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STABLE IDENTIFICATION OF A SEMILINEAR TERM IN A PARABOLIC EQUATION

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Abstract. We consider a semilinear parabolic equation in a rectangular domain $\Omega \subset \mathbb{R}^n$: $(\partial_t u)(x,t) = \Delta u(x,t) + a(u(x,t))$ with the zero initial value and suitable Dirichlet data. We discuss an inverse problem of determining the nonlinear term $a(\cdot)$ from Neumann data $\frac{\partial u}{\partial n}$ on $\partial\Omega \times (0,T)$. Under appropriate Dirichlet data, we prove conditional stability of the Hölder type in this inverse problem within a suitable admissible set of unknown functions $a(\cdot)$.

§1. Introduction.

Let $\Omega \subset \mathbb{R}^n$, $n \ge 1$, be a rectangular domain: $\Omega = (0,\ell_1) \times \cdots \times (0,\ell_n)$ with $\ell_1, ..., \ell_n > 0$. We consider an initial/boundary value problem:

$$(\partial_t u)(x,t) = \Delta u(x,t) + a(u(x,t)), \quad x \in \Omega,\ 0 < t < T, \tag{1.1}$$

$$u(x,0) = 0, \qquad x \in \Omega, \tag{1.2}$$

$$u(x,t) = \varphi(x,t), \qquad x \in \partial\Omega,\ 0 < t < T. \tag{1.3}$$

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Under suitable conditions on a and φ, we can prove the unique existence of the solution to (1.1) - (1.3) (e.g., Henry [5], Ladyženskaja, Solonnikov and Ural'ceva [10]) and we denote the solution by $u_a(x,t)$ for specifying the dependence on the semilinear term $a(\cdot)$. In this paper, we discuss

Inverse Problem. Determine $a = a(\cdot)$ from the boundary observations $\frac{\partial u_a}{\partial n}|_{\partial\Omega\times(0,T)}$.

The semilinear parabolic equation of form (1.1) appears, for example, in modelling enzyme kinetics (e.g., Kernevez [8]). See also [1], [13] for other models. In our inverse problem, we are requested to model nonlinear dynamics in order to match with the boundary output.

More precisely, we are concerned with

Uniqueness. Does

$$\frac{\partial u_a}{\partial n} = \frac{\partial u_b}{\partial n} \quad \text{on } \partial\Omega \times (0,T)$$

imply $a(\eta) = b(\eta)$ for $\eta \in I$: some interval?

Stability. With suitable norms, estimate $a - b$ by $\frac{\partial u_a}{\partial n} - \frac{\partial u_b}{\partial n}$.

For theoretical results for inverse problems for parabolic equations of determining a semilinear term, we refer to DuChateau and Rundell [4], Klibanov [9], Lorenzi [11], Pilant and Rundell [14], Yamamoto [16]. Lorenzi [11] established the uniqueness and the stability in the inverse problem in the case where $\Omega = (0,\infty)$ and φ satisfies

$$\frac{d\varphi}{dt} \ge a(\varphi(t)), \qquad t \ge 0. \tag{1.4}$$

The paper [14] proved the existence of a semilinear term realizing given boundary observations with numerical examples. In [16], determination of scalar parameters in a semilinear term a is discussed and the stability was proved by the one-point observation at a fixed moment. In particular, DuChateau and Rundell [4] proved the uniqueness by means of an interesting lower inequality of the difference $u_a - u_b$.

In the above papers, we do not change Dirichlet boundary data φ. On the other hand, by changing the Dirichlet boundary input arbitrarily and observing the corresponding Neumann data, we can consider determination of a semilinear term a which is called a formulation by Dirichlet-to-Neumann map. As such papers, we refer to Isakov [6], Nakamura [12].

In this paper, we will prove conditional stability for our inverse problem (not by the Dirichlet-to-Neumann map) without assumptions such as (1.4).

The present paper is composed of 5 sections.

Section 1. Introduction.

Section 2. Main results.

Section 3. Proof of Theorem 1.

Section 4. Proof of Theorem 2.

Section 5. Proof of Theorem 3.

§2. Main results.

Let $\alpha \in (0,1)$ be arbitrarily fixed. Henceforth $H^{\alpha+2,\frac{\alpha+2}{2}}(\overline{\Omega}\times[0,T])$, $H^{\alpha+2,\frac{\alpha+2}{2}}(\partial\Omega\times[0,T])$ and $H^{\alpha+1}(\overline{\Omega})$, etc. denote the Hölder spaces (e.g., [10]).

In (1.3), we assume that $\varphi \in H^{\alpha+2,\frac{\alpha+2}{2}}(\partial\Omega\times[0,T])$ and that

$$\varphi \ge 0, \quad \varphi(\cdot,0) = 0, \quad \varphi \not\equiv 0 \quad \text{on } \partial\Omega\times[0,T]. \tag{2.1}$$

We set

$$L = \max_{(x,t)\in\partial\Omega\times[0,T]} \varphi. \tag{2.2}$$

Then, by Lemma 1 (i) below, we see that $0 \le u_a(x,t) \le L$ for $(x,t) \in \overline{\Omega} \times [0,T]$. For arbitrarily fixed $\alpha \in (0,1)$, $M_0 > 0$ and $M > 0$, we define an admissible set $\mathcal{U}$ of unknown semilnear terms a's by:

$$
\begin{aligned}
\mathcal{U} =&\{a \in H^{1+\alpha}[0,L];\ \|a\|_{H^{1+\alpha}[0,L]} \le M_0, \quad a(\eta) \le 0 \quad \text{for } \eta \in [0,L],\\
&a(0) = 0 \text{ and there exists a unique solution}\\
&u_a \in H^{\alpha+2,\frac{\alpha+2}{2}}(\overline{\Omega} \times [0,T]) \text{ such that} \|u_a\|_{H^{\alpha+2,\frac{\alpha+2}{2}}(\overline{\Omega}\times[0,T])} \le M\}.
\end{aligned}
\tag{2.3}
$$

In definition (2.3), the uniform boundedness of $\|u_a\|_{H^{\alpha+2,\frac{\alpha+2}{2}}(\overline{\Omega}\times[0,T])}$ requires boundedness and regularity of a and φ, the compatibility conditions (e.g., [10]). Moreover, since Ω is a rectangular domain (although Ω is not smooth), we can explicitly give the fundamental solution of $\partial_t - \Delta$ with a boundary condition (cf. (3.13)), we can discuss estimates of u_a by the boundary data φ and the semilinear term a by following the arguments in Chapter IV of [10]. However we will not here state them explicitly, in order to concentrate on discussions of the inverse problem. In other words, we a priori assume the existence of u_a such that $\|u_a\|_{H^{\alpha+2,\frac{\alpha+2}{2}}(\overline{\Omega}\times[0,T])} \le M$.

