(-T,T;L^2(\Gamma))} + \|h\|_{L^2(-T,T;H^{\frac{1}{2}}(\Gamma))}.$$

Next we will discuss an inverse problem: In (1.1), we assume that the zeroth order coefficient $r = r(x)$ is t-independent. Then determine $r(x)$ by means of lateral Cauchy data on $\Gamma \times (-T,T)$.

Inverse Problem. Determine $r = r(x)$ in some neighbourhood of $x_0 \in \Gamma$ by $u|_{\Gamma\times(-T,T)}$ and $\frac{\partial u}{\partial \nu}|_{\Gamma\times(-T,T)}$ where u satisfies $Au = 0$ in $D^+ \times (-T,T)$, and $u(\cdot,0)$ and $\partial_t u(\cdot,0)$ are given suitably in D^+.

This kind of inverse problem is related with the unique continuation and the paper by Bukhgeim and Klibanov [9] is the first work, where a Carleman estimate and an inequality for a Volterra integral operator in t are essential. After Bukhgeim and Klibanov [9], there are many papers with similar methodology concerning determination of coefficients in hyperbolic or ultrahyperbolic equations by lateral Cauchy data; Amirov [2] - [4], Bellassoued [6], Bellassoued and Yamamoto [7], Bukhgeim [8], Imanuvilov and Yamamoto [14], [15], [16], Isakov [19], Khaĭdarov [20], [21], Klibanov [22], Klibanov and Timonov [23], Klibanov and Yamamoto [24], Yamamoto [31]. As for similar inverse problems for a Schrödinger equation and a parabolic equation, we refer to Baudouin and Puel [5], and Imanuvilov, Isakov and Yamamoto [13], Imanuvilov and Yamamoto [17], respectively.

In all the papers treating hyperbolic inverse problems except for Amriov [3], [4], we have to assume that D^+ is convex near Γ, because the grounding Carleman estimate requires the convexity of D^+. Therefore the uniqueness in the inverse problem has been not studied for non-convex D^+. We will solve this open problem: the uniqueness and the conditional stability, which are local around x_0.

Theorem 3. *Let $x_0 \in \Gamma \setminus \partial\Gamma$, and let us assume that (1.3) and (1.4) hold, and let $p = p(x) \in C^1(\overline{D^+})$, $q_j, \partial_t q_j \in L^\infty(D^+ \times (-T,T))$, $1 \le j \le n+1$. Let $u_\ell \in H^2(D^+ \times (-T,T))$, $\ell = 1, 2$, satisfy*

$$\partial_t u_\ell \in H^2(D^+ \times (-T,T)) \cap L^\infty(D^+ \times (-T,T)), \tag{1.11}$$

$$\begin{aligned} &p(x)\partial_t^2 u_\ell(x,t) = \Delta u_\ell(x,t) \\ &+\sum_{k=1}^{n} q_k(x,t)\partial_k u_\ell(x,t) + q_{n+1}(x,t)\partial_t u_\ell(x,t) + r_\ell(x)u_\ell(x,t), \end{aligned} \tag{1.12}$$

$$u_\ell(x,0) = a(x), \quad \partial_t u_\ell(x,0) = b(x), \quad x \in D^+ \tag{1.13}$$

and

$$\begin{aligned} &\|\partial_t u_\ell\|_{L^\infty(D^+\times(-T,T))}, \|u_\ell\|_{H^2(D^+\times(-T,T))}, \|\partial_t u_\ell\|_{H^2(D^+\times(-T,T))}, \\ &\|r_\ell\|_{L^\infty(D^+)} \le M, \qquad \ell = 1,2. \end{aligned} \tag{1.14}$$

We assume that

$$|a(x)| > 0 \qquad \text{on } \overline{D^+}. \tag{1.15}$$

Then there exist a neighbourhood $\mathcal{V}$ of x_0 and constants $C > 0$, $\theta \in (0,1)$ which *are dependent on* M, a, b, p, q_j, $1 \le j \le n+1$, *such that*

$$\begin{aligned}
&\|r_1 - r_2\|_{L^2(\mathcal{V}\cap D^+)} \\
\le& C\Bigg\{\sum_{k=0}^{1}\Bigg(\|\partial_t^k(u_1-u_2)\|_{H^{\frac{3}{2}}(\Gamma\times(-T,T))} \\
+&\|\partial_t^k(u_1-u_2)\|_{H^2(-T,T;L^2(\Gamma))} + \left\|\partial_t^k\left(\frac{\partial}{\partial\nu}(u_1-u_2)\right)\right\|_{H^2(-T,T;L^2(\Gamma))} \\
+&\left\|\partial_t^k\left(\frac{\partial}{\partial\nu}(u_1-u_2)\right)\right\|_{L^2(-T,T;H^{\frac{1}{2}}(\Gamma))}\Bigg\}^{\theta}. \qquad (1.16)
\end{aligned}$$

The proofs of our main theorems are based on a Carleman estimate with an uncommon choice of a weight function whose derivation is, however, quite conventional. Our grounding Carleman estimate is proved Section 2, where the weight function is same as in Amirov [2] and different from Isakov's one in [19], and our Carleman estimate is suitable for treating non-convex D^+.

This paper is composed of four sections. In Secion 2, we will establish a key Carleman estimate and in Section 3, we will complete the proofs of Theorems 1 and 2. In Section 4, we will prove Theorem 3.

§2. A key Carleman estimate.

Let $\Gamma \subset \mathbb{R}^n$ be a C^2-hypersurface such that $0 = (0, ..., 0) \in \Gamma \setminus \partial\Gamma$ and $\nu(0) = (1, 0, ..., 0)$. Near 0, we will parametrize Γ by

$$x_1 = \gamma(x_2, ..., x_n), \quad |x_2|^2 + \cdots + |x_n|^2 < \rho^2. \qquad (2.1)$$

We assume that

$$-\alpha_0 \equiv (\partial_1 p)(0,0) < 0 \qquad (2.2)$$

$$\kappa < \frac{\alpha_0}{4(\|p\|_{L^\infty(B_\rho(0,0))} + 1)} \qquad (2.3)$$

and

$$-\kappa\sum_{j=2}^{n}|x_j|^2 < \gamma(x_2,...,x_n) \quad \text{if} \sum_{j=2}^{n}|x_j|^2 < \rho^2. \tag{2.4}$$

Here and henceforth we set

$$B_\rho(0,0) = \{(x,t) \in \mathbb{R}^{n+1};\ |x|^2 + t^2 < \rho^2\}, \quad B_\rho(0) = \{x \in \mathbb{R}^n;\ |x| < \rho\}.$$

Furthermore we set

$$M_1 = \max\{\|p\|_{C^1(B_\rho(0,0))}, 1\} \tag{2.5}$$

Let

$$D^- = \{x \in B_\rho(0) \subset \mathbb{R}^n;\ x_1 < \gamma(x_2,...,x_n)\}$$

and

$$D^+ = B_\rho(0) \setminus \overline{D^-}.$$

First let us choose $\alpha > 0$ arbitrarily such that $\alpha_0 > \alpha$. Then there exists a sufficiently small $\delta_0 > 0$ such that $0 < \delta_0 < \min\{1, \rho^2\}$ and

$$(\partial_1 p)(x,t) < -\alpha \quad \text{if } |x|^2 + t^2 \le \delta_0. \tag{2.6}$$

This is possible by (2.2).

