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On the determination of a sound speed and a damping coefficient by two measurements

by

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Abstract. We discuss an inverse problem of finding a sound speed $c(x)$ and a damping coefficient $\sigma(x)$ in a second order hyperbolic equation from two boundary observations. The coefficients are assumed to be unknown inside a ball in $\mathbb{R}^n$ with $n \geq 2$. On a suitable bounded part of the cylindrical surface, Cauchy data for solutions to a hyperbolic equation with zero initial data and a source located on the plane $\{(x,t) \in \mathbb{R}^{n+1} | \, x \cdot \nu = 0, t = 0\}$, are supposed to be given for two different unit vectors $\nu = \nu^{(k)}$, $k = 1, 2$. We obtain a conditional stability estimate under a priori assumptions on smallness of $c(x) - 1$ and $\sigma(x)$.

§1. Statement of the inverse problem and main results

In Glushkova [3], Romanov [11] - [13], Romanov and Yamamoto [14], [15], a new method for obtaining conditional stability estimates in determining coefficients for linear hyperbolic equations, has been proposed. This method uses a single observation for finding one unknown coefficient. An analysis shows that the problem with several unknown coefficients under the derivatives of the first order can also be successfully studied by this method (see [12, 13, 15]). However, when we apply such a method to determination of multiple coefficients under derivatives of different orders, we meet some difficulties.

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Recently the problem of finding a damping coefficient and a potential from two measurements was considered in Glushkova and Romanov [4].

In this article, we consider an inverse problem of determining unknown coefficients of the leading term and the term of the first order derivative in time, by means of two measurements. The technique of this article is essentially different from the former one but keeps some common features with the papers [11], [14].

Let $u = u(x,t)$, $x \in \mathbb{R}^n$, $n \geq 2$, be the solution to the equation

$$u_{tt} - c^2 \Delta u + \sigma u_t = 2\delta(t)\,\delta(x \cdot \nu), \quad (x,t) \in \mathbb{R}^{n+1}, \tag{1.1}$$

satisfying the zero initial condition:

$$u|_{t<0} = 0. \tag{1.2}$$

Here δ is the Dirac delta function supported on the specified point or the hyperplane, ν is a unit vector and the symbol $x \cdot \nu$ means the scalar product of the vectors x and ν. The solution to problem (1.1) and (1.2) depends on the parameter ν, i.e., $u = u(x,t,\nu)$. Physically $c(x)$ describes the sound speed, while $\sigma(x)$ is the damping coefficient of the medium which is inhomogeneous in x. In (1.1), the external force term is impulsive which we operate for determining c and σ.

Let us set

$$B = \{x \in \mathbb{R}^n |\, |x - x^0| < r\}, \quad \partial B = \{x \in \mathbb{R}^n |\, |x - x^0| = r\},$$

and assume that $B \subset \{x \in \mathbb{R}^n |\, x \cdot \nu > 0\}$. Throughout this paper, we set

$$m = \left[\frac{n-1}{2}\right] + 4,$$

and $[(n-1)/2]$ denotes the integer part of $(n-1)/2$.

Furthermore we assume that the supports of coefficients $\sigma(x)$, $c(x)-1$ are located strictly inside the ball B. Suppose also that $\sigma(x)$, $c(x)>0$ and $\sigma \in \mathbf{C}^{3m+2}(\mathbb{R}^n)$, $c \in \mathbf{C}^{3m+4}(\mathbb{R}^n)$.

Introduce the function $\tau(x,\nu)$ as the solution to the following problem for the eikonal equation:

$$|\nabla\tau(x,\nu)|^2 = c^{-2}(x), \quad \tau|_{x\cdot\nu=0} = 0. \tag{1.3}$$

Let $G(\nu)$ be the cylindrical domain:

$$G(\nu) := \{(x,t)|\, x\in B, \tau(x,\nu) < t < T+\tau(x,\nu)\},$$

where T be a positive number. Denote by $S(\nu)$ the lateral boundary of this domain and by $\Sigma_0(\nu)$ and $\Sigma_T(\nu)$ the lower and upper basements, respectively, i.e.,

$$\begin{aligned}
S(\nu) &:= \{(x,t)|\, x\in \partial B, \tau(x,\nu) \le t \le T+\tau(x,\nu)\},\\
\Sigma_0(\nu) &:= \{(x,t)|\, x\in B, t = \tau(x,\nu)\},\\
\Sigma_T(\nu) &:= \{(x,t)|\, x\in B, t = T+\tau(x,\nu)\}.
\end{aligned}$$

Consider the problem of determination of $\sigma(x)$ and $c(x)$. Let the following information be known. We choose two unit vectors $\nu^{(1)}$ and $\nu^{(2)}$ arbitrarily such that $B\subset \{x\in\mathbb{R}^n|\, x\cdot\nu^{(k)} > 0\}$ for $k=1,2$. For $\nu=\nu^{(k)}$, $k=1,2$, the traces of functions $\tau(x,\nu^{(k)})$ on ∂B and the traces on $S(\nu^{(k)})$ of solutions and its normal derivatives to problem (1.1) - (1.2) are given, that is,

$$\begin{aligned}
&u(x,t,\nu^{(k)}) = f^{(k)}(x,t), \quad \frac{\partial}{\partial n}u(x,t,\nu^{(k)}) = g^{(k)}(x,t), \quad (x,t)\in S(\nu^{(k)});\\
&\tau(x,\nu^{(k)}) = \tau^{(k)}(x), \quad x\in\partial B; \quad k=1,2.
\end{aligned} \tag{1.4}$$

Here and henceforth $\mathbf{n}$ denotes the unit outward normal vector to ∂B and we set $\frac{\partial u}{\partial n} = \nabla u \cdot \mathbf{n}$.

We discuss

Inverse Problem. Find $\sigma(x)$ and $c(x)$ from data $f^{(k)}$, $g^{(k)}$, $\tau^{(k)}$, $k = 1, 2$.

For constants $q_0 > 0$ and $d > 0$, let $\Lambda(q_0, d)$ be the set of functions (σ, c) satisfying the following two conditions:

$$\begin{aligned}&\text{There exists a domain } \Omega \text{ such that}\\ &\operatorname{supp} \sigma \cup \operatorname{supp} (c-1) \subset \Omega \subset B, \quad \operatorname{dist}(\partial B, \Omega) \geq d,\end{aligned} \tag{1.5}$$

and

$$\|\sigma\|_{\mathbf{C}^{3m+2}(\mathbb{R}^n)} \leq q_0, \quad \|c-1\|_{\mathbf{C}^{3m+4}(\mathbb{R}^n)} \leq q_0. \tag{1.6}$$

Throughout this article, we regard $\Lambda(q_0, d)$ as an admissible set where unknown σ and c are assumed to vary.

Let $(\sigma_j, c_j) \in \Lambda(q_0, d)$ and $\{f_j^{(k)}, g_j^{(k)}, \tau_j^{(k)}\}$ be the data of the corresponding solutions to (1.1) - (1.2) with $\sigma = \sigma_j(x)$, $c = c_j(x)$ and $\nu = \nu^{(k)}$, $k, j = 1, 2$. We here prove the following stability and uniqueness theorems.