For $a, b \in C[0,L]$, we further set

$$
m(a,b) = \begin{cases} \sup\{\eta \in [0,L];\ a \ge b \text{ or } b \ge a \text{ on } [0,\eta]\}, \\ \text{if } a - b \text{ changes signs finite times near } 0, \\ 0, \qquad \text{otherwise.} \end{cases}
\tag{2.4}
$$

We note that $a(m(a,b)) = b(m(a,b))$.

Here and henceforth by the finiteness of changes of signs we mean: for any $a, b \in C[0,L]$, there exist $N \in \mathbb{N}$ and a partition $0 \equiv d_0 < d_1 < \cdots < d_N < d_{N+1} \equiv L$

such that

$$\begin{cases} a-b \ge 0 \text{ (respectively } \le 0) \text{ on } [d_0,d_1]\cup[d_2,d_3]\cup\cdots[d_{N_1-1},d_{N_1}], \\ \text{where} \quad N_1 = \begin{cases} N+1 & \text{if } N \text{ is even,} \\ N, & \text{if } N \text{ is odd,} \end{cases} \\ a-b \le 0 \text{ (respectively } \ge 0) \text{ on } [d_1,d_2]\cup[d_3,d_4]\cup\cdots[d_{N_2-1},d_{N_2}], \\ \text{where} \quad N_2 = \begin{cases} N, & \text{if } N \text{ is even,} \\ N+1, & \text{if } N \text{ is odd.} \end{cases} \end{cases}$$

Now we are ready to state our first result on conditional stability between 0 and the first point where $a-b$ changes signs.

Theorem 1. *We arbitrarily fix $\rho > 2$. There exists a constant $C = C(\mathcal{U},\Omega,T,\rho) > 0$ such that*

$$\|a-b\|_{C[0,m(a,b)]} \le C \left\| \frac{\partial u_a}{\partial n} - \frac{\partial u_b}{\partial n} \right\|^{\frac{1}{n+2}}_{L^\rho(0,T;L^\infty(\partial\Omega))} \tag{2.5}$$

for any $a,b\in\mathcal{U}$.

This theorem yields the uniqueness of a and b on $[0,L]$ if a and b are analytic functions, which can be regarded as a special case of Corollary in DuChateau and Rundell [4] although the paper takes a different formulation.

Remark. Our proof is based on pointwise lower bound (3.14) of the fundamental solution of $\partial_t - \Delta$ in Ω with the homogeneous Neumann boundary condition. To the authors' knowledge, suitable lower bounds are not proved for a general parabolic operator in a general bounded domain (see Chapter 3 in Davies [3] for $\Omega = \mathbb{R}^n$). In the case of rectangular domain Ω, we directly have a suitable lower bound, because the fundamental solution is explicitly given.

We do not look for the optimal exponent at the right hand side of (2.5). However, the exponent $\frac{1}{n+2}$ is optimal as long as we apply our argument of Section 3.

Theorem 1 does not assert the stability on $[0, L]$. However, within analytic functions, we can prove a more global estimate.

Theorem 2. *For arbitrarily given $M_1, M_2, M_3 > 0$ such that $M_1 \ge L$ and $M_2 > \frac{L}{2}$, we set*

$$\begin{aligned}\widehat{\mathcal{U}} = &\{a \in \mathcal{U};\ \ a \text{ can be extended analytically to the rectangle} \\ &D_0 \equiv \{z \in \mathbb{C}; 0 < Re\, z < M_1,\ |Im\, z| < M_2\}, \\ &a \in C(\overline{D_0}) \text{ and } \|a\|_{C(\overline{D_0})} \le M_3\}\ .\end{aligned}$$

Let $\rho > 2$ be arbitrary. Then there exists a constant $C = C(\widehat{\mathcal{U}}, \Omega, T, \rho) > 0$ such that

$$\|a - b\|_{C[0,L]} \le C \left\| \frac{\partial u_a}{\partial n} - \frac{\partial u_b}{\partial n} \right\|_{L^\rho(0,T;L^\infty(\partial\Omega))}^{\frac{\mu(m(a,b))}{n+2}} \tag{2.6}$$

for any $a, b \in \widehat{\mathcal{U}}$. Here we set

$$\begin{cases} \mu(m(a,b)) = \dfrac{1}{3}\left(1 - \dfrac{L}{M_1}\right)^{\frac{1}{\alpha}}, \\ \alpha = \alpha(m(a,b)) = \dfrac{2}{\pi} \arctan \dfrac{m(a,b)}{2\sqrt{3}M_1 - \sqrt{3}m(a,b)}. \end{cases}$$

Inequality (2.6) is not uniform in $a, b \in \widehat{\mathcal{U}}$, because $\mu(m(a,b)) \longrightarrow 0$ as $m(a,b) \longrightarrow 0$. In the case where we can take $M_1 = \infty$ in $\widehat{\mathcal{U}}$, we have

Corollary 1. *For arbitrarily given $M_2, M_3 > 0$ with $M_2 > \frac{L}{2}$, we set*

$$\begin{aligned}\widehat{\mathcal{U}_1} = &\{a \in \mathcal{U};\ \ a \text{ can be extended analytically to} \\ &D_1 \equiv \{z \in \mathbb{C}; Re\, z > 0,\ |Im\, z| < M_2\}, \\ &a \in C(\overline{D_1}) \text{ and } \|a\|_{C(\overline{D_1})} \le M_3\}\ .\end{aligned}$$

Let $\rho > 2$ be arbitrary. Then, for any $\delta \in \left(0, \frac{1}{3}\right)$, there exists a constant $C = C(\widehat{\mathcal{U}_1}, \Omega, T, \rho, \delta) > 0$ such that

$$\|a - b\|_{C[0,km(a,b)]} \le C \left\| \frac{\partial u_a}{\partial n} - \frac{\partial u_b}{\partial n} \right\|_{L^\rho(0,T;L^\infty(\partial\Omega))}^{\frac{1}{n+2}\left(\frac{1}{3}-\delta\right)\exp\left(\sqrt{3}\pi\left(\frac{1}{2}-k\right)\right)}$$

for any $a, b \in \widehat{\mathcal{U}_1}$ and any $k \in \mathbb{N}$, provided that $km(a,b) \le L$.

