UTMS 2004–17 May 26, 2004

The second main theorem for holomorphic curves into semi-abelian varieties II

by

J. Noguchi, J. Winkelmann and K. Yamanoi

UNIVERSITY OF TOKYO

GRADUATE SCHOOL OF MATHEMATICAL SCIENCES

KOMABA, TOKYO, JAPAN

The Second Main Theorem for Holomorphic Curves into Semi-Abelian Varieties II

Junjiro Noguchi, Jörg Winkelmann and Katsutoshi Yamanoi

Abstract

We establish the second main theorem with the best truncation level one

$$T_f(r; L(\bar{D})) \leqq N_1(r; f^*D) + \epsilon T_f(r)||_\epsilon$$

for an entire holomorphic curve $f : \mathbf{C} \to A$ into a semi-abelian variety A and an arbitrary effective reduced divisor D on A; the low truncation level is important for applications. We will actually prove this for the jet lifts of f. Finally we give some applications, including the solution of a problem posed by Mark Green.

1 Introduction and main result

Let $f : \mathbf{C} \to V$ be a holomorphic curve into a complex projective manifold V with Zariski dense image and let D be an effective reduced divisor on V. Under some ampleness condition for the space $H^0(V, \Omega^1_V(\log D))$ of logarithmic 1-forms along D we proved in [N77], [N81] the following inequalities of the second main theorem type,

$$\kappa T_f(r) \leqq N(r; f^*D) + O(\log r) + O(\log T_f(r))||,$$
$$\kappa' T_f(r) \leqq N_1(r; f^*D) + O(\log r) + O(\log T_f(r))||,$$

where $T_f(r)$ denotes the order function of f, $N(r; f^*D)$ (resp. $N_l(r; f^*D)$) the counting function (resp. truncated to level l) of the pull-backed divisor f^*D, and κ and κ' are positive constants (cf. §2). It is an interesting and fundamental problem to determine the constant κ or κ'. In the case where V is the compactification of a semi-abelian variety A this problem is related to what kind of compactification V of A we take. In our former paper [NWY02] we proved that for a holomorphic curve $f : \mathbf{C} \to A$ into a semi-abelian variety A and an algebraic divisor D on A,

$$T_f(r; L(\bar{D})) \leqq N_l(r; f^*D) + O(\log r) + O(\log T_f(r; L(\bar{D})))||. \tag{1.1}$$

Here we used a compactification $\bar{A}$ of A such that the maximal affine subgroup $(\mathbf{C}^*)^t$ of A was compactified by $(\mathbf{P}^1(\mathbf{C}))^t$, and we assumed a boundary condition (Condition 4.11 in [NWY02]) for the closure $\bar{D}$ of D in $\bar{A}$; this roughly meant the divisor $\bar{D} + (\bar{A} \setminus A)$ to be in general position and has been expected to be removed by a suitable choice of a compactification of A. It is an important and very interesting problem to take the truncation level l as small as possible.

Let $X_k(f)$ denote the Zariski closure of the image of the k-jet lift of f in the k-jet space $J_k(A)$ over A. The purpose of this paper is to prove (cf. §§2, 3 for notation)

Main Theorem. *Let A be a semi-abelian variety. Let $f : \mathbf{C} \to A$ be a holomorphic curve with Zariski dense image. Let D be an effective reduced Cartier divisor on $X_k(f)$ $(k \geqq 0)$. Then there exists a compactification $\bar{X}_k(f)$ of $X_k(f)$ such that*

$$T(r; \omega_{\bar{D}, J_k(f)}) \leqq N_1(r; J_k(f)^* D) + \epsilon T_f(r)||_\epsilon, \quad \forall \epsilon > 0, \tag{1.2}$$

where $\bar{D}$ is the closure of D in $\bar{X}_k(f)$.

In the case of $k = 0$ the compactification $\bar{A}$ of A can be chosen as smooth, equivariant with respect to the A-action, and independent of f; furthermore, (1.2) *takes the form*

$$T_f(r; L(\bar{D})) \leqq N_1(r; f^* D) + \epsilon T_f(r; L(\bar{D}))||_\epsilon, \quad \forall \epsilon > 0. \tag{1.3}$$

Note that in the above estimate (1.2) or (1.3) the small error term "$\epsilon T_f(r)$" cannot be replaced by "$O(\log r) + O(\log T_f(r))$" (see [NWY02] Example (5.36)).

The Main Theorem is an advancement of [NWY02] and [Y04]. When A is an abelian variety, the above Main Theorem was proved by [Y04], where the case of $k > 0$ was implicit (see [Y04] (3.1.8)). There is a related result due to Siu-Yeung [SY03]. They obtained (1.1) with a truncation level $l = l(D)$ dependent only on the Chern numbers of D; in [SY03] the key was Claim 1 at p. 443, same as [NYW02] Lemma 5.6 in the abelian case but for the improved dependence of the order $l(D)$ of jets.

It is interesting to observe that the error term being "$O(\log r) + O(\log T_f(r; L(\bar{D})))||$", the truncation level l in (1.1) has to depend on D, but the error term being allowed to be "$\epsilon T_f(r; L(\bar{D})))||_\epsilon$", l can be one, the smallest possible.

To deal with semi-abelian varieties the main difficulties are caused by the following two points:

(i) Semi-abelian varieties are not compact and need some good compactifications.

(ii) There is no Poincaré reducibility theorem for semi-abelian varieties.

It is also noted that a part of the proof of the Main Theorem for abelian varieties in [Y04] does not hold for semi-abelian varieties ([Y04] §3 Claim), and that a different and considerably simpler proof for that part will be provided (see Lemma 6.1).

In §7 we will give two applications of the Main Theorem. The first is a complete affirmative answer to a conjecture of M. Green [G74] pp. 229–230 (cf. Theorem 7.2). The second is a non-existence theorem for some differential equations defined over semi-abelian varieties (cf. Theorem 7.6).

Acknowledgement. We learned the conjecture of M. Green [G74] from Professor A.E. Eremenko, to whom we are very grateful.

2 Notation

The notation here follows that of [NWY02]. For a general reference of this section, cf. [NO$\frac{84}{90}$]. For convenience we recall some of definitions. Let M be a compact complex manifold and let ω be a smooth (1,1)-form on M. Let $f : \mathbf{C} \to M$ be a holomorphic curve into M. We define the order function of f with respect to ω by

$$T_f(r;\omega) = \int_1^r \frac{dt}{t} \int_{|z|<t} f^*\omega \qquad (r > 1). \tag{2.1}$$

If M is Kähler and $d\omega = 0$,

$$T_f(r;\omega) = T_f(r;\omega') + O(1)$$

for a d-closed (1,1)-form ω' in the same cohomology class $[\omega] \in H^2(M, \mathbf{R})$. Therefore we set, up to $O(1)$-term,

$$T_f(r;[\omega]) = T_f(r;\omega). \tag{2.2}$$

Let $L \to M$ be a hermitian line bundle with Chern class $c_1(L)$. Then we set

$$T_f(r;L) = T_f(r;c_1(L)),$$

which is defined again up to $O(1)$-term.

For a divisor D on M we denote by $L(D)$ the line bundle determined by D.

Let $E = \sum_{\mu=1}^\infty \nu_\mu z_\mu$ be a divisor on $\mathbf{C}$ with distinct $z_\mu \in \mathbf{C}$. Then we set

$$\mathrm{ord}_z E = \begin{cases} \nu_\mu, & z = z_\mu, \\ 0, & z \notin \{z_\mu\}. \end{cases}$$

We define the counting functions of E truncated to $l \leqq \infty$ by

$$n_l(t; E) = \sum_{\{|z_\mu|<t\}} \min\{\nu_\mu, l\},$$
$$N_l(r; E) = \int_1^r \frac{n_l(t; E)}{t} dt.$$

We define the counting functions of E by

$$n(t; E) = n_\infty(t; E), \qquad N(r; E) = N_\infty(r; E).$$

Definition of small terms. (i) For a line bundle $L \to M$ and a holomorphic curve $f : \mathbf{C} \to M$ we denote by $S_f(r; L)$ such a small term as

$$S_f(r; L) = O(\log r) + O(\log^+ T_f(r; L))||,$$

where "$||$" stands for the inequality to hold for every $r > 1$ outside a Borel set of finite Lebesgue measure.

(ii) Let $h(r)$ $(r > 1)$ be a real valued function. We write

$$h(r) \leqq \epsilon T_f(r; L)||_\epsilon, \quad \forall \epsilon > 0,$$

if the stated inequality holds for every $r > 1$ outside a Borel set of finite Lebesgue measure, dependent on an arbitrarily given $\epsilon > 0$.

Definition. When M is an algebraic variety, we say that $f : \mathbf{C} \to M$ is *algebraically (resp. non-) degenerate* if the image $f(\mathbf{C})$ is (resp. not) contained in a proper algebraic subset of M.

The following follows from general properties of order functions ([NO$\frac{84}{90}$]).

Lemma 2.3 *Let $f : \mathbf{C} \to M$ be a holomorphic curve into a complex projective manifold M and H a line bundle on M. Assume that H is big, and that f is algebraically non-degenerate. Then*

$$T_f(r, L) = O(T_f(r, H))$$

for every line bundle L on M.

If $f : \mathbf{C} \to M$ is algebraically degenerate, we may consider the Zariski closure N of $f(\mathbf{C})$ and a desingularization $\tau : \tilde{N} \to N$. Then f lifts to a map to $\tilde{N}$ and $\tau^*(H|_N)$ is big on $\tilde{N}$ for every ample line bundle H on M. As a consequence we obtain:

Lemma 2.4 *Let $f : \mathbf{C} \to M$ be a holomorphic curve into a complex projective manifold M. Let $h(r)$ be a non-negative valued function in $r > 1$. Then $h(r) = S_f(r; H)$ holds for every ample line bundle if and only if it holds for at least one ample line bundle.*

Similarily the statement $h(r) \leqq \epsilon T_f(r; H)||_\epsilon, \forall \epsilon > 0$, respectively $h(r) = O(T_f(r; H))$ holds for every ample line bundle H if and only if it holds for at least one ample line bundle.

If one of these conditions holds for one and therefore for all ample line bundles H, we simply write $h(r) = S_f(r)$ (resp. $h(r) \leqq \epsilon T_f(r)||_\epsilon$, $h(r) = O(T_f(r))$).

For a quasi-projective manifold V and for a holomorphic curve $f : \mathbf{C} \to V$ we write simply $T_f(r) = T_f(r; H)$ for the order function with respect to an ample line bundle H over a projective compactification $\bar{M}$ of M if the choice of $\bar{M}$ and H do not matter.

The following related property of order functions will be frequently used ([NO$\frac{84}{90}$] Lemma (6.1.5)).

Lemma 2.5 *Let $\eta : V \to W$ be a rational mapping between quasi-projective manifolds V and W. Then for an algebraically non-degenerate holomorphic curve $f : \mathbf{C} \to V$*

$$T_{\eta \circ f}(r) = O(T_f(r)).$$

Moreover, if η is generically finite, then

$$T_f(r) = O(T_{\eta \circ f}(r)).$$

We define the proximity function $m_f(r; \mathcal{I})$ not only for divisors but also for a coherent ideal sheaf $\mathcal{I}$ of the structure sheaf $\mathcal{O}_M$ over M. Let $\{U_j\}$ be a finite open covering of M such that

(i) there is a partition of unity $\{c_j\}$ associated with $\{U_j\}$,

(ii) there are finitely many sections $\sigma_{jk} \in \Gamma(U_j, \mathcal{I}), k = 1, 2, \ldots,$ generating every fiber $\mathcal{I}_x$ over $x \in U_j$.

Setting $\rho_{\mathcal{I}}(x) = \left(\sum_j c_j(x) \sum_k |\sigma_{jk}(x)|^2\right)^{1/2}$, we take a positive constant C so that

$$C\rho_{\mathcal{I}}(x) \leqq 1, \qquad x \in M.$$

Using the compactness of M, one easily verifies that, up to addition by a bounded continuous function on M, $\log \rho_{\mathcal{I}}$ is independent of the choices of the open covering, the partition of unity, the local generators of the ideal sheaf $\mathcal{I}$, and the constant C.

We define the proximity function of f for $\mathcal{I}$ or for the subspace (may be non-reduced) $Y = (\operatorname{Supp}\mathcal{O}_M/\mathcal{I}, \mathcal{O}/\mathcal{I})$ by

$$(2.6) \qquad m_f(r; Y) = m_f(r; \mathcal{I}) = \int_{|z|=r} \log \frac{1}{C\rho_{\mathcal{I}}(f(re^{i\theta}))} \frac{d\theta}{2\pi} \quad (\geqq 0),$$

provided that $f(\mathbf{C}) \not\subset \operatorname{Supp} Y$. Note that if $\mathcal{I}$ is the ideal sheaf defined by an effective divisor D on M, $m_f(r; \mathcal{I})$ coincides $m_f(r; D)$ defined in [NWY02] up to $O(1)$-term. The function $\rho_{\mathcal{I}} \circ f(z)$ is smooth over $\mathbf{C} \setminus f^{-1}(\operatorname{Supp} Y)$. For $z_0 \in f^{-1}(\operatorname{Supp} Y)$ choose an open neighborhood U of z_0 and a positive integer ν such that $f^*\mathcal{I} = ((z - z_0)^\nu)$. Then

$$\log \rho_{\mathcal{I}} \circ f(z) = \nu \log |z - z_0| + \psi(z), \quad z \in U.$$

for some smooth function $\psi(z)$ defined on U. We define the counting function $N(r; f^*\mathcal{I})$ and $N_l(r; f^*\mathcal{I})$ by using ν in the same way as using $\operatorname{ord}_{z_0}(E)$ in the definition of $N(r; E)$ and $N_l(r; E)$. Moreover we define

$$(2.7) \qquad \begin{aligned} \omega_{\mathcal{I},f} = \omega_{Y,f} &= -dd^c\psi(z) = -\frac{i}{2\pi}\partial\bar{\partial}\psi(z) \\ &= dd^c \log \frac{1}{\rho_{\mathcal{I}} \circ f(z)} \quad (z \in U), \end{aligned}$$

which is well-defined on $\mathbf{C}$ as a smooth (1,1)-form. The order function of f for $\mathcal{I}$ or Y is defined by

$$(2.8) \qquad T(r; \omega_{\mathcal{I},f}) = T(r; \omega_{Y,f}) = \int_1^r \frac{dt}{t} \int_{|z|<t} \omega_{\mathcal{I},f}.$$

When $\mathcal{I}$ defines a divisor D on M, we see that

$$T(r; \omega_{\mathcal{I},f}) = T_f(r; L(D)) + O(1).$$

Let $\mathcal{I}_i$ $(i = 1, 2)$ be coherent ideal sheaves of $\mathcal{O}_M$ and let Y_i be the subspace defined by $\mathcal{I}_i$. We write $Y_1 \supset Y_2$ if $\mathcal{I}_1 \subset \mathcal{I}_2$.

