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EXISTENCE OF NAVIER-STOKES FLOW IN THE HALF SPACE WITH NON-DECAYING INITIAL DATA

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Abstract. In this paper consider the Dirichlet problem of the Navier-Stokes equation in the half space. We show the existence and the uniqueness of the mild solution when the initial data can be represented as $u_0 = \Lambda_{\alpha,j} v_0$ with a function v_0 coupled with the pseudo-differential operator $\Lambda_{\alpha,j} = \partial_j(-\Delta')^{-\alpha/2}$, where Δ' denotes the Laplace operator in the tangential space $\mathbb{R}^{n-1}$.

§1 Introduction

We consider the initial-boundary problem of the Navier-Stokes equation in the half space $\mathbb{R}^n_+ = \{x = (x_1, \cdots, x_n) \in \mathbb{R}^n : x_n > 0\}$ $(n \geq 2)$:

$$
\begin{aligned}
(1.1) \qquad u_t - \Delta u + u \cdot \nabla u + \nabla p = 0,\ \mathrm{div} u = 0 \text{ in } \mathbb{R}^n_+ \times (0, \infty),\\
u = 0 \text{ on } \partial\mathbb{R}^n_+ \times (0, \infty),\\
u = u_0 \text{ at } t = 0.
\end{aligned}
$$

Here $u = (u^1, \ldots, u^n)$ is the unknown velocity field and p is the unknown pressure field. The initial data u_0 is assumed to satisfy the *compatibility condition* : $\mathrm{div} u_0 = 0$ in $\mathbb{R}^n_+$ and the normal component of u_0 equals zero on $\partial\mathbb{R}^n_+ = \{x_n = 0\}$.

To solve the equation, we consider the following integral equation equivalent to (1.1);

$$
(1.2) \qquad u(t, x) = [e^{-tA} u_0](x) - \int_0^t e^{-(t-s)A} P \nabla (u \otimes u)(s, x) ds,
$$

where $u \otimes u = (u^i u^j)_{1 \leq i,j \leq n}$ is a tensor matrix, P is the projection operator from $L^p(\mathbb{R}^n_+)$ onto $L^p_\sigma(\mathbb{R}^n_+) = \{u \in L^p(\mathbb{R}^n_+); \mathrm{div} u = 0 \text{ in } \mathbb{R}^n_+,\ u^n|_{\partial\mathbb{R}^n_+} = 0\}$, and A is the

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Stokes operator that generates the semigroup e^{-tA}, i.e., $v(t) = e^{-tA}u_0$ solves the non-stationary Stokes problem

$$
\begin{aligned}
&v_t - \Delta v + \nabla q = 0, \ \mathrm{div} u = 0 \text{ in } \mathbb{R}^n_+ \times (0,\infty), \\
&v = 0 \text{ on } \partial\mathbb{R}^n_+ \times (0,\infty), \\
&v = u_0 \text{ at } t = 0.
\end{aligned} \tag{1.3}
$$

The purpose of this paper is to show existence of local smooth solutions with non-decaying initial data at infinity. To this goal, we need estimates of each term in (1.2) in L^∞. First, we establish the estimate of spacial structure $e^{-tA}P\partial_j$ related to the non-linear term. Shimizu [15] showed that the first derivatives of the solution of (1.3) decays in L^∞ like $t^{-1/2}$, that is,

$$
\|\nabla v(t)\|_{L^\infty(\mathbb{R}^n_+)} \le Ct^{-1/2}\|u_0\|_{L^\infty(\mathbb{R}^n_+)} \text{ for } t > 0. \tag{1.4}
$$

Later on, Shimizu [16] proved that the L^∞-norm in right-hand side of (1.4) can be replaced by the BMO-norm. However in our problem, we need L^∞-estimates of the solution of (1.3) with $u_0 = P\partial_j v_0$ because (1.2) has the structure $e^{-(t-s)A}P\nabla(u \otimes u)$ of the non-linear term. In this paper, we show $L^\infty - \mathrm{BMO}$ estimate of the solution of (1.3) with $u_0 = Q_j v_0$, where $Q_j = ((Q_j\cdot)', (Q_j\cdot)^n)$ is the combination of the Helmholtz projection with the first-order derivative in space, which can be expressed as

$$
(Q_j u)' = \begin{cases} \partial_j \tilde{u'} + & R'(R' \cdot \partial_j \tilde{u'}) + R'(R_n \partial_j \bar{u^n}) \\ & \text{for } 1 \le j \le n-1, \\ \partial_n \bar{u'} + & R'(R' \cdot \partial_n \bar{u'}) + R'(R_n \partial_n \tilde{u^n}) \\ & \text{for } j = n, \end{cases}
$$

$$
(Q_j u)^n = \begin{cases} \partial_j \tilde{u^n} + & R_n(R' \cdot \partial_j \bar{u'}) + R_n^2 \partial_j \tilde{u^n} \\ & \text{for } 1 \le j \le n-1, \\ \partial_n \bar{u^n} + & R_n(R' \cdot \partial_n \tilde{u'}) + R_n^2 \partial_n \bar{u^n} \\ & \text{for } j = n. \end{cases}
$$

Here $R_j = \partial_j(-\Delta)^{-1/2}$ is the Riesz transform, $R' = (R_1, \dots, R_{n-1})$, and $\tilde{u}$ (resp. $\bar{u}$) denotes the odd (resp. even) extension of u for $x_n < 0$;

$$
\tilde{u}(x) = \begin{cases} u(x) & \text{for } x_n > 0, \\ -u(x', -x_n) & \text{for } x_n < 0, \end{cases}
$$

$$
\bar{u}(x) = \begin{cases} u(x) & \text{for } x_n > 0, \\ u(x', -x_n) & \text{for } x_n < 0. \end{cases}
$$

Roughly speaking, we see that Q_j is an extension of $P\partial_j$ to the whole space $\mathbb{R}^n$ (c.f. Proposition 2.2).

Our first result now reads as follows.