Finally we show

Theorem 3. *We arbitrarily fix $\rho > 2$. There exists a constant $C = C(\mathcal{U}, \Omega, T, \rho) > 0$ such that*

$$\|a - b\|_{C[0,L]} \le C \left\| \frac{\partial u_a}{\partial n} - \frac{\partial u_b}{\partial n} \right\|_{L^\rho(0,T;L^\infty(\partial\Omega))}^{\frac{1}{(n+2)^N}}$$

provided that $a, b \in \mathcal{U}$ and $a - b$ changes signs N-times over $[0, L]$.

Applying Theorem 3 to a class of piecewise fractional functions, we can directly obtain the following corollary.

Corollary 2. *Let us set*

$$\begin{aligned}
&\mathcal{P}_{N,n_1,n_2} = \{a \in \mathcal{U}; \text{there exist}\\
&N_1 \in \{1, ..., N\} \text{ and a partition } 0 \equiv d_0 < d_1 < ... < d_{N_1-1} < d_{N_1} \equiv L\\
&\text{such that } a|_{[d_j,d_{j+1}]} = \frac{p_j}{q_j},\ 0 \le j \le N_1 - 1,\\
&p_j \text{ and } q_j \text{ are polynomials whose orders are at most } n_1 \text{ and } n_2 \text{ respectively}\\
&q_j(\eta) \ne 0 \quad \text{for } \eta \ge 0\}
\end{aligned}$$

and let us arbitrarily fix $\rho > 2$. There exists a constant $C = C(\mathcal{P}_{N,n_1,n_2}, \Omega, T, \rho) > 0$ such that

$$\|a - b\|_{C[0,L]} \le C \left\| \frac{\partial u_a}{\partial n} - \frac{\partial u_b}{\partial n} \right\|_{L^\rho(0,T;L^\infty(\partial\Omega))}^{\frac{1}{(n+2)^{Nn_1n_2}}}$$

for any $a, b \in \mathcal{P}_{N,n_1,n_2}$.

§3. Proof of Theorem 1.

First we will show

Lemma 1. *Let* $a \in \mathcal{U}$ *and let* $\varphi \in H^{\alpha+2,\frac{\alpha+2}{2}}(\overline{\Omega} \times [0,T])$ *satisfy (2.1).*

(i) $0 \le u_a(y,s) \le \max_{\partial\Omega\times[0,t]} \varphi$ *for* $y \in \overline{\Omega}$ *and* $0 \le s \le t$.

(ii) For any $\eta \in u_a(\overline{\Omega} \times [0,T])$, *there exist* $y_0 \in \partial\Omega$ *and* $s_0 \in [0,T]$ *such that* $\varphi(y_0,s_0) = \eta$.

Proof of Lemma 1. By the mean value theorem and $a(0) = 0$, we have

$$a(u(x,t)) = a(u(x,t)) - a(0) = a'(\lambda(x,t))u(x,t),$$

where $\lambda(x,t)$ is some value between $u(x,t)$ and 0, and we can take $\lambda(x,t)$ as a continuous function. Therefore

$$\Delta u_a + a'(\lambda)u_a - \partial_t u_a = 0 \quad \text{in } \Omega \times (0,T).$$

Setting $v = e^{-M_0 t}u_a$, we have

$$\Delta v + (a'(\lambda) - M_0)v - \partial_t v = 0 \quad \text{in } \Omega \times (0,T).$$

Let us assume contrarily that $\inf_{(x,t)\in\Omega\times(0,T)} v(x,t) < 0$. Then, since $v|_{\partial\Omega\times(0,T)} = e^{-M_0 t}\varphi \ge 0$ and $v(\cdot,0) = 0$, we see that v attains the minimum at $(x_0,t_0) \in \Omega \times (0,T]$. By $a \in \mathcal{U}$, we have $a'(\lambda) - M_0 \le 0$. Therefore the strong maximum principle (e.g., Renardy and Rogers [15, p.122]), we see that v is constant in $\Omega \times (0,T)$. By $v(\cdot,0) = 0$, we arrive at $v \equiv 0$, which contradicts that $\varphi \not\equiv 0$. Hence $\inf_{(x,t)\in\Omega\times(0,T)} v(x,t) \ge 0$, and the first inequality in (i) follows.

Next we will prove the second inequality in (i). Let us set $P_0 v = \Delta v - \partial_t v$. Then, by $a \le 0$, we have

$$(P_0 u_a)(x,s) \ge 0, \qquad x \in \overline{\Omega},\ 0 \le s \le t.$$

Therefore the weak maximum principle (e.g., p.121 in [15]) yields the second inequality in (i). Thus the proof of (i) is complete.

Finally we will complete the proof of (ii). Let $\eta \in u_a(\overline{\Omega} \times [0,T])$. Then (i) yields that $\eta \in [0, \max_{\partial\Omega\times[0,T]} \varphi]$. Since $\varphi \ge 0$ and φ is continuous, the set $\varphi(\partial\Omega \times [0,t])$ is an interval, so that there exists $(y_0, s_0) \in \partial\Omega \times [0,T]$ such that $\eta = \varphi(y_0, s_0)$.

Henceforth C, C_0, etc. denote generic positive constants depending only on $\mathcal{U}$, $\widehat{\mathcal{U}}$, Ω, T, but independent of choices of a and b.

Proof of Theorem 1. We set $m = m(a,b)$. We may assume that $\|a-b\|_{C[0,m]} > 0$. Otherwise conclusion (2.5) is trivial by $a(0) = b(0) = 0$. Without loss of generality, we may assume that

$$a - b > 0 \qquad \text{on } (0,m). \tag{3.1}$$

Then we can choose $T_m \in (0,T]$ such that $m \in \varphi(\partial\Omega \times [0,T_m])$, that is,

$$[0,m] = \varphi(\partial\Omega \times [0,T_m]). \tag{3.2}$$

Let $|(a-b)(\eta)|$ attain the maximum $\|a-b\|_{C[0,m]}$ at η_0. Then $0 < \eta_0 < m$ by $a(0) = b(0)$ and $\|a-b\|_{C[0,m]} > 0$. By (3.2) and Lemma 1 (ii), we choose $y_0 \in \partial\Omega$ and $s_0 \in (0,T_m)$ such that

$$\varphi(y_0, s_0) = \eta_0. \tag{3.3}$$

In terms of $\varphi(\cdot,0) = 0$ and $0 < \eta_0 < m$, we note that we can choose s_0 such that

$$s_0 < T_m. \tag{3.4}$$

In fact, let us assume contrarily that $\varphi(y,s) < \eta_0$ for any $y \in \partial\Omega$ and $s_0 < T_m$. Then, by (3.2), we have $m \le \eta_0$, which contradicts that $m > \eta_0$. Hence there exist $y_0 \in \partial\Omega$ and $s_0 < T_m$ such that $\varphi(y_0, s_0) \ge \eta_0$. If $\varphi(y_0, s_0) = \eta_0$, then (3.3) is clearly seen. Let $\varphi(y_0, s_0) > \eta_0$. Since $\varphi(y_0, s)$ is continuous in s and $\varphi(y_0, 0) = 0$, the intermediate value theorem yields the existence $s_1 \in (0, s_0)$ such that $\varphi(y_0, s_1) = \eta_0$. Thus (3.4) follows.

