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by

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An inverse problem in periodic diffractive optics: Global uniqueness with a single wave number

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Abstract. We consider the problem of recovering a perfectly reflecting two-dimensional diffraction grating from the knowledge of one wave number, one incident direction and the total field measured above the grating. We prove a global uniqueness result within the class of polygonal grating profiles. The proof relies on the analyticity of solutions to the Helmholtz equation and the Rayleigh expansion of the scattered field.

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1. Introduction

The problem of recovering a periodic structure from knowledge of the scattered field occurs in many applications, e.g., in diffractive optics, radar imaging and nondestructive testing. In this paper we consider the scattering of monochromatic plane waves by a perfectly reflecting diffraction grating and restrict ourselves to the transverse electric polarization (the TE mode) in an isotropic lossless medium. Our goal is to prove global uniqueness in determining polygonal periodic grating profiles by near field observations with a single wave number.

Let the profile of the diffraction grating be given by the curve

$$\Lambda_f := \{(x_1, x_2) \in \mathbb{R}^2 : x_2 = f(x_1)\}$$

where f is a 2π-periodic Lipschitz function. Suppose that a plane wave given by

$$u^{in} := \exp(i\alpha x_1 - i\beta x_2), \quad (\alpha, \beta) = k(\sin\theta, \cos\theta)$$

is incident on Λ_f from the top, where the wave number k is a positive constant and $\theta \in (-\pi/2, \pi/2)$ is the incident angle. The domain above the curve is denoted by

$$\Omega_f := \{x \in \mathbb{R}^2 : x_2 > f(x_1)\}.$$

Then the total field $u = u(x_1, x_2)$, which is the sum of u^{in} and the scattered field, satisfies the Dirichlet problem

$$\Delta u + k^2 u = 0 \quad \text{in} \quad \Omega_f\,, \quad u = 0 \quad \text{on} \quad \Lambda_f\,, \tag{1.1}$$

and is assumed to be α-quasiperiodic in x_1:

$$u(x_1 + 2\pi, x_2) = \exp(i2\alpha\pi)u(x_1, x_2)\,. \tag{1.2}$$

Moreover, u is required to satisfy the radiation condition

$$u(x) = u^{in} + \sum_{n\in\mathbb{Z}} A_n \exp(i(n+\alpha)x_1 + i\beta_n x_2)\,,\; x_2 > \max(f) \tag{1.3}$$

with the Rayleigh coefficients $A_n \in \mathbb{C}$ and

$$\beta_n := \begin{cases} (k^2 - (n+\alpha)^2)^{1/2} & \text{if} \quad |n+\alpha| \le k\,, \\ i((n+\alpha)^2 - k^2)^{1/2} & \text{if} \quad |n+\alpha| > k\,. \end{cases} \tag{1.4}$$

Since β_n is real for at most a finite number of indices, we notice that only a finite number of plane waves in the sum (1.4) propagate into the far field, with the remaining evanescent waves decaying exponentially as $x_2 \to \infty$.

It is known that there exists exactly one solution $u \in H^1_{\text{loc}}(\Omega_f)$ of the *direct diffraction problem* (1.1)–(1.3); see [9] for sufficiently smooth ($f \in C^2$) and [7] for Lipschitz profiles.

The *inverse problem* or the *profile reconstruction problem* can now be formulated as follows.

(IP): Determine the profile function f from the knowledge of one wave number k, one incident direction θ and the total field $u|_{x_2=b}$ on a straight line $\{x \in \mathbb{R}^2 : x_2 = b\}$ with $b > \max(f)$.

Note that this problem also involves near field measurements since the evanescent modes cannot be measured far away from the grating profile.

The global uniqueness in problem (IP) is known if the wave number or the amplitude of the grating are sufficiently small [8]; see also [1] in the case of a lossy medium (i.e., $\operatorname{Im} k > 0$). For related stability results we refer to [4], [5]. A recent review on uniqueness results in scattering theory for periodic structures can be found in [2].