Next by (2.3), we can choose $N > 0$ such that

$$\kappa < \frac{1}{2N} < \frac{\alpha}{4(M_0+1)}, \tag{2.7}$$

where we set

$$M_0 = \|p\|_{L^\infty(B_\rho(0,0))}.$$

For κ and N, we will further choose sufficiently small $\varepsilon \in (0,1)$ such that

$$\varepsilon^2\left|\max\left\{\frac{\kappa}{1-2N\kappa}, \frac{1}{2N}\right\}\right|^2 + \frac{\varepsilon}{1-2N\kappa} + \varepsilon + \frac{2\kappa N\varepsilon}{1-2N\kappa} \le \delta_0 \tag{2.8}$$

and

$$
\begin{aligned}
&\alpha N - 2(M_0^2 + M_0) > 2(M_1^2 + M_1) \\
&\times \left\{ \varepsilon^2 \left| \max\left\{ \frac{\kappa}{1-2N\kappa}, \frac{1}{2N} \right\} \right|^2 + \frac{\varepsilon}{1-2N\kappa} + \varepsilon + \frac{2\kappa N \varepsilon}{1-2N\kappa} \right\}^{\frac{1}{2}}, \\
&N^2 > M_1 \left\{ \varepsilon^2 \left| \max\left\{ \frac{\kappa}{1-2N\kappa}, \frac{1}{2N} \right\} \right|^2 + \frac{\varepsilon}{1-2N\kappa} + \varepsilon + \frac{2\kappa N \varepsilon}{1-2N\kappa} \right\}.
\end{aligned}
\tag{2.9}
$$

Here we note that (2.7) implies that $1 - 2N\kappa > 0$ and $\alpha N - 2(M_0 + 1) > 0$. We define a weight function by

$$
\psi(x,t) = N x_1 + \frac{1}{2} \sum_{j=2}^{n} |x_j|^2 + \frac{1}{2} t^2 + \frac{1}{2}\varepsilon \tag{2.10}
$$

and

$$
Q_\mu = \left\{ (x,t) \in \mathbb{R}^{n+1};\ x_1 > -\kappa \sum_{j=2}^{n} |x_j|^2,\ \sum_{j=2}^{n} |x_j|^2 < \delta_0,\ \psi(x,t) < \mu \right\} \tag{2.11}
$$

with $\frac{\varepsilon}{2} < \mu$.

We note that

$$
\psi(x,t) > \frac{\varepsilon}{2} \quad \text{if } x_1 > -\kappa \sum_{j=2}^{n} |x_j|^2. \tag{2.12}
$$

In fact, by $x_1 > -\kappa \sum_{j=2}^{n} |x_j|^2$, we have

$$
-N\kappa \sum_{j=2}^{n} |x_j|^2 + \frac{1}{2} \sum_{j=2}^{n} |x_j|^2 + \frac{\varepsilon}{2} \le N x_1 + \frac{1}{2} \sum_{j=2}^{n} |x_j|^2 + \frac{1}{2} t^2 + \frac{\varepsilon}{2} = \psi(x,t).
$$

By (2.7), we obtain

$$
\frac{\varepsilon}{2} \le \frac{\varepsilon}{2} + \left(\frac{1}{2} - N\kappa \right) \sum_{j=2}^{n} |x_j|^2 \le \psi(x,t).
$$

In particular, we see by (2.12) that

$$
Q_\mu \ne \emptyset \quad \text{if } \mu > \frac{\varepsilon}{2}.
$$

Then we show our key Carleman estimate:

Lemma 1. *Let $\|q_j\|_{L^\infty(B_\rho(0,0))}$, $\|r\|_{L^\infty(B_\rho(0,0))} \le M_2$ for $1 \le j \le n+1$. Under the above assumptions, there exist constants $C = C(p, \varepsilon, M_2) > 0$, $\eta = \eta(p, \varepsilon, M_2) > 0$ and $s_0 = s_0(p, \varepsilon, M_2) > 0$ such that*

$$\begin{aligned}&\int_{Q_\varepsilon} (s|\nabla u|^2 + s|\partial_t u|^2 + s^3 u^2) \exp(2s\psi^{-\eta}) dxdt \\ \le & C \int_{Q_\varepsilon} |Au|^2 \exp(2s\psi^{-\eta}) dxdt \end{aligned} \tag{2.13}$$

for all $u \in H_0^2(Q_\varepsilon)$ and $s \ge s_0$.

Remark 3. In our Carleman estimate (2.13), choice (2.10) of the weight function is a key and was established in Amirov [2]. In fact, ψ is same as in a Carleman estimate for a parabolic operator (p.73 in Lavrent'ev, Romanov and Shishat·skiĭ[26]), which is not conventional for the hyperbolic operator. For example, for the unique continuation across flat Γ, Isakov [19] uses the weight function

$$\exp(2s \exp[\eta(-2(x_1 - \beta_1)^2 - \sum_{j=2}^n |x_j|^2 - \theta^2 t^2 + \beta_2)])$$

where $\beta_1 > 0$, β_2, $\theta > 0$ are constants. His weight function is isotropic with respect to t and all the components $x_1, ..., x_n$. With our choice, we can prove the unique continuation whose character has a similarity to the parabolic case.

Proof of Lemma 1. Let us set

$$t = x_{n+1}, \quad \zeta = (\zeta_1, ..., \zeta_{n+1}), \quad \xi = (\xi_1, ..., \xi_{n+1}),$$

$$\zeta' = (\zeta_1, ..., \zeta_n), \quad \xi' = (\xi_1, ..., \xi_n), \quad \nabla = (\partial_1, ..., \partial_n), \quad \nabla_{x,t} = (\partial_1, ..., \partial_n, \partial_t),$$

$$A_0 = p(x,t)\partial_t^2 - \Delta, \quad A(x,t,\zeta) = p(x,t)\zeta_{n+1}^2 - \sum_{k=1}^n \zeta_k^2.$$

Then it is sufficient to prove

$$\begin{aligned}&\int_{Q_\varepsilon} (s|\nabla u|^2 + s|\partial_t u|^2 + s^3 u^2) \exp(2s\psi^{-\eta}) dxdt \\ \le & C \int_{Q_\varepsilon} |A_0 u|^2 \exp(2s\psi^{-\eta}) dxdt \end{aligned} \tag{2.14}$$

for all $u \in C_0^\infty(Q_\varepsilon)$ and for all sufficiently large $s > 0$.

In fact, since

$$|Au|^2 \le |A_0 u|^2 + C(|u|^2 + |\nabla u|^2 + |\partial_t u|^2)$$

in Q_ε by (1.2), estimate (2.14) implies conclusion (2.13) for all $u \in C_0^\infty(Q_\varepsilon)$ by taking s sufficiently large. Since $C_0^\infty(Q_\varepsilon)$ is dense in $H^2(Q_\varepsilon)$, a usual density argument completes the proof.

In order to prove (2.14), we can apply a general result by Hörmander [11], Isakov [18], [19], which gives a sufficient condition on $\psi^{-\eta}$ and A_0 in order that a Carleman estimate holds true. Here we use the version by Isakov (e.g., Theorem 3.2.1 in [19]). We set

$$\varphi = \psi^{-\eta}, \qquad A = A(x,t,\zeta).$$

By [19], we have to verify: If

$$A(x,t,\zeta) = 0, \quad \zeta = \xi + is\nabla_{x,t}\varphi, \quad \zeta \ne 0,\, \xi \in \mathbb{R}^{n+1},\, (x,t) \in \overline{Q_\varepsilon}, \tag{2.15}$$

then

$$J(x,t,\zeta) \equiv \sum_{j,k=1}^{n+1} (\partial_j\partial_k\varphi)\frac{\partial A}{\partial \zeta_j}\overline{\frac{\partial A}{\partial \zeta_k}} + \frac{1}{s}\Im\left(\sum_{k=1}^{n+1}(\partial_k A)\overline{\frac{\partial A}{\partial \zeta_k}}\right) > 0, \quad (x,t) \in \overline{Q_\varepsilon}. \tag{2.16}$$

By J_1 and J_2, we denote the first and the second terms at the right hand side of (2.16) respectively. First we have

$$\begin{cases} \partial_j\varphi = -\eta(\partial_j\psi)\psi^{-\eta-1}, \\ \partial_j\partial_k\varphi = \eta(\eta+1)(\partial_j\psi)(\partial_k\psi)\psi^{-\eta-2} \\ -\eta(\partial_j\partial_k\psi)\psi^{-\eta-1}, \quad 1 \le j,k \le n+1, \\ \zeta = \xi - is\eta\psi^{-\eta-1}\nabla_{x,t}\psi. \end{cases} \tag{2.17}$$