Theorem 2.9 *Let $f : \mathbf{C} \to M$ and $\mathcal{I}$ be as above. Then we have the following:*

(i) (First Main Theorem)

$$T(r; \omega_{\mathcal{I},f}) = N(r; f^*\mathcal{I}) + m_f(r; \mathcal{I}) - m_f(1; \mathcal{I}).$$

(ii) *If M is projective, $m_f(r, \mathcal{I}) = O(T_f(r))$.*

(iii) *Let $\mathcal{I}_i$ $(i = 1, 2)$ be coherent ideal sheaves of $\mathcal{O}_M$ and let Y_i be the subspace defined by $\mathcal{I}_i$. If $\mathcal{I}_1 \subset \mathcal{I}_2$ or equivalently $Y_1 \supset Y_2$, then*

$$m_f(r; \mathcal{I}_2) \leqq m_f(r; \mathcal{I}_1) + O(1),$$

or equivalently,

$$m_f(r;Y_2) \leqq m_f(r;Y_1) + O(1).$$

(iv) *Let $\phi : M_1 \to M_2$ be a holomorphic mappings between compact complex manifolds. Let $\mathcal{I}_2 \subset \mathcal{O}_{M_2}$ be a coherent ideal sheaf and let $\mathcal{I}_1 \subset \mathcal{O}_{M_1}$ be the coherent ideal sheaf generated by $\phi^*\mathcal{I}_2$. Then*

$$m_f(r;\mathcal{I}_1) = m_{\phi\circ f}(r;\mathcal{I}_2) + O(1).$$

(v) *Let $\mathcal{I}_i$, $i = 1,2$ be two coherent ideal sheaves of $\mathcal{O}_M$. Suppose that $f(\mathbf{C}) \not\subset$* Supp $(\mathcal{O}_M/\mathcal{I}_1 \otimes \mathcal{I}_2)$*. Then we have*

$$T(r;\omega_{\mathcal{I}_1\otimes\mathcal{I}_2,f}) = T(r;\omega_{\mathcal{I}_1,f}) + T(r;\omega_{\mathcal{I}_2,f}) + O(1).$$

Proof. (i) This immediately follows from the well-known Jensen formula (cf. [NO$\frac{84}{90}$] Theorem (5.2.15)).

(ii) Let Y be the subvariety defined by $\mathcal{I}$. There is an ample divisor D on M such that $D \supset Y$ (counting multiplicities). It follows from Theorem (2.9) (iii) that

$$m_f(r;Y) \leqq m_f(r;D) \leqq T_f(r;L(D)) = O(T_f(r)).$$

(iii) (iv) (v) These are immediate by definition. *Q.E.D.*

3 General position

Convention 3.1 Unless explicitly stated otherwise, all varieties, morphisms, group actions, compactifications, divisors etc. are assumed to be algebraic.

3.1 General position

Let A be a semi-abelian variety and let X be a complex algebraic variety (possibly singular) on which A acts:

$$(a,x) \in A \times X \to a \cdot x \in X.$$

Let Y be a subvariety embedded into a Zariski open subset of X.

Definition 3.2 We say that Y is *generally positioned in* X if the closure $\bar{Y}$ of Y in X contains no A-orbit. If the support of a divisor E on a Zariski open subset of X is generally positioned in X, then E is said to be generally positioned in X.

Let $\pi : X_1 \to X$ be a blow-up of smooth projective manifolds on which A acts. Let D be a divisor on X and let D_1 be its strict transform. Then $D_1 \sim \pi^*D - E$, where E is an effective divisor with support contained in the exceptional locus of the blow-up. If π is the blow-up along a smooth connected submanifold $C \subset X$, then E is empty unless $C \subset D$.

Lemma 3.3 *Assume that D is generally positioned in X. Let $\pi : X_1 \to X$ be an equivariant blow-up. Then $D_1 = \pi^*D$, i.e., E is empty.*

Proof. Since the blow-up is assumed to be equivariant, its center C must be an invariant subset, i.e., C is a union of A-orbits. Now D is assumed to be generally positioned in X. This implies that D contains no A-orbit. Therefore no irreducible component of C is contained in D. *Q.E.D.*

Corollary 3.4 *Assume that D is big and generally positioned in X. Then D_1 is big, too.*

Proof. This is immediate from $D_1 = \pi^*D$. *Q.E.D.*

Unfortunately the assumption of being generally positioned can not be dropped. For example, let us consider $X = \mathbf{P}^2(\mathbf{C})$. Let D be a line and let $X_1 \to X$ be the blow-up of a point p on the line D. Then X_1 is a ruled surface. It admits a fibration $\tau : X_1 \to \mathbf{P}^1(\mathbf{C})$ which arises as follows: We may identify $\mathbf{P}^1(\mathbf{C})$ with $\mathbf{P}(T_p\mathbf{P}^2(\mathbf{C}))$. Then for $x \in \mathbf{P}^2(\mathbf{C}) \setminus \{p\}$ we set $\tau(x)$ to be the tangent line at p of the unique line in $\mathbf{P}^2(\mathbf{C})$ connecting p and x. Now the strict transform D_1 of D turns out to be a fiber of τ. As a fiber of a holomorphic map, it can not be big. However, D, as an effective divisor on $\mathbf{P}^2(\mathbf{C})$, is big.

To give another example, consider the blow-up of $\mathbf{P}^2(\mathbf{C})$ in two points $p, q \in D$. A blow-up decreases the self-intersection number of a curve by 1. Therefore the self-intersection number of the strict transform D_2 of D under this blow-up $X_2 \to X$ is a curve with self-intersection number -1. As a consequence we have $\dim H^0(X_2, L(nD_2)) = 1$ for all $n \in \mathbf{N}$.

Note that these examples are equivariant for a suitably chosen action of $A = (\mathbf{C}^*)^2$, but D is not generally positioned in $\mathbf{P}^2(\mathbf{C})$.

On the other hand, bigness can only be destroyed, not created via blow-up. This follows from the following fact: $D_1 = \pi^*D - E$ where E is effective. Thus fixing a section $\sigma \in H^0(X_1, E)$ we obtain an injection

$$H^0(X_1, L(nD_1)) \overset{\alpha}{\hookrightarrow} H^0(X_1, L(n\pi^*D)) \cong H^0(X, L(nD)) \quad (\forall n \in \mathbf{N})$$

given by mapping a section to its tensor product with σ^n. Therefore the Iitaka D-dimension can only decrease ([I71]).

Lemma 3.5 *Let $\pi : X_1 \to X$ be an equivariant blow-up, let D be a divisor on X which is generally positioned in X, and let D_1 be its strict transform. Then D_1 is generally positioned in X_1, too.*

Proof. If D_1 would contain an A-orbit Ω, we could infer that $\pi(\Omega) \subset \pi(D_1) = D$. Since π is assumed to be equivariant, this would imply that D contains an A-orbit, namely $\pi(\Omega)$. *Q.E.D.*

3.2 Stabilizer

Let A be a semi-abelian variety such that

$$0 \to T \to A \xrightarrow{\pi} A_0 \to 0, \tag{3.6}$$

where $T \cong (\mathbf{C}^*)^t$ and A_0 is an abelian variety. Let D be a divisor on A. The stabilizer of D is defined by

$$\mathrm{St}(D) = \{a \in A : a + D = D\}^0, \tag{3.7}$$

where $\{\cdot\}^0$ denotes the identity component.

Lemma 3.8 *Let D be an effective divisor on A and let $\bar{D}$ be its closure in an equivariant compactification $\bar{A}$ of A. Let $L_0 \in \mathrm{Pic}(A_0)$ and let E be an A-invariant divisor on $\bar{A}$ such that $L(\bar{D}) \cong L(E) \otimes \pi^* L_0$. Assume that $\mathrm{St}(D)$ is contained in T. Then L_0 is ample on A_0.*

Proof. By [NW04] Lemma 5.2 we obtain $c_1(L_0) \geqq 0$. We may regard $c_1(L_0)$ as a bilinear form on a vector space V which can be interpreted as the Lie algebra $\mathrm{Lie}(A_0)$ or the dual of cotangent bundle $\Omega^1(A_0)^*$ over A_0. Assume that L_0 is not ample. Then there is a vector $v \in V \setminus \{0\}$ such that $c_1(L_0)|_{\mathbf{C}v} \equiv 0$. Choose a direct sum decomposition (orthogonal with respect to $c_1(L_0)$) $V = \mathbf{C}v \oplus V'$ and let ω be a $(1,1)$-form which is positive on V', but annihilates $\mathbf{C}v$. Then $c_1(L) \wedge \omega^{g-1} = 0$ where $g = \dim A_0 = \dim V$. Let Ω be a $(1,1)$-form on $\bar{A}$ which is positive along the fibers of $\bar{A} \to A_0$ as constructed in [NW03] Lemma 5.1. Then

$$0 = \int_{\bar{A}} \Omega^s \wedge \pi^* \left(c_1(L_0) \wedge \omega^{g-1} \right) = \int_D \Omega^s \wedge \pi^* \left(\omega^{g-1} \right)$$

By construction of ω this implies that v is everywhere tangent to D. But in this case $v \in \mathrm{Lie}(A_0)$ is in the Lie algebra of the stabilizer $\mathrm{St}(D)$. This is a contradiction. *Q.E.D.*

Proposition 3.9 *Let $\bar{A}$ be a smooth equivariant compactification of a semi-abelian variety A. Let D be an effective divisor on A and let $\bar{D}$ be its closure in $\bar{A}$. Then the following properties hold.*

(i) *$\bar{A} \setminus A$ is a divisor with only simple normal crossings.*

(ii) *If $\mathrm{St}(D) = \{0\}$, then $\bar{D}$ is big on $\bar{A}$.*

Proof. (i) This is [NW04] Lemma 3.4.

(ii) Due to [NW04] there is a line bundle L_0 on A_0 and an A-invariant divisor E on $\bar{A}$ such that $L(\bar{D}) \cong L(E) \otimes \pi^* L_0$. By Lemma 3.8 the triviality of $\mathrm{St}(D)$ implies the ampleness of L_0.

Now consider the T-action. Evidently E is T-invariant. Since T acts only along the fibers of $\pi : \bar{A} \to A_0$, the line bundle $\pi^* L_0$ is also T-invariant. It follows that for every $g \in T$ the pull-back $g^* D$ is linearly equivalent to D.[1] Next we define sets S_x for $x \in A$ as follows:

$$S_x = \cap_{g \in T : g(x) \in D} \, g^* D.$$

By this definition we know that for every $y \notin S_x$ there is a section σ in $L(D)$ such that $\sigma(x) = 0 \neq \sigma(y)$. From the definition it follows furthermore that S_x is an algebraic subvariety of A. Using the A-invariant trivialization of the tangent bundle $TA \cong A \times \mathrm{Lie}(A)$ we can identify $T_x(S_x)$ with a vector subspace of $\mathrm{Lie}(A)$. In this identification we obtain

$$T_x(S_x) = \cap_{g \in T : g(x) \in D} \, g^* D = \cap_{g \in T : g(x) \in D} \, T_{g(x)} D = \cap_{y \in \pi^{-1}(\pi(x)) \cap D} \, T_y(D).$$

Thus $T_x(S_x)$ depends only on $\pi(x)$. Let $F_x = \pi^{-1}(\pi(x))$. Then all the points in $F_x \cap S_x$ have the same tangent space. It follows that $F_x \cap S_x$ is an orbit under a Lie subgroup of T. On the other hand, $F_x \cap S_x$ is an algebraic subvariety. Therefore $F_x \cap S_x$ is an orbit under an algebraic subgroup of T. A priori this subgroup may depend on the point x. However, $T \cong (\mathbf{C}^*)^s$ contains only countably many algebraic subgroups. For this reason it follows that this algebraic subgroup must be the same for almost all points $x \in A$. Thus there is an algebraic subgroup $H \subset T$ such that each connected component of $S_x \cap F_x$ is a H-orbit for almost all $x \in A$. But this implies that D is invariant under H. Since $\mathrm{St}(D) = \{0\}$, H is finite. Thus $S_x \to A_0$ is generically finite for almost all $x \in A$. Combined with the ampleness of L_0 this implies that D is big. *Q.E.D.*

[1] Actually $g^* D \sim D$ holds for every $g \in T$ and every $T \cong (\mathbf{C}^*)^s$-action on a projective manifold. This can be deduced from the fact that the Picard variety of a projective manifold contains no rational curves.