Theorem 1.1. *Let the condition $4r/T < 1$ be satisfied. Then there exist positive numbers q_0 and C, depending on T, r and $|\nu^{(1)} - \nu^{(2)}|$, such that the following inequality holds:*

$$\begin{aligned} & \|\sigma_1 - \sigma_2\|^2_{\mathbf{L}^2(B)} + \|c_1 - c_2\|^2_{\mathbf{H}^1(B)} \\ \leq \quad & C \sum_{k=1}^{2} \Big(\|\widehat{f}_1^{(k)} - \widehat{f}_2^{(k)}\|^2_{\mathbf{H}^2(\partial B \times \{0\})} + \|(\widehat{f}_1^{(k)} - \widehat{f}_2^{(k)})_t\|^2_{\mathbf{H}^1(\partial B \times (0,T))} \\ + \quad & \|(\widehat{g}_1^{(k)} - \widehat{g}_2^{(k)})_t\|^2_{\mathbf{L}^2(\partial B \times (0,T))} + \|\tau_1^{(k)} - \tau_2^{(k)}\|^2_{\mathbf{H}^3(\partial B)} \Big), \end{aligned}$$

for any $(\sigma_1, c_1), (\sigma_2, c_2) \in \Lambda(q_0, d)$*. Here*

$$\widehat{f}_j^{(k)}(x,t) = f_j^{(k)}(x, t + \tau_j^{(k)}(x)), \quad \widehat{g}_j^{(k)}(x,t) = g_j^{(k)}(x, t + \tau_j^{(k)}(x)).$$

Theorem 1.2. *Let the conditions in Theorem 1.1 be fulfilled. Then one can find a number* $q_0 > 0$ *such that if* $(\sigma_j, q_j) \in \Lambda(q_0, d)$*,* $j = 1, 2$*, and the corresponding data partly coincide, namely,*

$$\tau_1^{(k)}(x) = \tau_2^{(k)}(x),\ x \in \partial B, \quad \widehat{f}_1^{(k)}(x,t) = \widehat{f}_2^{(k)}(x,t),\ (x,t) \in \partial B \times (0,T)$$

for $k = 1, 2$*, then* $\sigma_1(x) = \sigma_2(x)$ *and* $c_1(x) = c_2(x)$*,* $x \in B$*.*

For establishing the stability in our inverse problem, we have to change suitable external forces twice. Our formulation is not overdetermining, unlike most of the Dirichlet-to-Neumann map approaches. As such a formulation, we can refer to the method by the Carleman estimate which proves the uniqueness and the conditional stability by a finite number of observations. See for example, Bukhgeim [1], Bukhgeim and Klibanov [2], Imanuvilov and Yamamoto [5], Isakov [6], Khaĭdarov [7], Klibanov [8], Klibanov and Timonov [9]. For applications of Carleman estimates to the inverse problems, we have to assume that the inputs such as initial values or external forces, should satisfy some positivity conditions, which means that we are required to control the unknown media, and these conditions are restrictive for many practical applications. Therefore it is very desirable to prove the uniqueness and the stability without any positivity of inputs. However this is an open problem. In our formulation given by (1.1), we need not assume any positivity of the external forces. On the other hand, for our method, it is necessary to assume the smallness, that is, unknown functions σ and $c-1$ are sufficiently small (see (1.6)). If we could omit such a smallness assumption, then the corresponding result would solve the open problem completely. For the moment, we do not know whether or not it is possible. As for other formulations of inverse problems, we refer to Isakov [6], Romanov [10], for example.

Theorem 1.1 and Theorem 1.2 are proven in Sections 2 and 4, respectively. In Section 3, we prove the following lemma which gives some properties of solution to problem (1.1), (1.2) and is used for proofs of the theorems.

Lemma 1.1. *For each fixed $T_0 > 0$, there exists a positive number $q_0 = q_0(T_0)$ such that for $(\sigma, c) \in \Lambda(q_0, d)$ the solution to problem (1.1), (1.2) in the domain*

$$K(T_0, \nu) := \{(x,t) | \, t \le T_0 - \tau(x,\nu)\}$$

can be represented in the form

$$u(x,t,\nu) = \sum_{k=0}^{m} \alpha_k(x,\nu)\, \theta_k(t - \tau(x,\nu)) + \widehat{u}_m(x,t,\nu). \tag{1.7}$$

Here $m = [(n-1)/2] + 4$, $\theta_0(t)$ is the Heaviside function: $\theta_0(t) = 1$ for $t \ge 0$ and $\theta_0(t) = 0$ for $t < 0$, $\theta_k(t) = t^k \theta_0(t)/k!$, the coefficients $\alpha_k(x,\nu)$ are given in the form

$$\alpha_0(x,\nu) = \exp(\varphi(x,\nu)), \; \varphi(x,\nu) = -\frac{1}{2} \int\limits_{\Gamma(x,\nu)} (\sigma(\xi) + c^2(\xi)\Delta\tau(\xi,\nu))\, ds,$$

$$\alpha_k(x,\nu) = \frac{\alpha_0(x,\nu)}{2} \int\limits_{\Gamma(x,\nu)} \frac{c^2(\xi)\Delta\alpha_{k-1}(\xi,\nu)}{\alpha_0(\xi,\nu)}\, ds, \quad k = 1, \dots, m, \tag{1.8}$$

$\Gamma(x,\nu)$ is the geodesic line joining x with the plane $\{y \in \mathbb{R}^n |\ y \cdot \nu = 0\}$, ds is the element of the Riemannian length: $ds = c^{-1}(x)(\sum_{k=1}^n dx_k^2)^{1/2}$, $\tau = \tau(x,\nu) \in \mathbf{C}^{3m+4}(\Omega(T_0,\nu))$, $\alpha_k = \alpha_k(x,\nu) \in \mathbf{C}^{3m+2-2k}(\Omega(T_0,\nu))$,

$$\Omega(T_0,\nu) := \{x \in \mathbb{R}^n |\ \tau(x,\nu) \le T_0/2\}$$

and the function $\widehat{u}_m(x,t,\nu)$ vanishes for $t \le \tau(x,\nu)$ and belongs to the function space $\mathbf{H}^{m+1}(K(T_0,\nu))$ for fixed ν.