Therefore (2.15) is equivalent to

$$p(\xi_{n+1}^2 - s^2\eta^2\psi^{-2\eta-2}(\partial_{n+1}\psi)^2) = |\xi'|^2 - s^2\eta^2\psi^{-2\eta-2}|\nabla\psi|^2 \tag{2.18}$$

and

$$p\xi_{n+1}\partial_{n+1}\psi = (\xi' \cdot \nabla\psi). \tag{2.19}$$

Then, by (2.17), we have

$$\begin{aligned}
&J_1(x,t,\zeta) = \sum_{j,k=1}^{n+1} \eta(\eta+1)(\partial_j\psi)(\partial_k\psi)\psi^{-\eta-2}\frac{\partial A}{\partial\zeta_j}\overline{\frac{\partial A}{\partial\zeta_k}}\\
&-\sum_{j,k=1}^{n+1}\eta(\partial_j\partial_k\psi)\psi^{-\eta-1}\frac{\partial A}{\partial\zeta_j}\overline{\frac{\partial A}{\partial\zeta_k}}\\
&=\eta(\eta+1)\psi^{-\eta-2}\left|\sum_{j=1}^{n+1}(\partial_j\psi)\frac{\partial A}{\partial\zeta_j}\right|^2 - \sum_{j=2}^{n+1}\eta\psi^{-\eta-1}\left|\frac{\partial A}{\partial\zeta_j}\right|^2\\
&\equiv J_{11} + J_{12}.
\end{aligned}$$

Here, by (2.17) and (2.19), we have

$$\begin{aligned}
&\sum_{j=1}^{n+1}(\partial_j\psi)\frac{\partial A}{\partial\zeta_j}\\
&=(2p(\partial_{n+1}\psi)\xi_{n+1} - 2(\nabla\psi\cdot\xi')) + 2is\eta\psi^{-\eta-1}(|\nabla\psi|^2 - p|\partial_{n+1}\psi|^2)\\
&=2is\eta\psi^{-\eta-1}(|\nabla\psi|^2 - p|\partial_{n+1}\psi|^2),
\end{aligned}$$

so that

$$J_{11}(x,t,\zeta) = 4s^2\eta^3(\eta+1)\psi^{-3\eta-4}(|\nabla\psi|^2 - p|\partial_{n+1}\psi|^2)^2.$$

Similarly we can calculate to obtain

$$\begin{aligned}
&J_{12}(x,t,\zeta) = -4\eta\psi^{-\eta-1}\left(\sum_{j=2}^{n}|\xi_j|^2 + p^2|\xi_{n+1}|^2\right) - 4s^2\eta^3\psi^{-3\eta-3}\left(\sum_{j=2}^{n}|\partial_j\psi|^2 + p^2|\partial_{n+1}\psi|^2\right)\\
&\ge -4\eta\psi^{-\eta-1}(|\xi'|^2 + p^2|\xi_{n+1}|^2) - 4s^2\eta^3\psi^{-3\eta-3}(|\nabla\psi|^2 + p^2|\partial_{n+1}\psi|^2).
\end{aligned}$$

Therefore, by (2.18) we obtain

$$
\begin{aligned}
&J_1(x,t,\zeta) \ge -4\eta\psi^{-\eta-1}(|\xi'|^2 + p^2|\xi_{n+1}|^2)\\
&+4s^2\eta^3\psi^{-3\eta-4}\left\{(\eta+1)\left(N^2+\sum_{j=2}^n |x_j|^2 - pt^2\right)^2 - \psi\left(N^2+\sum_{j=2}^n |x_j|^2 + p^2t^2\right)\right\}\\
&= -4\eta\psi^{-\eta-1}(p+1)p\xi_{n+1}^2\\
&+4s^2\eta^3\psi^{-3\eta-4}\Bigg\{(\eta+1)\left(N^2+\sum_{j=2}^n |x_j|^2 - pt^2\right)^2\\
&-\psi\left(2N^2+2\sum_{j=2}^n |x_j|^2 + (p^2-p)t^2\right)\Bigg\}. \qquad (2.20)
\end{aligned}
$$

Next we will calculate J_2. For $1 \le k \le n$, we have

$$
\begin{aligned}
&(\partial_k A)\overline{\frac{\partial A}{\partial \zeta_k}} = (\partial_k p)\zeta_{n+1}^2(-2)\overline{\zeta_k}\\
&= -2(\partial_k p)\{(\xi_{n+1}^2 - s^2\eta^2\psi^{-2\eta-2}|\partial_{n+1}\psi|^2) - 2is\eta\psi^{-\eta-1}(\partial_{n+1}\psi)\xi_{n+1}\}\\
&\times\{\xi_k + is\eta\psi^{-\eta-1}(\partial_k\psi)\}
\end{aligned}
$$

and

$$
\begin{aligned}
&\Im\left((\partial_k A)\overline{\frac{\partial A}{\partial \zeta_k}}\right)\\
&=2s\eta\psi^{-\eta-1}(\partial_k p)\{2(\partial_{n+1}\psi)\xi_{n+1}\xi_k - (\partial_k\psi)(\xi_{n+1}^2 - s^2\eta^2\psi^{-2\eta-2}|\partial_{n+1}\psi|^2)\}.
\end{aligned}
$$

Moreover we have

$$
\begin{aligned}
&(\partial_t A)\overline{\frac{\partial A}{\partial \zeta_{n+1}}}\\
&=2p(\partial_t p)\{(\xi_{n+1}^2 - s^2\eta^2\psi^{-2\eta-2}|\partial_{n+1}\psi|^2) - 2is\eta\psi^{-\eta-1}(\partial_{n+1}\psi)\xi_{n+1}\}\\
&\times\{\xi_{n+1} + is\eta\psi^{-\eta-1}(\partial_{n+1}\psi)\}
\end{aligned}
$$

and

$$
\begin{aligned}
&\Im\left((\partial_t A)\overline{\frac{\partial A}{\partial \zeta_{n+1}}}\right)\\
&=2p(\partial_t p)s\eta\psi^{-\eta-1}\{(\partial_{n+1}\psi)(\xi_{n+1}^2 - s^2\eta^2\psi^{-2\eta-2}|\partial_{n+1}\psi|^2) - 2(\partial_{n+1}\psi)\xi_{n+1}^2\}.
\end{aligned}
$$

Therefore we obtain

$$J_2(x,t,\zeta) = 2\eta\psi^{-\eta-1}[-\{(\nabla p\cdot\nabla\psi) + p(\partial_t p)(\partial_{n+1}\psi)\}\xi_{n+1}^2 + 2(\nabla p\cdot\xi')(\partial_{n+1}\psi)\xi_{n+1}]$$

$$+2s^2\eta^3\psi^{-3\eta-3}|\partial_{n+1}\psi|^2\{(\nabla p\cdot\nabla\psi) - p(\partial_t p)\partial_{n+1}\psi\}. \tag{2.21}$$

On the other hand, let $(x,t)\in\overline{Q_\varepsilon}$. Then

$$-\kappa\sum_{j=2}^n |x_j|^2 \le x_1 < -\frac{1}{2N}\sum_{j=2}^n |x_j|^2 - \frac{1}{2N}t^2 + \frac{\varepsilon}{2N} \le \frac{\varepsilon}{2N}, \tag{2.22}$$

so that

$$\frac{1-2N\kappa}{2N}\sum_{j=2}^n |x_j|^2 < \frac{\varepsilon}{2N},$$

that is,

$$\sum_{j=2}^n |x_j|^2 \le \frac{\varepsilon}{1-2N\kappa}. \tag{2.23}$$

By (2.22), we have

$$|x_1| \le \max\left\{\frac{\varepsilon\kappa}{1-2N\kappa}, \frac{\varepsilon}{2N}\right\}. \tag{2.24}$$