Theorem 1.1. *Let $1 \leq j \leq n$. Assume that $v_0 = (v_0^1, \ldots, v_0^n)$ is in* $\mathrm{BMO}(\mathbb{R}^n_+)^n$. *Then there exists a function $U_j = U_j(t, x)$ such that $U_j(t) \in L^\infty(\mathbb{R}^n)$ for all $t > 0$ and $U_j|_{\mathbb{R}^n_+}$ gives the solution of the Stokes equation* (1.3) *in $\mathbb{R}^n_+$ with the initial data $u_0 = Q_j v_0$. Moreover, U_j fulfills the estimate*

$$\Big|\int_{\mathbb{R}^n} U_j(t,x)\cdot\phi(x)dx\Big| \leq Ct^{-1/2}[v_0]_{\mathrm{BMO}}\|\phi\|_{L^1(\mathbb{R}^n)},\ t>0 \tag{1.5}$$

for all ϕ in $C_0^\infty(\mathbb{R}^n)$, where C is a constant independent of ϕ and v_0.

Here $[\cdot]_{\mathrm{BMO}}$ denotes the semi-norm of BMO-space in $\mathbb{R}^n_+$.

Next, we consider the estimate of the linear term in (1.2). Ukai [18] showed that for any p and q with $1 < q \leq p < \infty$,

$$\|e^{-tA}u_0\|_{L^p(\mathbb{R}^n_+)} \leq C_{p,q}t^{-n(q^{-1}-p^{-1})/2}\|u_0\|_{L^q(\mathbb{R}^n_+)} \tag{1.6a}$$

$$\|\nabla e^{-tA}u_0\|_{L^p(\mathbb{R}^n_+)} \leq C_{p,q}t^{-n(q^{-1}-p^{-1})/2-1/2}\|u_0\|_{L^q(\mathbb{R}^n_+)} \tag{1.6b}$$

holds for all $u_0 \in L^q(\mathbb{R}^n_+)$ and for all $t > 0$. Ukai [18] established a representation formula to the solution of (1.3) in terms of the Riesz transform. Applying L^p-boundedness of the Riesz transform, he obtained (1.6). We have difficulty to show (1.6) for $p = q = \infty$, because the Riesz transform is not bounded in L^∞. However, Shimizu [15] showed (1.6b) for $p = q = \infty$, which is equivalent to (1.4). For such a case, the first-order derivatives make the spacial decay of solution more rapidly at infinity and lead the boundedness of the integral kernels in the Hardy space $\mathcal{H}^1$. So we can expect that when $0 < \alpha < 1$, the α-th order derivatives of solution of (1.3) will decay in L^∞ like $t^{-\alpha/2}u_0$. In this paper, we show the $L^\infty-\mathrm{BMO}$ estimate of the solution when the initial data can be represented as $u_0 = \Lambda_{\alpha,j}v_0$ with a function v_0 coupled with the pseudo-differential operator $\Lambda_{\alpha,j} = \partial_j(-\Delta')^{-\alpha/2}$, where Δ' denotes the Laplace operator in the tangential space $\mathbb{R}^{n-1}$.

Indeed, we have the following result.

Theorem 1.2. *Let $1 \leq j \leq n-1$. Assume that $v_0 = (v_0^1, \ldots, v_0^n)$ is in* $\mathrm{BMO}(\mathbb{R}^n_+)^n$ *and satisfies* $\mathrm{div}v_0 = 0$. *Then there exists a function $V_{\alpha,j} = V_{\alpha,j}(t, x)$ such that $V_{\alpha,j}(t) \in L^\infty(\mathbb{R}^n)$ for all $t > 0$ and $V_{\alpha,j}|_{\mathbb{R}^n_+}$ gives to the solution of the Stokes equation* (1.3) *in $\mathbb{R}^n_+$ with the initial data $u_0 = \Lambda_{\alpha,j}v_0$. Moreover, $V_{\alpha,j}$ satisfies the estimate*

$$\Big|\int_{\mathbb{R}^n} V_{\alpha,j}(t,x)\cdot\phi(x)dx\Big| \leq Ct^{-(1-\alpha)/2}[v_0]_{\mathrm{BMO}}\|\phi\|_{L^1(\mathbb{R}^n)},\ t>0 \tag{1.7}$$

for all ϕ in $C_0^\infty(\mathbb{R}^n)$, where C is a constant independent of ϕ and v_0.

As an application of Theorem 1.1 and 1.2, we can solve the original non-liniar equation (1.1). To this end, we consider the following integral equation which is equivalent to (1.2):

$$u(t,x) = [e^{-tA}u_0](x) - \sum_{i=1}^{n}\int_0^t e^{-(t-s)A}Q_i(u_iu)(s,x)ds. \tag{1.8}$$

This formulation was first proposed by Shimizu [15] and we shall explain more precisely in Proposition 2.2. To solve (1.8), we define a function space $BC_{q,T}$, $0 < q < \infty$ by

$$\begin{aligned} &BC_{q,T} \\ =&\{f \in C((0,T); L^\infty(\mathbb{R}^n_+)); \|f\|_{BC_{q,T}} := \sup_{0<t<T} t^q \|f\|_{L^\infty(\mathbb{R}^n_+)} < \infty\}. \end{aligned}$$

Based on the estimates (1.5) and (1.7), we have the following existence theorem of the time-local smooth solution for (1.8) in $BC_{(1-\alpha)/2,T}$:

Theorem 1.3. *Assume that v_0 is in* $\mathrm{BMO}(R^n_+)$. *Let $u_0 = \Lambda_{\alpha,j} v_0$, $j = 1, \dots, n-1$, for $0 < \alpha < 1$. Then there exist $T > 0$ and a unique solution $u \in BC_{(1-\alpha)/2,T}$ of* (1.8). *The existence time interval can be written as $T = C/[v_0]^{2/\alpha}_{\mathrm{BMO}}$ with C depending only on α.*

This result may be regarded as treatment of the maginal case of Cannone [4] to the half space. In [4], he showed local existence of smooth solutions in the homogeneous Besov space $\dot{B}_p^{-1+n/p,\infty}(\mathbb{R}^n)$. In fact, the above theorem deals with the case when $p = \infty$.