By the mean value theorem and $a, b \in \mathcal{U}$, we have

$$\begin{aligned} \|a-b\|_{C[0,m]} &= |(a-b)(\eta_0)| \\ &= |(a-b)(\eta_0) - (a-b)(0)| \le C\eta_0. \end{aligned} \tag{3.5}$$

By (3.3), (1.2), (1.3) and $\varphi(y_0, 0) = 0$, we apply the mean value theorem again, so that

$$\eta_0 = |\varphi(y_0, s_0) - \varphi(y_0, 0)| = |(\partial_t \varphi)(y_0, \theta) s_0| \le C s_0,$$

where θ is some value in $(0, s_0)$. Therefore, by (3.5), we obtain

$$\|a-b\|_{C[0,m]} \le C s_0. \tag{3.6}$$

Let us set

$$d = \left\| \frac{\partial(u_a - u_b)}{\partial n} \right\|_{L^\rho(0,T;L^\infty(\partial\Omega))}. \tag{3.7}$$

If $s_0 \le d^{\frac{1}{n+2}}$, then (3.6) finishes the proof of (2.5). Hence we can assume that

$$d^{\frac{1}{n+2}} < s_0. \tag{3.8}$$

Let us set $v = u_a - u_b$ on $\overline{\Omega} \times [0, T]$. Then

$$\partial_t v - \Delta v - q(x,t)v = a(u_b(x,t)) - b(u_b(x,t)), \quad x \in \Omega,\ 0 < t < T_m \tag{3.9}$$

$$v(x, 0) = 0, \qquad x \in \Omega, \tag{3.10}$$

$$v = 0 \qquad \text{on } \partial\Omega \times (0, T_m), \tag{3.11}$$

where

$$q(x,t) = \begin{cases} \frac{a(u_a(x,t)) - a(u_b(x,t))}{u_a(x,t) - u_b(x,t)}, & \text{if } u_a(x,t) \neq u_b(x,t), \\ a'(u_a(x,t)), & \text{if } u_a(x,t) = u_b(x,t). \end{cases}$$

Then

$$a(u_b(x,t)) - b(u_b(x,t)) \geq 0 \quad \text{on } \Omega \times [0, T_m] \tag{3.12}$$

by (3.1).

Let $G = G(t, x, s, y)$ be the fundamental solution to $\partial_t - \Delta - q\cdot$ with the homogeneous Neumann condition.

We will prove

Lemma 2. *There exists a constant $\mu_0 > 0$ such that*

$$\liminf_{r\to 0} \frac{1}{r^{n+1}} \int_{s_0 - r}^{s_0} \int_{|y-y_0|<r} G(T_m, y_0, s, y) dy ds \geq \mu_0$$

for every $y_0 \in \overline{\Omega}$ and $s_0 \in (0, T_m)$. Here the constant $\mu_0 > 0$ is independent of T_m, y_0, s_0.

Proof. By the mean value theorem, for any $(x,t) \in \overline{\Omega} \times [0,T]$, by Lemma 1, there exists $\lambda = \lambda(x,t)$ such that $0 \leq \lambda(x,t) \leq L$ and $q(x,t) = a'(\lambda(x,t))$ for $(x,t) \in \overline{\Omega} \times [0,T]$. Therefore, by $a \in \mathcal{U}$, we have

$$q(x,t) \leq \|a'\|_{C[0,L]} \leq M_0, \qquad (x,t) \in \overline{\Omega} \times [0,T].$$

Let $G_0 = G_0(t,x,s,y)$ be the fundamental solution to $\partial_t - \Delta + M_0$ with the homogeneous Neumann boundary condition. Then the comparison theorem for the fundamental solution yields

$$G(t,x,s,y) \geq G_0(t,x,s,y), \qquad t > s > 0,\ x, y \in \overline{\Omega}$$

(e.g., Theorem 11.1 (p.85) in Itô [7]). Hence, by noting that $G_0(t,x,s,y) = G_0(t-s,x,0,y)$ for $0<s<t$ and $x,y\in\overline{\Omega}$, it suffices to prove

$$\liminf_{r\to 0}\frac{1}{r^{n+1}}\int_{s_0-r}^{s_0}\int_{|y-y_0|<r} G_0(T_m-s,y_0,0,y)dyds \ge \mu_0.$$

Since we can directly verify that $e^{M_0 t}G_0(t,x,s,y)$ is the fundamental solution for $\partial_t-\Delta$ with the homogeneous Neumann condition, we can assume that $M_0=0$ without loss of generality. For $M_0=0$, we can directly prove that

$$\begin{aligned} G_0(t,x,s,y) =& \frac{1}{(4\pi(t-s))^{\frac{n}{2}}} \\ &\times \Pi_{j=1}^n\sum_{k=-\infty}^{\infty}\left\{\exp\left(-\frac{(x_j-y_j+2k\ell_j)^2}{4(t-s)}\right)+\exp\left(-\frac{(x_j+y_j+2k\ell_j)^2}{4(t-s)}\right)\right\}, \\ &\quad 0<s<t,\ x,y\in\overline{\Omega}, \end{aligned} \tag{3.13}$$

so that we have

$$G_0(t,x,s,y)\ge \frac{1}{(4\pi(t-s))^{\frac{n}{2}}}e^{-\frac{|x-y|^2}{4(t-s)}},\quad 0<s<t,\ x,y\in\overline{\Omega}. \tag{3.14}$$

Therefore we use the polar coordinate to obtain

$$\begin{aligned} &\frac{1}{r^{n+1}}\int_{s_0-r}^{s_0}\int_{|y-y_0|<r} G_0(T_m-s,y_0,0,y)dyds \\ \ge&\frac{1}{r^{n+1}}\int_{T_m-s_0}^{T_m-s_0+r}\int_{|y-y_0|<r}\frac{1}{(4\pi\eta)^{\frac{n}{2}}}e^{-\frac{|y-y_0|^2}{4\eta}}dyd\eta \\ =&\frac{c_0}{r^{n+1}}\int_{T_m-s_0}^{T_m-s_0+r}\int_0^r\frac{1}{\eta^{\frac{n}{2}}}e^{-\frac{\xi^2}{4\eta}}\xi^{n-1}d\xi d\eta. \end{aligned}$$