Moreover, by (2.22) and (2.23), we obtain

$$-\kappa\frac{\varepsilon N}{1-2N\kappa} + \frac{1}{2}t^2 < Nx_1 + \frac{1}{2}t^2 + \sum_{j=2}^n |x_j|^2 < \frac{\varepsilon}{2},$$

that is,

$$t^2 < \varepsilon + \frac{2\kappa N\varepsilon}{1-2N\kappa}. \tag{2.25}$$

Therefore, in terms of (2.8), we have

$$|x|^2 + t^2 \le \varepsilon^2\left|\max\left\{\frac{\kappa}{1-2N\kappa}, \frac{1}{2N}\right\}\right|^2 + \frac{\varepsilon}{1-2N\kappa} + \varepsilon + \frac{2\kappa N\varepsilon}{1-2N\kappa}$$

$$\equiv \mu_0(\varepsilon) \le \delta_0. \tag{2.26}$$

Hence, by (2.18) and the Schwarz inequality, we obtain

$$
\begin{aligned}
&- \{(\nabla p \cdot \nabla\psi) + p(\partial_t p)(\partial_{n+1}\psi)\}\xi_{n+1}^2 + 2(\nabla p \cdot \xi')(\partial_{n+1}\psi)\xi_{n+1} \\
\geq &- \{(\nabla p \cdot \nabla\psi) + p(\partial_t p)(\partial_{n+1}\psi)\}\xi_{n+1}^2 - |\nabla p||\partial_{n+1}\psi|(|\xi'|^2 + |\xi_{n+1}|^2) \\
= &- \{(\nabla p \cdot \nabla\psi) + p(\partial_t p)(\partial_{n+1}\psi) + |\nabla p||\partial_{n+1}\psi|(p+1)\}\xi_{n+1}^2 \\
&-|\nabla p||\partial_{n+1}\psi|s^2\eta^2\psi^{-2\eta-2}(|\nabla\psi|^2 - p|\partial_{n+1}\psi|^2).
\end{aligned}
$$

Therefore, in terms of (2.26), inequality (2.21) yields

$$
\begin{aligned}
&J_2(x,t,\zeta) \geq -2\eta\psi^{-\eta-1}\{(\nabla p \cdot \nabla\psi) + p(\partial_t p)(\partial_{n+1}\psi) + |\nabla p||\partial_{n+1}\psi|(p+1)\}\xi_{n+1}^2 \\
&+2s^2\eta^3\psi^{-3\eta-3}\{((\nabla p \cdot \nabla\psi) - p(\partial_t p)\partial_{n+1}\psi)(\partial_{n+1}\psi)^2 - |\nabla p||\partial_{n+1}\psi|(|\nabla\psi|^2 - p|\partial_{n+1}\psi|^2)\} \\
&\geq - 2\eta\psi^{-\eta-1}\{N(\partial_1 p) + 2(M_1^2 + M_1)\sqrt{\mu_0(\varepsilon)}\}\xi_{n+1}^2 - 2s^2\eta^3\psi^{-3\eta-3} \times C(N, M_1, \delta_0).
\end{aligned}
\tag{2.27}
$$

Here and henceforth $C(N, M_1, \delta_0) > 0$ denotes generic constants which are independent of $\eta > 0$ and $s > 0$. Similarly, by (2.26), we have

$$
(p+1)\xi_{n+1}^2 < (M_0+1)\xi_{n+1}^2
$$

and

$$
\begin{aligned}
&(\eta+1)\left(N^2 + \sum_{j=2}^n |x_j|^2 - pt^2\right)^2 - \left(2N^2 + 2\sum_{j=2}^n |x_j|^2 + (p^2-p)t^2\right)\psi \\
\geq&(\eta+1)(N^2 - M_1\mu_0(\varepsilon))^2 - C(N, M_1, \delta_0),
\end{aligned}
$$

so that (2.20) implies

$$
\begin{aligned}
&J_1(x,t,\zeta) \geq 4s^2\eta^3\psi^{-3\eta-4}\{\eta(N^2 - M_1\mu_0(\varepsilon))^2 - C(N, M_1, \delta_0)\} \\
&-4\eta\psi^{-\eta-1}(M_0^2 + M_0)\xi_{n+1}^2.
\end{aligned}
\tag{2.28}
$$

Estimates (2.27) and (2.28) yield

$$J(x,t,\zeta) \geq 2\eta\psi^{-\eta-1}\xi_{n+1}^2(-N(\partial_1 p) - 2(M_0^2 + M_0) - 2(M_1^2 + M_1)\sqrt{\mu_0(\varepsilon)})$$

$$+4s^2\eta^3\psi^{-3\eta-3}\left\{\eta(N^2 - M_1\mu_0(\varepsilon))^2 - (1+\varepsilon)C(N, M_1, \delta_0)\right\}.$$

By the first inequality in (2.9) and (2.6), we have

$$- N(\partial_1 p) - 2(M_0^2 + M_0) - 2(M_1^2 + M_1)\sqrt{\mu_0(\varepsilon)}$$

$$> \alpha N - 2(M_0^2 + M_0) - 2(M_1^2 + M_1)\sqrt{\mu_0(\varepsilon)} \equiv \mu_1(N, M_1, \delta_0, \varepsilon) > 0.$$

Moreover, by the second inequality in (2.9), we choose $\eta > 0$ sufficiently large, so that

$$\eta(N^2 - M_1\mu_0(\varepsilon))^2 - (1+\varepsilon)C(N, M_1, \delta_0) \equiv \mu_2(N, M_1, \delta_0, \varepsilon) > 0.$$

Hence we obtain

$$J(x,t,\zeta) \geq 2\eta\psi^{-\eta-1}\xi_{n+1}^2\mu_1(N, M_1, \delta, \varepsilon) + 4s^2\eta^3\psi^{-3\eta-3}\mu_2(N, M_1, \delta_0, \varepsilon)$$

for $(x,t) \in \overline{Q_\varepsilon}$ if (2.15) holds. Thus the proof of Lemma 1 is complete.

§3. Proof of Theorem 2.

It is sufficient to prove Theorem 2 because Theorem 1 follows directly from Theorem 2. On the basis of Lemma 1, a Carleman estimate, we introduce a cut-off function and apply a usual argument (e.g., Chapter VII in Hörmander [11], Chapter 3 in Isakov [19]).

Since Δ is invariant with respect to rotations, translation and symmetric transforms of the coordinate system, without loss of generality, we may assume that $x_0 = 0 = (0, ..., 0)$, $\nu(x_0) = (1, 0, ..., 0)$ and that Γ is given by (2.1) near 0. Therefore for $\kappa > 0$ satisfying (1.4), we choose δ_0, N, ε such that (2.6) - (2.9) hold. Let ψ be defined by (2.10) and let us set $\varphi = \psi^{-\eta}$ for sufficiently large $\eta > 0$.

First we will determine the boundary of Q_ε. By (2.10) and (2.11), for $0 < \mu \le \varepsilon$, we have

$$\begin{aligned}\partial Q_\mu = &\left\{(x,t) \in \mathbb{R}^{n+1};\, x_1 = \gamma(x_2,...,x_n), \sum_{j=2}^n |x_j|^2 < \delta_0,\, \psi(x,t) < \mu\right\}\\ \bigcup&\left\{(x,t) \in \mathbb{R}^{n+1};\, x_1 > \gamma(x_2,...,x_n), \sum_{j=2}^n |x_j|^2 < \delta_0,\, \psi(x,t) = \mu\right\}\\ \bigcup&\left\{(x,t) \in \mathbb{R}^{n+1};\, x_1 > \gamma(x_2,...,x_n), \sum_{j=2}^n |x_j|^2 = \delta_0,\, \psi(x,t) < \mu\right\}\\ \equiv&\partial Q_\mu^1 \cup \partial Q_\mu^2 \cup \partial Q_\mu^3. \end{aligned} \tag{3.1}$$

We can prove that

$$\partial Q_\mu^3 = \emptyset.$$

In fact, since $x_1 > -\kappa \sum_{j=2}^n |x_j|^2$ and $\sum_{j=2}^n |x_j|^2 \le \delta_0$ by (2.23), we have

$$-2N\kappa \sum_{j=2}^n |x_j|^2 + \sum_{j=2}^n |x_j|^2 + t^2 < 2Nx_1 + \sum_{j=2}^n |x_j|^2 + t^2 = 2\psi(x,t) - \varepsilon < 2\mu - \varepsilon \le \varepsilon,$$

that is, $(1-2N\kappa)\delta_0 + t^2 < \varepsilon$ by $1 - 2N\kappa > 0$. Moreover (2.8) implies $\frac{\varepsilon}{1-2N\kappa} < \delta_0$, so that $\varepsilon + t^2 < \varepsilon$, which is impossible.

