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by

Naoto Kumano-go and Daisuke Fujiwara

UNIVERSITY OF TOKYO
GRADUATE SCHOOL OF MATHEMATICAL SCIENCES
KOMABA, TOKYO, JAPAN

Smooth Functional Derivatives in Feynman Path Integral

Naoto Kumano-go* and Daisuke Fujiwara†

Abstract

Based on Feynman and Hibbs's idea, we give a mathematically rigorous definition of functional derivatives in Feynman path integral. In this definition, the functionals which belong to Fujiwara's class, can be differentiated as many times as we want. Furthermore, the Taylor expansion formula and the integration by parts hold with respect to the functional derivatives.

1 Introduction

In this paper, using the theory of the time slicing approxiamtion [4], we give a mathematically rigorous definition of functional derivatives in Feynman path integral [1]. Feynman and Hibbs [2] explained functional derivatives in Feynman path integral, using broken line paths. Based on Feynman and Hibbs's idea, we "restrict the directions of functional derivatives" to broken line paths. By this restriction, the functionals which belong to Fujiwara's class, can be differentiated as many times as we want. Furthermore, we prove the invariance of Feynman path integral with respect to translations, the Taylor expansion formula with respect to the functional derivatives, and the integration by parts with respect to the functional derivatives.

In order to state the results of this paper, we recall the notation in [4].

For a path $\gamma : [0,T] \to \mathbf{R}^d$ which starts from $x_0 \in \mathbf{R}^d$ at time 0 and reaches $x \in \mathbf{R}^d$ at time T, i.e., $\gamma(0) = x_0$ and $\gamma(T) = x$, we define the action $S[\gamma]$ by

$$S[\gamma] = \int_0^T \frac{1}{2}\Big|\frac{d\gamma}{dt}\Big|^2 - V(t,\gamma)dt\,. \tag{1.1}$$

Let $F[\gamma]$ be a functional on the path space $C([0,T] \to \mathbf{R}^d)$ whose domain contains all of broken line paths at least. Let $0 < \hbar < 1$ be a parameter.

Let $\Delta_{T,0}$ be an arbitrary division of the interval $[0,T]$ into subintervals, i.e.,

$$\Delta_{T,0} : T = T_{J+1} > T_J > \cdots > T_1 > T_0 = 0\,. \tag{1.2}$$

*Department of Mathematics, Kogakuin University, 1-24-2 Nishishinjuku Shinjuku-ku Tokyo 163-8677, JAPAN, e-mail:ft24343@ns.kogakuin.ac.jp

†Department of Mathematics, Gakushuin University, 1-5-1 Mejiro Toshima-ku Tokyo 171-8588, JAPAN, e-mail:daisuke.fujiwara@gakushuin.ac.jp

Set $x_{J+1} = x$. Let x_J, x_{J-1}, $\cdots$, x_1 be arbitrary points of $\mathbf{R}^d$. Let $\gamma_{\Delta_{T,0}}$ be the broken line path which connects (T_j, x_j) and (T_{j-1}, x_{j-1}) by a line segment for any $j = 1, 2, \ldots, J, J+1$, i.e., $\gamma_{\Delta_{T,0}}(T_j) = x_j$. Set $t_j = T_j - T_{j-1}$. Let $|\Delta_{T,0}|$ be the size of the division defined by $|\Delta_{T,0}| = \max_{1\le j\le J+1} t_j$. Using the oscillatory integrals, we define the Feynman path integral by

$$\int e^{\frac{i}{\hbar}S[\gamma]} F[\gamma]\mathcal{D}[\gamma]$$
$$= \lim_{|\Delta_{T,0}|\to 0} \prod_{j=1}^{J+1} \left(\frac{1}{2\pi i\hbar t_j}\right)^{d/2} \int_{\mathbf{R}^{dJ}} e^{\frac{i}{\hbar}S[\gamma_{\Delta_{T,0}}]} F[\gamma_{\Delta_{T,0}}] \prod_{j=1}^{J} dx_j \,. \qquad (1.3)$$

Our assumption for the potential $V(t,x)$ through this paper is the following.

Assumption 1 *$V(t,x)$ is a real-valued function of $(t,x) \in \mathbf{R}\times\mathbf{R}^d$, and for any multi-index α, $\partial_x^\alpha V(t,x)$ is continuous in $\mathbf{R}\times\mathbf{R}^d$. For any $|\alpha| \ge 2$, there exists a positive constant A_α such that*

$$|\partial_x^\alpha V(t,x)| \le A_\alpha \,. \qquad (1.4)$$

We defined Fujiwara's class $\mathcal{F}$ of functionals, using Assumption 5 in [4]. The assumption of this type was first found by Fujiwara [3]. Furthermore, we proved the following results. See Theorems 1,2 in [4]. For simplicity, when a functional $F[\gamma]$ belongs to Fujiwara's class $\mathcal{F}$, we write $F[\gamma] \in \mathcal{F}$.