Here $c_0>0$ is depends only on n. Noting that

$$\begin{aligned} \frac{1}{\eta^{\frac{n}{2}}}e^{-\frac{\xi^2}{4\eta}} \ge& \frac{1}{(T_m-s_0+r)^{\frac{n}{2}}}e^{-\frac{\xi^2}{4(T_m-s_0)}} \\ \ge&\frac{1}{(T+r)^{\frac{n}{2}}}e^{-\frac{\xi^2}{4(T_m-s_0)}}, \end{aligned}$$

we have

$$
\begin{aligned}
&\liminf_{r\to 0} \frac{1}{r^{n+1}} \int_{s_0-r}^{s_0} \int_{|y-y_0|<r} G_0(T_m - s, y_0, 0, y)dyds \\
\ge &\liminf_{r\to 0} \frac{c_0}{(T+r)^{\frac{n}{2}}} \frac{1}{r^n} \int_0^r e^{-\frac{\xi^2}{4(T_m-s_0)}} \xi^{n-1} d\xi \\
= &\frac{c_0}{nT^{\frac{n}{2}}}
\end{aligned}
$$

by the de L'Hôpital theorem. Thus the proof of Lemma 2 is complete.

On the other hand,

$$
\begin{aligned}
v(x,t) = &\int_0^t \int_\Omega G(t,x,s,y)\{a(u_b(y,s)) - b(u_b(y,s))\}dyds \\
&+ \int_0^t \int_{\partial\Omega} G(t,x,s,y) \frac{\partial(u_a - u_b)}{\partial n}(y,s)d\sigma_y ds, \quad x \in \overline{\Omega},\, t > 0
\end{aligned}
\tag{3.15}
$$

(e.g., Theorem 9.1 (pp.68-69) in [7]). Here $\int_{\partial\Omega} \cdots d\sigma_y$ is the surface integral. By (3.11), we have

$$
\begin{aligned}
&\int_0^t \int_\Omega G(t,x,s,y)\{a(u_b(y,s)) - b(u_b(y,s))\}dyds \\
= &\int_0^t \int_{\partial\Omega} G(t,x,s,y) \frac{\partial(u_b - u_a)}{\partial n}(y,s)d\sigma_y ds, \quad x \in \partial\Omega,\, t > 0.
\end{aligned}
\tag{3.16}
$$

We denote the right and left hand sides of (3.16) respectively by $I_1(x,t)$ and $I_2(x,t)$. We will first estimate $|I_1(x,T_m)|$. We recall that $\rho > 2$ and we set $\rho' = \frac{\rho}{\rho-1}$. Then $1 < \rho' < 2$. By

$$
\int_{\partial\Omega} |G(t,x,s,y)|d\sigma_y \le C(t-s)^{-\frac{1}{2}}, \qquad x \in \overline{\Omega},\, t,s \in [0,T]
$$

(e.g., (7.10) (p.53) in [7]), in terms of the Hölder inequality, we have

$$
\begin{aligned}
|I_1(x,T_m)| \le &\int_0^{T_m} \left\| \frac{\partial(u_a - u_b)}{\partial n}(\cdot, s)\right\|_{L^\infty(\partial\Omega)} \int_{\partial\Omega} |G(T_m,x,s,y)|d\sigma_y ds \\
\le &C\left(\int_0^{T_m} (T_m - s)^{-\frac{\rho'}{2}}\right)^{\frac{1}{\rho'}} \left\| \frac{\partial(u_a - u_b)}{\partial n}\right\|_{L^\rho(0,T;L^\infty(\partial\Omega))} \\
= &C\left(\frac{2}{2-\rho'} T_m^{\frac{2-\rho'}{2}}\right)^{\frac{1}{\rho'}} \left\| \frac{\partial(u_a - u_b)}{\partial n}\right\|_{L^\rho(0,T;L^\infty(\partial\Omega))}.
\end{aligned}
$$

Consequently

$$|I_1(x,T_m)| \le C_1 d, \qquad x \in \overline{\Omega}. \tag{3.17}$$

Here we recall that d is defined by (3.7).

We set

$$\begin{aligned} B_0 =& \{y \in \overline{\Omega};\ |y-y_0| < d^{\frac{1}{n+2}}\} \\ & \times\{s \in (0,T_m);\ s_0 - d^{\frac{1}{n+2}} < s < s_0\} \equiv B_{y_0} \times B_{s_0}. \end{aligned}$$

Since $s_0 > d^{\frac{1}{n+2}}$ by (3.8), we have

$$|B_{s_0}| = d^{\frac{1}{n+2}}. \tag{3.18}$$

Next we will establish a lower estimate of $I_2(y_0,T_m)$. In terms of (3.12), Lemma 2 and (3.18), for sufficiently small $d>0$, we have

$$\begin{aligned} I_2(y_0,T_m) &= \int_0^{T_m}\int_\Omega G(T_m,y_0,s,y)\{a(u_b(y,s)) - b(u_b(y,s))\}dyds \\ &\ge \int_{B_0} G(T_m,y_0,s,y)\{a(u_b(y,s)) - b(u_b(y,s))\}dyds \\ &\ge \min_{(y,s)\in B_0}\{a(u_b(y,s)) - b(u_b(y,s))\}\int_{B_0} G(T_m,y_0,s,y)dyds \\ &\ge \mu_0 d^{\frac{n+1}{n+2}} \min_{(y,s)\in B_0}\{a(u_b(y,s)) - b(u_b(y,s))\}. \end{aligned} \tag{3.19}$$

By (3.17) and (3.19), we obtain

$$\min_{(y,s)\in B_0}\{a(u_b(y,s)) - b(u_b(y,s))\} \le C_2 d^{\frac{1}{n+2}}. \tag{3.20}$$

On the other hand, for $(y,s) \in B_0$, by (3.18), we have

$$
\begin{aligned}
&|a(u_b(y,s)) - b(u_b(y,s))| \\
=&|a(u_b(y_0,s_0)) - b(u_b(y_0,s_0)) + a(u_b(y,s)) - a(u_b(y_0,s_0)) \\
&+b(u_b(y_0,s_0)) - b(u_b(y,s))| \\
\geq&\|a-b\|_{C[0,m]} - (\|a'\|_{L^\infty(0,M)} + \|b'\|_{L^\infty(0,M)}) \\
&\times \sup_{(y,s)\in B_0} |u_b(y,s) - u_b(y_0,s_0)| \\
\geq&\|a-b\|_{C[0,m]} - C_3 M_0 M d^{\frac{1}{n+2}} \qquad (3.21)
\end{aligned}
$$

Therefore (3.21) and (3.20) yield

$$\|a-b\|_{C[0,L]} \leq C_4 d^{\frac{1}{n+2}}.$$

Thus the proof of Theorem 1 is complete.