Moreover

$$\partial Q_\mu^j \subset \overline{Q_\varepsilon^j}, \qquad j = 1, 2,$$

and it follows from (2.25) that $(x,t) \in \overline{Q_\varepsilon}$ implies

$$|t| \le \left(\varepsilon + \frac{2\kappa N\varepsilon}{1-2N\kappa}\right)^{\frac{1}{2}} \equiv t_0, \tag{3.2}$$

so that

$$\begin{aligned} &\partial Q_\mu^1 \subset \{x;\, x_1 = \gamma(x_2,...,x_n)\} \times \{|t| \le t_0\},\\ &\partial Q_\mu^2 \subset \{x;\, \psi(x,t) = \mu\} \quad \text{for } 0 < \mu \le \varepsilon. \end{aligned} \tag{3.3}$$

Now we will proceed to the proof of Theorem 2. By the extension theorem, there exists $F \in H^2(D^+ \times (-T,T))$ such that

$$
\begin{cases}
F = g, \quad \dfrac{\partial F}{\partial \nu} = h \quad \text{on } \Gamma \times (-T,T), \\
\|F\|^2_{H^2(D^+\times(-T,T))} \le C\Big(\|g\|^2_{H^{\frac{3}{2}}(\Gamma\times(-T,T))} + \|g\|^2_{H^2(-T,T;L^2(\Gamma))} \\
+\|h\|^2_{H^2(-T,T;L^2(\Gamma))} + \|h\|^2_{L^2(-T,T;H^{\frac{1}{2}}(\Gamma))}\Big) \equiv C\mathcal{D}.
\end{cases} \tag{3.4}
$$

Set $u - F = v$, and we have

$$
\begin{cases}
Av = f - AF \qquad \text{in } D^+ \times (-T,T), \\
v = \dfrac{\partial v}{\partial \nu} = 0 \quad \text{on } \Gamma \times (-T,T).
\end{cases} \tag{3.5}
$$

Let us fix $0 < \varepsilon_0 < \frac{\varepsilon}{8}$ arbitrarily and let us introduce a cut-off function $\chi = \chi(x,t) \in C_0^\infty(\mathbb{R}^{n+1})$ such that $0 \le \chi \le 1$ and

$$
\chi(x,t) = \begin{cases} 1, & \psi(x,t) \le \varepsilon - 2\varepsilon_0, \\ 0, & \varepsilon - \varepsilon_0 \le \psi(x,t) \le \varepsilon. \end{cases} \tag{3.6}
$$

We set

$$
w = \chi v.
$$

Then, by the choice of ε, N, κ, noting (3.2) - (3.4), we see that

$$
w \in H_0^2(Q_\varepsilon).
$$

By (3.5), we have

$$
\begin{aligned}
Aw = 2p(\partial_t p)(\partial_t \chi) + pv(\partial_t^2 \chi) \\
-2\nabla v \cdot \nabla \chi - v\Delta\chi - \sum_{j=1}^{n+1}(q_j \partial_j \chi)v + \chi(f - AF) \quad \text{in } Q_\varepsilon.
\end{aligned}
$$

Henceforth $C > 0$ denotes generic constants which are independent of $s > 0$. Therefore we can apply Lemma 1 to Aw, so that

$$\begin{aligned}
&\int_{Q_\varepsilon} (s^3|w|^2 + s|\nabla w|^2 + s|\partial_t w|^2)e^{2s\varphi}dxdt \\
\le& C\int_{Q_\varepsilon} \left| 2p(\partial_t p)(\partial_t \chi) + pv(\partial_t^2\chi) - 2\nabla v\cdot\nabla\chi - v\Delta\chi - \sum_{j=1}^{n+1}(q_j\partial_j\chi)v \right|^2 e^{2s\varphi}dxdt \\
+& C\int_{Q_\varepsilon} |f - AF|^2 e^{2s\varphi}dxdt.
\end{aligned}$$

By (3.6), the first integral at the right hand side is not zero only if $\varepsilon - 2\varepsilon_0 \le \psi(x,t) \le \varepsilon - \varepsilon_0$, that is, $\psi(x,t)^{-\eta} \le (\varepsilon - 2\varepsilon_0)^{-\eta}$. Hence (3.4) yields

$$\begin{aligned}
&\int_{Q_\varepsilon} (s^3|w|^2 + s|\nabla w|^2 + s|\partial_t w|^2)e^{2s\varphi}dxdt \\
\le& C\|u\|^2_{H^1(Q_\varepsilon)} \exp(2s(\varepsilon - 2\varepsilon_0)^{-\eta}) + Ce^{2sC}(\|f\|^2_{L^2(Q_\varepsilon)} + \mathcal{D})
\end{aligned}$$

for all large $s > 0$. Since

$$\begin{aligned}
&\int_{Q_\varepsilon} (s^3|w|^2 + s|\nabla w|^2 + s|\partial_t w|^2)e^{2s\varphi}dxdt \\
\ge& \int_{Q_{\varepsilon-3\varepsilon_0}} (s^3|v|^2 + s|\nabla v|^2 + s|\partial_t v|^2)e^{2s\varphi}dxdt \\
\ge& \exp(2s(\varepsilon - 3\varepsilon_0)^{-\eta}) \int_{Q_{\varepsilon-3\varepsilon_0}} (s^3|v|^2 + s|\nabla v|^2 + s|\partial_t v|^2)dxdt,
\end{aligned}$$

by means of (3.6), we obtain

$$\begin{aligned}
&\exp(2s(\varepsilon - 3\varepsilon_0)^{-\eta}) \int_{Q_{\varepsilon-3\varepsilon_0}} (s^3|v|^2 + s|\nabla v|^2 + s|\partial_t v|^2)dxdt \\
\le& C\|u\|^2_{H^1(Q_\varepsilon)} \exp(2s(\varepsilon - 2\varepsilon_0)^{-\eta}) + Ce^{2sC}(\|f\|^2_{L^2(D^+\times(-T,T))} + \mathcal{D}),
\end{aligned}$$

that is, there exists a constant $s_0 > 0$ such that

$$\|v\|^2_{H^1(Q_{\varepsilon-3\varepsilon_0})} \le C\|u\|^2_{H^1(Q_\varepsilon)} e^{-s\mu_3} + Ce^{2sC}\mathcal{D}_1 \tag{3.7}$$

for all $s \ge s_0$. Here we set $\mu_3 = 2((\varepsilon - 3\varepsilon_0)^{-\eta} - (\varepsilon - 2\varepsilon_0)^{-\eta}) > 0$ and $\mathcal{D}_1 = \mathcal{D} + \|f\|^2_{L^2(D^+\times(-T,T))}$.

In (3.7), setting $s + s_0$ by s, we replace C by $C' = Ce^{2s_0C}$, so that we see that (3.7) holds for all $s \ge 0$. If $\mathcal{D}_1 = 0$ in (3.7), then $u = v$ and

$$\|u\|^2_{H^1(Q_{\varepsilon-3\varepsilon_0})} \le C\|u\|^2_{H^1(Q_\varepsilon)}e^{-s\mu_3}$$

for all $s > 0$, so that letting $s \longrightarrow \infty$, we have $u = 0$ in $Q_{\varepsilon-3\varepsilon_0}$. Therefore conclusion (1.10) holds. Next let $\mathcal{D}_1 > 0$. If $\|u\|^2_{H^1(Q_\varepsilon)} \le \mathcal{D}_1$, then conclusion (1.10) is obtained already.

If $\|u\|^2_{H^1(Q_\varepsilon)} > \mathcal{D}_1$, then we can set

$$s = \frac{1}{2C + \mu_3}\log\frac{\|u\|^2_{H^1(Q_\varepsilon)}}{\mathcal{D}_1} > 0.$$

Then (3.7) yields

$$\|v\|^2_{H^1(Q_{\varepsilon-3\varepsilon_0})} \le 2C\mathcal{D}_1^{\frac{\mu_3}{2C+\mu_3}}\|u\|^{\frac{4C}{2C+\mu_3}}_{H^1(Q_\varepsilon)}.$$

By definition (2.11) of $Q_{\varepsilon-3\varepsilon_0}$ and $\varepsilon - 3\varepsilon_0 > \frac{1}{2}\varepsilon$, we see that $Q_{\varepsilon-3\varepsilon_0}$ is a non-empty open set. Hence (1.10) follows. Thus the proof of Theorem 2 is complete.

§4. Proof of Theorem 3.