Moreover there exists a positive number C, which depends on T, r and q_0 and does not increase if q_0 decreases, such that the following inequalities hold:

$$\|u-1\|_{\mathbf{H}^{m+1}(G(\nu))} \le Cq_0, \quad \|\tau(x,\nu) - x\cdot\nu\|_{\mathbf{C}^{3m+3}(B)} \le Cq_0. \tag{1.9}$$

Corollary. *Let $q_0 > 0$ be sufficiently small. If $(\sigma, c) \in \Lambda(q_0, d)$, then the function $u(x,t,\nu)$ is continuous on the closure of domain $G(\nu)$ together with all the derivatives up to the third order.*

§2. Proof of Theorem 1.1

Introduce a function $v(x,t,\nu) := u(x, t+\tau(x,\nu), \nu)$. Then the function $v(x,t,\nu)$ for $(x,t) \in B \times (0,T]$, satisfies

$$\begin{aligned} &c^2(2\nabla v_t \cdot \nabla\tau - \Delta v) + (\sigma + c^2\Delta\tau)v_t = 0, \\ &v|_{t=+0} = \alpha_0(x,\nu), \quad v_t|_{t=+0} = \alpha_1(x,\nu), \end{aligned} \tag{2.1}$$

where $\nabla = (\partial/\partial x_1, \ldots, \partial/\partial x_n)$ and we recall that $\varphi(x,\nu) = \ln\alpha_0(x,\nu)$ from (1.8).

Then we have differential equations of the first order (compare with corresponding formulae (3.3) below):

$$\begin{aligned} &2c^2\nabla\varphi\cdot\nabla\tau+\sigma+c^2\Delta\tau=0,\\ &2c^2\nabla\alpha_1\cdot\nabla\tau+\alpha_1(\sigma+c^2\Delta\tau)-c^2\Delta\alpha_0=0. \end{aligned} \tag{2.2}$$

The second equation can be rewritten in the form

$$2(\nabla\alpha_1\cdot\nabla\tau-\alpha_1\nabla\varphi\cdot\nabla\tau)-\alpha_0(\Delta\varphi+|\nabla\varphi|^2)=0 \tag{2.3}$$

and equations (2.1) as follows

$$\begin{aligned} 2\nabla v_t\cdot\nabla\tau-\Delta v-2(\nabla\varphi\cdot\nabla\tau)v_t=0,\quad (x,t)\in B\times(0,T],\\ v|_{t=+0}=\alpha_0(x,\nu),\quad v_t|_{t=+0}=\alpha_1(x,\nu), \end{aligned} \tag{2.4}$$

Let $(\sigma_j,c_j)\in\Lambda(q_0,d)$ for $j=1,2$. Denote the functions u, v, φ, α_0, α_1, τ with the coefficients (σ_j,c_j) by u_j, v_j, φ_j, $\alpha_0^{(j)}$, $\alpha_1^{(j)}$, τ_j and introduce the differences:

$$\begin{aligned} &\widetilde{u}=u_1-u_2,\ \widetilde{v}=v_1-v_2,\ \widetilde{\varphi}=\varphi_1-\varphi_2,\ \widetilde{\alpha}=\alpha_0^{(1)}-\alpha_0^{(2)},\\ &\widetilde{\beta}=\alpha_1^{(1)}-\alpha_1^{(2)},\ \widetilde{\tau}=\tau_1-\tau_2,\ \widetilde{c}=c_1-c_2,\ \widetilde{\sigma}=\sigma_1-\sigma_2. \end{aligned}$$

Then we can obtain

$$\begin{aligned} &2\nabla\widetilde{v}_t\cdot\nabla\tau_1-\Delta\widetilde{v}+a_1\widetilde{v}_t+a_2\cdot\nabla\widetilde{\tau}+a_3\cdot\nabla\widetilde{\varphi}=0,\quad (x,t)\in B\times(0,T],\\ &\widetilde{v}|_{t=+0}=\widetilde{\alpha}(x,\nu),\quad \widetilde{v}_t|_{t=+0}=\widetilde{\beta}(x,\nu), \end{aligned} \tag{2.5}$$

where

$$a_1=-2(\nabla\varphi_1\cdot\nabla\tau_1),\quad a_2=2\nabla(v_2)_t-2(v_2)_t\nabla\varphi_1,\quad a_3=-2(v_2)_t\nabla\tau_2.$$

It follows from equations (2.2) and (2.3) that the functions $\widetilde{\alpha}(x,\nu)$, $\widetilde{\beta}(x,\nu)$, $\widetilde{\varphi}(x,\nu)$ satisfy the relations

$$\begin{aligned} &\widetilde{\alpha} = b_1\widetilde{\varphi}, \quad \nabla\widetilde{\varphi}\cdot b_2 + \nabla\widetilde{\tau}\cdot b_3 + \widetilde{\sigma} + c_1^2\Delta\widetilde{\tau} + b_4\widetilde{c} = 0, \\ &\Delta\widetilde{\varphi} + \nabla\widetilde{\varphi}\cdot h_1 = \nabla\widetilde{\beta}\cdot h_2 + \widetilde{\beta}h_3 + \nabla\widetilde{\tau}\cdot h_4 + \widetilde{\alpha}h_5, \end{aligned} \tag{2.6}$$

where

$$\begin{aligned} &b_1 = \int\limits_0^1 \exp[\varphi_2(1-s)+\varphi_1 s]\,ds, \quad b_2 = 2c_1^2\nabla\tau_1, \quad b_3 = 2c_1^2\nabla\varphi_2, \\ &b_4 = (c_1+c_2)[2\nabla\varphi_2\cdot\nabla\tau_2 + \Delta\tau_2], \quad h_1 = \nabla(\varphi_1+\varphi_2) + 2\alpha_1^{(2)}\nabla\tau_1/\alpha_0^{(2)}, \\ &h_2 = 2\nabla\tau_1/\alpha_0^{(2)}, \quad h_3 = -2\nabla\varphi_1\cdot\nabla\tau_1/\alpha_0^{(2)}, \\ &h_4 = 2(\nabla\alpha_1^{(2)} - \alpha_1^{(2)}\nabla\varphi_2)/\alpha_0^{(2)}, \quad h_5 = -(\Delta\varphi_1 + |\nabla\varphi_1|^2)/\alpha_0^{(2)}. \end{aligned}$$

Introduce the function $w(x,t,\nu) := \widetilde{v}_t(x,t,\nu)$. This function for $(x,t)\in B\times(0,T]$ satisfies

$$\begin{aligned} &2\nabla w_t\cdot\nabla\tau_1 - \Delta w + a_1\,w_t + (a_2)_t\cdot\nabla\widetilde{\tau} + (a_3)_t\cdot\nabla\widetilde{\varphi} = 0, \\ &w|_{t=+0} = \widetilde{\beta}(x,\nu). \end{aligned} \tag{2.7}$$

In view of Lemma 1.1 and the embedding theorem, we have

$$\begin{aligned} &\max(\|a_1\|_{\mathbf{C}(B)}, \|(a_2)_t\|_{\mathbf{C}(B\times(0,T])}, \|(a_3)_t\|_{\mathbf{C}(B\times(0,T])}) \le Cq_0, \\ &\max(\|b_1\|_{\mathbf{C}(B)}, \|b_2\|_{\mathbf{C}(B)}) \le C, \quad \max(\|b_3\|_{\mathbf{C}(B)}, \|b_4\|_{\mathbf{C}(B)}) \le Cq_0, \\ &\|h_2\|_{\mathbf{C}(B)} \le C, \quad \max(\|h_1\|_{\mathbf{C}(B)}, \|h_3\|_{\mathbf{C}(B)}, \|h_4\|_{\mathbf{C}(B)}, \|h_5\|_{\mathbf{C}(B)}) \le Cq_0. \end{aligned} \tag{2.8}$$