The idea of this paper is to apply the modified version of Ukai's formula to $Q_j v_0$ or $\Lambda_{\alpha,j} v_0$ obtained by Shimizu [15]. By the duality arguement, it is sufficient to estimate the test function coupled with the adjoint operator of solution in the Hardy space $\mathcal{H}^1$. In [15], the integral kernel involving the square root of the tangential Laplacian $\Lambda := (-\sum_{i=1}^{-n} \partial^2/\partial x_i^2)^{1/2}$ does not contain the characteristic function $\chi_+ := \chi_{\{x_n>0\}}$, which causes singularities on $\{x_n = 0\}$. However in our case, we have to deal with the term which contain χ_+ with the pseudo-differential operator $\Lambda_{\alpha,j}$. By investigating the integral kernels of $(-\Delta')^{-\alpha/2}$ and $(-\Delta)^{-1}$ precisely, we are able to contlol the singluler behavior of the kernel functions in $\mathcal{H}^1$.

The proofs of our theorems are divided to five sections. In Section 2, we introduce our main tools and represent the solution formula by Shimizu [15] which is a refined version of Ukai[18]'s. We can eliminate some of these singularities from Ukai's formula by extending solutions u to $\{x_n < 0\}$ as the odd function, so that the integral kernels involving Λ have no singularities on $x_n = 0$. We also show a similar property to the adjoint operator of the solution. In Section 3, we define the Hardy space and the BMO space in $\mathbb{R}^n_+$. We also recall the duality relation between $\mathcal{H}^1$ and BMO. In Section 4, we prove Theorem 1.1 and give the estimates related to the non-linear part. The difficulty appears when we estimate the kernel functions which contain χ_+. Since χ_+ and ∂_n do not commute, we cannot apply the well-known identity $\sum_{j=1}^n R_j^2 = -I$ for the Riesz transforms. However, if we divide the domain of integration into two parts according to the distance from the singularity of the integral kernel of $(-\Delta)^{-1}$, then we can handle each integration in $\mathcal{H}^1$. This procedure then provides an $\mathcal{H}^1$-boundedness for integral kernels which are necessary for the L^∞-BMO estimate of the non-linear part. In section 5, we prove Theorem 1.2 and give the estimates of linear part. By duality arguement, it is sufficient to show the $\mathcal{H}^1$-estimate of the integral kernels in the representation formula. The difficulty of our problem is that the kernel functions which have singularities on $\{x_n = 0\}$ contain both $(-\Delta)^{-1}$ and $\Lambda_{\alpha,j}$, which are represented as the integral operators. However, if we divide the domain of integration into two parts according to the distance from the singularity of the integral kernel of $(-\Delta)^{-1}$ and $\Lambda^{-\alpha}$, then we can handle each integration in $\mathcal{H}^1$. This procedure then provide $\mathcal{H}^1$-estimates for integral kernels which are necessary for the L^∞-BMO estimate of the linear part. Finally in section 6, as an application of our main theorems, we will show the time-local existence of solution to the Navier-Stokes equation in the half space.

§2 Solution formula

In this section, we recall a solution formula of (1.2) by Shimizu [15].

First, we fix some notations. For an n-dimensional vector a, we denote the tangential component $(a_1, \ldots, a_{n-1})$ by $a' \in \mathbb{R}^{n-1}$, so that $a = (a', a_n)$. We set $\partial_j = \partial/\partial x_j$ and let $\nabla' = (\partial_1, \cdots, \partial_{n-1})$. Hereafter, C denotes a positive constant which may differ from one occasion to another.

Let $\mathcal{F}$ be the Fourier transform in $\mathbb{R}^n$

$$\mathcal{F}f(\xi) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x)dx.$$

The Riesz operators R_j $(j = 1, \ldots, n)$, the operator Λ and $\Lambda_{\alpha,j}$ are defined by

$$\begin{aligned}
\mathcal{F}(R_j f)(\xi) &= \frac{i\xi_j}{|\xi|}\mathcal{F}f(\xi),\\
\mathcal{F}(\Lambda f)(\xi) &= |\xi'|\mathcal{F}f(\xi),\\
\mathcal{F}(\Lambda_{\alpha,j} f)(\xi) &= i\xi_j|\xi'|^{-\alpha}\mathcal{F}f(\xi),
\end{aligned}$$

We set $R' = (R_1, \ldots, R_{n-1})$ and $R = (R_1, \cdots, R_n)$.

We also define the operator $E(t)$ and $H(t)$ by

$$\begin{aligned}
[E_t f](x) &= \int_{\mathbb{R}^n_+} \{G_t(x-y) - G_t(x'-y', x_n + y_n)\} f(y)dy,\\
[H_t f](x) &= \int_{\mathbb{R}^n} G_t(x-y) f(y)dy,
\end{aligned}$$

where G_t is the Gauss kernel $G_t(x) = (4\pi t)^{-n/2}e^{-|x|^2/4t}$. Furthermore, we define the operator E_{t+} by

$$[E_{t+}f](x) = \begin{cases} [E_t f](x) & \text{for } x_n > 0,\\ [E_t f](x', -x_n) & \text{for } x_n < 0.\end{cases}$$

Note that $z = E_t f$ solves the heat equation in $\mathbb{R}^n_+$ with zero-Dirichlet boundary condition;

$$\begin{aligned}
z_t - \Delta z &= 0 \text{ in } \mathbb{R}^n_+ \times (0, T),\\
z|_{t=0} &= f,\\
z|_{x_n=0} &\equiv 0. \text{ (resp. } \partial_n z|_{x_n=0} = 0.)
\end{aligned}$$

Moreover note that the functions $[E_t f](x)$ can be defined for all x in $\mathbb{R}^n$.