We follow the argument by Imanuvilov and Yamamoto [14], [15], and the new ingredient is our Carleman estimate Lemma 1. Let us set

$$u = u_1 - u_2, \quad d = r_1 - r_2.$$

Then

$$\begin{aligned}&(Au)(x,t) \equiv p(x)\partial_t^2u(x,t) - \Delta u(x,t)\\ -&\sum_{k=1}^n q_k(x,t)\partial_k u(x,t) - q_{n+1}(x,t)\partial_t u(x,t) - r_1(x)u(x,t) = d(x)u_2(x,t)\\ &\qquad \text{in } D^+\times(-T,T),\end{aligned} \tag{4.1}$$

$$u(x,0) = \partial_t u(x,0) = 0, \quad x \in D^+ \tag{4.2}$$

and

$$\begin{aligned} &u(x,t) = (u_1 - u_2)(x,t) \equiv g(x,t), \\ &\frac{\partial u}{\partial \nu}(x,t) = \frac{\partial}{\partial \nu}(u_1 - u_2)(x,t) \equiv h(x,t), \quad (x,t) \in \Gamma \times (-T,T). \end{aligned} \tag{4.3}$$

Here, by the smoothness of $u_1 - u_2$, we note that $\partial_t^k u = \partial_t^k \frac{\partial u}{\partial \nu} = 0$, $k = 0,1$ on $\Gamma \times \{t = 0\}$.

Similarly to (3.4), we can choose $F \in H^2(D^+ \times (-T,T)) \cap H^3(-T,T;L^2(D^+))$ such that

$$F = g, \quad \frac{\partial F}{\partial \nu} = h \quad \text{on } \Gamma \times (-T,T), \tag{4.4}$$

$$\begin{aligned} &\sum_{k=0}^{1} \|\partial_t^k F\|^2_{H^2(D^+\times(-T,T))} \le C\Bigg(\sum_{k=0}^{1}(\|\partial_t^k g\|^2_{H^{\frac{3}{2}}(\Gamma\times(-T,T))} + \|\partial_t^k g\|^2_{H^2(-T,T;L^2(\Gamma))} \\ &+\|\partial_t^k h\|^2_{H^2(-T,T;L^2(\Gamma))} + \|\partial_t^k h\|^2_{L^2(-T,T;H^{\frac{1}{2}}(\Gamma))})\Bigg) \equiv C\mathcal{D}_2. \end{aligned} \tag{4.5}$$

Set

$$v = u - F.$$

Then we have

$$Av = d(x)u_2(x,t) - AF \quad \text{in } D^+ \times (-T,T) \tag{4.6}$$

$$v(x,0) = -F(x,0), \quad \partial_t v(x,0) = -(\partial_t F)(x,0), \qquad x \in D^+ \tag{4.7}$$

and

$$v = \frac{\partial v}{\partial \nu} = 0 \quad \text{on } \Gamma \times (-T,T). \tag{4.8}$$

Similarly to Section 3, we can assume that $x_0 = (0,...,0)$, $\nu(x_0) = (1,0,...,0)$ and that Γ is given by (2.1) near 0. For $\kappa > 0$ satisfying (1.4), we can choose δ_0, N, ε such that (2.6) - (2.9) hold. We note (3.3).

For fixed $\varepsilon_0 \in \left(0, \frac{\varepsilon}{8}\right)$, we choose the cut-off function χ defined by (3.6). We set $t = x_{n+1}$, $\partial_t = \partial_{n+1}$,

$$\varphi(x,t) = \psi(x,t)^{-\eta} \tag{4.9}$$

with $\eta > 0$ given in Lemma 1, and

$$z = (\partial_t v)e^{s\varphi}\chi \qquad \text{in } Q_\varepsilon. \tag{4.10}$$

Then, by (3.3) and (3.6), we see that

$$z \in H_0^2(Q_\varepsilon). \tag{4.11}$$

Moreover, by (4.6), we can verify

$$\begin{aligned} Az = d(\partial_t u_2)e^{s\varphi}\chi - \partial_t(AF)e^{s\varphi}\chi + \sum_{j=1}^{n+1}(\partial_t q_j)(\partial_j v)e^{s\varphi}\chi \\ +s\{-2\nabla\varphi\cdot\nabla z + 2p(\partial_t\varphi)(\partial_t z) + (A\varphi + r_1\varphi)z\} - s^2(p|\partial_t\varphi|^2 - |\nabla\varphi|^2)z \\ +2e^{s\varphi}\{p(\partial_t^2 v)(\partial_t\chi) - (\nabla(\partial_t v)\cdot\nabla\chi)\} + (\partial_t v)e^{s\varphi}(A\chi + r_1\chi) \quad \text{in } Q_\varepsilon. \end{aligned} \tag{4.12}$$

In fact,

$$\partial_j z = (\partial_t\partial_j v)e^{s\varphi}\chi + (\partial_t v)s(\partial_j\varphi)e^{s\varphi}\chi + (\partial_t v)e^{s\varphi}\partial_j\chi, \tag{4.13}$$

that is,

$$(\partial_t\partial_j v)e^{s\varphi}\chi = \partial_j z - (\partial_t v)s(\partial_j\varphi)e^{s\varphi}\chi - (\partial_t v)e^{s\varphi}\partial_j\chi, \quad 1 \le j \le n+1. \tag{4.14}$$

Therefore, by (4.13) and (4.14), we have

$$\partial_j^2 z = (\partial_t\partial_j^2 v)e^{s\varphi}\chi + 2(\partial_t\partial_j v)s(\partial_j\varphi)e^{s\varphi}\chi + 2(\partial_t\partial_j v)e^{s\varphi}\partial_j\chi$$

$$+2(\partial_t v)s(\partial_j\varphi)e^{s\varphi}\partial_j\chi + (\partial_t v)s(\partial_j^2\varphi)e^{s\varphi}\chi + (\partial_t v)s^2|\partial_j\varphi|^2 e^{s\varphi}\chi + (\partial_t v)e^{s\varphi}\partial_j^2\chi$$

$$=(\partial_t\partial_j^2 v)e^{s\varphi}\chi + 2s(\partial_j\varphi)\{\partial_j z - (\partial_t v)s(\partial_j\varphi)e^{s\varphi}\chi - (\partial_t v)e^{s\varphi}\partial_j\chi\}$$

$$+2(\partial_t\partial_j v)e^{s\varphi}\partial_j\chi + 2(\partial_t v)s(\partial_j\varphi)e^{s\varphi}\partial_j\chi + (\partial_t v)s(\partial_j^2\varphi)e^{s\varphi}\chi$$

$$+(\partial_t v)s^2|\partial_j\varphi|^2 e^{s\varphi}\chi + (\partial_t v)e^{s\varphi}\partial_j^2\chi$$

$$=(\partial_t\partial_j^2 v)e^{s\varphi}\chi + 2s(\partial_j\varphi)\partial_j z + (s(\partial_j^2\varphi) - s^2|\partial_j\varphi|^2)(\partial_t v)e^{s\varphi}\chi$$

$$+2(\partial_t\partial_j v)e^{s\varphi}\partial_j\chi + (\partial_t v)e^{s\varphi}\partial_j^2\chi, \quad 1 \le j \le n+1.$$

Therefore direct substitution yields (4.12).

Moreover we set

$$w = (\partial_t v)\chi.$$

Then, setting $s = 0$ in (4.12), we have

$$\begin{aligned} Aw &= d(\partial_t u_2)\chi - \partial_t(AF)\chi + \sum_{j=1}^{n+1}(\partial_t q_j)(\partial_j v)\chi \\ &+2\{p(\partial_t^2 v)(\partial_t\chi) - (\nabla(\partial_t v)\cdot\nabla\chi)\} + (\partial_t v)(A\chi + r_1\chi) \quad \text{in } Q_\varepsilon \end{aligned} \tag{4.15}$$

and

$$w \in H_0^2(Q_\varepsilon). \tag{4.16}$$

Consequently we apply Lemma 1 to w:

$$\int_{Q_\varepsilon}(s^3w^2 + s|\nabla w|^2 + s|\partial_t w|^2)e^{2s\varphi}dxdt$$

$$\le C\int_{Q_\varepsilon}(|d(\partial_t u_2)\chi|^2 + |\partial_t(AF)\chi|^2)e^{2s\varphi}dxdt + C\sum_{j=1}^{n+1}\int_{Q_\varepsilon}|(\partial_t q_j)(\partial_j v)\chi|^2 e^{2s\varphi}dxdt$$

$$+C\int_{Q_\varepsilon}|2\{p(\partial_t^2 v)(\partial_t\chi) - (\nabla(\partial_t v)\cdot\nabla\chi)\} + (\partial_t v)(A\chi + r_1\chi)|^2 e^{2s\varphi}dxdt$$

for all large $s > 0$.