Proposition 1 *If $F[\gamma] \in \mathcal{F}$ and $G[\gamma] \in \mathcal{F}$, then the sum $F[\gamma] + G[\gamma] \in \mathcal{F}$ and the product $F[\gamma]G[\gamma] \in \mathcal{F}$.*

Proposition 2 *Assume that T is sufficiently small. Then, for any $F[\gamma] \in \mathcal{F}$, the right-hand side of (1.3) really converges uniformly on any compact set of the configuration space $(x, x_0) \in \mathbf{R}^{2d}$.*

2 Results

Now we are ready to state the results of this paper. First we explain our definition of the functional derivatives whose directions are restricted to broken line paths.

Let $\gamma : [0,T] \to \mathbf{R}^d$ be any broken line path with $\gamma(0) = x_0$ and $\gamma(T) = x$, and let $\eta : [0,T] \to \mathbf{R}^d$ be any broken line path with $\eta(0) = \eta(T) = 0 \in \mathbf{R}^d$.
Then there exists a division

$$\Delta_{T,0} : T = T_{J+1} > T_J > \cdots > T_1 > T_0 = 0\,, \qquad (2.1)$$

such that $\Delta_{T,0}$ contains all of end points of the broken line paths γ and η. Furthermore, we have

$$\gamma = \gamma_{\Delta_{T,0}}\,, \qquad (2.2)$$

where $\gamma_{\Delta_{T,0}}$ connects (T_j, x_j) and (T_{j-1}, x_{j-1}) with a line segment for any $j =$ 1,2,..., J,$J+1$, i.e., $\gamma_{\Delta_{T,0}}(T_j) = x_j$. Set $\eta(T_j) = y_j$ for $j = 0, 1, \ldots, J, J+1$ and

$$F[\gamma_{\Delta_{T,0}}] = F_{\Delta_{T,0}}(x_{J+1}, x_J, \cdots, x_1, x_0)\,. \qquad (2.3)$$

Theorem 1 *For any $F[\gamma] \in \mathcal{F}$, we define the functional derivatives of $F[\gamma]$ whose directions are restricted to broken line paths η by*

$$(\delta F)[\gamma][\eta] = (\delta F)[\gamma_{\Delta_{T,0}}][\eta] = \sum_{j=1}^{J} (\partial_{x_j} F_{\Delta_{T,0}})(x_{J+1}, x_J, \cdots, x_1, x_0) \cdot y_j \,. \tag{2.4}$$

Then, $(\delta F)[\gamma][\eta]$ is well-defined, i.e., (2.4) is independent of the choice of $\Delta_{T,0}$. Furthermore, we have

$$(\delta F)[\gamma][\eta] \in \mathcal{F}\,. \tag{2.5}$$

Theorem 2 *$F[\gamma] \in \mathcal{F}$ can be differentiated as many times as we want. Furthermore, if $\Delta_{T,0}$ contains all of end points of the broken line paths γ and η^k with $\eta^k(0) = \eta^k(T) = 0$, $k = 1, 2, \ldots, K$, then we have*

$$\begin{aligned}&(\delta^K F)[\gamma][\eta^1][\eta^2] \cdots [\eta^K] \\ &= \sum_{j_1, j_2, \cdots, j_K = 1}^{J} \left((\prod_{k=1}^{K} y_j^k \cdot \partial_{x_{j_k}}) F_{\Delta_{T,0}} \right) (x_{J+1}, x_J, \cdots, x_1, x_0) \in \mathcal{F}\,,\end{aligned} \tag{2.6}$$

with $\eta^k(T_j) = y_j^k$, $j = 1, 2, \ldots, J$, $k = 1, 2, \ldots, K$.

Our result about the invariance of Feynman path integral with respect to translations is the following.

Theorem 3 *Let $\eta : [0, T] \to \mathbf{R}^d$ be any broken line path with $\eta(0) = \eta(T) = 0$. Let $F[\gamma] \in \mathcal{F}$. Then we have*

$$F[\gamma + \eta] \in \mathcal{F}\,. \tag{2.7}$$

Furthermore, assume that T is sufficiently small. Then we have

$$\int e^{\frac{i}{\hbar} S[\gamma+\eta]} F[\gamma + \eta] \mathcal{D}[\gamma] = \int e^{\frac{i}{\hbar} S[\gamma]} F[\gamma] \mathcal{D}[\gamma]\,. \tag{2.8}$$

Here we can choose T independent of $F[\gamma]$, of η and of $0 < \hbar < 1$.

Our result about the Taylor expansion formula with respect to the functional derivatives in Feynman path integral is the following.

Theorem 4 *Assume that T is sufficiently small. Then we have*

$$\begin{aligned}&\int e^{\frac{i}{\hbar} S[\gamma]} F[\gamma + \eta] \mathcal{D}[\gamma] - \sum_{k=0}^{K} \frac{1}{k!} \int e^{\frac{i}{\hbar} S[\gamma]} (\delta^k F)[\gamma][\eta][\eta] \cdots [\eta] \mathcal{D}[\gamma] \\ &= \int e^{\frac{i}{\hbar} S[\gamma]} \left(\int_0^1 \frac{(1-\theta)^K}{K!} (\delta^{K+1} F)[\gamma + \theta\eta][\eta][\eta] \cdots [\eta] d\theta \right) \mathcal{D}[\gamma]\,,\end{aligned} \tag{2.9}$$

for any $F[\gamma] \in \mathcal{F}$, any broken line path η with $\eta(0) = \eta(T) = 0$, and any $0 < \hbar < 1$.