Let $f(x)$ be a function defined in $\mathbb{R}^n_+$. Then we denote the odd (resp. even) extension of f by $\tilde{f}$ (resp. $\bar{f}$), i.e.

$$\tilde{f}(x) = \begin{cases} f(x) & \text{for } x_n > 0, \\ -f(x', -x_n) & \text{for } x_n < 0, \end{cases}$$
$$\bar{f}(x) = \begin{cases} f(x) & \text{for } x_n > 0, \\ f(x', -x_n) & \text{for } x_n < 0. \end{cases}$$

We define the Helmholtz-type operators $\tilde{P}$, $\bar{P}$ and P as

$$(\bar{P}u)' = \bar{u'} + R'(R' \cdot \bar{u'}) + R'(R_n \tilde{u^n}), \tag{2.1a}$$
$$(\bar{P}u)^n = \bar{u^n} + R_n(R' \cdot \tilde{u'}) + R_n^2 \bar{u^n}, \tag{2.1b}$$
$$(\tilde{P}u)' = \tilde{u'} + R'(R' \cdot \tilde{u'}) + R'(R_n \bar{u^n}), \tag{2.1c}$$
$$(\tilde{P}u)^n = \tilde{u^n} + R_n(R' \cdot \bar{u'}) + R_n^2 \tilde{u^n}, \tag{2.1d}$$
$$Pu = ((\tilde{P}u)', (\tilde{P}u)^n). \tag{2.1e}$$

Moreover, we define the operators Q_j by

$$(Q_j u)' = \begin{cases} \partial_j \tilde{u'} + & R'(R' \cdot \partial_j \tilde{u'}) + R'(R_n \partial_j \bar{u^n}) \\ & \quad \text{for } 1 \leq j \leq n-1, \\ \partial_n \bar{u'} + & R'(R' \cdot \partial_n \bar{u'}) + R'(R_n \partial_n \tilde{u^n}) \\ & \quad \text{for } j = n, \end{cases} \tag{2.2a}$$

$$(Q_j u)^n = \begin{cases} \partial_j \tilde{u^n} + & R_n(R' \cdot \partial_j \bar{u'}) + R_n^2 \partial_j \tilde{u^n} \\ & \quad \text{for } 1 \leq j \leq n-1, \\ \partial_n \bar{u^n} + & R_n(R' \cdot \partial_n \tilde{u'}) + R_n^2 \partial_n \bar{u^n} \\ & \quad \text{for } j = n, \end{cases} \tag{2.2b}$$

$$Q_j u = ((Q_j u)', (Q_j u)^n). \tag{2.2c}$$

We note that Q_j is the combination of P defined in (2.1) with the first-order derivative toward x_j-axis. Finally, we denote the characteristic function of $\{x_n > 0\}$ (resp. $\{x_n < 0\}$) by χ_+ (resp. χ_-), i.e.

$$\chi_+(x_n) = \begin{cases} 1 & \text{for } x_n > 0, \\ 0 & \text{for } x_n < 0, \end{cases}$$
$$\chi_-(x_n) = 1 - \chi_+(x_n).$$

To show our main theorem, we apply the modified Ukai's formula obtained by Shimizu [15]:

Proposition 2.1(Shimizu [15]). *Assume that u_0 is in $L^p(\mathbb{R}^n_+)$, $1 \le p \le \infty$. Let*

(2.3a)
$$\begin{aligned} U^n(t) =& -\Lambda(-\Delta)^{-1}\nabla' \cdot E_t u_0' + \partial_n(-\Delta)^{-1}\nabla' \cdot E_{t+}u_0' \\ & - (-\Delta)^{-1}\Delta' E_{t+}u_0^n - \partial_n(-\Delta)^{-1}\Lambda E_t u_0^n, \end{aligned}$$

(2.3b)
$$\begin{aligned} U'(t) =& E_t u_0' + \Lambda^{-1}\nabla' E_t u_0^n \\ & + \nabla'(-\Delta)^{-1}(\nabla' \cdot E_{t+}u_0') - \partial_n(-\Delta)^{-1}\Lambda^{-1}\nabla'(\nabla' \cdot E_t u_0') \\ & - \nabla'(-\Delta)^{-1}\Lambda E_t u_0^n + \partial_n(-\Delta)^{-1}\nabla' E_{t+}u_0^n. \end{aligned}$$

Then U is a function defined in $\mathbb{R}^n$ and $U|_{\mathbb{R}^n_+}$ satisfies (1.3) *with an initial data u_0.*

Before applying Proposition 2.1 to our problem, we recall the properties of the operator P:

Proposition 2.2.

(1) *If $u = (u^1, \dots, u^n) \in (L^2(\mathbb{R}^n_+))^n$ satisfies* $\mathrm{div}u = 0$ *in $\mathbb{R}^n_+$ and $u^n(x', 0) = 0$ on $\mathbb{R}^{n-1}$, then $Pu = u$ holds in $\mathbb{R}^n_+$.*
(2) *If $p \in L^2_+(\mathbb{R}^n)$ satisfies $p(x', 0) = 0$ on $\mathbb{R}^{n-1}$, then $P\nabla p = 0$ and holds in $\mathbb{R}^n_+$.*
(3)
$$\begin{aligned} \mathrm{div}Pu =& \nabla' \cdot (\tilde{u'} - \bar{u'}) + \partial_n(\tilde{u^n} - \bar{u^n}) \\ & + (R' \cdot R')\nabla \cdot (\tilde{u'} - \bar{u'}) + R_n^2\partial_n(\tilde{u^n} - \bar{u^n}) \end{aligned}$$
holds for $u \in (L^2(\mathbb{R}^n))^n$.

Proposition 2.2 shows that P is an orthogonal projection operator from $(L^2(\mathbb{R}^n_+))^n$ onto $(L^2_\sigma(\mathbb{R}^n_+))^n := \{u \in (L^2(\mathbb{R}^n_+))^n : \mathrm{div}u = 0 \text{ in } \mathbb{R}^n_+, u(x', 0) = 0\}$ and the operator Q_j is well-defined.

Now We set $u_0 = Q_j v_0$. Computing the initial data, we have the following corollary:

Corollary 2.3. *Let $1 \le j \le n$ and assume that v_0 is in $L^p(\mathbb{R}^n_+)$ $(1 \le p \le \infty)$. Let*

(2.4a)
$$\begin{aligned} U_j^n(t) =& -\Lambda(-\Delta)^{-1}\nabla' \cdot E_t(Q_j v_0)' + \partial_n(-\Delta)^{-1}\nabla' \cdot E_{t+}(Q_j v_0)' \\ & - (-\Delta)^{-1}\Delta' E_{t+}(Q_j v_0)^n - \partial_n(-\Delta)^{-1}\Lambda E_t(Q_j v_0)^n, \end{aligned}$$