Henceforth C, C_j denote generic positive costants which depend on p, q_j, $1 \le j \le n+1$, r_1, M_1, N, η, ρ, ε, ε_0, but independent of s.

The third term at the right hand side contains derivatives of χ as factors, and so it is not zero only if $\varphi \le (\varepsilon - 2\varepsilon_0)^{-\eta}$ by means of (3.6). Therefore, noting for the second integral that $\partial_j(v\chi) = (\partial_j v)\chi + v\partial_j\chi$, $1 \le j \le n+1$, in terms of (1.14), we obtain

$$\begin{aligned}
&\int_{Q_\varepsilon} (s^3 w^2 + s|\nabla w|^2 + s|\partial_t w|^2) e^{2s\varphi} dxdt \\
\le& C_1 \int_{Q_\varepsilon} |d(\partial_t u_2)\chi|^2 e^{2s\varphi} dxdt + C_1 e^{C_2 s} \|\partial_t(AF)\|^2_{L^2(Q_\varepsilon)} \\
+& C \int_{Q_\varepsilon} (|\nabla(v\chi)|^2 + |\partial_t(v\chi)|^2 + |v\chi|^2) e^{2s\varphi} dxdt \\
+& C_1 \exp(2s(\varepsilon - 2\varepsilon_0)^{-\eta})
\end{aligned} \tag{4.17}$$

for all large $s > 0$.

On the other hand, setting $y = v\chi \in H_0^2(Q_\varepsilon)$, we have

$$Ay = du_2\chi - (AF)\chi + 2p(\partial_t v)(\partial_t \chi) - 2\nabla v \cdot \nabla \chi + v(A\chi + r_1\chi) \quad \text{in } Q_\varepsilon.$$

Therefore, similarly to (4.17), Lemma 1 yields

$$\begin{aligned}
&\int_{Q_\varepsilon} (s^3 |v\chi|^2 + s|\nabla(v\chi)|^2 + s|\partial_t(v\chi)|^2) e^{2s\varphi} dxdt \\
\le& C_1 \int_{Q_\varepsilon} |d\chi|^2 e^{2s\varphi} dxdt + C_1 e^{C_2 s} \|AF\|^2_{L^2(Q_\varepsilon)} + C_1 \exp(2s(\varepsilon - 2\varepsilon_0)^{-\eta})
\end{aligned} \tag{4.18}$$

for all large $s > 0$. Substitution of (4.18) into (4.17) yields

$$\begin{aligned}
&\int_{Q_\varepsilon} (s^3 w^2 + s|\nabla w|^2 + s|\partial_t w|^2) e^{2s\varphi} dxdt \\
\le& C_1 \int_{Q_\varepsilon} |d\chi|^2 e^{2s\varphi} dxdt + C_1 e^{C_2 s} \|AF\|^2_{H^1(-T,T;L^2(D^+))} + C_1 \exp(2s(\varepsilon - 2\varepsilon_0)^{-\eta})
\end{aligned} \tag{4.19}$$

for all large $s > 0$.

Noting $s^3 w^2 e^{2s\varphi} = s^3 z^2$ and

$$s|\partial_j z|^2 = s|\partial_j w + s(\partial_j\varphi)w|^2 e^{2s\varphi} \le C(s|\nabla_{x,t} w|^2 e^{2s\varphi} + s^3 w^2 e^{2s\varphi}),$$

we see from (4.19) that

$$\begin{aligned}
&\int_{Q_\varepsilon} (s^3 z^2 + s|\nabla z|^2 + s|\partial_t z|^2) dxdt \\
\le& C_1 \int_{Q_\varepsilon} |d\chi|^2 e^{2s\varphi} dxdt + C_1 e^{C_2 s} \|AF\|^2_{H^1(-T,T;L^2(D^+))} + C_1 \exp(2s(\varepsilon - 2\varepsilon_0)^{-\eta})
\end{aligned} \tag{4.20}$$

for all large $s > 0$.

Set $Q_\varepsilon^- = \{(x,t) \in Q_\varepsilon;\ t < 0\}$. Multiply (4.12) by $\partial_t z$ and integrate over Q_ε^-:

$$\begin{aligned}
&\int_{Q_\varepsilon^-} (Az)(\partial_t z) dxdt = \int_{Q_\varepsilon^-} \{d(\partial_t u_2) e^{s\varphi} \chi \partial_t z - \partial_t(AF) e^{s\varphi} \chi \partial_t z\} dxdt \\
&+ \int_{Q_\varepsilon^-} (\partial_t z) \sum_{j=1}^{n+1} (\partial_t q_j)(\partial_j v) e^{s\varphi} \chi dxdt \\
&+ \int_{Q_\varepsilon^-} \Bigg[s\{-2\nabla\varphi \cdot \nabla z + 2p(\partial_t\varphi)(\partial_t z) + (A\varphi + r_1\varphi) z\} \\
&- s^2 (p|\partial_t\varphi|^2 - |\nabla\varphi|^2) z \Bigg] \partial_t z dxdt \\
&+ \int_{Q_\varepsilon^-} \left[2e^{s\varphi}\{p(\partial_t^2 v)(\partial_t\chi) - (\nabla(\partial_t v) \cdot \nabla\chi)\} + (\partial_t v) e^{s\varphi} (A\chi + r_1\chi) \right] \partial_t z dxdt.
\end{aligned}$$

By I_1 and I_2 we denote the left and the right hand sides respectively. Then, by integration by parts, $z \in H_0^2(Q_\varepsilon)$ and the Schwarz inequality, we have

$$\begin{aligned}
I_1 \ge& \frac{1}{2} \int_{Q_\varepsilon^-} \partial_t(p|\partial_t z|^2) dxdt + \frac{1}{2} \int_{Q_\varepsilon^-} \partial_t(|\nabla z|^2) dxdt \\
&- C_3 \int_{Q_\varepsilon^-} (|\nabla z|^2 + |\partial_t z|^2 + |z|^2) dxdt \\
=& \frac{1}{2} \int_{Q_\varepsilon \cap \{t=0\}} (p(x)|(\partial_t z)(x,0)|^2 + |\nabla z(x,0)|^2) dxdt \\
&- C_3 \int_{Q_\varepsilon^-} (|\nabla z|^2 + |\partial_t z|^2 + |z|^2) dxdt.
\end{aligned}$$

Furthermore by (1.15), (4.5) - (4.7) and (4.10), we obtain

$$|(\partial_t z)(x,0)| = \left|(\partial_t v)(x,0)\frac{\partial(e^{s\varphi}\chi)}{\partial t}(x,0) + (\partial_t^2 v)(x,0)e^{s\varphi(x,0)}\chi(x,0)\right|$$

$$\ge |p(x)^{-1}e^{s\varphi(x,0)}d(x)a(x)\chi(x,0) - e^{s\varphi(x,0)}\chi(x,0)(\partial_t^2 F)(x,0)|$$

$$-|(\partial_t F)(x,0)||s(\partial_t\varphi)(x,0)\chi(x,0) + (\partial_t\chi)(x,0)|e^{s\varphi(x,0)}$$

$$\ge C_4|d(x)||e^{s\varphi(x,0)}\chi(x,0)| - C_5e^{C_6 s}\mathcal{D}_2.$$

We can estimate $|\nabla z(x,0)|^2$ similarly, so that

$$I_1 \ge C_7\int_{Q_\varepsilon\cap\{t=0\}} |d(x)|^2|\chi(x,0)|^2e^{2s\varphi(x,0)}dx - C_7e^{sC_8}\mathcal{D}_2$$