§4. Proofs of Theorem 2 and Corollary 1.

We may assume that

$$d = \left\| \frac{\partial(u_a - u_b)}{\partial n} \right\|_{L^\rho(0,T;L^\infty(\partial\Omega))}$$

is sufficiently small. First we show

Lemma 3. *Let $0 < m < \frac{M_1}{2}$, $m \leq L \leq M_1$, $\varepsilon \leq 1$ and $C > 1$. Let $f = f(z)$ be analytic in $D_0 = \{z \in \mathbb{C}; 0 < Re\, z < M_1, |Im\, z| < M_2\}$ with $M_2 > \frac{m}{2}$ and $f \in C(\overline{D_0})$. Suppose that $|f| \leq C$ on $\overline{D_0}$ and that $|f(x)| \leq \varepsilon$ for $x \in [0,m]$. Then*

$$|f(x)| \leq C^{1-\mu}\varepsilon^\mu, \qquad 0 \leq x \leq L, \tag{4.1}$$

where we set

$$
\begin{cases}
\mu = \mu(m) = \dfrac{1}{3}\left(1 - \dfrac{L}{M_1}\right)^{\frac{1}{\alpha}}, \\
\alpha = \alpha(m) = \dfrac{2}{\pi} \arctan \dfrac{m}{2\sqrt{3}M_1 - \sqrt{3}m}.
\end{cases} \tag{4.2}
$$

Proof of Lemma 3. We will argue similarly to the proof of Lemma 10.6.6 (pp.124-125) in Cannon [2]. Let $0<\delta<\gamma<m-\delta$ and let us set $A=(\gamma-\delta,0)$, $B=(\gamma+\delta,0)$ and $P=(\gamma,\sqrt{3}\delta)$. Applying the Lindelöf theorem (e.g., Lemma 10.6.4 (p.123) in [2]) to f in the regular triangle $\triangle ABP$ and noting that $\varepsilon\le C$, we obtain

$$\left|f\left(\gamma+\frac{\sqrt{-1}}{\sqrt{3}}\delta\right)\right|\le C^{\frac{2}{3}}\varepsilon^{\frac{1}{3}}$$

if $0<\delta<\gamma<m-\delta$. Set $O=(0,0)$, $P_1=(m,0)$, $P_2=\left(\frac{m}{2},\frac{1}{2\sqrt{3}}m\right)$ and $P_3=\left(\frac{m}{2},-\frac{1}{2\sqrt{3}}m\right)$. Changing (γ,δ) such that $0<\delta<\gamma<m-\delta$, we see that

$$|f(z)|\le C^{\frac{2}{3}}\varepsilon^{\frac{1}{3}},\quad z\in\overline{\triangle OP_1P_2}.$$

Arguing similarly in the case of $\operatorname{Im} z<0$, we obtain

$$|f(z)|\le C^{\frac{2}{3}}\varepsilon^{\frac{1}{3}},\quad z\in\ \widehat{D_0}\equiv\overline{\triangle OP_1P_2\cup\triangle OP_1P_3}. \tag{4.3}$$

Here and henceforth we identify $z=z_1+\sqrt{-1}z_2$, $z_1,z_2\in\mathbb{R}$ with $z\in\mathbb{R}^2$.

We consider the circle Γ whose centre is $(M_1,0)$ and passes P_2 and P_3. That is, Γ: $(z_1-M_1)^2+z_2^2=R^2$, where

$$R=R(M_1)=\sqrt{\frac{m^2-3mM_1+3M_1^2}{3}}.$$

The z_1-coordinate of the rest intersection points of Γ with the infinite straight line OP_2, is $\frac{3M_1-m}{2}$, and the assumption $m<\frac{M_1}{2}$ implies that $\frac{3M_1-m}{2}>\frac{m}{2}$, so that the inferior arc P_2P_3 of Γ is included in $\widehat{D_0}$. Therefore (4.3) yields

$$|f(z)|\le C^{\frac{2}{3}}\varepsilon^{\frac{1}{3}},\quad z\in\text{inferior arc } P_2P_3 \text{ of } \Gamma. \tag{4.4}$$

On the other hand, we introduce the sector defined by

$$\left\{z\in\mathbb{C};\,0<|z-M_1|<R,\quad -\frac{\alpha\pi}{2}<\arg(z-M_1)<\frac{\alpha\pi}{2}\right\}.$$

By the assumption that $\varepsilon < C$ and $M_2 > \frac{m}{2}$, we see that $|f| \le C$ on the closure of the sector, and $C^{\frac{2}{3}}\varepsilon^{\frac{1}{3}} < C$. Therefore we can apply a theorem by Carleman (e.g., p.121 in [2]), so that

$$|f(x)| \le C^{1-\left(\frac{M_1-x}{R}\right)^{\frac{1}{\alpha}}} \left(C^{\frac{2}{3}}\varepsilon^{\frac{1}{3}}\right)^{\left(\frac{M_1-x}{R}\right)^{\frac{1}{\alpha}}}$$

for $\frac{m}{2} \le x \le M_1$. Therefore, for $\frac{m}{2} \le x \le L$, noting that $M_1 - x \ge M_1 - L$, $\varepsilon \le 1$ and $C \ge 1$, we have

$$|f(x)| \le C^{1-\frac{1}{3}\left(\frac{M_1-L}{R}\right)^{\frac{1}{\alpha}}} \varepsilon^{\frac{1}{3}\left(\frac{M_1-L}{R}\right)^{\frac{1}{\alpha}}}, \qquad \frac{m}{2} \le x \le L. \tag{4.5}$$

Since $\varepsilon < C^{1-\theta}\varepsilon^{\theta}$ for $\varepsilon < C$ and $0 < \theta < 1$, inequality (4.5) and $|f(x)| \le \varepsilon$ for $0 \le x \le m$ imply (4.5) for all $x \in [0, L]$.

For $0 < m < \frac{M_1}{2}$, we have $R \le M_1$ and, by $\varepsilon \le 1$, we see that

$$\varepsilon^{\frac{1}{3}\left(\frac{M_1-L}{R}\right)^{\frac{1}{\alpha}}} \le \varepsilon^{\frac{1}{3}\;1-\frac{L}{M_1}\;{}^{\frac{1}{\alpha}}}.$$

Thus the proof of Lemma 3 is complete.