(2.4b)
$$\begin{aligned} U_j'(t) =& E_t(Q_j v_0)' + \Lambda^{-1}\nabla' E_t(Q_j v_0)^n \\ & + \nabla'(-\Delta)^{-1}\{\nabla' \cdot E_{t+}(Q_j v_0)'\} \\ & - \partial_n(-\Delta)^{-1}\Lambda^{-1}\nabla'\{\nabla' \cdot E_t(Q_j v_0)'\} \\ & - \nabla'(-\Delta)^{-1}\{\Lambda E_t(Q_j v_0)^n\} + \partial_n(-\Delta)^{-1}\nabla' E_{t+}\{(Q_j v_0)^n\}. \end{aligned}$$

Then U_j is a function defined in $\mathbb{R}^n$ and $U_j|_{\mathbb{R}^n_+}$ satisfies (1.3) *with an initial data $u_0 = P_j v_0$.*

By duality arguement, we have the following corollary:

Corollary 2.4. *Let U_j $(1 \leq j \leq n)$ be the function in Corollary 2.3 and let ϕ be in $C_0^\infty(\mathbb{R}^n)^n$.*

(1) *If $1 \leq j \leq n-1$, then*

$$
\begin{aligned}
&\int_{\mathbb{R}^n} U_j(t,x)\cdot\phi(x)dx \\
&= -\int_{\mathbb{R}^n} \Big\{ (\tilde{P}v_0)'(x)\cdot R' R_j \Lambda[H_t\phi^n](x) \\
&\quad +(\tilde{P}v_0)'(x)\cdot[\partial_j H_t(\chi_+\nabla'\partial_n(-\Delta)^{-1}\phi^n)](x) \\
&\quad -(\tilde{P}v_0)'(x)\cdot[\partial_j H_t(\chi_-\nabla'\partial_n(-\Delta)^{-1}\phi^n)](x) \\
&\quad +(\tilde{P}v_0)^n(x)[\partial_j H_t(\chi_+\Delta'(-\Delta)^{-1}\phi^n)](x) \\
&\quad -(\tilde{P}v_0)^n(x)[\partial_j H_t(\chi_-\Delta'(-\Delta)^{-1}\phi^n)](x) \\
&\quad -(\tilde{P}v_0)^n(x)\Lambda R_j R_n[H_t\phi^n](x) \\
&\quad -(\tilde{P}v_0)'(x)\partial_j[H_t\phi'](x) + (\tilde{P}v_0)^n(x)\nabla'\cdot\partial_j\Lambda^{-1}[H_t\phi'](x) \\
&\quad +(\tilde{P}v_0)'(x)\cdot[\partial_j H_t\{\chi_+\nabla'(-\Delta)^{-1}(\nabla'\cdot\phi')\}](x) \\
&\quad -(\tilde{P}v_0)'(x)\cdot[\partial_j H_t\{\chi_-\nabla'(-\Delta)^{-1}(\nabla'\cdot\phi')\}](x) \\
&\quad +(\tilde{P}v_0)'(x)\cdot\nabla'(R_j R_n\Lambda^{-1}\nabla'\cdot[H_t\phi'](x) \\
&\quad +(\tilde{P}v_0)^n(x)\Lambda R_j R'\cdot[H_t\phi'](x) \\
&\quad +(\tilde{P}v_0)^n(x)[\partial_j H_t\{\chi_+\nabla'\cdot(\nabla'\cdot(-\Delta)^{-1}\partial_n\phi')\}](x) \\
&\quad -(\tilde{P}v_0)^n(x)[\partial_j H_t\{\chi_-\nabla'\cdot(\nabla'\cdot(-\Delta)^{-1}\partial_n\phi')\}](x)\Big\} dx.
\end{aligned}
\tag{2.5a}
$$

(2) *If $j = n$, then*

$$
\begin{aligned}
&\int_{\mathbb{R}^n} U_j(t,x)\cdot\phi(x)dx \\
&= -\int_{\mathbb{R}^n} \Big\{ (\bar{P}v_0)'(x)\cdot R' R_n \Lambda[H_t\phi^n](x) \\
&\quad +(\bar{P}v_0)'(x)\cdot[\partial_n H_t(\chi_+\nabla'\partial_n(-\Delta)^{-1}\phi^n)](x) \\
&\quad -(\bar{P}v_0)'(x)\cdot[\partial_n H_t(\chi_-\nabla'\partial_n(-\Delta)^{-1}\phi^n)](x) \\
&\quad +(\bar{P}v_0)^n(x)[\partial_n H_t(\chi_+\Delta'(-\Delta)^{-1}\phi^n)](x) \\
&\quad -(\bar{P}v_0)^n(x)[\partial_n H_t(\chi_-\Delta'(-\Delta)^{-1}\phi^n)](x) \\
&\quad -(\bar{P}v_0)^n(x)\Lambda R_n^2[H_t\phi^n](x) \\
&\quad -(\bar{P}v_0)'(x)\partial_n[H_t\phi'](x) - (\tilde{P}v_0)'(x)\cdot\nabla'(\nabla'\cdot\Lambda^{-1}[H_t\phi'](x) \\
&\quad +(\bar{P}v_0)'(x)\cdot[\partial_n H_t\{\chi_+\nabla'(-\Delta)^{-1}(\nabla'\cdot\phi')\}](x) \\
&\quad -(\bar{P}v_0)'(x)\cdot[\partial_n H_t\{\chi_-\nabla'(-\Delta)^{-1}(\nabla'\cdot\phi')\}](x)
\end{aligned}
\tag{2.5b}
$$

$$
\begin{aligned}
&+(\bar{P}v_0)'(x)\cdot\nabla'(R_n^2\Lambda^{-1}\nabla'\cdot[H_t\phi'](x)\\
&+(\bar{P}v_0)^n(x)\Lambda R_n R'\cdot[H_t\phi'](x)\\
&+(\bar{P}v_0)^n(x)[\partial_n H_t\{\chi_+\nabla'\cdot(\nabla'\cdot(-\Delta)^{-1}\partial_n\phi')\}](x)\\
&-(\bar{P}v_0)^n(x)[\partial_n H_t\{\chi_-\nabla'\cdot(\nabla'\cdot(-\Delta)^{-1}\partial_n\phi')\}](x)\Big\}dx.
\end{aligned}
$$

Next we set $u_0=\Lambda_{\alpha,j}v_0$. Computing the initial data carefully, we have the following corollary:

Corollary 2.5. *Let $1\le j\le n-1$ and assume that v_0 is in $L^p(\mathbb{R}^n_+)$, $1\le p\le\infty$. Let*

(2.6a)
$$
\begin{aligned}
V^n_{\alpha,j}(t)=&-\Lambda^{1-\alpha}R_jR'\cdot E_tv_0'+\partial_n(-\Delta)^{-1}\nabla'\cdot E_{t+}\Lambda_{\alpha,j}v_0'\\
&-(-\Delta)^{-1}\Delta'E_{t+}\Lambda_{\alpha,j}v_0^n-\Lambda^{1-\alpha}R_jR_nE_tv_0^n,
\end{aligned}
$$

(2.6b)
$$
\begin{aligned}
V'_{\alpha,j}(t)=&\Lambda_{\alpha,j}E_tv_0'+\Lambda_{\alpha+1,j}\nabla'E_tv_0^n\\
&+\nabla'(-\Delta)^{-1}(\nabla'\cdot E_{t+}\Lambda_{\alpha,j}v_0')-R_nR_j\Lambda^{-\alpha-1}\nabla'(\nabla'\cdot E_tv_0')\\
&-R'R_j\Lambda^{-\alpha+1}E_tv_0^n+\partial_n(-\Delta)^{-1}\nabla'E_{t+}\Lambda_{\alpha,j}v_0^n.
\end{aligned}
$$

Then $V_{\alpha,j}$ is a function defined in $\mathbb{R}^n$ and $V_{-\alpha,j}|_{\mathbb{R}^n_+}$ satisfies (1.3) *with an initial data $u_0=\Lambda_{-\alpha,j}v_0$.*

By duality arguement, we have the following corollary:

Corollary 2.6. *Let $V_{\alpha,j}$ $(1\le j\le n-1)$ be the function in Corollary 2.6 and let ϕ be in $C_0^\infty(\mathbb{R}^n)^n$. Then*

(2.7)
$$
\begin{aligned}
&\int_{\mathbb{R}^n}V_{\alpha,j}(t,x)\cdot\phi(x)dx\\
=&-\int_{\mathbb{R}^n}\Big\{\tilde{v_0'}(x)\cdot R'R_j\Lambda^{1-\alpha}[H_t\phi^n](x)\\
&+\tilde{v_0'}(x)\cdot[\Lambda_{\alpha,j}H_t(\chi_+\nabla'\partial_n(-\Delta)^{-1}\phi^n)](x)\\
&-\tilde{v_0'}(x)\cdot[\Lambda_{\alpha,j}H_t(\chi_-\nabla'\partial_n(-\Delta)^{-1}\phi^n)](x)\\
&+\tilde{v_0^n}(x)[\Lambda_{\alpha,j}H_t(\chi_+\Delta'(-\Delta)^{-1}\phi^n)](x)\\
&-\tilde{v_0^n}(x)[\Lambda_{\alpha,j}H_t(\chi_-\Delta'(-\Delta)^{-1}\phi^n)](x)\\
&-\tilde{v_0^n}(x)\Lambda^{1-\alpha}R_jR_n[H_t\phi^n](x)\\
&-\tilde{v_0'}(x)\Lambda_{\alpha,j}[H_t\phi'](x)+\tilde{v_0^n}(x)\nabla'\cdot\Lambda_{\alpha+1,j}[H_t\phi'](x)\\
&+\tilde{v_0'}(x)\cdot[\Lambda_{\alpha,j}H_t\{\chi_+\nabla'(-\Delta)^{-1}(\nabla'\cdot\phi')\}](x)\\
&-\tilde{v_0'}(x)\cdot[\Lambda_{\alpha,j}H_t\{\chi_-\nabla'(-\Delta)^{-1}(\nabla'\cdot\phi')\}](x)\\
&+\tilde{v_0'}(x)\cdot\nabla'(R_jR_n\Lambda^{-\alpha}\nabla'\cdot[H_t\phi'](x)
\end{aligned}
$$

$$
\begin{aligned}
&+\tilde{v_0^n}(x)\Lambda^{1-\alpha}R_jR'\cdot[H_t\phi'](x)\\
&+\tilde{v_0^n}(x)[\Lambda_{\alpha,j}H_t\{\chi_+\nabla'\cdot(\nabla'\cdot(-\Delta)^{-1}\partial_n\phi')\}](x)\\
&-\tilde{v_0^n}(x)[\Lambda_{\alpha,j}H_t\{\chi_-\nabla'\cdot(\nabla'\cdot(-\Delta)^{-1}\partial_n\phi')\}](x)\Big\}dx.
\end{aligned}
$$

§3 Study of bouded mean oscillation spaces

In this section, we introduce two function spaces that appear in our theorem and proof. First, we introduce the Hardy space $\mathcal{H}^1$ that is a subspace of L^1.

Definition 3.1. A function $f \in L^1(\mathbb{R}^n)$ belongs to the Hardy space $\mathcal{H}^1 = \mathcal{H}^1(\mathbb{R}^n)$ if

$$
f^+(x) = \sup_{s>0} |G_s * f(x)| \in L^1(\mathbb{R}^n), \tag{3.1}
$$

where the symbol $*$ denotes the convolution with respect to the space variable x. The norm of $f \in \mathcal{H}^1(\mathbb{R}^n)$ is defined by

$$
\|f\|_{\mathcal{H}^1} := \|f^+\|_{L^1(\mathbb{R}^n)} \tag{3.2}
$$

Next, we define the space of "Bounded Mean Oscillation" BMO.

Definition 3.2. A function g belongs to the space of bounded mean oscillation BMO if $g \in L^1_{loc}(\mathbb{R}^n)$ and

$$
[g]_{\mathrm{BMO}} = \sup_{Q:\text{ cube}} \frac{1}{|Q|}\int_Q |g(x) - g_Q|dx < \infty, \tag{3.3}
$$

where $|Q|$ means the lebesgue measure of Q and where g_Q means the mean of g on Q such that

$$
g_Q = \frac{1}{|Q|}\int_Q g(x)dx. \tag{3.4}
$$

Note that $[\cdot]_{\mathrm{BMO}}$ is semi-norm because $[C]_{\mathrm{BMO}} = 0$ for any constant function C. So we usually consider the quotient space BMO/$\mathbb{R}$.