Proposition 3.10 *Let D be an effective divisor on A and let $\bar{D}$ be its closure in a smooth equivariant compactification $\bar{A}$ of A. If $\mathrm{St}(D) = \{0\}$, then there is an equivariant blow-up $\bar{A}^\dagger \to \bar{A}$ such that the strict transform of $\bar{D}$ is generally positioned in $\bar{A}^\dagger$.*

In particular, there exists a smooth equivariant compactification of A in which D is generally positioned.

Proof. Using a result of Vojta ([V99] Theorem 2.4 (2)) we obtain a (possibly singular) completion $\hat{\imath} : A \hookrightarrow \hat{A}$ such that D is generally positioned in $\hat{A}$. Consider the diagonal embedding $j : A \hookrightarrow \bar{A} \times \hat{A}$ given by $j = (i, \hat{\imath})$ and let $\bar{A}'$ denote the closure of the image $j(A)$. Let $\bar{A}^\dagger \to \bar{A}'$ be an equivariant desingularization (cf. [Hi64], [BM97]). Then the composed map $\bar{A}^\dagger \to \bar{A}$ is a blow-up of $\bar{A}$. Considering the natural projection $\bar{A}^\dagger \to \hat{A}$, we conclude, as in Lemma 3.5, that D is generally positioned in $\bar{A}^\dagger$. *Q.E.D.*

Proposition 3.11 *Let A be a semi-abelian variety, let $A \to \bar{A}$ be an equivariant compactification and let D be a divisor on A. Then there is an equivariant blow-up $\tilde{A} \to \bar{A}$ such that the quotient $\tilde{A}/\mathrm{St}(D)$ exists.*

Proof. $\mathrm{St}(D)$ is an algebraic subgroup of A. Hence there is a quotient morphism $q : A \to A/\mathrm{St}(D)$. Let $A/\mathrm{St}(D) \subset Z$ be an A-equivariant smooth compactification. Then q is a morphism from an Zariski open subset of $\bar{A}$ to Z and thus defines a rational map from $\bar{A}$ to Z. Now we just blow up $\bar{A}$ and Z to remove the indeterminacies and obtain a regular morphism. Since $q : A \to A/\mathrm{St}(D)$ is equivariant, it is clear that the indeterminacies on $\bar{A}$ are A-invariant subvarieties. Therefore the blow-up can be done equivariantly. *Q.E.D.*

3.3 Finitely many orbits

We will need the following auxiliary result.

Lemma 3.12 *Let A be a semi-abelian variety and $A \hookrightarrow \bar{A}$ a smooth equivariant algebraic compactification. Then there are only finitely many A-orbits in $\bar{A}$.*

Proof. Let $\tau : \mathbf{C}^n \to A$ denote the universal covering. Then $A = \mathbf{C}^n/\Gamma$, where $\Gamma = \tau^{-1}\{0\}$. Note that Γ generates $\mathbf{C}^n$ as complex vector space.

Let H be an algebraic subgroup of A. Then H is a semi-abelian variety, too. It follows that the connected component $\hat{H}$ of $\tau^{-1}(H)$ coincides with the complex vector subspace of $\mathbf{C}^n$ generated by $\hat{H} \cap \Gamma$. Evidently there are only countably many finitely generated subgroups of Γ. It follows that there are only countably many algebraic subgroups H of A.

Let p be a point in $\bar{A}$ and let $H = A_p$ be its isotropy group. Let Ap denote the A-orbit through p. Let $\bar{A}^H$ denote the fixed point set of H-action, i.e., $\bar{A}^H = \{x \in \bar{A} : ax = x, \forall a \in H\}$. Then $\bar{A}^H$ is a closed algebraic subvariety of $\bar{A}$. Let $T_p(\bar{A}^H)$ be its Zariski tangent space at p. Because H is reductive, the H-action on $T_p(\bar{A})$ is almost effective. On the other hand, because H acts trivially on $\bar{A}^H$, the action on $T_p(\bar{A}^H)$ is likewise trivial. Therefore there is an almost effective H-action on the quotient vector space $T_p(\bar{A})/T_p(\bar{A}^H)$. Since H is abelian, this implies $\dim H \leqq \dim \big(T_p(\bar{A})/T_p(\bar{A}^H)\big)$. From this we deduce

$$\dim(Ap) = \dim A - \dim H \geqq \dim X - \dim \big(T_p(\bar{A})/T_p(\bar{A}^H)\big) = \dim T_p(\bar{A}^H)$$

Since $Ap \subset \bar{A}^H$, it follows that $\bar{A}^H$ is smooth at p and Ap is open in $\bar{A}^H$. In particular, there is an open neighborhood W of p in $\bar{A}$ such that Ap is the only A-orbit in W with H as isotropy group. Using algebraicity it follows that there are only finitely many A-orbits in $\bar{A}$ with H as isotropy group.

Since there are only countably many algebraic subgroups of A, we obtain as a consequence that there are only countably many A-orbits in $\bar{A}$.

Thus A is an algebraic group acting on an algebraic variety $\bar{A}$ with only countably many orbits. This implies that there are actually only finitely many orbits. *Q.E.D.*

3.4 Action

Let A be a semi-abelian variety and let $\mathbf{P}^N(\mathbf{C})$ be the complex projective N-space. Then A acts on the product $A \times \mathbf{P}^N(\mathbf{C})$ by the group action of the first factor:

$$(a, (b, x)) \in A \times (A \times \mathbf{P}^N(\mathbf{C})) \to a \cdot (b, x) = (a + b, x) \in A \times \mathbf{P}^N(\mathbf{C}).$$

Let $p : A \times \mathbf{P}^N(\mathbf{C}) \to A$ be the first projection. Let X be an irreducible algebraic subset of $A \times \mathbf{P}^N(\mathbf{C})$ such that $p(X) = A$. We set

$$B = \mathrm{St}(X) = \{a \in A; a \cdot X = X\}^0,$$

and assume that $\dim B > 0$. Set $C = A/B$.

Taking direct products with $\mathbf{P}^N(\mathbf{C})$ the projection $A \to C$ extends to $\tau : A \times \mathbf{P}^N(\mathbf{C}) \to C \times \mathbf{P}^N(\mathbf{C})$. This is a B-principal bundle. The subvariety X of $A \times \mathbf{P}^N(\mathbf{C})$ is B-invariant; therefore $X = \tau^{-1}(\tau(X))$. It follows that $\tau(X)$ is a closed subvariety of $C \times \mathbf{P}^N(\mathbf{C})$ which we can regard as the quotient X/B of X with respect to the B-action. In particular $\pi = \tau|_X : X \to Y = \tau(X)$ is a B-principal bundle such that the B-action on X is simply the principal right action of B for this bundle structure.

Let $\hat{B}$ be a smooth equivariant compactification of B. Then we have a relative compactification $\hat{A} \to C$ of $A \to C$ arising as the $\hat{B}$-bundle associated to the B-principal bundle $A \to C$. In other words: $\hat{A} = A \times_B \hat{B}$ where $A \times_B \hat{B}$ denotes the quotient of $A \times \hat{B}$ with respect to the equivalence relation for which $(a, b) \sim (a', b')$ if and only if there exists an element $g \in B$ such that $ag = a'$ and $b = gb'$. The projection map p extends to $\hat{p} : \hat{A} \times \mathbf{P}^N(\mathbf{C}) \to \hat{A}$. Let $\hat{X}$ be the closure of X in $\hat{A}$. Then $\hat{X} = X \times_B \hat{B}$. The compactness of $\hat{B}$ implies that the projection map $\hat{\pi} : \hat{X} \to Y$ is proper.

Let $E \subset X$ be an irreducible algebraic subset such that

$$B \cap \mathrm{St}(E) = \{0\}. \tag{3.13}$$

Proposition 3.14 *Let $\hat{X}$, X, E, etc. be as above. Assume in addition that E is of codimension one, i.e., a divisor. Then there is a B-equivariant blow-up*

$$\psi : X^\dagger \to \hat{X}$$

with center in $\hat{X} \setminus X$ such that $X^\dagger$ has a stratification by B-invariant strata

$$X^\dagger = \cup_\lambda \Gamma_\lambda$$

satisfying the following properties:

(i) $\Gamma_\lambda \cong X/B_x \quad (x \in \Gamma_\lambda)$ *where $B_x = \{b \in B : b \cdot x = x\}$ is the isotropy group at x.*

(ii) *The closure of E in $X^\dagger$ contains none of the strata Γ_λ.*

(iii) *The open subset X of $X^\dagger$ coincides with one of the strata Γ_λ.*

Proof. Before starting the proof we make a remark: Since $X \to Y$ is a B-principal bundle, we can define quotient varieties X/H for all algebraic subgroups H of B. Therefore statement (i) of the proposition makes sense.

Now we start the proof. We will only consider blow-ups $X^\dagger \to \hat{X}$ which arise in the following way: We take an equivariant blow-up $B^\dagger \to \hat{B}$ and define $X^\dagger = X \times_B B^\dagger$. We recall that there are only finitely many B-orbits in $B^\dagger$ (Lemma 3.12) and that $X \times_B B^\dagger$ is defined as a quotient of $X \times B^\dagger$. Let $\{\Omega_\lambda\}_\lambda$ be the family of B-orbits in $B^\dagger$. Then a stratification $\{\Gamma_\lambda\}_\lambda$ of $X^\dagger$ is induced as follows: For each λ we define Γ_λ is the image of $X \times \Omega_\lambda$ under the projection $X \times B^\dagger \to X \times_B B^\dagger = X^\dagger$. Each of these B-orbits Ω_λ can be written as quotient of B by some closed algebraic subgroup H_λ:

$$\Omega_\lambda \cong B/H_\lambda.$$

Then H_λ is the isotropy group of the B-action on Γ_λ at any point $x \in \Gamma_\lambda$ and $\Gamma_\lambda = X/H_\lambda$. Thus the stratification $\{\Gamma_\lambda\}_\lambda$ of $X^\dagger$ has the properties required by (i), for every choice of an equivariant blow-up $B^\dagger \to \hat{B}$.

By construction, the open subset X of $X^\dagger$ coincides with the open B-orbit in $B^\dagger$, hence (iii).

Let us now verify that $B^\dagger \to B$ can be chosen in such a way that property (ii) holds, too. For $y \in Y$ let E_y be defined as $E_y = \{p \in E : \pi(p) = y\}$. We observe that $\bar{E}_y = \pi^{-1}(y) \cap \bar{E}$ for almost all $y \in \pi(E)$. Using [N81], Lemma 4.1., we infer from (3.13) that for a generic point $y \in \pi(E)$ the fiber E_y has a discrete stabiliser with respect to the B-action on X. Thus we may invoke Proposition 3.10 and deduce that there exists an equivariant blow-up $B^\dagger \to \hat{B}$ such that E_y is generally positioned in $B^\dagger$. Let $X^\dagger \to \hat{X}$ be the associated blow-up of $\hat{X}$. Now E_y being generally positioned in $B^\dagger$ implies that the closure of E in $X^\dagger$ contains none of the strata Γ_λ. *Q.E.D.*

4 Second main theorem for jet lifts

Let A be a semi-abelian variety of dimension n and let T be the maximal affine subgroup of A. Then $T \cong (\mathbf{C}^*)^t$ and there is an exact sequence of rational homomorphisms

$$0 \to T \to A \to A_0 \to 0,$$

where A_0 is an abelian variety. Let $\bar{A}$ be a smooth equivariant compactification of A. Set $\partial A = \bar{A} \setminus A$ and let $J_k(\bar{A}, \log \partial A)$ be the logarithmic k-jet bundle along ∂A (cf. [N86]). Then A acts on $J_k(\bar{A}, \log \partial A)$ and there is an equivariant trivialization

$$J_k(\bar{A}, \log \partial A) \cong \bar{A} \times J_{k,A},$$

where A acts trivially on the second factor $J_{k,A} = \mathbf{C}^{kn}$. Let $\bar{J}_{k,A}$ be a projective compactification of $J_{k,A}$. With the trivial action of A on $\bar{J}_{k,A}$ and the usual action on A (by translations) and $\bar{A}$ this yields an A-equivariant compactification

$$\bar{J}_k(\bar{A}, \log \partial A) = \bar{A} \times \bar{J}_{k,A}$$

of $J_k(A)$ with an open A-invariant subset

$$\tilde{J}_k(A) = A \times \bar{J}_{k,A}.$$

For example, we may set $\bar{J}_{k,A} = \mathbf{P}^{nk}(\mathbf{C})$ or $\bar{J}_{k,A} = (\mathbf{P}^n(\mathbf{C}))^k$. Then $J_k(A) = J_k(\bar{A}, \log \partial A)|_A$ is a Zariski open subset of $\bar{J}_k(\bar{A}, \log \partial A)$ and

$$J_k(A) \cong A \times J_{k,A}.$$

We set

$$J_k^{\mathrm{reg}}(\bar{A}, \log \partial A) = \left\{ j_k(g) \in J_k(\bar{A}, \log \partial A); j_1(g) \neq 0 \right\} \cong \bar{A} \times J_{k,A}^{\mathrm{reg}},$$
$$J_k^{\mathrm{reg}}(A) = J_k^{\mathrm{reg}}(\bar{A}, \log \partial A)|_A \cong A \times J_{k,A}^{\mathrm{reg}},$$

of which elements are called *regular jets*.