In general, global uniqueness may not be true when k is real. This can be seen from the simple counterexample of the scattering of $u^{in} = \exp(-ikx_2)$ when one moves the flat grating in certain multiples of the wavelength. Even though we exclude this flat case, the global uniqueness is not known with a single general $k > 0$. The purpose of this paper is to solve this open problem in some case, that is, to establish the global uniqueness within piecewise linear profiles for any fixed k. Notice that a class of piecewise linear profiles is acceptable from a practical viewpoint (e.g., [10]). Now we state our main result.

Theorem. *Exclude the Rayleigh frequencies by assuming*

$$\beta_n \neq 0\,, \quad \textit{i.e.}, \quad k^2 \neq (n+\alpha)^2 \quad \textit{for all} \quad n \in \mathbb{Z}\,. \tag{1.5}$$

Let f_1 and f_2 be continuous 2π-periodic piecewise linear profile functions consisting of finitely many segments, where the case $f_1 \equiv \mathrm{const}$, $f_2 \equiv \mathrm{const}$ is excluded. Let u_1 and u_2 solve the corresponding direct diffraction problem (1.1)–(1.3) in Ω_{f_1} and Ω_{f_2}, respectively, and let $b > \max(f_1, f_2)$. Then the relation

$$u_1(x_1, b) = u_2(x_1, b) \quad \textit{for all} \quad x_1 \in \mathbb{R}$$

implies $f_1 = f_2$.

Remark 1. If condition (1.5) is violated, one obtains further non-uniqueness examples for (IP) (in addition to the case of parallel half-planes).

(i) Let $k = 1$ and $\alpha = 0$ (orthogonal incidence). Then we have $\beta = 1$, $\beta_{\pm 1} = 0$, and the finite Rayleigh expansion

$$u(x) = \exp(-ix_2) + \exp(ix_2) - \exp(-ix_1) - \exp(ix_1)$$

satisfies the Helmholtz equation in the whole plane and vanishes on the lines $\{x_2 = x_1\}$, $\{x_2 = -x_1\}$ and on the (quadratic) grids obtained by the 2π-periodic extensions of $\{x_2 = x_1\}$, $\{x_2 = -x_1\}$.

(ii) For $k = 2$ and $\theta = \pi/6$ we have $\alpha = 1$, $\beta = \sqrt{3}$, $\beta_1 = \beta_{-3} = 0$. Then the finite sum of plane waves

$$u(x) = \exp(ix_1 - i\sqrt{3}x_2) + \exp(-ix_1 + i\sqrt{3}x_2) - \exp(2ix_1) - \exp(-2ix_1)$$

satisfies the Helmholtz equation and vanishes on the lines $\{x_2 = \sqrt{3}x_1\}$, $\{x_2 = -x_1/\sqrt{3}\}$ and on the grids generated by the 2π-periodic extensions of these lines.

The proof of the theorem is based on the analyticity of solutions to the Helmholtz equation (e.g., [6]) and the Rayleigh expansion (1.3). The following section is devoted to this proof, which is divided into several steps.

2. Proof of the theorem

2.1. Reflection argument

We need the following auxiliary lemma. Consider a triangular or quadrilateral domain $\Omega_1 \subset \mathbb{R}^2$ which is symmetric with respect to a line L. Furthermore let Γ_0 and Γ_1 be the two segments of $\partial\Omega_1$, which are not perpendicular to L and where Γ_0 is the reflection of Γ_1 with respect to L. Finally, set $\Gamma_2 = \Omega_1 \cap L$ and let Ω_2 be the subdomain of Ω_1 lying between Γ_0 and Γ_2; see Fig. 1.

Lemma 1. *Let $u_j \in H^2(\Omega_j)$ satisfy the Helmholtz equation $(\Delta + k^2)u = 0$ in Ω_j such that $u_j|_{\Gamma_j} = 0$ $(j = 1, 2)$, and assume that $u_1 = u_2$ in Ω_2. Then $u_1|_{\Gamma_0} = 0$.*

Proof. Since the Laplace operator is invariant under translation and rotation, we may assume that Ω_1 is symmetric with respect to the x_1 axis. Setting $v = u_1$ in Ω_2' (the