The definition of BMO seems to be complicated to apply. However, we do not have to use that definition directly because we have the duality characterization that is easy to apply for our problem.

Proposition 3.3(Fefferman-Stein[8]**).** *Assume that $f \in \mathcal{H}^1$ and $g \in \mathrm{BMO}$. Then*

$$
\Big|\int_{\mathbb{R}^n} f(x)g(x)dx\Big| \le C\|f\|_{\mathcal{H}^1}[g]_{\mathrm{BMO}}. \tag{3.5}
$$

Corollary 3.4. *Assume that* $f \in \mathcal{H}^1$ *and* $g \in \mathrm{BMO}$. *Then the convolution function* $f * g$ *is in* L^∞ *and*

$$\|f * g\|_{L^\infty} \le C\|f\|_{\mathcal{H}^1}[g]_{\mathrm{BMO}}. \tag{3.6}$$

Moreover, we note an estimate introduced by Giga-Matsui-Shimizu[9].

Lemma 3.5(Giga-Matsui-Shimizu[9]). *Let* K *be an integral operator of form*

$$Kf(x) = \int_{\mathbb{R}^n} k(x,y)f(y)dy \text{ for } x \in \mathbb{R}^n. \tag{3.7}$$

If the kernel $k(x,y)$ *satisfies that*

$$\sup_{y\in\mathbb{R}^n} \|k(\cdot,y)\|_{\mathcal{H}^1} = k_0 < \infty,$$

then K *is a bounded operator from* $L^1(\mathbb{R}^n)$ *to* $\mathcal{H}^1(\mathbb{R}^n)$ *i.e.*

$$\|Kf\|_{\mathcal{H}^1} \le k_0\|f\|_{L^1(\mathbb{R}^n)}. \tag{3.8}$$

We note that the Riesz operators R_j are bounded in $\mathcal{H}^1$ and BMO, i.e.

$$\|R_j f\|_{\mathcal{H}^1} \le C_1\|f\|_{\mathcal{H}^1},$$
$$[R_j g]_{\mathrm{BMO}} \le C_2[g]_{\mathrm{BMO}}.$$

Remark. We remark the BMO space in the half space. Assume that g is a function defined in the half space. A function g belongs to BMO if there exists an extension function over the whole space which is equal to g in the half space and belongs to BMO. The norm of g is defined as

$$[g]_{\mathrm{BMO}(\mathbb{R}^n_+)} := \inf_{G:\text{extension}} [G]_{\mathrm{BMO}}.$$

§4 Proof of Theorem 1.1

Now we are ready to prove our theorem. In this section, we show Theorem 1.1, the estimates of non-linear part of Navier-Stokes equation. By (3.8), we have that the terms with v_0 in (2.5) are bounded with $[u]_{\mathrm{BMO}}$. Now there remains $\mathcal{H}^1$-estimates of the terms with ϕ.

First we estimate the kernels without $\chi_\pm$. Note that the estimates of Gauss kernel in Hardy space has been obtained by Giga-Matsui-Shimizu [10].

Lemma 4.1. *Let* G_t *be the Gauss kernel. Then*

$$\|\partial_i G_t\|_{\mathcal{H}^1} \le Ct^{-1/2} \text{ for } 1 \le i \le n, \tag{4.1a}$$

$$\|\Lambda G_t\|_{\mathcal{H}^1} \le Ct^{-1/2}, \tag{4.1b}$$

$$\|\partial_j\partial_k\Lambda^{-1}G_t\|_{\mathcal{H}^1} \le Ct^{-1/2} \text{ for } 1 \le j,k \le n-1. \tag{4.1c}$$

By Lemma 4.1 and Corollary 3.4, we have

$$\begin{aligned}\|\partial_j H_t\phi\|_{\mathcal{H}^1} &\le \|\phi\|_{L^1(\mathbb{R}^n)}\|\partial_j G_t\|_{\mathcal{H}^1} \\ &\le Ct^{-1/2}\|\phi\|_{L^1(\mathbb{R}^n)},\end{aligned} \tag{4.2a}$$

$$\|\partial_j\partial_k\Lambda^{-1}E(t)\phi\|_{\mathcal{H}^1} \le Ct^{-1/2}\|\phi\|_{L^1(\mathbb{R}^n)}, \tag{4.2b}$$

$$\|\partial_j\partial_k\Lambda^{-1}f(t)\phi\|_{\mathcal{H}^1} \le Ct^{-1/2}\|\phi\|_{L^1(\mathbb{R}^n)}. \tag{4.2c}$$

By the boundedness of the Riesz operators in the Hardy space, we have

$$\begin{aligned}\|R_jR_k\Lambda H_tu_0\|_{\mathcal{H}^1} &\le \|\phi\|_{L^1(\mathbb{R}^n)}\|R_jR_k\Lambda G_t\|_{\mathcal{H}^1} \\ &\le C\|\phi\|_{L^1(\mathbb{R}^n)}\|\Lambda G_t\|_{\mathcal{H}^1} \\ &\le Ct^{-1/2}\|\phi\|_{L^1(\mathbb{R}^n)}.\end{aligned} \tag{4.3}$$

Next, we estimate the terms containing $\chi_\pm$, i.e.

$$\partial_j[H_t(\chi_\pm(-\Delta)^{-1}\partial_i\partial_k\phi)](x),$$

where $1 \le i \le n-1$ and $1 \le j,k \le n$.

Remark. Shimizu [15] showed the $\mathcal{H}$-L^1 estimate of $[H_t(\chi_\pm\partial_iR_jR_k\phi)](x)$ with $1 \le i \le n-1$ and $1 \le j,k \le n$. In this case, we may assume that $j \ne n$, because if $j = k = n$, then we can reduce to $j \ne n$ by using the property of the Riesz kernels, i.e.

$$\sum_{\alpha=1}^{n} R_\alpha^2 = -I. \tag{4.4}$$

However in our problem, we cannot apply (4.4) when $j = k = n$, because $\chi_\pm$ cause the singularities on $x_n = 0$.