Let $f : \mathbf{C} \to A$ be a holomorphic curve and $J_k(f) : \mathbf{C} \to J_k(A)$ be the k-jet lift of f. We denote by $X_k(f)$ (resp. $\tilde{X}_k(f)$) the Zariski closure of the image $J_k(f)(\mathbf{C})$ in $J_k(A)$ (resp. $\tilde{J}_k(A)$):

$$X_k(f) \subset J_k(A), \qquad \tilde{X}_k(f) \subset \tilde{J}_k(A). \tag{4.1}$$

Theorem 4.2 (Second Main Theorem) *Let $f : \mathbf{C} \to A$ be an algebraically non-degenerate holomorphic curve. Let D be an effective reduced Cartier divisor on $X_k(f)$. Then there exists a natural number l_0 and a compactification $\bar{X}_k(f)$ of $X_k(f)$ such that for the closure $\bar{D}$ of D in $\bar{X}_k(f)$*

$$m_{J_k(f)}(r; \bar{D}) = S_f(r), \tag{4.3}$$
$$T(r; \omega_{\bar{D}, J_k(f)}) \leqq N_{l_0}(r; J_k(f)^* D) + S_f(r). \tag{4.4}$$

In the case of $k = 0$ the compactification $\bar{A}$ of A can be chosen smooth, equivariant, and independent of f; moreover, (4.3) *and* (4.4) *take the following forms, respectively:*

$$m_f(r; \bar{D}) = S_f(r; L(\bar{D})), \tag{4.5}$$
$$T_f(r; L(\bar{D})) \leqq N_{l_0}(r; f^* D) + S_f(r; L(\bar{D})). \tag{4.6}$$

Proof. Since the very basic idea of the proof is the same as that of the Main Theorem of [NWY03], it will be helpful to confer it.

We extend the divisor D to the closure in $\tilde{X}_k(f)$ which is denoted by the same D.

We first prove (4.3) and (4.5). Set $B = \mathrm{St}(X_k(f))$. Then we have the quotient maps:

$$q^B : A \to A/B = C,$$
$$q_k^B : J_k(A) \to J_k(A)/B \cong C \times J_{k,A},$$
$$\tilde{q}_k^B : \tilde{J}_k(A) \to C \times \bar{J}_{k,A}.$$

By [N98] and [NW03] Lemma 2.3

$$\dim B > 0, \quad T_{q_k^B \circ J_k(f)}(r) = S_f(r). \tag{4.7}$$

Setting $\tilde{Y}_k = \tilde{X}_k(f)/B$, we have a quotient map:

$$\tilde{\pi}_k : \tilde{X}_k(f) \to \tilde{Y}_k \subset C \times \bar{J}_{k,A}.$$

Let $\bar{B}$ be a smooth equivariant compactification of B. Define $\hat{A}$, $\hat{X}_k(f)$, $\hat{D}$, etc. as the partial compactifications of A, $\tilde{X}_k(f)$, D, etc. as in subsection 3.3. We then have proper maps,

$$\hat{q}_k^B : \hat{A} \times \bar{J}_{k,A} \to C \times \bar{J}_{k,A},$$
$$\hat{\pi}_k = \hat{q}_k^B|_{\hat{X}_k(f)} : \hat{X}_k(f) \to \tilde{Y}_k \subset C \times \bar{J}_{k,A},$$

whose fibers are isomorphic to $\bar{B}$.

There are two cases, $B \subset \mathrm{St}(D)$ and $B \not\subset \mathrm{St}(D)$, which we consider separately.

(a) Suppose that $B \subset \mathrm{St}(D)$. Set $\hat{F} = \hat{\pi}_k(\hat{D}) = \hat{D}/B$. Then $\hat{F}$ is of codimension one in $\tilde{Y}_k$. Let $T \cong (\mathbf{C}^*)^t$ be the maximal affine subgroup of A and let S be that of B. Then S is a subgroup of T and there is a splitting, $T \cong S \times S'$. Take an equivariant compactification $\bar{S}'$ of S' and set

$$\bar{A} = \hat{A} \times_{S'} \bar{S}'.$$

Then $\bar{A}$ is an equivariant compactification of A and $\hat{A}$. We have an algebraic exact sequence

$$0 \to S' \to C \to C_0 \to 0,$$

where C_0 is an abelian variety, and an equivariant compactification $\bar{C} = C \times_{S'} \bar{S}'$. Thus $\hat{q}_k^B$ extends to

$$\bar{q}_k^B : \bar{A} \times J_{k,A} \to \bar{C} \times J_{k,A},$$

Let $\bar{X}_k(f)$ (resp. $\bar{Y}_k$, $\bar{F}$) be the closure of $\hat{X}_k(f)$ (resp. $\hat{Y}_k$, $\hat{F}$) in $\bar{A} \times \bar{J}_{k,A}$ (resp. $\bar{C} \times \bar{J}_{k,A}$). Thus we have the restriction

$$\bar{\pi}_k = \bar{q}_k^B|_{\bar{X}_k(f)} : \bar{X}_1(f) \to \bar{Y}_k.$$

Note that $\bar{\pi}_k$ is surjective and

$$\bar{F} \neq \bar{Y}_k. \tag{4.8}$$

It follows from Theorem 2.9 (ii) and (4.7) that

$$\begin{aligned} m_{J_k(f)}(r; \bar{D}) &\leqq m_{\bar{\pi}_k \circ J_k(f)}(r; \bar{F}) + O(1) \\ &= O(T_{\bar{\pi}_k \circ J_k(f)}(r)) = S_f(r). \end{aligned} \tag{4.9}$$

(b) Suppose that $B \not\subset \mathrm{St}(D)$. We set

$$B' = B \cap \mathrm{St}(D), \quad D' = D/B', \quad \tilde{X}'_k(f) = \tilde{X}_k(f)/B', \quad A' = A/B', \quad B'' = B/B'.$$

Moreover, we define F as the image of D under the quotient $\tilde{X}'_k(f) \to \tilde{X}'_k(f)/B'' = \tilde{Y}_k$. We have the following commutative diagram and quotient maps:

$$\begin{array}{ccccc}
D & \underset{(\mathrm{codim}=1)}{\subsetneqq} & \tilde{X}_k(f) & \subset & A \times \bar{J}_{k,A} \\
\downarrow & & \downarrow & & \downarrow q_k^{B'} \\
D' & \underset{(\mathrm{codim}=1)}{\subsetneqq} & \tilde{X}'_k(f) & \subset & A' \times \bar{J}_{k,A} \\
\downarrow \tilde{\pi}'_k|_{D'} & & \downarrow \tilde{\pi}'_k & & \downarrow q_k^{B''} \\
F & \subset & \tilde{Y}_k & \subset & C \times \bar{J}_{k,A}
\end{array}$$

Since $\mathrm{codim}_{\tilde{X}_k(f)} D = 1$, F is Zariski dense in $\tilde{Y}_k$. Note that

$$\mathrm{St}(X'_k(f)) = B'', \qquad \mathrm{St}(D') \cap B'' = \{0\}. \tag{4.10}$$

Let $\bar{B}''$ be a smooth equivariant compactification of B''. We have

$$\begin{aligned}
\hat{A}' &= A' \times_{B''} \bar{B}'', \\
\hat{\partial} A' &= \hat{A}' \setminus A', \\
\hat{X}'_k(f) &= \tilde{X}'_k(f) \times_{B''} \bar{B}'', \\
\hat{D}' &= \bar{D}' \quad \text{(the closure of } D' \text{ in } \hat{X}'_k(f)), \\
\hat{\partial} X'_k(f) &= \hat{X}'_k(f) \setminus \tilde{X}'_k(f).
\end{aligned} \tag{4.11}$$

Note that the boundary divisor $\hat{\partial} A'$ has only normal crossings (Proposition 3.9 (i)). We obtain proper maps

$$\begin{array}{ccccc}
\hat{D}' & \underset{(\mathrm{codim}=1)}{\subsetneqq} & \hat{X}'_k(f) & \subset & \hat{A}' \times \bar{J}_{k,A} \\
\downarrow \hat{\pi}'_k|_{\hat{D}'} & & \downarrow \hat{\pi}'_k & & \downarrow \hat{q}_k^{B''} \\
\hat{F} & = & \tilde{Y}_k & \subset & C \times \bar{J}_{k,A}\ ,
\end{array}$$

where $\hat{F} = \hat{\pi}'_k(\hat{D}')$. By Proposition 3.14 we have a blow-up

$$\psi : \hat{X}'^{\dagger}_k(f) \to \hat{X}'_k(f)$$

with center in $\hat{\partial} X'_k(f)$, the strict transform $\hat{D}'^{\dagger}$ of $\hat{D}'$ and the boundary

$$\Gamma = \hat{X}'^{\dagger}_k(f) \setminus \tilde{X}'_k(f)$$

with stratification $\Gamma = \cup_\lambda \Gamma_\lambda$ such that

$$\Gamma_\lambda \cong \tilde{X}'_k(f)/\mathrm{Iso}_x(B'') \quad (x \in \Gamma_\lambda), \tag{4.12}$$

$$\Gamma_\lambda \cap \hat{D}'^{\dagger} \neq \Gamma_\lambda. \tag{4.13}$$

Here, if $k = 0$, we use Proposition 3.10 in place of Proposition 3.14, and deduce the stated property for $\bar{A}$.

Let $\psi_{*l} : J_l(\hat{X}'^{\dagger}_k(f), \log \Gamma) \to J_l(\hat{X}'_k(f), \log \hat{\partial} X'_k(f))$ be the morphism naturally induced by ψ. We consider a sequence of morphisms

$$\begin{aligned} J_l(\hat{D}'^{\dagger}, \log \Gamma) \subset & J_l(\hat{X}'^{\dagger}_k(f), \log \Gamma) \overset{\psi_{*l}}{\to} J_l(\hat{X}'_k(f), \log \hat{\partial} X'_k(f)) \\ & \hookrightarrow J_l(\hat{A}' \times \bar{J}_{k,A}, \log(\hat{\partial} A' \times \bar{J}_{k,A})) \\ & \cong J_l(\hat{A}', \log \hat{\partial} A') \times J_l(\bar{J}_{k,A}) \\ & \cong \hat{A}' \times J_l(J_{k,A'}) \times J_l(\bar{J}_{k,A}) \\ & \overset{\text{proj.}}{\to} J_l(J_{k,A'}) \times J_l(\bar{J}_{k,A}). \end{aligned}$$

Thus we have a morphism

$$\beta_l : J_l(\hat{X}'^{\dagger}_k(f), \log \Gamma) \to J_l(J_{k,A'}) \times J_l(\bar{J}_{k,A}).$$

Let $p_l : J_l(\hat{X}'^{\dagger}_k(f)) \to \hat{X}'^{\dagger}_k(f)$ be the projection to the base space. Henceforth we obtain a proper morphism

$$\gamma_l = (\hat{\pi}'_k \circ \psi \circ p_l) \times \beta_l : J_l(\hat{X}'^{\dagger}_k(f), \log \Gamma) \to \tilde{Y}_k \times J_l(J_{k,A'}) \times J_l(\bar{J}_{k,A}).$$

We claim that for some $l_0 \geqq 1$

$$\gamma_{l_0}(J_{l_0}(\hat{D}')) \neq \gamma_{l_0}(J_{l_0}(\hat{X}'_k(f))). \tag{4.14}$$

Assume contrarily that $\gamma_l(J_l(\hat{D}')) = \gamma_l(J_l(\hat{X}'_k(f)))$ for all $l \geqq 1$. Then for an arbitrary $z \in \mathbf{C}$

$$J_l(q_1^{B'} \circ J_k(f))(z) \in \gamma_l(J_l(\hat{D}'^{\dagger}, \log \Gamma)). \tag{4.15}$$

Take $z_0 \in \mathbf{C}$ so that $\hat{\pi}_k \circ J_k(f)(z_0) \in \tilde{Y}^{\circ}_k$ and set

$$\xi_l = J_l(q_1^{B'} \circ J_k(f))(z_0) \in \gamma_l(J_l(\hat{D}'^{\dagger}, \log \Gamma)), \qquad l \geqq 1.$$

Set $\Xi_l = \gamma_l^{-1}(\xi_l)$ for $l \geqq 0$. Then the restriction $p_l|_{\Xi_l}$ is proper and $p_l|_{\Xi_l} : \Xi_l \to p_l(\Xi_l)$ is an isomorphism. We set

$$\Lambda_l = p_l(\Xi_l), \qquad l = 1, 2, \dots .$$

The sequence of $\Lambda_l \supset \Lambda_{l+1}$, $l = 1, 2, \dots$ terminates to $\Lambda_\infty = \Lambda_{l_0} = \Lambda_{l_0+1} = \cdots$ $(\subset \hat{X}'^{\dagger}_k(f))$ for some l_0. Then $\Lambda_\infty \neq \emptyset$. If $\Lambda_\infty \cap \tilde{X}'_k(f) \neq \emptyset$, there is an element $a \in A'$ such that

$$a \cdot (J_l(q_1^{B'} \circ J_k(f))(z_0)) \in J_l(D'), \quad \forall l \geqq 0.$$

By the identity principle we deduce that $a \cdot \tilde{X}'_k(f) \subset D'$; this is absurd.

Now assume that $\Lambda_\infty \cap \Gamma \neq \emptyset$. There is a point $x_0 \in \Lambda_\infty \cap \Gamma$ such that

$$(x_0, \xi_l) \in J_l(\hat{D}'^{\dagger})_{x_0}, \qquad l \geqq 1.$$

Let Γ_{λ_0} be the boundary stratum containing x_0. Let $\alpha : \tilde{X}'_k(f) \to \tilde{X}'_k(f)/\mathrm{Iso}_{x_0}(B'') \cong \Gamma_{\lambda_0}$ be the quotient map. Then there exists an element $a_0 \in A$ such that

$$a \cdot (\alpha \circ q_1^{B'} \circ J_k(f)(z)) \in \Gamma_{\lambda_0} \cap \hat{D}'^{\dagger}$$

in a neighborhood of z_0 and hence for all $z \in \mathbf{C}$. Henceforth a contradiction follows from this, (4.13) and the image $J_k(f)(\mathbf{C})$ being Zariski dense in $X_k(f)$.

This proves the claim.