Our result about the characterization of the functional derivatives in Feynman path integral is the following.

Theorem 5 *Assume that T is sufficiently small. Then we have*

$$\frac{d}{ds}\int e^{\frac{i}{\hbar}S[\gamma]}F[\gamma+s\eta]\mathcal{D}[\gamma]\bigg|_{s=0} = \int e^{\frac{i}{\hbar}S[\gamma]}(\delta F)[\gamma][\eta]\mathcal{D}[\gamma]\,, \tag{2.10}$$

for any $F[\gamma]\in\mathcal{F}$, any broken line path η with $\eta(0)=\eta(T)=0$, and any $0<\hbar<1$.

Our result about the integration by parts with respect to the functional derivatives in Feynman path integral is the following.

Theorem 6 *If we define $(\delta S)[\gamma][\eta]$ by (2.4), then we have*

$$(\delta S)[\gamma][\eta]\in\mathcal{F}\,. \tag{2.11}$$

Furthermore, assume that T is sufficiently small. Then we have

$$\int e^{\frac{i}{\hbar}S[\gamma]}(\delta F)[\gamma][\eta]\mathcal{D}[\gamma] = -\frac{i}{\hbar}\int e^{\frac{i}{\hbar}S[\gamma]}(\delta S)[\gamma][\eta]F[\gamma]\mathcal{D}[\gamma]\,, \tag{2.12}$$

for any $F[\gamma]\in\mathcal{F}$, any broken line path η with $\eta(0)=\eta(T)=0$, and any $0<\hbar<1$.

Corollary 1 *Assume that T is sufficiently small. Then we have*

$$\int e^{\frac{i}{\hbar}S[\gamma]}\int_0^T\left(\frac{d\gamma}{dt}\cdot\frac{d\eta}{dt}-(\partial_x V)(t,\gamma(t))\eta(t)\right)dt\mathcal{D}[\gamma]=0\,, \tag{2.13}$$

for any broken line path η with $\eta(0)=\eta(T)=0$ and any $0<\hbar<1$.

3 Examples

First we give the functional derivatives of the examples of Theorem 3 in [4].

Assumption 2 *Let m be non-negative integer. $B(t,x)$ is a function of $(t,x)\in\mathbf{R}\times\mathbf{R}^d$. For any multi-index α, $\partial_x^\alpha B(t,x)$ is continuous on $[0,T]\times\mathbf{R}^d$, and there exists a positive constant C_α such that*

$$|\partial_x^\alpha B(t,x)|\le C_\alpha(1+|x|)^m\,. \tag{3.1}$$

Theorem 7 *Under Assumption 2, let $0\le T'\le T''\le T$. Let $\rho(\tau)$ be a function of bounded variation on $[T',T'']$. Then we have the following.*

(1) *If $F[\gamma]=\int_{T'}^{T''}B(\tau,\gamma(\tau))d\rho(\tau)$, then*

$$(\delta F)[\gamma][\eta]=\int_{T'}^{T''}(\partial_x B)(\tau,\gamma(\tau))\cdot\eta(\tau)d\rho(\tau)\,. \tag{3.2}$$

(2) *If $F[\gamma]=B(\tau,\gamma(\tau))$ with $0\le\tau\le T$, then*

$$(\delta F)[\gamma][\eta]=(\partial_x B)(\tau,\gamma(\tau))\cdot\eta(\tau)\,. \tag{3.3}$$

Next we give the functional derivative of the example of Theorem 1 in [5].

Assumption 3 *Let m be non-negative integer. $Z(t,x)$ is a vector-valued function of $(t,x) \in \mathbf{R} \times \mathbf{R}^d$ into $\mathbf{R}^d$. For any multi-index α, $\partial_x^\alpha Z(t,x)$ and $\partial_x^\alpha \partial_t Z(t,x)$ are continuous on $[0,T] \times \mathbf{R}^d$, and there exists a positive constant C_α such that*

$$|\partial_x^\alpha Z(t,x)| + |\partial_x^\alpha \partial_t Z(t,x)| \le C_\alpha (1+|x|)^m \,. \tag{3.4}$$

Furthermore, $\partial_x Z(t,x)$ is a symmetric matrix, i.e., ${}^t(\partial_x Z) = \partial_x Z$.

Theorem 8 *Under Assumption 3, let $0 \le T' \le T'' \le T$.*
If $F[\gamma] = \int_{T'}^{T''} Z(\tau,\gamma(\tau)) \cdot d\gamma(\tau)$, then

$$\begin{aligned}(\delta F)[\gamma][\eta] \;&=\; Z(T'',\gamma(T'')) \cdot \eta(T'') - Z(T',\gamma(T')) \cdot \eta(T') \\ &\quad - \int_{T'}^{T''} (\partial_t Z)(\tau,\gamma(\tau)) \cdot \eta(\tau) d\tau \,.\end{aligned} \tag{3.5}$$

4 Proofs

We use the following notation in [4].