Here and henceforth $C>0$ denotes a generic constant which depends on T, r, q_0 and does not increase when q_0 decreases. Therefore relations (2.6) –

(2.8) lead to the following inequalities:

$$
\begin{aligned}
\|\widetilde{\alpha}\|^2_{\mathbf{L}^2(B)} &\le C\|\widetilde{\varphi}\|^2_{\mathbf{L}^2(B)},\\
\|\Delta\widetilde{\varphi}\|^2_{\mathbf{L}^2(B)} &\le C\left(q_0^2(\|\widetilde{\varphi}\|^2_{\mathbf{H}^1(B)} + \|\widetilde{\tau}\|^2_{\mathbf{H}^1(B)}) + \|\widetilde{\beta}\|^2_{\mathbf{H}^1(B)}\right),\\
\|2\nabla w_t\cdot\nabla\tau_1 - \Delta w\|^2_{\mathbf{L}^2(B\times(0,T])} &\le Cq_0^2\Big(\|w\|^2_{\mathbf{H}^1(B\times(0,T])}\\
&\quad +\|\widetilde{\tau}\|^2_{\mathbf{H}^1(B)} + \|\widetilde{\varphi}\|^2_{\mathbf{H}^1(B)}\Big). \qquad (2.9)
\end{aligned}
$$

Let us use the following lemma which is derived from Lemma 4.3.6 in [13] (see also [11]).

Lemma 2.1. *Let $c \in \Lambda(q_0, d)$, $4r/T < 1$ and $z \in \mathbf{H}^2(B\times(0,T])$. Then for sufficiently small q_0, there exists a positive constant C, depending on r, T, q_0, such that the following inequality holds:*

$$
\begin{aligned}
&\|z\|^2_{\mathbf{H}^1(B\times(0,T])} + \|z\|^2_{\mathbf{H}^1(B\times\{0\})} \le C(\|2\nabla z_t\cdot\nabla\tau - \Delta z\|^2_{\mathbf{L}^2(B\times(0,T])}\\
&+\|z\|^2_{\mathbf{H}^1(\partial B\times[0,T])} + \|\nabla z\cdot\mathbf{n}\|^2_{\mathbf{L}^2(\partial B\times[0,T])}). \qquad (2.10)
\end{aligned}
$$

Applying this lemma to the function $w(x,t)$ with $\tau = \tau_1$ and using the third inequalities in (2.9), we obtain

$$
\begin{aligned}
&\|w\|^2_{\mathbf{H}^1(B\times(0,T])} + \|w\|^2_{\mathbf{H}^1(B\times\{0\})}\\
&\le C[q_0^2(\|w\|^2_{\mathbf{H}^1(B\times(0,T])} + \|\widetilde{\tau}\|^2_{\mathbf{H}^1(B)} + \|\widetilde{\varphi}\|^2_{\mathbf{H}^1(B)}) + \varepsilon^2(\nu)], \qquad (2.11)
\end{aligned}
$$

where

$$
\begin{aligned}
&\varepsilon^2(\nu) = \|(\widehat{f}_1 - \widehat{f}_2)_t\|^2_{\mathbf{H}^1(\partial B\times[0,T])} + \|(\widehat{g}_1 - \widehat{g}_2)_t\|^2_{\mathbf{L}^2(\partial B\times[0,T])},\\
&\widehat{f}_j(x,t) = f_j(x, t+\tau_j(x,\nu)), \quad \widehat{g}_j(x,t) = g_j(x, t+\tau_j(x,\nu)) \qquad (2.12)
\end{aligned}
$$

for $j = 1, 2$. By relation (2.11) for sufficiently small q_0, we derive the inequality

$$\|w\|^2_{\mathbf{H}^1(B\times\{0\})} \le C[q_0^2(\|\widetilde{\tau}\|^2_{\mathbf{H}^1(B)} + \|\widetilde{\varphi}\|^2_{\mathbf{H}^1(B)}) + \varepsilon^2(\nu)]. \tag{2.13}$$

Since $w(x, +0) = \widetilde{\beta}(x, \nu)$ by (2.7), we find

$$\|\widetilde{\beta}\|^2_{\mathbf{H}^1(B)} \le C[q_0^2(\|\widetilde{\tau}\|^2_{\mathbf{H}^1(B)} + \|\widetilde{\varphi}\|^2_{\mathbf{H}^1(B)}) + \varepsilon^2(\nu)]. \tag{2.14}$$

Then from the second inequality in (2.9), it follows that

$$\|\Delta\widetilde{\varphi}\|^2_{\mathbf{L}^2(B)} \le C\left(q_0^2(\|\widetilde{\varphi}\|^2_{\mathbf{H}^1(B)} + \|\widetilde{\tau}\|^2_{\mathbf{H}^1(B)}) + \varepsilon^2(\nu)\right). \tag{2.15}$$

Let us estimate the $H^2(B)$-norm of the function $\widetilde{\varphi}(x, \nu)$ through the $L^2(B)$-norm of its Laplacian and boundary values of the second order derivatives. For this, use the identity:

$$(\Delta\widetilde{\varphi}\,)^2 = \sum_{i,j=1}^{n} \widetilde{\varphi}_{x_ix_i}\widetilde{\varphi}_{x_jx_j} = \sum_{i,j=1}^{n} \left[(\widetilde{\varphi}_{x_i}\widetilde{\varphi}_{x_jx_j})_{x_i} - (\widetilde{\varphi}_{x_i}\widetilde{\varphi}_{x_ix_j})_{x_j} + \widetilde{\varphi}^2_{x_ix_j}\right]. \tag{2.16}$$

Integrating this identity over the domain B and applying the divergence formula, we obtain

$$\int_B \sum_{i,j=1}^{n} \widetilde{\varphi}^2_{x_ix_j}\, dx = \int_B (\Delta\widetilde{\varphi}\,)^2\, dx - \int_{\partial B} \left[(\Delta\widetilde{\varphi}\,)(\nabla\widetilde{\varphi}\cdot\mathbf{n}) - (\nabla|\nabla\widetilde{\varphi}\,|^2\cdot\mathbf{n})/2\right] dS. \tag{2.17}$$

Hence the following estimate follows:

$$\|\widetilde{\varphi}\,\|^2_{\mathbf{H}^2(B)} \le C\left(\|\Delta\widetilde{\varphi}\,\|^2_{\mathbf{L}^2(D)} + \sum_{|\gamma|\le 2} \|\partial^\gamma\widetilde{\varphi}\,\|^2_{\mathbf{L}^2(\partial B)}\right), \tag{2.18}$$

where

$$\partial^\gamma = \frac{\partial^{|\gamma|}}{\partial x^{\gamma_1}\ldots\partial x^{\gamma_n}}, \quad |\gamma| = \gamma_1 + \ldots + \gamma_n.$$