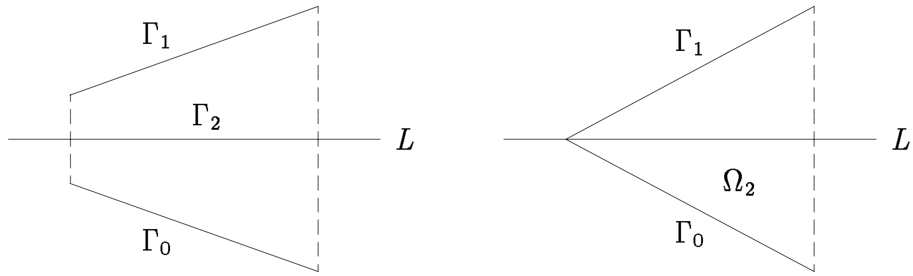

Figure 1. Symmetric domain Ω_1

reflection of Ω_2 with respect to the x_1 axis) and $v(x_1, x_2) = -u_1(x_1, -x_2)$ in Ω_2, we obviously have

$$(\Delta + k^2)v = -(\Delta + k^2)u_1 = 0 \quad \text{in} \quad \Omega_2.$$

Moreover, since $u_1 = u_2$ in Ω_2 and $u_2|_{\Gamma_2} = 0$, we obtain $v|_{\Gamma_2} = 0$, and the jump across Γ_2 of the normal derivative $\partial_2 v = \partial v / \partial x_2$ (which is defined in the sense of $H^{1/2}(\Gamma_2)$) is zero. Hence v is a distributional solution of the Helmholtz equation in $\Omega_1 = \Omega_2' \cup \Gamma_2 \cup \Omega_2$, which belongs to $H^2(\Omega_1)$ by the elliptic regularity. Since $v = u_1$ in Ω_2', we also have $v = u_1$ in Ω_1 by unique continuation. This implies $u_1|_{\Gamma_0} = u_1|_{\Gamma_1} = 0$ by the definition of v. ∎

2.2. Construction of an "exit direction"

To prove the theorem, let $f_1 \neq f_2$ be 2π-periodic continuous piecewise linear profile functions, with the case $f_1 = \text{const}$, $f_2 = \text{const}$ excluded. Consider the solutions $u_j \in H^1_{\text{loc}}(\Omega_{f_j})$ of the corresponding direct diffraction problems, and define

$$f(t) := \max(f_1(t), f_2(t)), \quad t \in \mathbb{R}.$$

Note that by the elliptic regularity for the Dirichlet problem (1.1), each function u_j is infinitely smooth up to the boundary, with the exception of the corner points of Λ_{f_j}. Moreover, since u_j satisfies the Helmholtz equation, u_j is real-analytic in Ω_{f_j} (e.g., [6]). Since $u_1 = u_2$ and then also $\partial_2 u_1 = \partial_2 u_2$ on $\{x_2 = b\}$, we have

$$u_1 = u_2 =: u \quad \text{in} \quad \Omega_f\,; \tag{2.6}$$

see, e.g., [1]. Henceforth by a ray we mean a straight line starting from one point and extended to the point at infinity.

Lemma 2. *There exists a ray S such that*

$$S \textit{ is not parallel to the coordinate axes}, \quad S \subset \Omega_f \quad \textit{and} \quad u|_S = 0\,. \tag{2.7}$$

Proof. We have to consider the following cases.
1) $f_2(t) > f_1(t)\,,\ t \in \mathbb{R}\,,$
a) $f_2 \neq \text{const}:$ Consider a segment of Λ_{f_2} with an endpoint of maximal x_2 coordinate,

which is not parallel to the x_1 direction, and extend it to a ray $S \subset \Omega_{f_1}$. Then $u|_S = u_1|_S = 0$ since u_1 vanishes on an open subset of S and is real-analytic in Ω_{f_1} (and thus real-analytic as a function of a single independent variable on $S \cap \Omega_{f_1}$).

b) $f_2 = \text{const}$: Consider the reflection of Λ_{f_1} with respect to Λ_{f_2}, which is denoted by Λ^*. Then $u|_{\Lambda^*} = 0$ by Lemma 1, and as in a) we can choose a ray $S \subset \Omega_{f_2}$ such that $u|_S = 0$, because $f_1 \neq \text{const}$ by the assumption.