Furthermore in the case $M_1 = \infty$, we can have

Lemma 4. *Let $m \le L$, $\varepsilon \le 1$ and $C > 1$. Let $f = f(z)$ be analytic in $D_1 = \{z \in \mathbb{C}; Re\, z > 0,\ |Im\, z| < M_2\}$ with $M_2 > \frac{m}{2}$ and $f \in C(\overline{D_1})$. Suppose that $|f| \le C$ on $\overline{D_1}$ and that $|f(x)| \le \varepsilon$ for $x \in [0, m]$. Then, for any μ_0 such that*

$$0 < \mu_0 < \frac{1}{3}\exp\left(\sqrt{3}\pi\left(\frac{1}{2} - \frac{L}{m}\right)\right), \tag{4.6}$$

we have

$$|f(x)| \le C^{1-\mu_0}\varepsilon^{\mu_0}, \qquad 0 \le x \le L. \tag{4.7}$$

Proof of Lemma 4. In terms of (4.2), it is sufficient to verify that

$$\lim_{M_1\to\infty}\frac{1}{3}\left(\frac{M_1-L}{R(M_1)}\right)^{\frac{1}{\alpha}}=\frac{1}{3}\exp\left(\sqrt{3}\pi\left(\frac{1}{2}-\frac{L}{m}\right)\right).$$

This limit can be verified directly by the de L'Hôpital theorem, for example.

Now we will complete

Proof of Theorem 2. First, since a and b are analytic, we conclude that $m(a,b)>0$ or $a\equiv b$ on $[0,L]$. In fact, let $m(a,b)=0$. Then we can choose an infinite number of distinct $m_k\in(0,L)$, $k\in\mathbb{N}$, such that $a(m_k)=b(m_k)$, $k\in\mathbb{N}$, and $\lim_{k\to\infty}m_k=0$. Hence, by the uniqueness of analytic functions, we have $a\equiv b$ on $[0,M]$. In the case of $a\equiv b$, conclusion (2.6) is trivial.

Consequently it suffices to consider the case of $m(a,b)>0$. In the case $m(a,b)<\frac{M_1}{2}$, we can directly apply Lemma 3 to $a-b$ in D_0 and in terms of (2.5), conclusion (2.6) is seen. Finally let $m(a,b)\ge\frac{M_1}{2}$. Since $\alpha(m)=\frac{2}{\pi}\arctan\frac{m}{2\sqrt{3}M_1-\sqrt{3}m}$ is monotone increasing in m, we have

$$\alpha(m(a,b))\ge\alpha\left(\frac{M_1}{2}\right). \tag{4.8}$$

On the other hand, by (2.5), we have

$$\|a-b\|_{C[0,M_1/2]}\le C\left\|\frac{\partial u_a}{\partial n}-\frac{\partial u_b}{\partial n}\right\|_{L^\rho(0,T;L^\infty(\partial\Omega))}^{\frac{1}{n+2}}$$

and Lemma 3 yields

$$\|a-b\|_{C[0,L]}\le C\left\|\frac{\partial u_a}{\partial n}-\frac{\partial u_b}{\partial n}\right\|_{L^\rho(0,T;L^\infty(\partial\Omega))}^{\frac{1}{3(n+2)}\,1-\frac{L}{M_1}\,\frac{1}{\alpha(M_1/2)}}.$$

Since we may assume that

$$\left\|\frac{\partial u_a}{\partial n}-\frac{\partial u_b}{\partial n}\right\|_{L^\rho(0,T;L^\infty(\partial\Omega))}\le 1,$$

inequality (4.8) yields conclusion (2.6). Thus the proof of Theorem 2 is complete.

Finally, setting $m = m(a, b)$ and $L = km$ with $k \in \mathbb{N}$ and applying Lemma 4, we can complete the proof of Corollary 1.

§5. Proof of Theorem 3.

Let us recall that $m(a, b)$ is defined by (2.4) and let us set $m_0 = m(a, b)$. By the finiteness of the number of zeros of $a - b$ on $[0, L]$, we can choose $m_1 \in (m_0, L)$ such that

$$\begin{aligned} &a(\eta) \ge b(\eta) \quad \text{for } \eta \in (0, m_0), \\ &a(\eta) \le b(\eta) \quad \text{for } \eta \in (m_0, m_1), \quad a(m_1) = b(m_1). \end{aligned} \tag{5.1}$$

We will prove

$$\|a - b\|_{C[m_0, m_1]} \le C d^{\frac{1}{(n+2)^2}}, \tag{5.2}$$

where d is defined by (3.7).

There exist $T_0 < T_1$ such that $T_0, T_1 \in (0, T]$ and

$$[0, m_j] = \varphi(\partial\Omega \times [0, T_j]), \qquad j = 1, 2. \tag{5.3}$$

By Lemma 1, we have

$$0 \le u_b(y, s) \le m_1, \qquad y \in \overline{\Omega},\ 0 \le s \le T_1, \tag{5.4}$$

and, in view of Theorem 1,

$$\|a - b\|_{C[0, m_0]} \le C_1 d^{p_0} \quad \text{where } p_0 = \frac{1}{n+2}. \tag{5.5}$$

Henceforth $C_j > 0$ denote constants which are dependent on $\mathcal{U}$ and φ, but independent of the choices $a, b \in \mathcal{U}$.

We set

$$Q_1 = \{(y,s) \in \overline{\Omega} \times [0,T_1];\ m_0 \le u_b(y,s) \le m_1\},$$

$$Q_2 = \{(y,s) \in \overline{\Omega} \times [0,T_1];\ 0 \le u_b(y,s) \le m_0\}.$$

By (5.4) we note that $\overline{Q_1 \cup Q_2} = \overline{\Omega} \times [0,T_1]$. Let us recall that $v = u_a - u_b$ satisfies (3.9) - (3.11), so that we have (3.16). We will rewrite (3.16) as

$$\begin{aligned}
&\int\int_{Q_2} G(T_1,x,s,y)\{a(u_b(y,s)) - b(u_b(y,s))\}dyds \\
&- \int_0^{T_1}\int_{\partial\Omega} G(T_1,x,s,y)\frac{\partial(u_a-u_b)}{\partial n}(y,s)d\sigma_y ds \\
&= \int\int_{Q_1} G(T_1,x,s,y)\{b(u_b(y,s)) - a(u_b(y,s))\}dyds, \qquad x \in \partial\Omega.
\end{aligned} \tag{5.6}$$

Let $|(a-b)(\eta)|$ attain the maximum $\|a-b\|_{C[m_0,m_1]}$ at $\eta_1 \in (m_0,m_1)$:

$$\|a-b\|_{C[m_0,m_1]} = (b-a)(\eta_1).$$

Note that we can assume that $m_0 < \eta_1 < m_1$. Otherwise $\eta_1 = m_0$ or $\eta_1 = m_1$, so that $\|a-b\|_{C[m_0,m_1]} = 0$. Then (5.2) is trivial.