$$-C_3\int_{Q_\varepsilon}(|\nabla z|^2 + |\partial_t z|^2 + |z|^2)dxdt$$

$$\ge C_7\int_{Q_\varepsilon\cap\{t=0\}} |d(x)|^2|\chi(x,0)|^2e^{2s\varphi(x,0)}dxdt - C_7\int_{Q(\varepsilon)}|d\chi|^2e^{2s\varphi}dxdt$$

$$-C_7e^{sC_8}\mathcal{D}_2 - C_7\exp(2s(\varepsilon-2\varepsilon_0)^{-\eta}) \tag{4.21}$$

by (4.20). Moreover, arguing similarly to the estimate of the last term at the right hand side of (4.17), by the Schwarz inequality, we obtain

$$I_2 \le C_9\int_{Q_\varepsilon}|d|^2e^{2s\varphi}\chi^2dxdt + C_9\int_{Q_\varepsilon}|\partial_t(AF)|^2e^{2s\varphi}\chi^2dxdt$$

$$+C_9\int_{Q_\varepsilon}(s^3|z|^2 + s|\nabla z|^2 + s|\partial_t z|^2)dxdt$$

$$+C_9\int_{Q_\varepsilon}\left(\sum_{j=1}^{n+1}|\partial_j v|^2 + |v|^2\right)\chi^2e^{2s\varphi}dxdt + C_9\exp(2s(\varepsilon-2\varepsilon_0)^{-\eta}).$$

Applying (4.5), (4.18) and (4.20), we have

$$I_2 \le C_9\int_{Q_\varepsilon}|d|^2e^{2s\varphi}\chi^2dxdt + C_9e^{C_{10}s}\mathcal{D}_2 + C_9\exp(2s(\varepsilon-2\varepsilon_0)^{-\eta}) \tag{4.22}$$

for all large $s > 0$. Hence (4.21) and (4.22) yield

$$\int_{Q_\varepsilon\cap\{t=0\}}|d(x)|^2|\chi(x,0)|^2e^{2s\varphi(x,0)}dx \le C_9\int_{Q_\varepsilon}|d|^2e^{2s\varphi}\chi^2dxdt$$

$$+C_9e^{sC_{10}}\mathcal{D}_2 + C_9\exp(2s(\varepsilon-2\varepsilon_0)^{-\eta})$$

for all large $s > 0$. Replacing the integral at the left hand side over $Q_\varepsilon \cap \{t=0\}$ by the one over $Q_{\varepsilon-3\varepsilon_0} \cap \{t=0\}$, in terms of (3.6), we have

$$\begin{aligned}
&\int_{Q_{\varepsilon-3\varepsilon_0}\cap\{t=0\}} |d(x)|^2 e^{2s\varphi(x,0)} dx \le C_9 \left(\int_{Q_{\varepsilon-3\varepsilon_0}} |d|^2 e^{2s\varphi} dxdt + \int_{Q_\varepsilon \setminus Q_{\varepsilon-3\varepsilon_0}} |d|^2 e^{2s\varphi} \chi^2 dxdt \right) \\
&+C_9 e^{sC_{10}} \mathcal{D}_2 + C_9 \exp(2s(\varepsilon - 2\varepsilon_0)^{-\eta}) \\
&\le C_9 \int_{Q_{\varepsilon-3\varepsilon_0}} |d|^2 e^{2s\varphi} dxdt + C_9 \exp(2s(\varepsilon - 3\varepsilon_0)^{-\eta}) \\
&+C_9 e^{sC_{10}} \mathcal{D}_2 + C_9 \exp(2s(\varepsilon - 2\varepsilon_0)^{-\eta})
\end{aligned} \tag{4.23}$$

for all large $s > 0$. Since $Q_{\varepsilon-3\varepsilon_0} \subset (Q_{\varepsilon-3\varepsilon_0} \cap \{t=0\}) \times (-T, T)$, we have

$$\begin{aligned}
&\int_{Q_{\varepsilon-3\varepsilon_0}} |d|^2 e^{2s\varphi} dxdt \\
\le &\int_{Q_{\varepsilon-3\varepsilon_0}\cap\{t=0\}} |d|^2 e^{2s\varphi(x,0)} \left(\int_{-T}^{T} e^{2s(\varphi(x,t)-\varphi(x,0))} dt \right) dx.
\end{aligned}$$

Here the mean value theorem implies

$$\begin{aligned}
&\varphi(x,t) - \varphi(x,0) = \psi(x,t)^{-\eta} - \psi(x,0)^{-\eta} \\
=&\eta\Lambda^{-\eta-1}(\psi(x,0) - \psi(x,t)) = \eta\Lambda^{-\eta-1}\left(-\frac{1}{2}t^2\right),
\end{aligned}$$

where Λ is a number such that $\psi(x,0) < \Lambda < \psi(x,t)$. Here $\psi(x,t) \le \frac{1}{2}\varepsilon + N|x_1| + \frac{1}{2}\left(\sum_{j=2}^n |x_j|^2 + t^2\right)$ and

$$\psi(x,t) \ge -\kappa N \sum_{j=2}^n |x_j|^2 + \frac{1}{2}\sum_{j=2}^n |x_j|^2 + \frac{1}{2}t^2 + \frac{\varepsilon}{2} \ge \frac{\varepsilon}{2}$$

for $(x,t) \in \overline{Q_\varepsilon}$. Therefore we apply (2.26) and can take constants $C'_{11} > 0$ and $C''_{11} > 0$ such that $C''_{11} \le \psi(x,t) \le C'_{11}$ for $(x,t) \in \overline{Q_\varepsilon}$. Therefore

$$\varphi(x,t) - \varphi(x,0) \le -\frac{1}{2}\eta\psi(x,t)^{-\eta-1}t^2 \le -\frac{1}{2}C_{11}t^2.$$

Hence

$$\int_{-T}^{T} e^{2s(\varphi(x,t)-\varphi(x,0))}dt \le \int_{-T}^{T} e^{-C_{11}st^2}dt = \frac{C_{12}}{\sqrt{s}}.$$

Substituting this inequality into (4.23), we obtain

$$\begin{aligned}&\left(1-\frac{C_{13}}{\sqrt{s}}\right)\int_{Q_{\varepsilon-3\varepsilon_0}\cap\{t=0\}} |d(x)|^2 e^{2s\varphi(x,0)}dx\\ \le& C_9\exp(2s(\varepsilon-3\varepsilon_0)^{-\eta}) + C_9 e^{sC_{10}}\mathcal{D}_2.\end{aligned}$$

Again replacing the integral over $Q_{\varepsilon-3\varepsilon_0}\cap\{t=0\}$ at the left hand side by the one over $Q_{\varepsilon-4\varepsilon_0}\cap\{t=0\}$, noting that $e^{2s\varphi(x,0)} \ge \exp(2s(\varepsilon-4\varepsilon_0)^{-\eta})$ in $Q_{\varepsilon-4\varepsilon_0}\cap\{t=0\}$, and $1-\frac{C_{13}}{\sqrt{s}} \ge \frac{1}{2}$ for sufficiently large $s>0$, we obtain

$$\begin{aligned}&\int_{Q_{\varepsilon-4\varepsilon_0}\cap\{t=0\}} |d(x)|^2 dx\\ \le& C_{14}\exp[-2s\{(\varepsilon-4\varepsilon_0)^{-\eta}-(\varepsilon-3\varepsilon_0)^{-\eta}\}] + C_{14}e^{sC_{10}}\mathcal{D}_2\end{aligned}$$

for all large $s>0$. Therefore we argue similarly to the derivation of (1.10) from (3.7) in the proof of Theorem 2.

We note that $(x,0)\in Q_{\varepsilon-4\varepsilon_0}$ implies that

$$-\kappa\sum_{j=2}^{n}|x_j|^2 < x_1 < -\frac{1}{2N}\sum_{j=2}^{n}|x_j|^2 + \frac{\varepsilon-8\varepsilon_0}{2N}.$$

By $0<\varepsilon_0<\frac{\varepsilon}{8}$ and $\kappa<\frac{1}{2N}$, in terms of definition (2.11) of $Q_{\varepsilon-4\varepsilon_0}$, we see that there exists a non-empty neighbourhood $\mathcal{V}$ of 0 such that $Q_{\varepsilon-4\varepsilon_0}\cap\{t=0\}\supset\mathcal{V}$. Thus the proof of Theorem 3 is complete.

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