2) The graphs of f_1 and f_2 do intersect:
Henceforth P_1P_2 denotes the open segment connecting points P_1 and P_2. By the periodicity of f_1 and f_2, we can choose two intersection points. Let Q_1 and Q_2 be intersection points of Λ_{f_1} and Λ_{f_2} such that $f_2(t) > f_1(t)$ between Q_1 and Q_2. Choose a directed segment P_1Q of Λ_{f_2} between Q_1 and Q_2 with origin P_1 and an endpoint Q of the maximal x_2 coordinate. Then the ray S extending P_1Q either satisfies (2.7) (cf. the case 1a)), or intersects Λ_f at some point P_2. In the latter case, we have $u|_{P_1P_2} = 0$, where the real analyticity of u_1 was used again.

Assume, for example, that the segment P_1P_2 has non-negative slope with respect to the positive x_1 direction. Here P_1 may coincide with Q_1. If P_2 is different from Q_2, then there exists a segment Σ of Λ_f lying below P_1P_2 and intersecting P_1P_2 at P_2 from the left, since Λ_f is the graph of a piecewise linear function. If P_2 coincides with Q_2, we can choose a segment $\Sigma \subset \Lambda_{f_1}$ with the same properties. In each case we have $u_1|_\Sigma = 0$.

By Lemma 1, we obtain $u = 0$ on the reflection of Σ with respect to P_1P_2. A possibly repeated application of the reflection argument and the above analyticity argument yield either a ray S satisfying (2.7), or a segment of angle $\gamma_2 > 0$ with the negative x_1 direction such that $u|_{P_2P_3} = 0$, $P_3 \in \Lambda_f$ and P_3 lies above P_2 and on the left from P_1; see Fig. 2.

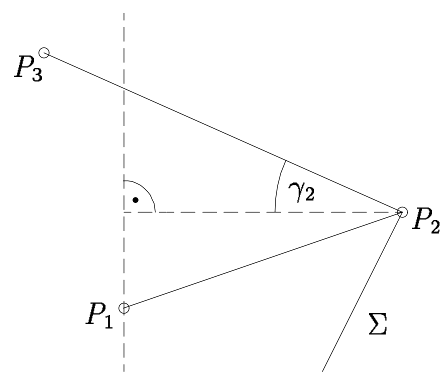

Figure 2. Reflection step at P_2

Then we repeat the construction with respect to the point P_3 and apply reflection at P_2P_3 etc., which gives either a ray S satisfying (2.7), or a segment P_3P_4 of angle $\gamma_3 > 0$ (with positive x_1 direction) such that $u|_{P_3P_4} = 0, P_4 \in \Lambda_f$ and P_4 is lying above P_3 and on the right from P_2.

Thus we obtain a sequence of segments P_nP_{n+1} with slopes γ_n such that $u|_{P_nP_{n+1}} = 0$ and $P_{n+1}P_{n+2}$ lies above P_nP_{n+1} for $n = 1, 2, \ldots$. This process must terminate with finding a ray S satisfying (2.7) unless $\gamma_n \to 0$ as $n \to \infty$. We finally show that this case is impossible.

Since there are only finitely many segments of $\Lambda_{f_1} \cup \Lambda_{f_2}$, there exists a minimal slope $\gamma^* > 0$ (with respect to the x_1 axis) among all segments not parallel to the x_1 axis. Therefore, if all γ_n $(n \geq n^*)$ are close to zero, then the reflection step at P_{n^*+1} yields a segment $P_{n^*+1}P_{n^*+2}$ of slope γ_{n^*+1} close to γ^*, which is a contradiction. ∎

2.3. Reduction to a finite sum of propagating waves

Consider the solution u of the Helmholtz equation in Ω_f defined by (2.6). Then we have the convergent Rayleigh series expansion

$$\begin{aligned} u(x) = &\left(A\exp(i\alpha x_1 - i\beta x_2) + \sum_{n\in P} A_n \exp(i(\alpha+n)x_1 + i\beta_n x_2)\right) \\ &+ \left(\sum_{n\in\mathbb{Z}\backslash P} A_n \exp(i(n+\alpha)x_1 + i\beta_n x_2)\right) := v + w\,,\ x_2 > \max(f)\,, \end{aligned} \tag{2.8}$$

where $A = 1$ and P denotes the finite set $\{n \in \mathbb{Z} : \beta_n \in \mathbb{R}\}$. Note that $-i\beta_n \geq C > 0$ for all $n \in \mathbb{Z}\backslash P$ and $\beta_n \sim |n|i$ as $|n| \to \infty$.