By the assumptions that $\operatorname{supp}\sigma_j \cup \operatorname{supp}(c_j - 1) \subset \Omega \subset B$ for $j = 1, 2$ and $\operatorname{dist}(\partial B, \Omega) \geq d$, we see that $\Gamma(x, \nu)$ does not intersect Ω if $x \notin \partial B_+ := \{x \in \partial B | \, \nu \cdot (x - x^0) > \sqrt{r^2 - (r-d)^2}\}$, so that the function $\widetilde{\varphi}(x, \nu)$ and all its derivatives vanish on $\partial B \setminus \overline{\partial B_+}$. Similarly we can verify that the function $\widetilde{\tau}(x, \nu)$ and its derivative vanish on $\partial B \setminus \overline{\partial B_+}$. By $\operatorname{supp}\sigma_j \cup \operatorname{supp}(c_j - 1) \subset B$ for $j = 1, 2$, (1.3) and (2.6), we have

$$2\nabla\widetilde{\varphi} \cdot \nabla\tau_1 + 2\nabla\widetilde{\tau} \cdot \nabla\varphi_2 + \Delta\widetilde{\tau} = 0$$

and

$$\nabla\widetilde{\tau} \cdot \nabla(\tau_1 + \tau_2) = 0$$

outside of B, so that all the derivatives of $\widetilde{\varphi}$ and $\widetilde{\tau}$ on ∂B_+ can be expressed by the corresponding derivatives along ∂B_+. Therefore we have

$$\begin{aligned} &\sum_{|\gamma| \leq 4} \| \partial^\gamma \widetilde{\tau} \|^2_{\mathbf{L}^2(\partial B)} \leq C \| \widetilde{\tau} \|^2_{\mathbf{H}^4(\partial B)}, \\ &\sum_{|\gamma| \leq 2} \| \partial^\gamma \widetilde{\varphi} \|^2_{\mathbf{L}^2(\partial B)} \leq C \left(\| \widetilde{\varphi} \|^2_{\mathbf{H}^2(\partial B)} + \| \widetilde{\tau} \|^2_{\mathbf{H}^4(\partial B)} \right). \end{aligned} \tag{2.19}$$

Moreover, since by (1.4) and (2.12), we have

$$\begin{aligned} \widetilde{v}(x, t, \nu) &= u_1(x, t + \tau_1(x, \nu), \nu) - u_2(x, t + \tau_1(x, \nu), \nu) \\ &= f_1(x, t + \tau_1(x, \nu)) - f_2(x, t + \tau_1(x, \nu)) = \widehat{f}_1(x, t) - \widehat{f}_2(x, t), \\ &\qquad\qquad (x, t) \in S(\nu), \end{aligned}$$

it follows from (2.1) and (2.6) that $\widetilde{\varphi} = \widetilde{\alpha}/b_1$ and $\widetilde{\alpha}(x, \nu) = \widetilde{v}(x, +0, \nu) = (\widehat{f}_1 - \widehat{f}_2)(x, 0)$. Therefore we obtain

$$\sum_{|\gamma| \leq 2} \| \partial^\gamma \widetilde{\varphi} \|^2_{\mathbf{L}^2(\partial B)} \leq C \varepsilon_1^2(\nu), \tag{2.20}$$

where

$$\varepsilon_1^2(\nu) := \|\widehat{f}_1 - \widehat{f}_2\|^2_{\mathbf{H}^2(\partial B\times\{0\})} + \|\widetilde{\tau}\|^2_{\mathbf{H}^4(\partial B)}.$$

Taking into account inequalities (2.15), (2.18) and (2.20), we find an estimate:

$$\|\widetilde{\varphi}\|^2_{\mathbf{H}^2(B)} \le C\left(q_0^2(\|\widetilde{\varphi}\|^2_{\mathbf{H}^1(B)} + \|\widetilde{\tau}\|^2_{\mathbf{H}^1(B)}) + \varepsilon^2(\nu) + \varepsilon_1^2(\nu)\right). \tag{2.21}$$

Hence, for sufficiently small q_0, we have

$$\|\widetilde{\varphi}\|^2_{\mathbf{H}^2(B)} \le C\left(q_0^2\|\widetilde{\tau}\|^2_{\mathbf{H}^1(B)} + \varepsilon^2(\nu) + \varepsilon_1^2(\nu)\right). \tag{2.22}$$

Consider inequality (2.22) and the second of relations (2.6) for $\nu = \nu^{(k)}$, $k = 1, 2$. Set

$$\widetilde{\varphi}_k(x,t) = \widetilde{\varphi}(x,t,\nu^{(k)}), \quad \widetilde{\tau}_k(x) = \tau_1(x,\nu^{(k)}) - \tau_2(x,\nu^{(k)}),$$

$$b_{jk}(x) = b_j(x,\nu^{(k)}), \quad \widetilde{\varepsilon}_k^2 = \varepsilon^2(\nu^{(k)}) + \varepsilon_1^2(\nu^{(k)}), \quad j = 1,2,3,4,\ k = 1,2.$$

Inequality (2.22) leads to

$$\|\widetilde{\varphi}_k\|^2_{\mathbf{H}^2(B)} \le C\left(q_0^2\|\widetilde{\tau}_k\|^2_{\mathbf{H}^1(B)} + \widetilde{\varepsilon}_k^2\right), \quad k = 1,2. \tag{2.23}$$

In terms of (2.6), we have

$$\widetilde{\sigma} = -\nabla\widetilde{\varphi}_k \cdot b_{2k} - \nabla\widetilde{\tau}_k \cdot b_{3k} - c_1^2\Delta\widetilde{\tau}_k - b_{4k}\widetilde{c}, \quad k = 1,2. \tag{2.24}$$

Furthermore, eikonal equation (1.3) yields

$$\nabla\widetilde{\tau}_k \cdot r_k + \widetilde{c} = 0, \quad r_k(x) = \frac{c_1^2c_2^2}{c_1+c_2}\nabla(\tau_1+\tau_2)(x,\nu^{(k)}), \quad k = 1,2. \tag{2.25}$$

Hence there exists a positive constant $C = C(q_0, r)$ such that

$$\|\widetilde{c}\|_{\mathbf{H}^1(B)} \le C\|\widetilde{\tau}_k\|_{\mathbf{H}^2(B)}, \quad k = 1,2. \tag{2.26}$$

By (2.24), (2.26) and (2.8), we obtain

$$\|\widetilde{\sigma}\|^2_{\mathbf{L}^2(B)} \le C\left(\|\widetilde{\varphi}_k\|^2_{\mathbf{H}^1(B)} + \|\widetilde{\tau}_k\|^2_{\mathbf{H}^2(B)}\right), \quad k = 1, 2, \tag{2.27}$$

with a positive constant $C = C(r, T, q_0)$. Taking into account estimate (2.23), we obtain

$$\|\widetilde{\sigma}\|^2_{\mathbf{L}^2(B)} \le C\left(\|\widetilde{\tau}_k\|^2_{\mathbf{H}^1(B)} + \widehat{\varepsilon}^2_k\right), \quad k = 1, 2. \tag{2.28}$$