By making use of the assumption for D to be Cartier, we infer (4.4) and (4.6) as in the proof of the Main Theorem of [NWY02] p. 152 (cf. [NWY02] (5.12)).

Let us now prove the additional statements for the case $k = 0$. In this case we take the quotient, $q : A \to A/\mathrm{St}(D)$ and we deal with the holomorphic curve $q \circ f : \mathbf{C} \to A/\mathrm{St}(D)$ and the divisor $D/\mathrm{St}(D)$.

In this way we may assume $\mathrm{St}(D) = \{0\}$. Then Proposition 3.9 (ii) implies that D is big and we can deduce (4.5) with the help of Lemma 2.4. *Q.E.D.*

5 Higher codimensional subvarieties of $X_k(f)$

Let $f : \mathbf{C} \to A$ be a holomorphic curve in a semi-abelian variety A. We use the same notation, $X_k(f)$, $\mathrm{St}(X_k(f))$, etc. as in the previous section.

The purpose of this section is to prove the following.

Theorem 5.1 *Let $f : \mathbf{C} \to A$ be a holomorphic curve and let $Z \subset X_k(f)$ be a subvariety of* $\mathrm{codim}_{X_k(f)} Z \geqq 2$. *Then*

$$N_1(r; J_k(f)^* Z) \leqq \epsilon T_f(r)||_\epsilon, \quad \forall \epsilon > 0.$$

Remark. For an abelian variety A this was proved by [Y04].

It suffices to prove Theorem 5.1 for irreducible Z. Hence, we assume throughout this section that Z is *irreducible*.

Our proof naturally divides into three steps (a)$\sim$(c). Before going to discuss the details, we give an outline of the proof.

(a) First, we reduce the case to the one that A admits a splitting $A = B \times C$ where B and C are semi-abelian varieties such that

$$B \subset \mathrm{St}(X_l(f)) \quad \text{for all } l \geqq 0 \tag{5.2}$$

and the composition of f and the second projection $q^B : A \to A/B = C$ satisfies

$$T_{q^B \circ f}(r) = S_f(r). \tag{5.3}$$

By this reduction, we may assume that the variety $X_l(f)$ has splitting $X_l(f) = B \times (X_l(f)/B)$ for all $l \geqq 0$.

We also make a reduction such that the image of Z under the second projection $\pi_k : X_k(f) \to X_k(f)/B$ has a Zariski dense image. Hence by the assumption $\mathrm{codim}_{X_k(f)} Z \geqq 2$, we may assume $\mathrm{codim}_{\pi_k^{-1}(x)} Z \cap \pi_k^{-1}(x) \geqq 2$ for general $x \in X_k(f)/B$.

(b) The second step is the main part of the proof. Using the above reduction, we shall construct auxiliary divisors $F_l \subset \bar{B} \times (X_{k+l}(f)/B)$ for all $l \geqq 0$ with the following properties:

(i) $(l+1)N_1(r; J_k(f)^*Z) \leqq N(r; J_{k+l}(f)^*F_l) + \epsilon T_f(r)||_\epsilon, \forall \epsilon > 0$:

(ii) $T_{J_{k+l}(f)}(r; L(F_l)) \leqq n(l) T_{\gamma \circ f}(r; D_B) + \epsilon T_f(r; D)||_\epsilon, \forall \epsilon > 0$,

where $\gamma : A \to B$ is the first projection, D is an ample line bundle over $\bar{A}$, D_B is an ample line bundle over $\bar{B}$ and $n(l)$ is a positive integer such that $\lim_{l\to\infty} n(l)/l = 0$.

(c) Finally, by (i) and (ii) above we have

$$\begin{aligned} N_1(r; J_k(f)^*Z) &\leqq \frac{1}{l+1} N(r; J_{k+l}(f)^*F_l) + \frac{\epsilon}{l+1} T_f(r; D)||_\epsilon \\ &\leqq \frac{n(l)}{l+1} T_{\gamma\circ f}(r; D_B) + \frac{\epsilon}{l+1} T_f(r; D)||_\epsilon \end{aligned}$$

for all $\epsilon > 0$ and all integer $l \geqq 0$. Since $n(l)/l \to 0$ $(l \to \infty)$, we have

$$N_1(r; J_k(f)^*Z) \leqq \epsilon (T_{\gamma\circ f}(r; D_B) + T_f(r; D))||_\epsilon, \quad \forall \epsilon > 0.$$

Since $T_{\gamma\circ f}(r; D_B) = O(T_f(r; D))$, the proof is completed.

(a) Reduction. Let $f : \mathbf{C} \to A$ be as above. Let $I_k : \hat{X}_k(f) \ (\hookrightarrow A \times J_{k,A}) \to J_{k,A}$ be the jet projection. It follows from [N77] (or [NWY02] Lemma 3.8) that

$$T_{I_k \circ J_k(f)}(r) = S_f(r). \tag{5.4}$$

We need the following.

Lemma 5.5 *Let the notation be as above. Let $G = \cap_{l \geqq 0} \mathrm{St}(X_l(f))$ and let $q^G : A \to A/G$ be the quotient map. Then*

$$T_{q^G \circ f}(r) = O(T_{I_k \circ J_k(f)}(r)) (= S_f(r)).$$

Proof. This is essentially the same as (4.7) and follows from the jet projection method; cf. [NW03] Lemma 2.4, [NWY02] Lemma 3.8 and their proofs. *Q.E.D.*

Lemma 5.6 *Let $B \subset A$ be a semi-abelian subvariety. Put $B' = B \cap (\cap_{l \geqq 0} \mathrm{St}(X_l(f))$. Let $q^B : A \to A/B$ and $q^{B'} : A \to A/B'$ be quotient mappings. Then we have*

$$T_{q^{B'} \circ f}(r) = O(T_{q^B \circ f}(r)) + S_f(r).$$

Proof. We write $G = \cap_{l \geqq 0} \mathrm{St}(X_l(f))$. Taking the natural embedding $A/B' \to (A/B) \times (A/G)$, we see that

$$T_{q^{B'} \circ f}(r) = O(T_{q^B \circ f}(r) + T_{q^G \circ f}(r)).$$

Thus the claim follows from Lemma 5.5. *Q.E.D.*

Lemma 5.7 *Let A and A' be semi-abelian varieties with a surjective homomorphism $p : A \to A'$. Let $g : \mathbf{C} \to A'$ be a holomorphic curve. Then we have a holomorphic curve $\hat{g} : \mathbf{C} \to A$ such that $p \circ \hat{g} = g$ and*

$$T_{\hat{g}}(r) = O(T_g(r)).$$

Proof. Set $n = \dim A$ and $n' = \dim A'$. Let $\varpi : \tilde{A} \cong \mathbf{C}^n \to A$ and $\tilde{A}' \cong \mathbf{C}^{n'} \to A'$ be the universal covering. Then there is a surjective linear homomorphism $\tilde{p} : \tilde{A} \to \tilde{A}'$. Let $\tilde{g} : \mathbf{C} \to \tilde{A}'$ be the lifting of g. Let $g(z) = \sum_{j=1}^{n'} g_j(z) e'_j$ with basis $\{e'_j\}$ of $\tilde{A}'$. Take a basis $\{e_j\}$ of $\tilde{A}$ such that $\tilde{p}(e_j) = e'_j$, $1 \leqq j \leqq n'$. Then we set $\hat{g}(z) = \varpi(\sum_{j=1}^{n'} g_j(z) e_j)$. It immediately follows from the definition of order functions (see [NWY02] §3) that $\hat{g}$ satisfies the requirement. *Q.E.D.*

Now we are going to reduce our proof to the case such that $A = B \times C$ and that B and C are semi-abelian subvarieties satisfying (5.2) and (5.3). Let $\mathcal{B}$ be the set of all semi-abelian subvarieties $B \subset A$ such that

$$T_{q^B \circ f}(r) = S_f(r).$$

Then since $\cap_{l \geqq 0} \mathrm{St}(X_l(f)) \in \mathcal{B}$, we have $\mathcal{B} \neq \emptyset$. Let $B \in \mathcal{B}$ be a minimal element of $\mathcal{B}$; i.e., if $B' \subset B$ and $B' \in \mathcal{B}$, then $B' = B$. If $B_i \in \mathcal{B}, i = 1, 2$, we deduce from Lemma 5.6 that $B_1 \cap B_2 \in \mathcal{B}$. Thus we get

$$B \subset \cap_{l \geqq 0} \mathrm{St}(X_l(f)).$$

Put $C = A/B$ and let $q^B : A \to C$ be the quotient map. By Lemma 5.7 we may take a holomorphic curve $g : \mathbf{C} \to A$ such that $q^B \circ g = q^B \circ f$ and

$$T_g(r) = S_f(r). \tag{5.8}$$

We may assume that the Zariski closure of the image $g(\mathbf{C})$ is a semi-abelian subvariety $C' \subset A$ ([N77], [N81]). Define the semi-abelian variety $\tilde{A}$ by the following pull-back.

$$\begin{array}{ccc} \tilde{A} & \xrightarrow{p_2} & A \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle q^B} \\ C' & \xrightarrow{q^B|_{C'}} & C \end{array}$$

Then $\tilde{A} = \{(c, a) \in C' \times A : q^B(c) = q^B(a)\}$. The inclusion map $i : C' \to A$ yields a map $\tau : C' \to \tilde{A}$ defined by $\tau(x) = (x, i(x))$. Note that this morphism τ is a section for $p_1 : \tilde{A} \to C'$. Hence this bundle is trivial, i.e. $\tilde{A} \cong B \times C'$ and $\tilde{A}/B = C'$.

Put $\tilde{f} = g \times f : \mathbf{C} \to \tilde{A}$. Then by (5.8) we have

$$T_f(r) = O(T_{\tilde{f}}(r)), \qquad T_{\tilde{f}}(r) = O(T_f(r)), \tag{5.9}$$

$$T_{p_1 \circ \tilde{f}}(r) = S_{\tilde{f}}(r). \tag{5.10}$$

Put

$$B' = B \cap \left(\cap_{l \geqq 0} \mathrm{St}(X_l(\tilde{f})) \right) \tag{5.11}$$

and $p_1' : \tilde{A} \to \tilde{A}/B'$ be the quotient map. By Lemma 5.6 and (5.10), we have

$$T_{p_1' \circ \tilde{f}}(r) = S_{\tilde{f}}(r). \tag{5.12}$$

Put $q^{B'} : A \to A/B'$ be the quotient map. Then we have

$$T_{q^{B'} \circ f}(r) = O(T_{p'_1 \circ \tilde{f}}(r)). \tag{5.13}$$

Hence by (5.9), (5.12) and (5.13) we conclude $B' \in \mathcal{B}$. Since B is minimal in $\mathcal{B}$, we get $B' = B$. By (5.11) we have $B \subset \cap_{l \geqq 0} \mathrm{St}(X_l(\tilde{f}))$. Let $p_{2,k} : X_k(\tilde{f}) \to X_k(f)$ be the morphism induced from $p_2 : \tilde{A} \to A$. Set

$$\tilde{Z} = p_{2,k}^{-1}(Z) \subset X_k(\tilde{f}).$$

Note that

$$N_1(r; J_k(f)^* Z) = N_1(r; J_k(\tilde{f})^* \tilde{Z})$$

and that (5.9) holds.

For the reduction we need $\mathrm{codim}_{X_k(\tilde{f})} \tilde{Z} \geqq 2$. By Lemma 5.6 we see that

$$B \subset \left(\cap_{l \geqq 0} \mathrm{St}(X_l(f))\right) \cap \left(\cap_{l \geqq 0} \mathrm{St}(X_l(\tilde{f}))\right).$$

Thus $p_{2,l} : X_l(\tilde{f}) \to X_l(f)$ is B-equivariant, and induces a morphism

$$p_{2,l}^B : X_l(\tilde{f})/B \to X_l(f)/B.$$

Let $\pi_l : X_l(f) \to X_l(f)/B$ be the quotient map. Then it follows from (5.3) and (5.4) that

$$T_{\pi_l \circ J_l(f)}(r) = S_f(r). \tag{5.14}$$

If the image $\pi_k(Z)$ is not Zariski dense in $X_k(f)/B$, there is a Cartier divisor H on $X_k(f)/B$ containing $\pi_k(Z)$. Then, making use of (5.14) and the natural embedding $X_k(f)/B \hookrightarrow (A/B) \times J_{k,A}$ we get

$$\begin{aligned} N_1(r; J_k(f)^* Z) &\leqq N(r; (\pi_k \circ J_k(f))^* H) = O(T_{\pi_k \circ J_k(f)}(r)) \\ &= S_f(r). \end{aligned} \tag{5.15}$$

Therefore the proof of Theorem 5.1 is finished in this case.

We assume that $\pi_k(Z)$ is Zariski dense in $X_k(f)$, and has a relative dimension at most $\dim B - 2$. Therefore the relative dimension of $\tilde{Z} \to X_k(\tilde{f})/B$ is at most $\dim B - 2$, and hence $\mathrm{codim}_{X_k(\tilde{f})} \tilde{Z} \geqq 2$.

Hence, by replacing A by $\tilde{A}$, C by C', f by $\tilde{f}$ and Z by $p_2^{-1}(Z)$, we may reduce our problem to the desired situation (5.2) and (5.3).

Therefore we assume the following in the sequel:

(i) Let $B \subset A$ be a semi-abelian subvariety satisfying

$$B \subset \cap_{l \geqq 0} \mathrm{St}(X_l(f)), \tag{5.16}$$

$$T_{q^B \circ f}(r) = S_f(r), \tag{5.17}$$

$$A \cong B \times (A/B), \tag{5.18}$$

where $q^B : A \to A/B$ is the quotient map.
(ii) $\pi_k(Z)$ is Zariski dense in $X_k(f)/B$.