From Lemma 2 we have a ray $S \subset \Omega_f$ such that $u|_S = 0$. The next lemma shows that then all Rayleigh coefficients of the evanescent modes must vanish.

Lemma 3. *Suppose there exists a ray S such that the conditions (2.7) hold. Then the Rayleigh expansion (2.8) of u satisfies*

$$A_n = 0 \quad \textit{for all} \quad n \in \mathbb{Z}\backslash P\,. \tag{2.9}$$

Proof. Since a translation of the x coordinates only amounts to different coefficients in the expansion (2.8) with $A \neq 0$, we can assume that $S = \{x_2 = ax_1 : x_1 > 0\}$ for some $a \in \mathbb{R}$, $a \neq 0$. Consider the case $a > 0$, for example.

Then, for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ sufficiently large so that

$$|w(t, at)| \leq \varepsilon\,, \quad t \geq N\,.$$

From the relation $u|_S = 0$ we then have

$$|v(t, at)| \leq \varepsilon\,, \quad t \geq N\,. \tag{2.10}$$

The finite sum $v_1(t) := v(t, at)$ is a special case of an almost periodic function on $\mathbb{R}$. Therefore the relation

$$\max\nolimits_{t\in\mathbb{R}}|v_1(t)| = \limsup\nolimits_{t\to+\infty}|v_1(t)|$$

holds (see, e.g., [3, p. 407]), which together with (2.10) implies $v_1 = 0$, hence $v|_S = 0$. From (2.7) and (2.8) we then obtain $w|_S = 0$.

It remains to verify that the last relation implies (2.9). Let

$$\beta^* = \min\{-i\beta_n : n \in \mathbb{Z}\backslash P\}\,.$$

There are at most two indices n, say n_0, n_1, such that $-i\beta_n = \beta^*$ and $n_1+\alpha = -(n_0+\alpha)$. We obtain the convergent series

$$\begin{aligned}&\exp(\beta^* at)w(t,at) = A_{n_0}\exp(i(n_0+\alpha)t)\\ &+ A_{n_1}\exp(-i(n_0+\alpha)t) + \sum_{\substack{n\in\mathbb{Z}\setminus P\\ n\neq n_0,n_1}} A_n\exp(i(n+\alpha)t + (i\beta_n+\beta^*)at) = 0 \qquad (2.11)\end{aligned}$$

for t sufficiently large. Since the infinite sum in (2.11) tends to zero as $t\to+\infty$, the sum of the first two terms on the right hand–side becomes arbitrarily small if t is large enough. This implies $A_{n_0} = A_{n_1} = 0$, using the above argument for almost periodic functions and the fact that the functions $\exp(i(n_0+\alpha)t)$, $\exp(-i(n_0+\alpha)t)$ are linearly independent. If there is only one index $n = n_0$ with $-i\beta_n = \beta^*$, then $A_{n_0} = 0$, of course, follows immediately. Repeating this reasoning, we obtain successively $A_n = 0$, $n\in\mathbb{Z}\setminus P$. ∎

2.4. End of proof of the theorem

From (2.8) and Lemma 3 we obtain the finite sum of plane waves,

$$v(x) = A\exp(i\alpha x_1 - i\beta x_2) + \sum_{n\in P} A_n\exp(i(\alpha+n)x_1 + i\beta_n x_2),\quad A\neq 0, \qquad (2.12)$$

so that $v(x)$ is analytically extended to any $x\in\mathbb{R}^2$ and satisfies the conditions $v|_L = 0$ by (2.6), where L is any straight line extending a segment of $\Lambda_{f_1}\cup\Lambda_{f_2}$. Here we note that v solves the Helmholtz equation $(\Delta+k^2)v = 0$ and is real-analytic in $\mathbb{R}^2$. Therefore, since both f_1 and f_2 cannot be constant, we find two lines L_0 and L_φ, $\varphi\in(0,\pi/2]$, so that