Hence, in view of (2.26) and (2.28), in order to complete the proof of Theorem 1.1, it suffices to estimate $\|\widetilde{\tau}_k\|^2_{\mathbf{H}^2(B)}$. For this goal, use (2.24) again. Subtracting the expression for $k = 2$ from the corresponding expression for $k = 1$, we can estimate:

$$\|\Delta(\widetilde{\tau}_1 - \widetilde{\tau}_2)\|^2_{\mathbf{H}^1(B)} \le C\sum_{k=1}^{2}\left(q_0^2(\|\widetilde{\tau}_k\|^2_{\mathbf{H}^2(B)} + \|\widetilde{c}\|^2_{\mathbf{H}^1(B)}) + \|\widetilde{\varphi}_k\|^2_{\mathbf{H}^2(B)}\right). \tag{2.29}$$

Using (2.23) and (2.26), we can rewrite (2.29) by

$$\|\Delta(\widetilde{\tau}_1 - \widetilde{\tau}_2)\|^2_{\mathbf{H}^1(B)} \le C\sum_{k=1}^{2}\left(q_0^2\|\widetilde{\tau}_k\|^2_{\mathbf{H}^2(B)} + \widehat{\varepsilon}^2_k\right). \tag{2.30}$$

On the other hand, we have

$$\nabla(\widetilde{\tau}_1 - \widetilde{\tau}_2)\cdot r_1 + \nabla\widetilde{\tau}_2 \cdot (r_1 - r_2) = 0. \tag{2.31}$$

By (1.9) and Lemma 1.1, for sufficiently small q_0, we have

$$\|\nabla\tau_j(x, \nu^{(k)}) - \nu^{(k)}\|_{\mathbf{C}^{3m+2}(B)} \le Cq_0, \qquad j, k = 1, 2.$$

Consequently, by the second inequality in (1.6), one can easily prove

$$\|(r_1 - r_2) - (\nu^{(1)} - \nu^{(2)})\|_{\mathbf{C}^{3m+2}(B)} \le Cq_0. \tag{2.32}$$

By $\nu^{(1)} \neq \nu^{(2)}$, inequality (2.32) implies that the function $(r_1 - r_2)(x)$ does not vanish anywhere in B if q_0 is small. For sufficiently small $q_0 > 0$ and $x \in B$, introduce the curve

$$L(x) := \{\xi \in \mathbb{R}^n | \xi = \xi(s,x), s \in (-\infty, 0)\},$$

where the function $\xi(s,x)$ is the solution of the following Cauchy problem for the ordinary differential equation:

$$\frac{d\xi}{ds} = \frac{(r_1 - r_2)(\xi)}{|(r_1 - r_2)(\xi)|}, \quad \xi|_{s=0} = x. \tag{2.33}$$

Here s is the Euclidean length of the curve starting at x. The function $\xi(s,x)$ possesses the same smoothness as the function $(r_1 - r_2)(x)$. It follows from (2.32) that for small q_0, the curve $L(x) \cap B$ is close to a segment of the straight line which passes the point x in the direction $\nu^{(1)} - \nu^{(2)}$. Hence each $L(x)$ intersects ∂B at some negative $s = s(x)$. By $x^* = \xi(s, s(x)) \in \partial B$ and $L'(x)$ respectively, we denote the intersection point and the segment of $L(x)$ which is in the domain B. Integrating equation (2.31) along $L'(x)$, we obtain

$$\widetilde{\tau}_2(x) = \widetilde{\tau}_2(x^*) - \int\limits_{L'(x)} \frac{\nabla(\widetilde{\tau}_1 - \widetilde{\tau}_2)(\xi) \cdot r_1(\xi)}{|(r_1 - r_2)(\xi)|}\, ds. \tag{2.34}$$

Therefore the estimate follows:

$$\|\widetilde{\tau}_2\|^2_{\mathbf{H}^2(B)} \leq C \left(\|\widetilde{\tau}_1 - \widetilde{\tau}_2\|^2_{\mathbf{H}^3(B)} + \|\widetilde{\tau}_2\|^2_{\mathbf{H}^2(\partial B)} \right) \tag{2.35}$$

with a constant $C = C(r, q_0, |\nu^{(1)} - \nu^{(2)}|)$. By $\widetilde{\tau}_1 = \widetilde{\tau}_2 + (\widehat{\tau}_1 - \widetilde{\tau}_2)$, a similar estimate is valid for $\widetilde{\tau}_1(x)$, so that

$$\|\widetilde{\tau}_k\|^2_{\mathbf{H}^2(B)} \leq C \left(\|\widetilde{\tau}_1 - \widetilde{\tau}_2\|^2_{\mathbf{H}^3(B)} + \|\widetilde{\tau}_2\|^2_{\mathbf{H}^2(\partial B)} \right), \quad k = 1, 2. \tag{2.36}$$

Substituting (2.36) in (2.30), we find

$$\|\Delta(\widetilde{\tau}_1-\widetilde{\tau}_2)\|^2_{\mathbf{H}^1(B)} \le C\left(q_0^2\|\widetilde{\tau}_2-\widetilde{\tau}_2\|^2_{\mathbf{H}^3(B)}+\widetilde{\varepsilon}_1^2+\widetilde{\varepsilon}_2^2\right). \tag{2.37}$$

Now we can estimate the $\mathbf{H}^3(B)$-norm of $(\widetilde{\tau}_1-\widetilde{\tau}_2)(x)$ through the $\mathbf{H}^1(B)$-norm of its Laplacian and boundary values similarly to the previous estimate of $\widetilde{\varphi}(x,\nu)$:

$$\|\widetilde{\tau}_1-\widetilde{\tau}_2\|^2_{\mathbf{H}^3(B)} \le C\left(\|\Delta(\widetilde{\tau}_1-\widetilde{\tau}_2)\|^2_{\mathbf{H}^1(B)}+\sum_{|\gamma|\le 3}\|\partial^\gamma(\widetilde{\tau}_1-\widetilde{\tau}_2)\|^2_{\mathbf{L}^2(\partial B)}\right)$$

$$\le \quad C\left(\|\Delta(\widetilde{\tau}_1-\widetilde{\tau}_2)\|^2_{\mathbf{H}^1(B)}+\sum_{k=1}^{2}\|\widetilde{\tau}_k\|^2_{\mathbf{H}^3(\partial B)}\right). \tag{2.38}$$

Using the same arguments which we have used for obtaining the first estimate in (2.19), we have

$$\sum_{|\gamma|\le 3}\|\partial^\gamma\widetilde{\tau}_k\|^2_{\mathbf{L}^2(\partial B)} \le C\|\widetilde{\tau}_k\|^2_{\mathbf{H}^3(\partial B)}, \quad k=1,2. \tag{2.39}$$