(b) Auxiliary divisor. Let the notation and the assumption be as above. Set $C = A/B$. We have

$$A \cong B \times C. \tag{5.19}$$

Then it naturally induces

$$X_l(f) \cong B \times (X_l(f)/B) \qquad (l \geqq 0).$$

Let $\bar{B}$ be an equivariant compactification of B and set $\hat{X}_l(f) = \bar{B} \times (X_l(f)/B)$. Let

$$\hat{\gamma}_l : \hat{X}_l(f) \to \bar{B},$$
$$\hat{\pi}_l : \hat{X}_l(f) \to X_l(f)/B$$

be the natural projections.

We denote by Z^{ns} the set of non-singular points of Z.

Lemma 5.20 *Let $L \to \bar{B}$ be an ample line bundle. Then there is a sequence of natural numbers $n(1), n(2), n(3), \ldots$ satisfying the following:*

(i) $\lim_{l\to\infty} \frac{n(l)}{l} = 0$.

(ii) *There exist effective Cartier divisors $F_l \subset \hat{X}_{k+l}(f)$ and line bundles M_l on $X_{k+l}(f)/B$ such that F_l is defined by a non-zero element of*

$$H^0(\hat{X}_{k+l}(f), \hat{\gamma}_{k+l}^* L^{\otimes n(l)} \otimes (\hat{\pi}_{k+l})^* M_l)$$

and that for every point $a \in \mathbf{C}$ with $J_k(f)(a) \in Z^{\mathrm{ns}}$

$$\mathrm{ord}_a J_{k+l}(f)^* F_l \geqq l + 1.$$

Proof. Let $f_B : \mathbf{C} \to B$ be the holomorphic curve defined by the composition of f and the first projection $A \to B$. Let $f_C : \mathbf{C} \to C$ be the holomorphic curve defined by the composition of f and the second projection $A \to C$. Then f_B and f_C have Zariski-dense images. Let $l \geqq 0$ be an integer, let $p_{k+l,k} : J_{k+l,A} \to J_{k,A}$ be the natural projection, and let

$$T \subset J_{k+l}(A) \times C \times J_{k,A} \cong B \times C \times J_{k+l,A} \times C \times J_{k,A}$$

be the Zariski closed subset defined by

$$T = \{(b, c, v, c', v') \in B \times C \times J_{k+l,A} \times C \times J_{k,A}; b = 0,\ c = c', v' = \ p_{k+l,k}(v)\}.$$

Let $\lambda : B \times C \times J_{k+l,A} \times C \times J_{k,A} \to C \times J_{k+l,A}$ be the product of the second projection and the third projection. We recall the following from [Y04] Proposition 2.1.1.

Lemma 5.21 *There exists a closed subscheme $\mathcal{T} \subset J_{k+l}(A) \times C \times J_{k,A}$ with the following properties:*

(i) $\operatorname{Supp} \mathcal{T} = T$.

(ii) *The restriction $\lambda' = \lambda|_{\mathcal{T}} : \mathcal{T} \to C \times J_{k+l,A}$ is a finite morphism. Furthermore the restriction of the direct image sheaf $\lambda'_*(\mathcal{O}_{\mathcal{T}})$ to $C \times J^{\mathrm{reg}}_{k+l,A}$ is a rank $l+1$ locally free $\mathcal{O}_{C \times J^{\mathrm{reg}}_{k+l,A}}$-module.*

(iii) *Let $f : \mathbf{C} \to A$ be a holomorphic curve such that $f_B(a) = 0$. Then*

$$\operatorname{ord}_a J_{k+l}(f)^* \mathcal{T}_{\rho \circ J_k(f)(a)} \geqq l + 1.$$

Let $r_1 : Z^\dagger \to \bar{Z}$ be a desingularization of $\bar{Z}$ such that r_1 gives an isomorphism over Z^{ns}. Put $Y_k = X_k(f)/B$. Consider the sequence of morphisms

$$Z^{\mathrm{ns}} \overset{r_0}{\hookrightarrow} Z^\dagger \overset{r_1}{\to} \bar{Z} \overset{r_2}{\hookrightarrow} \hat{X}_k(f) \overset{\hat{\pi}_k}{\to} Y_k. \tag{5.22}$$

Here r_0, $r_1 \circ r_0$ are open immersions and r_2 is a closed immersion. Put the composition of morphisms to be $r = \hat{\pi}_k \circ r_2 \circ r_1 : Z^\dagger \to Y_k$. Let Y_k^{fl} be a Zariski open subset of Y_k such that Y_k^{fl} is non-singular and the fibers of $r : Z^\dagger \to Y_k$ over Y_k^{fl} are all of the same dimension $\dim Z^\dagger - \dim Y_k$. Then the restriction of the family $r : Z^\dagger \to Y_k$ to Y_k^{fl} is a flat family.

Consider the pull back of the sequence of morphisms (5.22) by the natural projection $B \times Y_k \to Y_k$:

$$B \times Z^{\mathrm{ns}} \overset{s_0}{\hookrightarrow} B \times Z^\dagger \overset{s_1}{\to} B \times \bar{Z} \overset{s_2}{\hookrightarrow} B \times \hat{X}_k(f) \overset{s_3}{\to} B \times Y_k.$$

Again put the composition of these morphisms to be $s = s_3 \circ s_2 \circ s_1 : B \times Z^\dagger \to B \times Y_k$. Then s maps as

$$s : (a, z) \in B \times Z^\dagger \to (a, r(z)) \in B \times Y_k.$$

Let L be an ample line bundle on $\bar{B}$ and set

$$\phi : (a, w) \in B \times \hat{X}_k(f) \to a + \gamma_k(w) \in \bar{B}. \tag{5.23}$$

Let $L_1^\dagger$ be the line bundle on $B \times Z^\dagger$ which is the pull back of L by the composition of morphisms

$$B \times Z^\dagger \overset{s_2 \circ s_1}{\to} B \times \hat{X}_k(f) \overset{\phi}{\to} \bar{B}.$$

Since the restriction of s over $B \times Y_k^{\mathrm{fl}}$ (i.e., $s|_{B \times Y_k^{\mathrm{fl}}} : B \times (Z^\dagger|_{Y_k^{\mathrm{fl}}}) \to B \times Y_k^{\mathrm{fl}}$) is a flat family, the semi-continuity theorem [H77] p. 288 implies that there is a Zariski open subset $U_n \subset B \times Y_k^{\mathrm{fl}}$ $(n > 0)$ such that $H^0((B \times Z^\dagger)|_P, L_{1,P}^{\dagger \otimes n})$ are all the same dimensional $\mathbf{C}$-vector spaces for $P \in U_n$. Put this dimension as G_n. Here $(B \times Z^\dagger)|_P$ denotes the fiber of the morphism $s : B \times Z^\dagger \to B \times Y_k$ over $P \in B \times Y_k$, and $L_{1,P}^{\dagger \otimes n}$ is the induced line bundle. Since the intersection $\cap_{n \geqq 1} U_n$ is non-empty, put $(a, w) \in \cap_{n \geqq 1} U_n$ and replacing L by the pull back by the morphism

$$B \ni x \mapsto x + a \in B$$

we may assume $a = 0 \in B$.

Now for a positive integer $l > 0$, let $\mathcal{T}_l^\dagger \subset A \times J_{k+l,A} \times C \times J_{k,A}$ be the closed subscheme, and let $\lambda : \mathcal{T}_l^\dagger \to C \times J_{k+l,A}$ be the morphism obtained in Lemma 5.21. Then λ has the following properties;

(i) λ is finite,

(ii) the direct image sheaf $\lambda_* \mathcal{O}_{\mathcal{T}_l^\dagger}$ is locally generated by $l + 1$ elements as $\mathcal{O}_{C \times J_{k+l,A}}$ module on $C \times J_{k+l,A}^{\mathrm{reg}}$,

(iii) λ induces an isomorphism of the underlying topological spaces of $\mathcal{T}_l^\dagger$ and $C \times J_{k+l,A}$.

Since Y_{k+l} is a Zariski closed subset of $C \times J_{k+l,A}$, we denote $\sigma_{k+l} : Y_{k+l} \to C$ for the composition with the first projection $C \times J_{k+l,A} \to C$ and denote $\eta_{k+l} : Y_{k+l} \to J_{k+l,A}$ for the composition with the second projection. We have the closed immersion

$$B \times Y_{k+l} \times Y_k \subset B \times C \times J_{k+l,A} \times C \times J_{k,A} \cong A \times J_{k+l,A} \times C \times J_{k,A}, \tag{5.24}$$

where the first inclusion is given by

$$B \times Y_{k+l} \times Y_k \ni (b, v, v') \mapsto (b, \sigma_{k+l}(v), \eta_{k+l}(v), \sigma_k(v'), \eta_k(v')) \in B \times C \times J_{k+l,A} \times C \times J_{k,A}$$

and the second identification is given by

$$B \times C \times J_{k+l,A} \times C \times J_{k,A} \ni (b, c, u, c', u') \mapsto ((b, c), u, c', u') \in A \times J_{k+l,A} \times C \times J_{k,A}.$$

Let $\mathcal{S}_l \subset B \times Y_{k+l} \times Y_k$ be the closed subscheme obtained by the pull-back of $\mathcal{T}_l^\dagger$ by (5.24). Let $q : \mathcal{S}_l \to Y_{k+l}$ be the composition with the second projection $B \times Y_{k+l} \times Y_k \to Y_{k+l}$. We put

$$Y_{k+l}^{\mathrm{reg}} = Y_{k+l} \cap (C \times J_{k+l,A}^{\mathrm{reg}}),$$

which is the Zariski open subset of Y_{k+l}. Then by the above properties of λ, we have the corresponding properties for q;

(i) q is finite,

(ii) the direct image image sheaf $q_* \mathcal{O}_{\mathcal{S}_l}$ is locally generated by $l+1$ elements as $\mathcal{O}_{Y_{k+l}}$-module on Y_{k+l}^{reg},

(iii) q gives the isomorphism of under lying topological spaces of $\mathcal{S}_l$ and Y_{k+l}.

We consider the following commutative diagram (5.25) obtained by the base change of (5.22) with a sequence of morphisms

$$\mathcal{S}_l \hookrightarrow B \times Y_{k+l} \times Y_k \to B \times Y_k \to Y_k.$$

Here $B \times Y_{k+l} \times Y_k \to B \times Y_k$ is the natural projection:

$$B \times Y_{k+l} \times Y_k \ni (a, w, w') \mapsto (a, w') \in B \times Y_k.$$

$$(5.25) \qquad \begin{array}{ccccccc}
\mathcal{Z}_l^{\mathrm{ns}} & \longrightarrow & B \times Y_{k+l} \times Z^{\mathrm{ns}} & \longrightarrow & B \times Z^{\mathrm{ns}} & \longrightarrow & Z^{\mathrm{ns}} \\
\downarrow{\scriptstyle u_0} & & \downarrow{\scriptstyle t_0} & & \downarrow{\scriptstyle s_0} & & \downarrow{\scriptstyle r_0} \\
\mathcal{Z}_l^\dagger & \longrightarrow & B \times Y_{k+l} \times Z^\dagger & \longrightarrow & B \times Z^\dagger & \longrightarrow & Z^\dagger \\
\downarrow{\scriptstyle u_1} & & \downarrow{\scriptstyle t_1} & & \downarrow{\scriptstyle s_1} & & \downarrow{\scriptstyle r_1} \\
\mathcal{Z}_l & \xrightarrow{v'} & B \times Y_{k+l} \times \bar{Z} & \longrightarrow & B \times \bar{Z} & \longrightarrow & \bar{Z} \\
\downarrow{\scriptstyle u_2} & & \downarrow{\scriptstyle t_2} & & \downarrow{\scriptstyle s_2} & & \downarrow{\scriptstyle r_2} \\
\cdot & \longrightarrow & B \times Y_{k+l} \times \hat{X}_k(f) & \longrightarrow & B \times \hat{X}_k(f) & \longrightarrow & \hat{X}_k(f) \\
\downarrow{\scriptstyle u_3} & & \downarrow{\scriptstyle t_3} & & \downarrow{\scriptstyle s_3} & & \downarrow{\scriptstyle \hat{\pi}_k} \\
\mathcal{S}_l & \xrightarrow{v} & B \times Y_{k+l} \times Y_k & \longrightarrow & B \times Y_k & \longrightarrow & Y_k
\end{array}$$

Let $\mathcal{L}_l^\dagger$ be the line bundle on $\mathcal{Z}_l^\dagger$ obtained by the pull back of $L_1^\dagger$ by the morphisms in the above diagram (5.25). Let $\mathcal{S}_{l,n}$ be the non-empty Zariski open subset of $\mathcal{S}_l$ obtained by the inverse image of U_n. Since $\dim H^0((B \times Z^\dagger)|_P, L_{1,P}^{\dagger\otimes n}) = G_n$ for $P \in U_n$, the direct image sheaf $s_* L_1^{\dagger\otimes n}$ is a locally free sheaf of rank G_n on U_n and the natural map

$$s_* L_1^{\dagger\otimes n} \otimes \mathbf{C}(P) \to H^0((B \times Z^\dagger)|_P, L_{1,P}^{\dagger\otimes n})$$

is an isomorphism for $P \in U_n$. This follows by the Theorem of Grauert [H77] p.288, since U_n is reduced and irreducible. Here $s : B \times Z^\dagger \to B \times Y_k$ is the natural map; i.e., $s = s_3 \circ s_2 \circ s_1$. Let u be the morphism $u : \mathcal{Z}_l^\dagger \to \mathcal{S}_l$ obtained by the composition $u = u_3 \circ u_2 \circ u_1$, where u_1,u_2,u_3 are the morphisms in the above diagram (5.25). Then the natural map

$$u_* \mathcal{L}_l^{\dagger\otimes n} \otimes \mathbf{C}(P) \to H^0(\mathcal{Z}_l^\dagger|_P, \mathcal{L}_{l,P}^{\dagger\otimes n})$$

is also surjective, so an isomorphism on $P \in \mathcal{S}_{l,n}$. This follows by the Theorem of Cohomology and Base Change [H77] p. 290. Hence $u_* \mathcal{L}_l^{\dagger\otimes n}$ is locally generated by G_n elements as an $\mathcal{O}_{\mathcal{S}_l}$-module on $\mathcal{S}_{l,n} \subset \mathcal{S}_l$. Let $Y_{k+l,n} = q(\mathcal{S}_{l,n})$ be a non-empty Zariski open subset of Y_{k+l} (note that the under lying topological spaces of $\mathcal{S}_l$ and Y_{k+l} are the same). Then by the above properties of q, the direct image sheaf $(q \circ u)_* \mathcal{L}_l^{\dagger\otimes n}$ is locally generated by $(l+1)G_n$ elements as a $\mathcal{O}_{Y_{k+l}}$-module on $Y_{k+l,n} \cap Y_{k+l}^{\mathrm{reg}}$. Here, note that Y_{k+l}^{reg} is non-empty (otherwise f must be constant) and Y_{k+l} is irreducible. Hence $Y_{k+l,n} \cap Y_{k+l}^{\mathrm{reg}}$ is also non-empty.