$$x_j \in \mathbf{R}^d\,, \quad j = 0,1,\ldots,J,J+1\,, \tag{4.1}$$

$$t_j > 0\,, \quad j = 1,2,\ldots,J,J+1\,, \tag{4.2}$$

$$T_j = t_j + t_{j-1} + \cdots + t_1\,. \tag{4.3}$$

For simplicity, for any $0 \le l \le L \le J+1$, we set

$$x_{L,l} = (x_L, x_{L-1}, \cdots, x_l)\,. \tag{4.4}$$

Only when the character is $T_{L,l}$ for any $1 \le l \le L \le J+1$, we set

$$T_{L,l} = t_L + t_{L-1} + \cdots + t_l\,. \tag{4.5}$$

Proof of Theorem 1.
(1) For any $1 \le n \le N \le J$, we define $x^\triangleleft_{N,n} = x^\triangleleft_{N,n}(x_{N+1}, x_{n-1})$ by

$$x_j^\triangleleft = \frac{T_{j,n}}{T_{N+1,n}} x_{N+1} + \frac{T_{N+1,n} - T_{j,n}}{T_{N+1,n}} x_{n-1}\,, \quad j = n, n+1, \ldots, N\,. \tag{4.6}$$

Let $(\Delta_{T,T_{N+1}}, \Delta_{T_{n-1},0})$ be the division defined by

$$T = T_{J+1} > T_J > \cdots > T_{N+1} > T_{n-1} > \cdots > T_1 > T_0 = 0\,. \tag{4.7}$$

First, for simplicity, we consider the case when $(\Delta_{T,T_{N+1}}, \Delta_{T_{n-1},0})$ contains all of end points of the broken line paths γ and η, and

$$\gamma = \gamma_{(\Delta_{T,T_{N+1}}, \Delta_{T_{n-1},0})} = \gamma_{\Delta_{T,0}}\,. \tag{4.8}$$

By Lemma 3.1 in [4], we have

$$\begin{aligned}&F_{\Delta_{T,0}}(x_{J+1,N+1}, x^\triangleleft_{N,n}, x_{n-1,0}) \\ &= F[\gamma_{\Delta_{T,0}}] = F[\gamma_{(\Delta_{T,T_{N+1}}, \Delta_{T_{n-1},0})}] \\ &= F_{(\Delta_{T,T_{N+1}}, \Delta_{T_{n-1},0})}(x_{J+1,N+1}, x_{n-1,0})\,.\end{aligned} \tag{4.9}$$

By (2.4), we have

$$\begin{aligned}
&(\delta F)[\gamma_{(\Delta_{T,T_{N+1}},\Delta_{T_{n-1},0})}][\eta] \\
&= \sum_{j=1}^{n-1} \left(\partial_{x_j} F_{(\Delta_{T,T_{N+1}},\Delta_{T_{n-1},0})}\right)(x_{J+1,N+1}, x_{n-1,0}) \cdot y_j \\
&+ \sum_{j=N+1}^{J} \left(\partial_{x_j} F_{(\Delta_{T,T_{N+1}},\Delta_{T_{n-1},0})}\right)(x_{J+1,N+1}, x_{n-1,0}) \cdot y_j \,. \qquad (4.10)
\end{aligned}$$

By (4.9), we have

$$\begin{aligned}
&= \sum_{j=1}^{n-1} \left(\partial_{x_j} F_{\Delta_{T,0}}\right)(x_{J+1,N+1}, x^{\triangleleft}_{N,n}, x_{n-1,0}) \cdot y_j \\
&+ \sum_{j=n}^{N} \left(\partial_{x_j} F_{\Delta_{T,0}}\right)(x_{J+1,N+1}, x^{\triangleleft}_{N,n}, x_{n-1,0}) \cdot y^{\triangleleft}_j \\
&+ \sum_{j=N+1}^{J} \left(\partial_{x_j} F_{\Delta_{T,0}}\right)(x_{J+1,N+1}, x^{\triangleleft}_{N,n}, x_{n-1,0}) \cdot y_j \,. \qquad (4.11)
\end{aligned}$$

Since the broken line path η is a straight line path on $[T_{n-1}, T_{N+1}]$, we have $\eta(T_j) = y_j = y^{\triangleleft}_j$ for $j = n, n+1, \ldots, N$. Therefore we have

$$\begin{aligned}
&= \sum_{j=1}^{J} \left(\partial_{x_j} F_{\Delta_{T,0}}\right)(x_{J+1,N+1}, x^{\triangleleft}_{N,n}, x_{n-1,0}) \cdot y_j \\
&= (\delta F)[\gamma_{\Delta_{T,0}}][\eta] \,. \qquad (4.12)
\end{aligned}$$

Next we consider the case when $\Delta_{T,0}$ contains all of end points of the broken line paths γ and η, $\Delta'_{T,0}$ is any refinement of $\Delta_{T,0}$, and

$$\gamma = \gamma_{\Delta_{T,0}} = \gamma_{\Delta'_{T,0}} \,. \qquad (4.13)$$

By a similar calculation, we get

$$(\delta F)[\gamma_{\Delta_{T,0}}][\eta] = (\delta F)[\gamma_{\Delta'_{T,0}}][\eta] \,. \qquad (4.14)$$