Then we can choose $y_1 \in \partial\Omega$ and $s_1 \in (0,T_1]$ such that

$$\varphi(y_1,s_1) = \eta_1.$$

Moreover, in terms of (5.3), similarly to (3.4), we can prove that we can choose s_1 such that

$$T_0 < s_1 < T_1. \tag{5.7}$$

Since $(a-b)(m_0) = 0$, by the mean value theorem, we have

$$\begin{aligned}
&\|a-b\|_{C[m_0,m_1]} = (b-a)(\eta_1) \\
&= (b-a)(\eta_1) - (b-a)(m_0) \le C_2(\eta_1 - m_0).
\end{aligned} \tag{5.8}$$

Similarly, by $(a-b)(m_1)=0$, we have

$$\|a-b\|_{C[m_0,m_1]} \le C_2(m_1-\eta_1). \tag{5.9}$$

Let $\nu \in (0,1)$ be chosen later and let us consider the two cases:

$$d^{\nu p_0} \le \min\left\{\frac{\eta_1-m_0}{2M}, \frac{m_1-\eta_1}{2M}, \frac{T_0}{2}\right\} \tag{5.10}$$

and

$$d^{\nu p_0} \ge \min\left\{\frac{\eta_1-m_0}{2M}, \frac{m_1-\eta_1}{2M}, \frac{T_0}{2}\right\}. \tag{5.11}$$

In case (5.11), by (5.8) and (5.9), we can immediately obtain

$$\|a-b\|_{C[m_0,m_1]} \le C_3 d^{\nu p_0}$$

or

$$\|a-b\|_{C[m_0,m_1]} \le 2M_0 \le 4M_0 T_0^{-1} d^{\nu p_0}.$$

Hence with (5.5), choosing $\nu = \frac{1}{n+2}$, we can complete the proof of (5.2).

Let us consider case (5.10). We set

$$B_1 = \{y \in \overline{\Omega} : |y-y_1| < d^{\nu p_0}\} \times \{s \in (0,T_1];\, s_1 - d^{\nu p_0} < s < s_1\}.$$

Then, by (5.7) and (5.10), we have

$$s_1 - d^{\nu p_0} > T_0 - d^{\nu p_0} \ge T_0 - \frac{T_0}{2} = \frac{T_0}{2} > 0. \tag{5.12}$$

Moreover, for any $(y,s) \in B_1$, in terms of the mean value theorem, we obtain

$$\begin{aligned}
&|u_b(y,s)-\eta_1| = |u_b(y,s) - u_b(y_1,s_1)| \\
\le&\|u_b\|_{C^1(\overline{\Omega}\times[0,T])}(|y-y_1| + |s-s_1|) \le 2Md^{\nu p_0} \\
\le&\min\{\eta_1-m_0, m_1-\eta_1\},
\end{aligned}$$

so that

$$m_0 \le u_b(y,s) \le m_1, \qquad (y,s) \in B_1. \tag{5.13}$$

Let us denote the left and the right hand sides of (5.6) by $J_1(x)$ and $J_2(x)$ respectively. By (5.7), (5.13) and Lemma 2, we obtain

$$\begin{aligned}
J_2(x) \ge & \int\int_{B_1} G(T_1,x,s,y)\{b(u_b(y,s)) - a(u_b(y,s))\}dyds \\
\ge & \int_{s_1-d^{\nu p_0}}^{s_1} \int_{|y-y_1|<d^{\nu p_0}} G(T_1,x,s,y)dyds \times \min_{(y,s)\in B_1}\{b(u_b(y,s)) - a(u_b(y,s))\} \\
\ge & C_4 d^{\nu p_0(n+1)} \min_{(y,s)\in B_1}\{b(u_b(y,s)) - a(u_b(y,s))\}.
\end{aligned}$$

On the other hand, similarly to (3.21), we see that

$$\begin{aligned}
& b(u_b(y,s)) - a(u_b(y,s)) \\
= & b(u_b(y_1,s_1)) - a(u_b(y_1,s_1)) + a(u_b(y_1,s_1)) - a(u_b(y,s)) + b(u_b(y,s)) - b(u_b(y_1,s_1)) \\
\ge & \|a-b\|_{C[m_0,m_1]} - C_5 d^{\nu p_0}, \quad (y,s) \in B_1.
\end{aligned}$$

Hence

$$J_2(x) \ge C_4 d^{\nu p_0(n+1)}(\|a-b\|_{C[m_0,m_1]} - C_5 d^{\nu p_0}). \tag{5.14}$$

Next we will estimate $J_1(x)$. By (5.5) and definition of Q_2, we use (2.5) which was already proved to have

$$\begin{aligned}
|J_1(x)| \le & C_1 d^{p_0} \int_0^{T_1}\int_\Omega G(T_1,x,s,y)dyds \\
& + \int_0^{T_1}\int_{\partial\Omega} G(T_1,x,s,y)\left|\frac{\partial(u_a-u_b)}{\partial n}(y,s)\right| d\sigma_y ds \\
\le & C_6 d^{p_0} + C_6 d.
\end{aligned} \tag{5.15}$$

Here the first term is estimated by Theorem 8.3 in Chapter 2 in [7] for example, while the second term is estimated in the same manner as (3.17). Estimates (5.14) and (5.15) imply

$$C_4(\|a-b\|_{C[m_0,m_1]} - C_5 d^{\nu p_0}) \le C_6 d^{p_0-\nu p_0(n+1)} + C_6 d^{1-\nu p_0(n+1)}. \tag{5.16}$$

Since $0 < p_0 < 1$, the choice ν such that $p_0 - \nu p_0(n+1) = \nu p_0$, gives the optimal rate. That is, setting $\nu = \frac{1}{n+2}$, we have the optimal rate $\frac{1}{(n+2)^2}$ in (5.16), namely,

$$\|a-b\|_{C[m_0,m_1]} \le C_7 d^{\frac{1}{n+2}} + C_7 d^{\frac{1}{(n+2)^2}}.$$

Therefore we can continue this argument to complete the proof of Therem 3.

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