$$v|_{L_0} = v|_{L_\varphi} = 0\,.$$

Here and in the following L_ψ denotes the line of polar angle ψ with respect to L_0. Without loss of generality, we take the intersection point of L_0 and L_φ as the origin O. After repeated application of the reflection argument of Lemma 1, only the following two cases may occur:
(i) Let $\varphi = \lambda\pi$, where $\lambda\in(0,1/2)$ is irrational. Then we have

$$v|_{L_{k\varphi}} = 0\,,\ k\in\mathbb{N} \qquad (2.13)$$

where the directions of $L_{k\varphi}$ with respect to L_0 are dense in $[0,2\pi)$.
(ii) The case of rational λ leads to the relations

$$v|_{L_{k\pi/N}} = 0\,,\ k = 0,1,\ldots,N-1\,,\quad \text{for some}\quad N\in\mathbb{N},\ N\geq 2 \qquad (2.14)$$

by the odd extensions by Lemma 1.

Proof in case (i). Let $\alpha\geq 0$. By (2.13) we find $a>1$ such that $v(t,-at) = 0\,,\ t\in\mathbb{R}$. There exists $n_0\in P$ such that

$$(i\alpha + i\beta a) = i(n_0+\alpha) - i\beta_{n_0}a. \qquad (2.15)$$

Otherwise the first term on the right hand side of (2.12) is linearly independent of the second term and the equality (2.12) on $\{x_2 = -ax_1\}$ and $A \neq 0$ lead us to a contradiction.

The relation (2.15) implies

$$a(k^2 - \alpha^2)^{1/2} + a(k^2 - (n_0 + \alpha)^2)^{1/2} = n_0\,,$$

hence either $n_0 = 0$, $k = \alpha$, or $n_0 > 0$. The first case is excluded by the condition $|\theta| < \pi/2$. For $n_0 > 0$, we obtain

$$n_0 - a\beta = a(\beta^2 - n_0^2 - 2\alpha n_0)^{1/2}\,.$$

Since $\beta^2 \geq n_0^2 + 2\alpha n_0^2 \geq n_0^2$, we have $\beta \geq |n_0|$ so that $n_0 - a\beta \leq n_0 - an_0 < 0$, which is a contradiction. Analogously, for $\alpha \leq 0$, we obtain a contradiction with $v(t, at) = 0\,,\ t \in \mathbb{R}$, for some $a > 1$. Thus the case (i) always leads to a contradiction, proving uniqueness of (IP).

Proof in case (ii). Let $\Omega_k\,,\ k = 0, 1, \ldots, N$, be the open two-sided sector of angle π/N lying between the lines $L_{k\pi/N}$ and $L_{(k+1)\pi/N}$, with the convention that $\Omega_N = \Omega_0$. We note that

$$\overline{\cup_{k=0}^{N-1}\Omega_k} = \mathbb{R}^2,$$

since we consider two-sided sectors. For a point $x \in \Omega_k$, let $\mathcal{R}_k x \in \Omega_{k+1}$ denote its reflection with respect to the line $L_{(k+1)\pi/N}$. From the reflection argument of Lemma 1 (or rather its proof), we now deduce the relation

$$v(\mathcal{R}_0 x) = -v(x)\,, \quad x \in \Omega_1\,. \tag{2.16}$$

In fact, the function v^* defined by

$$v^*(x) = v(x)\,,\ x \in \Omega_0\,; \quad v^*(x) = -v(\mathcal{R}_0^{-1}x)\,,\ x \in \Omega_1$$

solves the homogeneous Helmholtz equation in $\Omega_0 \cup \Omega_1$ and vanishes on the line $L_{\pi/N}$, and the jump of its normal derivative across $L_{\pi/N}$ is zero. Hence v^* solves the homogeneous Helmholtz equation in $\Omega_0 \cup L_{\pi/N} \cup \Omega_1$, and (2.16) follows by unique continuation.