Therefore (2.4) is well-defined.
(2) Since $F[\gamma] \in \mathcal{F}$, $F_{\Delta_{T,0}}(x_{J+1}, x_{J,1}, x_0)$ satisfies Assumption 5 in [4]. We fix the broken line path η. Then, there exists a positive constant C such that

$$|y_j| = |\eta(T_j)| \leq C \,, \qquad (4.15)$$

for $j = 1, 2, \ldots, J$. Furthermore, the number of end points of the broken line path η is also fixed. Hence, for $j = 1, 2, \ldots, J$,

$$(\partial_{x_j} F_{\Delta_{T,0}})(x_{J+1}, x_{J,1}, x_0) \cdot y_j \,, \qquad (4.16)$$

also satisfies Assumption 5 in [4]. Therefore, $(\delta F)[\gamma_{\Delta_{T,0}}][\eta]$ satisfies Assumption 5 in [4]. Hence $(\delta F)[\gamma][\eta] \in \mathcal{F}$. $\square$

Proof of Theorem 2. By Theorem 1, if $(\delta^K F)[\gamma][\eta^1][\eta^2]\cdots[\eta^K] \in \mathcal{F}$, then $(\delta^{K+1}F)[\gamma][\eta^1][\eta^2]\cdots[\eta^K][\eta^{K+1}] \in \mathcal{F}$. □

Proof of Theorem 3. We fix the broken line path η. Let

$$T = \tau_{K+1} > \tau_K > \cdots > \tau_1 > \tau_0 = 0\,, \tag{4.17}$$

be all of end points of the broken line path η.
(1) Set $G[\gamma] = F[\gamma + \eta]$. Let $\gamma_{\Delta_{T,0}}$ be any broken line path with the division

$$\Delta_{T,0} : T = T_{J+1} > T_J > \cdots > T_1 > T_0 = 0\,. \tag{4.18}$$

Let $\Delta'_{T,0}$ be the division which consists of all points of $\Delta_{T,0}$ and all of end points of the broken line path η. Then the path $\gamma_{\Delta_{T,0}} + \eta$ is also a broken line path with the division $\Delta'_{T,0}$. Since $F[\gamma] \in \mathcal{F}$, $F[\gamma_{\Delta'_{T,0}}]$ satisfies Assumption 5 in [4]. Since η is fixed, there exists a positive constant C such that

$$|\eta(\tau)| \le C\,, \tag{4.19}$$

for any $0 \le \tau \le T$. Furthermore, for any $j = 1, 2, \cdots, J+1$, the number of end points of the broken line path η between T_j and T_{j-1}, is less than $K+3$. Hence, $G[\gamma_{\Delta_{T,0}}] = F[\gamma_{\Delta_{T,0}} + \eta]$ satisfies Assumption 5 in [4].
Therefore we have

$$F[\gamma + \eta] = G[\gamma] \in \mathcal{F}\,. \tag{4.20}$$

(2) We fix $0 < \hbar < 1$ and consider $e^{\frac{i}{\hbar}(S[\gamma+\eta]-S[\gamma])}$. We note that

$$\begin{aligned} S[\gamma+\eta] - S[\gamma] \;&=\; \int_0^T \frac{1}{2}\left|\frac{d\eta}{dt}\right|^2 dt + \int_0^T \frac{d\eta}{dt}\cdot\frac{d\gamma}{dt}dt \\ &\quad - \int_0^T \int_0^1 (\partial_x V)(\tau, \gamma + \theta\eta(\tau))d\theta \cdot \eta(\tau)d\tau\,. \end{aligned} \tag{4.21}$$

$(2-1)$ It is clear that

$$e^{\frac{i}{\hbar}\int_0^T \frac{1}{2}\left|\frac{d\eta}{dt}\right|^2 dt} \in \mathcal{F}\,. \tag{4.22}$$

$(2-2)$ In a similar way to the proof of Theorem 5 in [4], we have

$$e^{-\frac{i}{\hbar}\int_0^T \int_0^1 (\partial_x V)(\tau,\gamma+\theta\eta(\tau))d\theta\cdot\eta(\tau)d\tau} \in \mathcal{F}\,. \tag{4.23}$$

$(2-3)$ Since η is fixed, there exists a positive constant C' such that

$$\left|\frac{d\eta}{dt}\right| + \left|\frac{d^2\eta}{dt^2}\right| = \left|\frac{y_k - y_{k-1}}{\tau_k - \tau_{k-1}}\right| + 0 \le C'\,, \tag{4.24}$$

on (τ_{k-1}, τ_k) for any $k = 1, 2, \ldots, K$. In a similar way to the proofs of Theorem 5 in [4] and Theorem 1 in [5]

$$e^{\frac{i}{\hbar}\int_{\tau_{k-1}}^{\tau_k} \frac{d\eta}{dt}\cdot\frac{d\gamma}{dt}dt} \in \mathcal{F}\,. \tag{4.25}$$

By Proposition 1, we have

$$e^{\frac{i}{\hbar}\int_0^T \frac{d\eta}{dt}\cdot\frac{d\gamma}{dt}dt} = \prod_{k=1}^{K+1} e^{\frac{i}{\hbar}\int_{\tau_{k-1}}^{\tau_k} \frac{d\eta}{dt}\cdot\frac{d\gamma}{dt}dt} \in \mathcal{F}\,. \tag{4.26}$$