Substitution of (2.39) into (2.38) yields

$$\|\widetilde{\tau}_1-\widetilde{\tau}_2\|^2_{\mathbf{H}^3(B)} \le C\left(\|\Delta(\widetilde{\tau}_1-\widetilde{\tau}_2)\|^2_{\mathbf{H}^1(B)}+\sum_{k=1}^{2}\|\widetilde{\tau}_k\|^2_{\mathbf{H}^3(\partial B)}\right). \tag{2.40}$$

Combining inequalities (2.37) and (2.40), we derive

$$\|\widetilde{\tau}_1-\widetilde{\tau}_2\|^2_{\mathbf{H}^3(B)} \le C\left(q_0^2\|\widetilde{\tau}_1-\widetilde{\tau}_2\|^2_{\mathbf{H}^3(B)}+\widehat{\varepsilon}^2\right), \tag{2.41}$$

where

$$\widehat{\varepsilon}^2=\widetilde{\varepsilon}_1^2+\widetilde{\varepsilon}_2^2+\sum_{k=1}^{2}\|\widetilde{\tau}_k\|^2_{\mathbf{H}^3(\partial B)}.$$

Hence, for q_0 small enough, we have

$$\|\widetilde{\tau}_1-\widetilde{\tau}_2\|^2_{\mathbf{H}^3(B)} \le C\widehat{\varepsilon}^2. \tag{2.42}$$

Then from (2.36) we obtain

$$\|\widetilde{\tau}_k\|^2_{\mathbf{H}^2(B)} \leq C\widehat{\varepsilon}^2, \quad k = 1, 2, \tag{2.43}$$

and from (2.26) and (2.28), we find the estimate

$$\|\widetilde{c}\|^2_{\mathbf{H}^1(B)} + \|\widetilde{\sigma}\|^2_{\mathbf{L}^2(B)} \leq C\widehat{\varepsilon}^2, \tag{2.44}$$

which is the conclusion of Theorem 1.1. Thus the proof is complete. □

§3. Proof of Lemma 1.1

Consider first problem (1.3). Set $p(x,\nu) := \nabla\tau(x,\nu)$. Without loss of generality, we here assume that $\nu = (1, 0, \ldots, 0)$. Then $\tau(x,\nu) = -x_1$ for $x_1 \leq 0$. To find $\tau(x,\nu)$ for $x_1 > 0$, consider equations for the geodesic lines. Let $x = F(\tau,\gamma)$ and $p = P(\tau,\gamma)$ be the solution to the following Cauchy problem for the system of ordinary differential equations:

$$\frac{dx}{ds} = p\, c^2(x)\,, \quad \frac{dp}{ds} = -\nabla \ln c(x), \quad x|_{s=0} = (0,\gamma), \quad p|_{s=0} = \nu, \tag{3.1}$$

where $\gamma = (\gamma_2, \ldots, \gamma_n)$ and s is the Riemannian length which coincides with the travelling time from the plane $x_1 = 0$ to the point x. The equality $x = F(s,\gamma)$ is the equation of the geodesic line crossing the plane $x_1 = 0$ orthogonally at the point $(0,\gamma)$. For each domain $D := \{(s,\gamma)|\, 0 \leq s \leq s_0, |\gamma| \leq R\}$ with given $s_0 > 0$ and $R > 0$, it is easy to prove that there exists a unique solution $x = F(s,\gamma)$ and $p = P(s,\gamma)$ to problem (3.1) in D, $F, P \in \mathbf{C}^{3m+3}(D)$ and one can find a positive constant $C = C(q_0, s_0, R)$ such that

$$\|F - (0,\gamma) - \nu s\|_{\mathbf{C}^{3m+3}(D)} \leq Cq_0, \quad \|P - \nu\|_{\mathbf{C}^{3m+3}(D)} \leq Cq_0. \tag{3.2}$$

It means that each geodesic line and p are close to a straight line and the unite vector ν respectively if q_0 is small. For any given bounded domain $\Omega_0 \subset \mathbb{R}^n$, there exist $s_0 > 0$, large $R > 0$ and small $q_0 > 0$ such that $\{x \in \mathbb{R}^n | \, x = F(s,\gamma), (s,\gamma) \in D\} \supset \overline{\Omega_0}$. By means of (3.2), one can check also that for small q_0, the Jacobian $\dfrac{\partial(x_1, x_2, \ldots, x_n)}{\partial(s, \gamma_2, \ldots, \gamma_n)}$ does not vanish anywhere in D. Therefore for q_0 small enough, the equation $x = F(s,\gamma)$ is solvable with respect to s and γ, i.e., $s = s(x)$ and $\gamma = \gamma(x)$, where $s = s(x)$, $\gamma = \gamma(x) \in \mathbf{C}^{3m+3}(\overline{\Omega_0})$. Moreover, because $\nabla\tau(x) := p(x) = P(s(x), \gamma(x))$ is in $\mathbf{C}^{3m+3}(\overline{\Omega_0})$, we find that $\tau \in \mathbf{C}^{3m+4}(\overline{\Omega_0})$. Note that for a given x, the geodesic line $\Gamma(x)$ passing through x and crossing the plane $x_1 = 0$ orthogonally, is given by the equation $\xi = F(s, \gamma(x)), s \in [0, s(x)]$. Estimate (1.9) for $\tau(x)$ follows immediately from the first inequality in (3.2)

Substituting representation (1.7) into equation (1.1) and equating the terms of $\delta(t - \tau(x,\nu))$, $\theta_k(t - \tau(x,\nu))$, $k = 0, 1, \ldots, m-1$, we find equations for $\alpha_k(x,\nu)$ and the initial data of the form:

$$\begin{aligned}
&2c^2\nabla\alpha_0 \cdot \nabla\tau + \alpha_0(\sigma + c^2\Delta\tau) = 0, \quad \alpha_0|_{x\cdot\nu=0} = 1, \\
&2c^2\nabla\alpha_k \cdot \nabla\tau + \alpha_k(\sigma + c^2\Delta\tau) - c^2\Delta\alpha_{k-1} = 0, \quad \alpha_k|_{x\cdot\nu=0} = 0, \\
&k = 1, \ldots, m.
\end{aligned} \tag{3.3}$$

Since along the geodesic lines, according to (3.1), one has $c^2\nabla\alpha_k \cdot \nabla\tau = d\alpha_k/d\tau$, the equations can be integrated in an explicit form. The resulting expressions are given by formulae (1.8). From the definition of the set $\Lambda(q_0, d)$ and formulae (1.8), one can obtain $\|\alpha_0 - 1\|_{\mathbf{C}^{3m+2}(\Omega(T_0,\nu))} \leq Cq_0$ and $\|\alpha_k\|_{\mathbf{C}^{3m+2-2k}(\Omega(T_0,\nu))} \leq Cq_0$, $k = 1, \ldots, m$, with a positive constant $C = C(q_0, T_0)$. Here we recall that $\Omega(T_0, \nu) := \{x \in \mathbb{R}^n | \, \tau(x,\nu) \leq T_0/2\}$.