Using the relations (2.14), noticing that

$$\mathcal{R}x := \mathcal{R}_{N-1}\mathcal{R}_{N-2}\ldots\mathcal{R}_0 x = -x\,, \quad x \in \Omega_0$$

and applying the above reflection argument N times, we then obtain

$$v(x) = (-1)^N v(\mathcal{R}x) = (-1)^N v(-x)\,, \quad x \in \Omega_0\,.$$

By unique continuation, the last relation holds for all $x \in \mathbb{R}^2$ and takes the form

$$\begin{aligned} &A\exp(i\alpha x_1 - i\beta x_2) + \sum_{n\in P} A_n \exp(i(n+\alpha)x_1 + i\beta_n x_2) \\ &= (-1)^N A\exp(-i\alpha x_1 + i\beta x_2) + (-1)^N \sum_{n\in P} A_n \exp(-i(n+\alpha)x_1 - i\beta_n x_2)\,, \end{aligned} \tag{2.17}$$

implying

$$(-1)^N A\exp(-i\alpha x_1 + i\beta x_2) = A_{n_0}\exp(i(n_0+\alpha)x_1 + i\beta_{n_0}x_2) \tag{2.18}$$

for some $n_0 \in P$ and all $x \in \mathbb{R}^2$. Now $A \neq 0$ and (2.18) give the equalities $n_0 = -2\alpha$, $\beta_{n_0} = \beta$ and $A(-1)^N = A_{n_0}$. Therefore (2.17) implies

$$\sum_{n \in P \setminus \{n_0\}} A_n \exp(i(n+\alpha)x_1 + i\beta_n x_2) = \sum_{n \in P \setminus \{n_0\}} A_n \exp(-i(n+\alpha)x_1 - i\beta_n x_2)$$

for $(x_1, x_2) \in \mathbb{R}^2$. Then, for $n \in P \setminus \{n_0\}$, we must have $\beta_n = 0$ (i.e., a Rayleigh frequency would occur contradicting (1.5)) or $A_n = 0$ in (2.17). Therefore, v takes the form

$$v(x) = A \exp(i\alpha x_1 - i\beta x_2) + A_{n_0} \exp(-i\alpha x_1 + i\beta x_2)\,, \; A \neq 0$$

and vanishes on two different lines passing through the origin, which is also impossible. This concludes the proof of the theorem. ∎

Remark 2. The proof of Lemma 2 can be easily extended to the case that vertical lines are present in the profiles Λ_{f_1} and Λ_{f_2}. In that case we always find a ray $S \subset \Omega_f$ with $u|_S = 0$ which may be parallel to x_1 or x_2 direction. If S is parallel to the x_2 axis, then a contradiction to the presence of the incoming wave can be directly obtained using the relation $v|_S = 0$. For S parallel to x_1 direction, it follows easily by taking the Fourier expansion on S that u (after a translation of the x coordinates) reduces to the expression

$$u(x) = A \exp(i\alpha x_1 - i\beta x_2) + A_0 \exp(i\alpha x_1 + i\beta x_2)\,, \; A \neq 0$$

which must vanish on the x_1 axis and another line passing through the origin, leading to a contradiction again. This proves our global uniqueness result within a more general class of polygonal grating profiles (which are not necessarily defined by the graph of a piecewise linear function).

Note that we can obtain the uniqueness in problem (IP) without excluding the Rayleigh frequencies (condition (1.5)) if there exists a ray S parallel to one of the coordinate axes on which the total field vanishes.

Remark 3. The case of transverse magnetic polarization leads to the Neumann problem, where the Dirichlet condition in (1.1) is replaced by the Neumann condition

$$\partial_\nu u = 0 \quad \text{on} \quad \Lambda_f$$

and ∂_ν denotes the normal derivative. Then the global uniqueness results of the theorem and Remark 2 carry over to the inverse Neumann problem if we require the additional condition that no segment of the profiles Λ_{f_1} , Λ_{f_2} is parallel to the direction θ of the incident wave. If the latter condition is not satisfied, then one obtains further non-uniqueness examples for the reconstruction problem. The proof is essentially parallel to that for the inverse Dirichlet problem, but needs some modifications related with the Neumann boundary condition. We do not present the details here and only note that the odd extension used in Subsections 2.1 and 2.4 has to be replaced by even extension.

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