(3) Therefore, by (1), (2) and Proposition 1, we have

$$e^{\frac{i}{\hbar}(S[\gamma+\eta]-S[\gamma])}F[\gamma+\eta]\in\mathcal{F}\,. \tag{4.27}$$

By Proposition 2, if T is sufficiently small,

$$\begin{aligned}&\int e^{\frac{i}{\hbar}S[\gamma+\eta]}F[\gamma+\eta]\mathcal{D}[\gamma]\\&=\lim_{|\Delta_{T,0}|\to 0}\prod_{j=1}^{J+1}\left(\frac{1}{2\pi i\hbar t_j}\right)^{d/2}\int_{\mathbf{R}^{dJ}}e^{\frac{i}{\hbar}S[\gamma_{\Delta_{T,0}}]}\\&\times e^{\frac{i}{\hbar}(S[\gamma_{\Delta_{T,0}}+\eta]-S[\gamma_{\Delta_{T,0}}])}F[\gamma_{\Delta_{T,0}}+\eta]\prod_{j=1}^{J}dx_j\,,\end{aligned} \tag{4.28}$$

converges uniquely. Here we can choose T independent of $F[\gamma]$, of η and of $0<\hbar<1$.
(4) Now we can assume that $\Delta_{T,0}$ contains all of end points of the broken line path η. Set $\gamma_{\Delta_{T,0}}(T_j)=x_j$, $\eta(T_j)=y_j$ for $j=0,1,\ldots,J,J+1$, and

$$S[\gamma_{\Delta_{T,0}}]=S_{\Delta_{T,0}}(x_{J+1},x_{J,1},x_0)\,. \tag{4.29}$$

The path $\gamma_{\Delta_{T,0}}+\eta$ is the broken line path which connects (T_j,x_j+y_j) and $(T_{j-1},x_{j-1}+y_{j-1})$ with a line segment for $j=1,2,\ldots,J,J+1$. Hence we have

$$\begin{aligned}&\int e^{\frac{i}{\hbar}S[\gamma+\eta]}F[\gamma+\eta]\mathcal{D}[\gamma]\\&=\lim_{|\Delta_{T,0}|\to 0}\prod_{j=1}^{J+1}\left(\frac{1}{2\pi i\hbar t_j}\right)^{d/2}\int_{\mathbf{R}^{dJ}}e^{\frac{i}{\hbar}S_{\Delta_{T,0}}(x_{J+1},x_{J,1}+y_{J,1},x_0)}\\&\times F_{\Delta_{T,0}}(x_{J+1},x_{J,1}+y_{J,1},x_0)\prod_{j=1}^{J}dx_j\,.\end{aligned} \tag{4.30}$$

By the change of variables: $x_j+y_j\to x_j$, $j=1,2,\ldots,J$, we have

$$\begin{aligned}&=\lim_{|\Delta_{T,0}|\to 0}\prod_{j=1}^{J+1}\left(\frac{1}{2\pi i\hbar t_j}\right)^{d/2}\int_{\mathbf{R}^{dJ}}e^{\frac{i}{\hbar}S_{\Delta_{T,0}}(x_{J+1},x_{J,1},x_0)}\\&\times F_{\Delta_{T,0}}(x_{J+1},x_{J,1},x_0)\prod_{j=1}^{J}dx_j\\&=\int e^{\frac{i}{\hbar}S[\gamma]}F[\gamma]\mathcal{D}[\gamma]\,.\quad\square\end{aligned} \tag{4.31}$$

Proof of Theorem 4. We fix the broken line path η. Let γ is any broken line path. We assume that $\Delta_{T,0}$ contains all of end points of the broken line paths γ and η. Set $\gamma(T_j)=x_j$ and $\eta(T_j)=y_j$ for $j=0,1,\ldots,J,J+1$.
For any $0\le\theta\le 1$, the path $\gamma+\theta\eta$ is also a broken line path and

$$F[\gamma+\theta\eta]=F_{\Delta_{T,0}}(x_{J+1}.x_{J,1}+\theta y_{J,1},x_0)\,. \tag{4.32}$$

By the usual Taylor expansion formula and Theorem 2, we have

$$
\begin{aligned}
&F[\gamma+\eta]-\sum_{k=0}^{K}\frac{1}{k!}(\delta^k F)[\gamma][\eta][\eta]\cdots[\eta] \\
&=\int_0^1 \frac{(1-\theta)^K}{K!}(\delta^{K+1}F)[\gamma+\theta\eta][\eta][\eta]\cdots[\eta]d\theta\,. \qquad (4.33)
\end{aligned}
$$

By Theorems 2, 3 and Proposition 1, we have

$$
\int_0^1 \frac{(1-\theta)^K}{K!}(\delta^{K+1}F)[\gamma+\theta\eta][\eta][\eta]\cdots[\eta]d\theta \in \mathcal{F}\,. \qquad (4.34)
$$

Since (4.33) holds for any broken line path γ, we have (2.9). □

Proof of Theorem 5.