Then the function $\widehat{u}_m(x,t,\nu)$ solves

$$(\widehat{u}_m)_{tt} - c^2\Delta\widehat{u}_m + \sigma(\widehat{u}_m)_t = F_m(x,t,\nu), \quad (x,t) \in \mathbb{R}^{n+1}; \quad \widehat{u}_m|_{t<0} = 0, \tag{3.4}$$

where

$$F_m(x,t,\nu) = c^2(x)\Delta\alpha_m(x,\nu)\,\theta_m(t-\tau(x,\nu)).$$

It is obvious that the function $F_m(x,t,\nu)$ vanishes for $t \le \tau(x,\nu)$ and belongs to $\mathbf{H}^{m+1}(K(T_0,\nu))$. Here we recall that $K(T_0,\nu) := \{(x,t)|\, t \le T_0 - \tau(x,\nu)\}$ for fixed T_0, ν. Then the energy estimates method implies that $\widehat{u}_m(x,t,\nu)$ belongs to $\mathbf{H}^{m+1}(K(T_0,\nu))$, vanishes for $t \le \tau(x,\nu)$ and the estimate $\|\widehat{u}_m\|_{\mathbf{H}^{m+1}(K(T_0,\nu))} \le Cq_0$ holds. Since $G(\nu) \subset K(T_0,\nu)$ for sufficiently large T_0, the first inequality in (1.9) follows. $\square$

§4. Proof of Theorem 1.2

Let $(\sigma_j, c_j) \in \Lambda(q_0,d)$ and $u_j(x,t,\nu)$, $\tau_j(x,\nu)$, $\alpha_0^{(j)}(x,\nu)$ for $j = 1,2$, be the same as in Section 2. In terms of Theorem 1.1, in order to complete the proof of Theorem 1.2, it suffices to prove

Lemma 4.1. *If $\tau_1(x,\nu) = \tau_2(x,\nu) := \tau(x,\nu)$ on ∂B and $u_1(x,t,\nu) = u_2(x,t,\nu)$ on $S(\nu) := \{(x,t)|\, x \in \partial B, \tau(x,\nu) \le t \le T + \tau(x,\nu)\}$, then $(\nabla u_1 \cdot \mathbf{n}) = (\nabla u_2 \cdot \mathbf{n})$ on $S(\nu)$. Here $\mathbf{n}$ is the unit outward normal vector to $S(\nu)$.*

Proof of Lemma 4.1. Under the assumption, we have $\tau_1(x,\nu) = \tau_2(x,\nu) := \tau(x,\nu)$ and $\alpha_0^{(1)}(x,\nu) = \alpha_0^{(2)}(x,\nu)$ outside of B. Hence $\widetilde{u} = u_1 - u_2$ is a continuous function across the characteristic surface $t = \tau(x,\nu)$ if $x \notin B$. It follows from Lemma 1.1 that $\widetilde{u} \in \mathbf{H}^1(P)$ for any compact domain P which is located outside of $\{(x,t)|\, x\in B, t>0\}$. Now apply the energy estimate method to demonstrate that $\widetilde{u}=0$ inside an external domain adjoining to $S(\nu)$.

Take again $\nu=(1,0,\dots,0)$, and assume that $x^0=(x_1^0,0,\dots,0)$ and $q_0\le 1/2$. Then the cone $\{(x,t)|\, t\le T_0-|x|\}$ contains $S(\nu)$ if $T_0\ge T+3(x_1^0+r)$. Suppose that this condition is satisfied. Consider the domain $P(T_0)=\{(x,t)|\, |x-x^0|>r, 0<t<\min(T+\tau(x,\nu),T_0-|x|)\}$. This domain is bounded by the piecewise smooth characteristic surface $S_1=\{(x,t)|\, |x-x^0|>r, t=\min(T+\tau(x,\nu),T_0-|x|)\}$, the lateral boundary $S_2=\{(x,t)|\, |x-x^0|=r, 0<t<T+\tau(x,\nu)\}$ and $S_3=\{(x,t)|\, |x|<T_0, |x-x^0|>r, t=0\}$. We have $\widetilde{u}(x,t,\nu)=0$ for $(x,t)\in S_2\cup S_3$. Moreover $\widetilde{u}(x,t,\nu)\in \mathbf{H}^1(P(T_0))$ and satisfies

$$\Box\,\widetilde{u}=0,\quad (x,t)\in P(T_0). \tag{4.1}$$

Let $t_0\in(0,T_0)$ be arbitrary. Set $S(t_0)=\{(x,t)\in P(T_0)|\, t=t_0\}$ and $S_k(t_0):=\{(x,t)\in S_k|\, 0<t<t_0\}$, $k=1,2$. Note that $S(t_0)$ is the intersection of $P(T_0)$ by the plane $t=t_0$ and is the empty set if t_0 is close to T_0. Set $T_0^*=\sup\{t_0\}$ over all t_0 such that $S(t_0)$ is not empty.

Multiplying the both sides of equation (4.1) by $2\widetilde{u}_t$, using the identity $2\widetilde{u}_t\Box\,\widetilde{u}\equiv(\widetilde{u}_t^2+|\nabla\widetilde{u}|^2)_t-\mathrm{div}(2\widetilde{u}_t\nabla\widetilde{u})$, integrating the resulting equality over the domain $P(T_0,t_0):=\{(x,t)\in P(T_0)|\, 0<t<t_0<T_0\}$ and applying the divergence formula, we obtain

$$\int\limits_{S(t_0)}(\widetilde{u}_t^2+|\nabla\widetilde{u}|^2)\,dx$$
$$+\int\limits_{S_1(t_0)\cup S_2(t_0)\cup S_3}\left[(\widetilde{u}_t^2+|\nabla\widetilde{u}|^2)\cos(\mathbf{n},t)-2\widetilde{u}_t\sum_{j=1}^n\widetilde{u}_{x_j}\cos(\mathbf{n},x_j)\right]dS=0,$$

where dS is the area element.

Because $\widetilde{u}$ vanishes on S_2 and $\widetilde{u} = 0$, $\widetilde{u}_t = 0$ for $(x,t) \in S_3$, the integrals over $S_2(t_0) \cup S_3$ vanish. One can easily check that the integral over $S_1(t_0)$ is nonnegative. Therefore

$$\int\limits_{S(t_0)} (\widetilde{u}_t^2 + |\nabla \widetilde{u}|^2)\, dx \leq 0, \quad t_0 \in (0, T_0^*).$$

Then we obtain $\widetilde{u}(x,t,\nu) = 0$ for $(x,t) \in P(T_0)$. Hence $\nabla \widetilde{u} \cdot \mathbf{n} = 0$ and $\nabla u_1 \cdot \mathbf{n} = \nabla u_2 \cdot \mathbf{n}$ on $S(\nu)$. Thus the proof of Lemma 4.1 is complete. $\square$

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