Now look at the following commutative diagram

$$\begin{tikzcd}
\mathcal{Z}_l^{\mathrm{ns}} \arrow[d, "u_0"] & & & \\
\mathcal{Z}_l^\dagger \arrow[r, "t_2 \circ v' \circ u_1"] \arrow[d, "q \circ u"] & B \times Y_{k+l} \times \hat{X}_k(f) \arrow[r, "\psi"] \arrow[d, "\text{2nd proj}"] & \bar{B} \times Y_{k+l} \arrow[r, "\rho"] \arrow[d, "\tau"] & \bar{B} \\
Y_{k+l} \arrow[r, equal] & Y_{k+l} \arrow[r, equal] & Y_{k+l} &
\end{tikzcd}$$

where ρ is the first projection, τ is the second projection and ψ is the morphism

$$\psi : B \times Y_{k+l} \times \hat{X}_k(f) \ni (a, v, w) \mapsto (a + \gamma_k(w), v) \in \bar{B} \times Y_{k+l}.$$

Since $(\rho \circ \psi \circ t_2 \circ v' \circ u_1)^* L = \mathcal{L}_l^\dagger$, we have a natural morphism

$$\tau_* \rho^* L^{\otimes n} = H^0(\bar{B}, L^{\otimes n}) \otimes_{\mathbf{C}} \mathcal{O}_{Y_{k+l}} \to (q \circ u)_* \mathcal{L}_l^{\dagger\otimes n}. \tag{5.26}$$

Here, note that $\rho \circ \psi = \phi \circ \beta$ where $\beta : B \times Y_{k+l} \times \hat{X}_k(f) \to B \times \hat{X}_k(f)$ is the morphism in the diagram (5.25) and ϕ was defined by (5.23).

Now put $I_n = \dim_{\mathbf{C}} H^0(\bar{B}, L^{\otimes n})$. Then there is a positive integer n_0 and positive constants C_1, C_2 such that

$$I_n > C_1 n^{\dim \bar{B}}, \quad G_n < C_2 n^{\dim \bar{B}-2} \quad \text{for } n > n_0.$$

Here note that $G_n = \dim_{\mathbf{C}} H^0(B \times Z^\dagger|_P, L_{1,P}^{\dagger\otimes n})$ for $P \in \cap_{n \geqq 1} U_n$, and $B \times Z^\dagger|_P = s^{-1}(P)$ has dimension $\leqq \dim \bar{B} - 2$, for $\operatorname{codim}_{\hat{X}_k(f)} \bar{Z} \geqq 2$ and $\hat{\pi}_k \circ r_2 : \bar{Z} \to Y_k$ is dominant. Hence for a positive integer l, we can take a positive integer $n(l)$ (e.g. $\sim l^{3/4}$) such that

$$I_{n(l)} > (l+1) G_{n(l)}, \qquad \lim_{l \to \infty} \frac{n(l)}{l} = 0.$$

Let $\mathcal{F}$ be the kernel of (5.26) for $n = n(l)$;

$$0 \to \mathcal{F} \to \tau_* \rho^* L^{\otimes n(l)} \to (q \circ u)_* \mathcal{L}_l^{\dagger \otimes n(l)} \text{ (exact).}$$

Then we have $\mathcal{F} \neq 0$. By taking the tensor of a sufficiently ample line bundle M_l on Y_{k+l} with $\mathcal{F}$, we may assume that $H^0(Y_{k+l}, \mathcal{F} \otimes M_l) \neq 0$. Since we have

$$\begin{aligned} H^0(Y_{k+l}, \mathcal{F} \otimes M_l) &\subset H^0(Y_{k+l}, (\tau_* \rho^* L^{\otimes n(l)}) \otimes M_l) \\ &= H^0(Y_{k+l}, \tau_*(\rho^* L^{\otimes n(l)} \otimes \tau^* M_l)) \\ &= H^0(\bar{B} \times Y_{k+l}, \rho^* L^{\otimes n(l)} \otimes \tau^* M_l), \end{aligned}$$

we may take a divisor $F_l \subset \bar{B} \times Y_{k+l}$ which is defined by a non-zero global section of $H^0(Y_{k+l}, \mathcal{F} \otimes M_l)$. Then we have

$$\mathcal{Z}_l^{\mathrm{ns}} \subset \psi^* F_l.$$

Here note that $\mathcal{Z}_l^{\mathrm{ns}} \subset \mathcal{Z}_l$ is an open immersion and $\mathcal{Z}_l \overset{t_2 \circ v'}{\hookrightarrow} B \times Y_{k+l} \times \hat{X}_k(f)$ is a closed subscheme.

Using the decomposition $A = B \times C$, we let $f_B : \mathbf{C} \to B$ be the holomorphic curve obtained by the composition of f and the first projection $A \to B$, and let $f_C : \mathbf{C} \to C$ be the holomorphic curve obtained by the composition of f and the second projection $A \to C$. Now let $a \in \mathbf{C}$ be a point such that $J_k(f)(a) \in Z^{\mathrm{ns}}$. Put $\tilde{f} : \mathbf{C} \to B \times Y_{k+l} \times \hat{X}_k(f)$ as

$$\tilde{f}(z) = (f_B(z) - f_B(a), \hat{\pi}_{k+l} \circ J_{k+l}(f)(z), J_k(f)(a)).$$

Then we have

$$\tilde{f}(\mathbf{C}) \subset B \times Y_{k+l} \times Z, \quad \tilde{f}(a) \in \operatorname{Supp} \mathcal{Z}_l^{\mathrm{ns}}, \quad \psi \circ \tilde{f} = J_{k+l}(f),$$

where the last equality holds under the identification $\bar{B} \times Y_{k+l} = \hat{X}_{k+l}(f)$.

Since v' is the base change of v in (5.25) and $\tilde{f}$ factors through t_2, we have

$$\mathrm{ord}_a \tilde{f}^* \mathcal{Z}_l = \mathrm{ord}_a (t_3 \circ \tilde{f})^* \mathcal{S}_l,$$

hence by the construction of $\mathcal{S}_l$ and Lemma 5.21, we have

$$\mathrm{ord}_a \tilde{f}^* \mathcal{Z}_l = \mathrm{ord}_a (J_{k+l}(f) - f(a))^* \mathcal{T}^{\dagger}_{l,(J_k(f)-f(a))(a)} \geqq l+1.$$

Hence we have

$$\mathrm{ord}_a J_{k+l}(f)^* F_l = \mathrm{ord}_a \tilde{f}^* \psi^* F_l \geqq \mathrm{ord}_a \tilde{f}^* \mathcal{Z}_l^{\mathrm{ns}} = \mathrm{ord}_a \tilde{f}^* \mathcal{Z}_l \geqq l+1.$$

Here note that we consider F_l as the divisor on $\hat{X}_{k+l}(f)$ by the identification $B \times Y_{k+l} \cong \hat{X}_{k+l}(f)$, and τ correspond to π_{k+l} by this identification. *Q.E.D.*

(c) The end of the proof. It suffices to show

$$N_1(r; J_k(f)^* Z^{\mathrm{ns}}) \leqq \epsilon T_f(r)||_\epsilon, \quad \forall \epsilon > 0. \tag{5.27}$$

For we have

$$N_1(r; J_k(f)^* Z) = N_1(r; J_k(f)^* Z^{\mathrm{ns}}) + N_1(r; J_k(f)^* (Z \setminus Z^{\mathrm{ns}}))$$

and the second term of the right hand side is estimated to be at most "$\epsilon T_f(r)||_\epsilon$" by induction on dimension of Z. Here note that $\dim Z > \dim(Z \setminus Z^{\mathrm{ns}})$.

It follows from Lemma 5.20 and (5.14) that

$$\begin{aligned} (l+1) N_1(r; J_k(f)^* Z^{\mathrm{ns}}) &\leqq N(r; J_{k+l}(f)^* F_l) \leqq T_{J_{k+l}(f)}(r; L(F_l)) \\ &= n(l) T_{\gamma_{k+l} \circ J_{k+l}(f)}(r; L) + T_{\pi_{k+l} \circ J_{k+l}(f)}(r; M_l) \\ &\leqq n(l) T_{f_B}(r; L) + S_f(r). \end{aligned} \tag{5.28}$$

Using $\lim_{l \to \infty} n(l)/(l+1) = 0$ and $T_{f_B}(r; L) = O(T_f(r; D))$, we obtain (5.27) and our Theorem 5.1.

6 Proof of Main Theorem

(a) Let the notation be as in the Main Theorem. Set $B = \mathrm{St}(X_{k+1}(f))$, which has a positive dimension (cf. (4.7)).

Lemma 6.1 *Assume that D is irreducible and $B \not\subset \mathrm{St}(D)$. Taking an embedding $X_{k+1}(f) \hookrightarrow J_1(X_k(f))$, we have*

$$\mathrm{codim}_{X_{k+1}(f)} (X_{k+1}(f) \cap J_1(D)) \geqq 2.$$

Proof. Let $k=0$. It is first noted that $J_1(A)$ is the holomorphic tangent bundle $\mathbf{T}(A)$ over A, and $X_1(f) \subset \mathbf{T}(A)$.

Assume that $\operatorname{codim}_{X_1(f)}(X_1(f) \cap J_1(D)) = 1$. Let Z be an irreducible component of codimension 1 of $X_1(f) \cap J_1(D)$.

Let $\pi_1 : X_1(f) \to A$ be the natural projection. Then Z is an irreducible component of $X_1(f) \cap \pi_1^{-1}(D)$. Notice that $B \cdot Z$ (resp. $B \cdot D$) contains an open subset of $X_1(f)$ (resp. A).

Let $p \in f(\mathbf{C})$ be a point with the property that the orbit $B \cdot p$ intersects $D \setminus \mathrm{Sing}(D)$ transversely in a point q. Then we choose an analytic 1-dimensional disk $\Delta \subset B$ which contains the unit element e_B of B and we choose a non-empty open subset U of the non-singular part D^{ns} of D containing q such that

(i) $\Delta \times U \hookrightarrow A$ is an open embedding.

(ii) The subbundle $\cup_{\zeta \in \Delta} \mathbf{T}(\{\zeta\} \times U) \subset \mathbf{T}(\Delta \times U)$ with $\mathbf{T}(\{\zeta\} \times U) \cong \mathbf{T}(U)$ gives rise to a holomorphic foliation.

(iii) The union $\cup_{\zeta \in \Delta} \mathbf{T}(\{\zeta\} \times U)$ contains an open subset of $X_1(f)$.

Consider $\hat{f}(z) = b \cdot f(z - z_0)$ with $b \in B$ such that $b \cdot p = q$ and $p = f(z_0)$. Since B stabilizes $X_1(f)$, there is an open neighbourhood W of 0 in $\mathbf{C}$ such that $\hat{f}'(z)$ is tangent to the leaves of the above defined foliation for all $z \in U$. This implies $\hat{f}(\mathbf{C}) = b \cdot f(\mathbf{C}) \subset D$ which is absurd, since f is algebraically non-degenerate.