$$
\begin{aligned}
&\frac{d}{ds}\int e^{\frac{i}{\hbar}S[\gamma]}F[\gamma+s\eta]\mathcal{D}[\gamma]\Big|_{s=0} \\
&=\lim_{s\to 0}\frac{1}{s}\left(\int e^{\frac{i}{\hbar}S[\gamma]}F[\gamma+s\eta]\mathcal{D}[\gamma]-\int e^{\frac{i}{\hbar}S[\gamma]}F[\gamma]\mathcal{D}[\gamma]\right)\,. \qquad (4.35)
\end{aligned}
$$

We assume that $|s|\le 1$. By Theorem 4 with $K=1$, we have

$$
\begin{aligned}
= \;& \lim_{s\to 0}\int e^{\frac{i}{\hbar}S[\gamma]}(\delta F)[\gamma][\eta]\mathcal{D}[\gamma] \\
&+\lim_{s\to 0} s\times\int e^{\frac{i}{\hbar}S[\gamma]}\int_0^1(1-\theta)(\delta^2 F)[\gamma+\theta s\eta][\eta][\eta]d\theta\mathcal{D}[\gamma]\,. \qquad (4.36)
\end{aligned}
$$

By the results of Chapter 3 in [4], there exists a positive constant C such that

$$
\left|\int e^{\frac{i}{\hbar}S[\gamma]}\int_0^1(1-\theta)(\delta^2 F)[\gamma+\theta s\eta][\eta][\eta]d\theta\mathcal{D}[\gamma]\right|\le C\,, \qquad (4.37)
$$

for any $|s|\le 1$. Therefore we get (2.10). □

Proof of Theorem 6.
(1) Set $G[\gamma]=\int_0^T\frac{1}{2}|\frac{d\gamma}{dt}|^2dt$. We define $(\delta G)[\gamma][\eta]$ by (2.4). We fix the broken line path η. Let $\Delta_{T,0}$ be any division which contains all of end points of the broken line paths γ and η. Set $\gamma(T_j)=x_j$ and $\eta(T_j)=y_j$ for any $j=0,1,\ldots,J,J+1$. Then we have

$$
\begin{aligned}
(\delta G)[\gamma][\eta] &= \sum_{j=1}^{J}\left(\frac{x_j-x_{j+1}}{t_{j+1}}+\frac{x_j-x_{j-1}}{t_j}\right)\cdot y_j \\
&= \sum_{j=1}^{J+1}\frac{y_j-y_{j-1}}{t_j}\cdot(x_j-x_{j-1})=\int_0^T\frac{d\eta}{dt}\cdot d\gamma\,. \qquad (4.38)
\end{aligned}
$$

Now, let $\{j_k\}_{k=0}^{K+1}$ be the subsequence of $j=0,1,\ldots,J,J+1$ such that

$$
T=T_{j_{K+1}}>T_{j_K}>\cdots>T_{j_1}>T_{j_0}=0\,. \qquad (4.39)
$$

are all of end points of the broken line path η. Since η is fixed, K is also fixed. Furthermore, there exists a positive constant C' such that

$$\left|\frac{d\eta}{dt}\right| + \left|\frac{d^2\eta}{dt^2}\right| = \left|\frac{y_j - y_{j-1}}{t_j}\right| + 0 \leq C' \,, \tag{4.40}$$

on $(T_{j_{k-1}}, T_{j_k})$ for any $k = 1, 2, \ldots, K, K+1$. By Theorem 1 in [5] and Proposition 1, we have

$$\int_0^T \frac{d\eta}{dt} \cdot d\gamma = \sum_{k=1}^{K+1} \int_{T_{j_{k-1}}}^{T_{j_k}} \frac{d\eta}{dt} \cdot d\gamma \in \mathcal{F} \,. \tag{4.41}$$

By Proposition 1, we have $(\delta S)[\gamma][\eta] \in \mathcal{F}$.
(2) By Theorem 3, we have

$$0 = \int e^{\frac{i}{\hbar}S[\gamma+\eta]} F[\gamma+\eta]\mathcal{D}[\gamma] - \int e^{\frac{i}{\hbar}S[\gamma]} F[\gamma]\mathcal{D}[\gamma] \,. \tag{4.42}$$

We assume that $\Delta_{T,0}$ contain all of end points of the broken line path η.
Set $\gamma_{\Delta_{T,0}}(T_j) = x_j$ for any $j = 0, 1, \ldots, J, J+1$.
Then we have

$$\begin{aligned}
= & \lim_{|\Delta_{T,0}|\to 0} \prod_{j=1}^{J+1} \left(\frac{1}{2\pi i\hbar t_j}\right)^{d/2} \\
& \times \int_{\mathbf{R}^{dJ}} \left(e^{\frac{i}{\hbar}S[\gamma_{\Delta_{T,0}}+\eta]} F[\gamma_{\Delta_{T,0}}+\eta] - e^{\frac{i}{\hbar}S[\gamma_{\Delta_{T,0}}]} F[\gamma_{\Delta_{T,0}}] \right) \prod_{j=1}^{J} dx_j \\
= & \lim_{|\Delta_{T,0}|\to 0} \prod_{j=1}^{J+1} \left(\frac{1}{2\pi i\hbar t_j}\right)^{d/2} \int_{\mathbf{R}^{dJ}} \int_0^1 e^{\frac{i}{\hbar}S[\gamma_{\Delta_{T,0}}+\theta\eta]} \\
& \times \Big(\frac{i}{\hbar}(\delta S)[\gamma_{\Delta_{T,0}}+\theta\eta][\eta] F[\gamma_{\Delta_{T,0}}+\theta\eta] + (\delta F)[\gamma_{\Delta_{T,0}}+\theta\eta][\eta] \Big) \\
& \times d\theta \prod_{j=1}^{J} dx_j \\
= & \lim_{|\Delta_{T,0}|\to 0} \int_0^1 \prod_{j=1}^{J+1} \left(\frac{1}{2\pi i\hbar t_j}\right)^{d/2} \int_{\mathbf{R}^{dJ}} e^{\frac{i}{\hbar}S[\gamma_{\Delta_{T,0}}+\theta\eta]} \\
& \times \Big(\frac{i}{\hbar}(\delta S)[\gamma_{\Delta_{T,0}}+\theta\eta][\eta] F[\gamma_{\Delta_{T,0}}+\theta\eta] + (\delta F)[\gamma_{\Delta_{T,0}}+\theta\eta][\eta] \Big) \\
& \times \prod_{j=1}^{J} dx_j d\theta.
\end{aligned} \tag{4.43}$$

The path $\gamma_{\Delta_{T,0}} + \theta\eta$ is the broken line path which connects $(T_j, x_j + \theta\eta(T_j))$ and $(T_{j-1}, x_{j-1} + \theta\eta(T_{j-1}))$ with a line segment for any $j = 1, 2, \ldots, J, J+1$.
By the change of variables: $x_j + \theta\eta(T_j) \to x_j$, $j = 1, 2, \ldots, J$, we have

$$\begin{aligned}
= & \lim_{|\Delta_{T,0}|\to 0} \int_0^1 \prod_{j=1}^{J+1} \left(\frac{1}{2\pi i\hbar t_j}\right)^{d/2} \int_{\mathbf{R}^{dJ}} e^{\frac{i}{\hbar}S[\gamma_{\Delta_{T,0}}]} \\
& \times \Big(\frac{i}{\hbar}(\delta S)[\gamma_{\Delta_{T,0}}][\eta] F[\gamma_{\Delta_{T,0}}] + (\delta F)[\gamma_{\Delta_{T,0}}][\eta] \Big) \prod_{j=1}^{J} dx_j d\theta
\end{aligned}$$

$$
\begin{aligned}
= \quad & \lim_{|\Delta_{T,0}|\to 0} \prod_{j=1}^{J+1} \left(\frac{1}{2\pi i\hbar t_j} \right)^{d/2} \int_{\mathbf{R}^{dJ}} e^{\frac{i}{\hbar}S[\gamma_{\Delta_{T,0}}]} \\
& \times \Big(\frac{i}{\hbar} (\delta S)[\gamma_{\Delta_{T,0}}][\eta] F[\gamma_{\Delta_{T,0}}] + (\delta F)[\gamma_{\Delta_{T,0}}][\eta] \Big) \prod_{j=1}^{J} dx_j \\
= \quad & \frac{i}{\hbar} \int e^{\frac{i}{\hbar}S[\gamma]} (\delta S)[\gamma][\eta] F[\gamma] \mathcal{D}[\gamma] + \int e^{\frac{i}{\hbar}S[\gamma]} (\delta F)[\gamma][\eta] \mathcal{D}[\gamma] \,. \quad \square \qquad (4.44)
\end{aligned}
$$

Proof of Theorem 7. We only prove (2). The proof of (1) is similar.
Let $\Delta_{T,0}$ be any division which contains all of end points of the broken line paths γ and η. There exists a number k such that $T_{k-1} < \tau \le T_k$. Then we have

$$
F[\gamma_{\Delta_{T,0}}] = B\left(\tau, \frac{\tau - T_{k-1}}{T_k - T_{k-1}} x_k + \frac{T_k - \tau}{T_k - T_{k-1}} x_{k-1} \right) . \qquad (4.45)
$$

By (2.4), we have

$$
\begin{aligned}
& (\delta F)[\gamma_{\Delta_{T,0}}][\eta] \\
& = (\partial_x B)\left(\tau, \frac{\tau - T_{k-1}}{T_k - T_{k-1}} x_k + \frac{T_k - \tau}{T_k - T_{k-1}} x_{k-1} \right) \cdot \frac{\tau - T_{k-1}}{T_k - T_{k-1}} y_k \\
& + (\partial_x B)\left(\tau, \frac{\tau - T_{k-1}}{T_k - T_{k-1}} x_k + \frac{T_k - \tau}{T_k - T_{k-1}} x_{k-1} \right) \cdot \frac{T_k - \tau}{T_k - T_{k-1}} y_{k-1} \\
& = (\partial_x B)(\tau, \gamma(\tau)) \cdot \eta(\tau) \,. \quad \square \qquad (4.46)
\end{aligned}
$$

Proof of Theorem 8. In a similar way to the proof of Theorem 1 in [5], we can get (3.5). □

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ADDRESS:
Graduate School of Mathematical Sciences, The University of Tokyo
3–8–1 Komaba Meguro-ku, Tokyo 153, JAPAN
TEL +81-3-5465-7001　　FAX +81-3-5465-7012