The proof for $k \geqq 1$ is similar to the above. *Q.E.D.*

(b) *Proof of the Main Theorem.* Let $D = \sum_i D_i$ be the irreducible decomposition. By making use of Theorem 4.2 we have

$$\begin{aligned} T(r; \omega_{\bar{D}, J_k(f)}) &\leqq N_{k_0}(r; J_k(f)^* D) + S_f(r) \\ &\leqq N_1(r; J_k(f)^* D) + k_0 \sum_{i<j} N_1(r; J_k(f)^*(D_i \cap D_j)) \\ &\quad + k_0 \sum_i N_1(r; J_{k+1}(f)^* J_1(D_i)) + S_f(r). \end{aligned} \tag{6.2}$$

Since $\operatorname{codim}_{X_k(f)} D_i \cap D_j \geqq 2$ for $i \neq j$, it follows from Theorem 5.1 that

$$k_0 \sum_{i<j} N_1(r; J_k(f)^*(D_i \cap D_j)) \leqq \epsilon T_f(r) ||_\epsilon, \quad \forall \epsilon > 0.$$

Note that $J_{k+1}(f)^* J_1(D_i) = J_{k+1}(f)^*(X_{k+1}(f) \cap J_1(D_i))$. If $B \subset \mathrm{St}(D_i)$, then the image of D_i by $X_k(f) \to X_k(f)/B$ is contained in a divisor on $X_k(f)/B$. Then as in (4.9) we infer that

$$N_1(r; J_{k+1}(f)^* J_1(D_i)) \leqq N(r; J_k(f)^* D_i) \leqq S_f(r).$$

Suppose that $B \not\subset \mathrm{St}(D_i)$. It follows from Lemma 6.1 and Theorem 5.1 that

$$N_1(r; J_{k+1}(f)^* J_1(D)) \leqq N_1(r; J_{k+1}(f)^*(X_{k+1}(f) \cap J_1(D_i)) \leqq \epsilon T_f(r)||_\epsilon, \quad \forall \epsilon > 0.$$

Combining these with (6.2), we obtain

$$T(r; \omega_{\bar{D}, J_k(f)}) \leqq N_1(r; f^* D) + \epsilon C T_f(r)||_\epsilon, \quad \forall \epsilon > 0,$$

where C is a positive constant independent of ϵ. Now the proof of the Main Theorem is completed. *Q.E.D.*

7 Applications

(a) In [G74] M. Green discussed the algebraic degeneracy of a holomorphic curve $f : \mathbf{C} \to \mathbf{P}^n(\mathbf{C})$ omitting an effective reduced divisor D on $\mathbf{P}^n(\mathbf{C})$ with normal crossings and of degree $\geqq n + 2$. He proved the following theorem and conjectured that it would hold without the condition of finite order for f:

Theorem 7.1 (M. Green [G74]) *Let $f : \mathbf{C} \to \mathbf{P}^2(\mathbf{C})$ be a holomorphic curve of finite order and let $[x_0, x_1, x_2]$ be the homogeneous coordinate system of $\mathbf{P}^2(\mathbf{C})$. Assume that f omits two lines $\{x_i = 0\}, i = 1, 2$, and the conic $\{x_0^2 + x_1^2 + x_2^2 = 0\}$. Then the image $f(\mathbf{C})$ lies in a line or a conic.*

Here we answer his conjecture in more general form:

Theorem 7.2 *Let $f : \mathbf{C} \to \mathbf{P}^n(\mathbf{C})$ be a holomorphic curve and let $[x_0, \ldots, x_n]$ be the homogeneous coordinate system of $\mathbf{P}^n(\mathbf{C})$. Assume that f omits hyperplanes given by*

(7.3) $$x_i = 0, \qquad 1 \leqq i \leqq n,$$

and a hypersurface defined by

$$x_0^q + \cdots + x_n^q = 0, \qquad q \geqq 2.$$

Then f is algebraically degenerate.

Proof. Let $f(z) = [f_0(z), \dots, f_n(z)]$ be a reduced representation of f. Then $f_i(z)$ have no zero for $1 \leqq i \leqq n$. The assumption implies the existence of an entire function $h(z)$ such that

$$f_0^q(z) + \cdots + f_n^q(z) = e^{h(z)}.$$

Write the above equation as

$$\left(f_0(z)e^{-h(z)/q}\right)^q + \cdots + \left(f_n(z)e^{-h(z)/q}\right)^q = 1.$$

Changing the reduced representation of f, we may have that

$$f_1^q(z) + \cdots + f_n^q(z) - 1 = -f_0^q(z). \tag{7.4}$$

Now we take a holomorphic curve into a semi-abelian variety $A = (\mathbf{C}^*)^n$ with the natural coordinate system $(x_1, \dots, x_n)$ defined by

$$g : z \in \mathbf{C} \to (f_1(z), \dots, f_n(z)) \in A.$$

Define a divisor D on A by

$$x_1^q + \cdots + x_n^q - 1 = 0.$$

Let $\bar{A}$ be a equivariant compactification in which D is generally positioned. Let $\bar{D}$ be the closure of D in $\bar{A}$. Note that $\mathrm{St}(D) = \{0\}$ and that $\mathrm{ord}_z g^*D \geqq 2$ for all $z \in g^{-1}(D)$ by (7.4). Combining this with the Main Theorem ($k = 0$), we see that for arbitrary $\epsilon > 0$

$$\begin{aligned} T_g(r; L(\bar{D})) &\leqq N_1(r; g^*D) + \epsilon T_g(r; L(\bar{D}))||_\epsilon \\ &\leqq \frac{1}{q} N(r; g^*D) + \epsilon T_g(r; L(\bar{D}))||_\epsilon \\ &\leqq \frac{1 + q\epsilon}{q} T_g(r; L(\bar{D}))||_\epsilon. \end{aligned}$$

This leads to a contradiction for $\epsilon < (q-1)/q$. *Q.E.D.*

Remark. The Zariski closure of the image $f(\mathbf{C})$ can be more specified in terms of g defined in the above proof. It follows from [N98] that the Zariski closure of $g(\mathbf{C})$ is a translate X of a proper semi-abelian subvariety of A such that $X \cap D = \emptyset$.

(b) Let A be a semi-abelian variety as above and let $X \subset J_k(A)$ be an irreducible algebraic subvariety. We consider the existence problem of an algebraically nondegenerate entire holomorphic curve $f : \mathbf{C} \to A$ such that $J_k(f)(\mathbf{C}) \subset X$ and $J_k(f)(\mathbf{C})$ is Zariski dense in X. This is a problem of a system of algebraic differential equations described by the equations defining the subvariety X.

The first necessary condition for the existence of such solution f is that $\mathrm{St}(X) \neq \{0\}$ (cf. (4.7)). Now we assume the existence of such f. Then we take a big line bundle $L \to X$ and a section $\sigma \in H^0(X, L)$ which defines a reduced divisor on X. We arbitrarily fix a trivialization

$$J_k(f)^* L \cong \mathbf{C} \times \mathbf{C}, \tag{7.5}$$

and regard $J_k(f)^*\sigma$ as an entire function.

Theorem 7.6 *Let the notation be as above. Then there is no entire function $\psi(z)$ such that every zero of $\psi(z)$ has degree $\geqq 2$ and*

$$J_k(f)^*\sigma(z) = \psi(z), \qquad z \in \mathbf{C}. \tag{7.7}$$

In particular, there is no entire function $\psi(z)$ satisfying

$$J_k(f)^*\sigma(z) = (\psi(z))^q, \qquad z \in \mathbf{C}, \tag{7.8}$$

where $q \geqq 2$ is an integer.

Remark. The property given by (7.7) or (7.8) is independent of the choice of the trivialization (7.5).

Proof. Suppose that there is an entire function $\psi(z)$ satisfying (7.7) or (7.8). Then it follows that

$$N_1(r; J_k(f)^* D) \leqq \frac{1}{2} N(r; J_k(f)^* D).$$

Combining this with the Main Theorem, we infer the following contradiction:

$$T_{J_k(f)}(r; L) \leqq \frac{1}{2} T_{J_k(f)}(r; L) + \epsilon T_{J_k(f)}(r; L)||_\epsilon.$$

Q.E.D.

(c) The truncation level one in the Second Main Theorem ((1.3)) allows the following immediate improvement of Theorem 6.1. in [NWY02].

Theorem 7.9 *Let A be a compact complex torus and D a divisor which contains no positive-dimensional translate of a subtorus of A. Let $\pi : X \to A$ be a finite ramified covering which is ramified at all points in $\pi^{-1}(D)$. Then X is Kobayashi hyperbolic.*

Further applications to Kobayashi hyperbolicity question will be discussed in a future article.

References

[BM97] Bierstone, E. and Milman, P.D., Canonical desingularization in characteristic zero by blowing up the maximum strata of a local invariant, Invent. Math. **128** (1997), no. 2, 207–302.

[DL01] Dethloff, G. and Lu, S.S.Y., Logarithmic jet bundles and applications, Osaka J. Math. **38** (2001), 185-237.

[G74] Green, M., On the functional equation $f^2 = e^{2\phi_1} + e^{2\phi_2} + e^{2\phi_3}$ and a new Picard theorem, Trans. Amer. Math. Soc. **195** (1974), 223–230.

[H77] Hartshorne, R., Algebraic Geometry, Springer-Verlag, Berlin, 1977.

[Hi64] Hironaka, H., Resolution of singularities of an algebraic variety over a field of characteristic zero. I, II, Ann. Math. (2) **79** (1964), 109–203; ibid. (2) **79** (1964) 205–326.

[I71] Iitaka, S., On D-dimensions of algebraic varieties, J. Math. Soc. Japan **23** (1971), 356–373.

[N77] Noguchi, J., Holomorphic curves in algebraic varieties, Hiroshima Math. J. **7** (1977), 833-853.

[N81] Noguchi, J., Lemma on logarithmic derivatives and holomorphic curves in algebraic varieties, Nagoya Math. J. **83** (1981), 213-233.

[N86] Noguchi, J., Logarithmic jet spaces and extensions of de Franchis' theorem, Contributions to Several Complex Variables, pp. 227-249, Aspects Math. No. **9**, Vieweg, Braunschweig, 1986.

[N96] Noguchi, J., On Nevanlinna's second main theorem, Geometric Complex Analysis, Proc. the Third International Research Institute, Math. Soc. Japan, Hayama, 1995, pp. 489-503, World Scientific, Singapore, 1996.

[N98] Noguchi, J., On holomorphic curves in semi-Abelian varieties, Math. Z. **228** (1998), 713-721.

[NO$\frac{84}{90}$] Noguchi, J. and Ochiai, T., Geometric Function Theory in Several Complex Variables, Japanese edition, Iwanami, Tokyo, 1984; English Translation, Transl. Math. Mono. **80**, Amer. Math. Soc., Providence, Rhode Island, 1990.

[NW03] Noguchi, J. and Winkelmann, J., A note on jets of entire curves in semi-Abelian varieties, Math. Z. **244** (2003), 705–710.

[NW04] Noguchi, J. and Winkelmann, J., Bounds for curves in abelian varieties, to appear in Crelle, preprint UTMS 2002-21.

[NWY00] Noguchi, J., Winkelmann, J. and Yamanoi, K., The value distribution of holomorphic curves into semi-Abelian varieties, C.R. Acad. Scie. Paris t. **331**, Série I (2000), 235–240.

[NWY02] Noguchi, J., Winkelmann, J. and Yamanoi, K., The second main theorem for holomorphic curves into semi-Abelian varieties, Acta Math. **188** no.1 (2002), 129–161.

[O85] Oda, T., Convex Bodies and Algebraic Geometry. An introduction to the theory of Toric Varieties, Erg. Math. **3/15**, Springer Verlag, Berlin-Tokyo, 1985.

[SY03] Siu, Y.-T. and Yeung, S.-K., Addendum to "Defects for ample divisors of abelian varieties, Schwarz lemma, and hyperbolic hypersurfaces of low degrees," *American Journal of Mathematics* **119** *(1977), 1139–1172*, Amer. J. Math. **125** (2003), 441–448.

[V99] Vojta, P., Integral points on subvarieties of semiabelian varieties, II, Amer. J. Math. **121** (1999), 283-313.

[Y04] Yamanoi, K., Holomorphic curves in abelian varieties and intersection with higher codimensional subvarieties, to appear in Forum Math., preprint RIMS-1436 (2003), Res. Inst. Math. Sci. Kyoto University.

J. Noguchi
Graduate School of Mathematical Sciences
University of Tokyo
Komaba, Meguro,Tokyo 153-8914
Japan
e-mail: noguchi@ms.u-tokyo.ac.jp

J. Winkelmann
Institut Élie Cartan
Université Nancy I
B.P.239
54506 Vandœuvre-les-Nancy Cedex
France
e-mail:jwinkel@member.ams.org

K. Yamanoi
Research Institute for Mathematical Sciences
Kyoto University
Oiwake-cho, Sakyoku, Kyoto 606-8502
Japan
e-mail: ya@kurims.kyoto-u.ac.jp

UTMS

2004–7 Masaaki Suzuki: *Twisted Alexander polynomial for the Lawrence-Krammer representation.*

2004–8 Masaaki Suzuki: *On the Kernel of the Magnus representation of the Torelli group.*

2004–9 Hiroshi Kawabi: *Functional inequalities and an application for parabolic stochastic partial differential equations containing rotation.*

2004–10 Takashi Taniguchi: *On the zeta functions of prehomogeneous vector spaces for pair of simple algebras.*

2004–11 Harutaka Koseki and Takayuki Oda : *Matrix coefficients of representations of* $SU(2,2)$*: — the case of* P_J*-principal series —.*

2004–12 Takao Satoh: *Twisted first homology groups of the automorphism group of a free group.*

2004–13 M. K. Klibanov and M. Yamamoto: *Lipschitz stability of an inverse problem for an acoustic equation.*

2004–14 Teruhisa Tsuda: *Universal characters, integrable chains and the Painlevé equations.*

2004–15 Shushi Harashita: *Ekedahl-Oort strata contained in the supersingular locus.*

2004–16 Mourad Choulli and Masahiro Yamamoto: *Stable identification of a semilinear term in a parabolic equation.*

2004–17 J. Noguchi, J. Winkelmann and K. Yamanoi: *The second main theorem for holomorphic curves into semi-abelian varieties II.*

The Graduate School of Mathematical Sciences was established in the University of Tokyo in April, 1992. Formerly there were two departments of mathematics in the University of Tokyo: one in the Faculty of Science and the other in the College of Arts and Sciences. All faculty members of these two departments have moved to the new graduate school, as well as several members of the Department of Pure and Applied Sciences in the College of Arts and Sciences. In January, 1993, the preprint series of the former two departments of mathematics were unified as the Preprint Series of the Graduate School of Mathematical Sciences, The University of Tokyo. For the information about the preprint series, please write to the preprint series office.

ADDRESS:
Graduate School of Mathematical Sciences, The University of Tokyo
3–8–1 Komaba Meguro-ku, Tokyo 153-8914, JAPAN
TEL +81-3-5465-7001 FAX +81-3-5465-7012