When $j \ne n$, we can reduce our problems to Shimizu[15]'s. Hereafter, we assume that $j = n$. Since the integral kernel of $(-\Delta)^{-1}$ is $c_n|x|^{-n+2}$, we have

$$\begin{aligned}&\partial_n[H_t(\chi_+(-\Delta)^{-1}\partial_i\partial_k\phi)](x) \\ =&-c_n\int_{\mathbb{R}^n}\partial_{y_n}\{G_t(x-y)\}\chi_+(y_n)\partial_{y_i}\partial_{y_k}\int_{\mathbb{R}^n}\frac{1}{|y-z|^{n-2}}\phi(z)dzdy \\ =&-c_n\int_{\mathbb{R}^n}\partial_{y_n}\{G_t(x-y)\}\chi_+(y_n)\partial_{y_i}\partial_{y_k}\int_{\mathbb{R}^n}\frac{\psi_1(y-z)}{|y-z|^{n-2}}\phi(z)dzdy \\ &-c_n\int_{\mathbb{R}^n}\partial_{y_n}\{G_t(x-y)\}\chi_+(y_n)\partial_{y_i}\partial_{y_k}\int_{\mathbb{R}^n}\frac{\psi_2(y-z)}{|y-z|^{n-2}}\phi(z)dzdy \\ =&-c_n(I_1(t,x)+I_2(t,x)),\end{aligned} \tag{4.5}$$

where ψ_1 is a smooth function with supp$\psi_1 \subset B(0,1)$, $0 \le \psi_1 \le 1$ and $\psi_1|_{B(0,1/2)} = 1$, and where $\psi_2 = 1 - \psi_1$.

By partial integral, we have

$$
\begin{aligned}
&I_1(t,x) \\
=&\delta_{kn}\int_{\mathbb{R}^n}\phi(z)\int_{\mathbb{R}^{n-1}}\frac{\psi_1(y'-z',-z)}{|(y'-z',-z_n)|^{n-2}}(\partial_i\partial_k G_t)(x'-y',x_n)dy'dz \\
&-\int_{\mathbb{R}^n}\phi(z)\int_{\mathbb{R}^n_+}\frac{\psi_1(y-z)}{|y-z|^{n-2}}\partial_{y_i}\partial_{y_k}\partial_{y_n}[G_t(x-y)]dydz \\
=&\delta_{kn}\int_{\mathbb{R}^n}\phi(z)I_{1,1}(t,x,z)dz-\int_{\mathbb{R}^n}\phi(z)I_{1,2}(t,x,z)dz,
\end{aligned}
\tag{4.6}
$$

$$
\begin{aligned}
&I_2(t,x) \\
=&\int_{\mathbb{R}^n}\phi(z)\int_{\mathbb{R}^{n-1}}\partial_i\partial_k\Big[\frac{\psi_2(y-z)}{|y-z|^{n-2}}\Big]_{y_n=0}G_t(x'-y',x_n)dy'dz \\
&-\int_{\mathbb{R}^n}\phi(z)\int_{\mathbb{R}^n_+}\partial_{y_i}\partial_{y_k}\partial_{y_n}\Big[\frac{\psi_2(y-z)}{|y-z|^{n-2}}\Big]G_t(x-y)dydz \\
=&\int_{\mathbb{R}^n}\phi(z)I_{2,1}(t,x,z)dz-\int_{\mathbb{R}^n}\phi(z)I_{2,2}(t,x,z)dz,
\end{aligned}
\tag{4.7}
$$

where δ_{kn} is Kronecker's delta. By Lemma 3.5, it is sufficient to estimate the integral kernels in $\mathcal{H}^1$ for x. First, we show the pointwise estimates of integral kernels.

Lemma 4.2.

(1) *Let $h=1,2$. Assume that real parameters α and β satisfy $0\le\alpha\le n$ and $\beta\ge 0$. Then there exists a constant $C=C_{n,\alpha,\beta}$ independent of $x\in\mathbb{R}^n$ and $t\ge 0$ such that*

$$|I_{h,1}(t,x,z)|\le Ct^{(\alpha+\beta-n-1)/2}|x'-z'|^{-\alpha}|x_n|^{-\beta} \tag{4.8a}$$

(2) *Let $h=1,2$. Assume that a real parameter γ satisfies $0\le\gamma\le n$. Then there exists a constant $C=C_{n,\gamma}$ independent of $x\in\mathbb{R}^n$ and $t\ge 0$ such that*

$$|I_{h,2}(t,x,z)|\le Ct^{(\gamma-n-1)/2}|x-z|^{-\gamma} \tag{4.8b}$$

Proof. By the parameter arguement, it suffices to show (4.8) for $t=1$.

Computing derivatives, we have

$$
\begin{aligned}
&|I_{1,1}(1,x,z)| \\
\le&C\int_{|y'-z',-z_n|<1}\frac{1}{|(y'-z',-z_n)|^{n-2}} \\
&(1+|x_i-y_i||x_k-\delta_{kn}y_k|)G_1(x'-y',x_n)dy',
\end{aligned}
\tag{4.9a}
$$

$$
\begin{aligned}
&|I_{1,2}(1,x,z)| \\
\le&C\int_{\mathbb{R}^n_+\cap\{|y-z|<1\}}\frac{1}{|y-z|^{n-2}}(|x-y|+|x-y|^3)G_t(x-y)dy,
\end{aligned}
\tag{4.9b}
$$

$$\begin{aligned}
&|I_{2,1}(1,x,z)| \\
\le& C\int_{1/2<|y'-z',-z_n|<1}[|y'-z',-z_n|^{-n+2}+|y'-z',-z_n|^{-n+1}] \\
&G_t(x'-y',x_n)dy' \\
&+C\int_{1/2<|y'-z',-z_n|}|y'-z',-z_n|^{-n}G_t(x'-y',x_n)dy',
\end{aligned} \tag{4.9c}$$

$$\begin{aligned}
&|I_{2,2}(1,x,z)| \\
\le& C\int_{\mathbb{R}^n_+\cap\{1/2<|y-z|<1\}}[|y-z|^{-n+2}+|y-z|^{-n+1}+|y-z|^{-n}] \\
&G_t(x-y)dy \\
&+C\int_{\mathbb{R}^n_+\cap\{1/2<|y-z|\}}|y-z|^{-n-2}G_t(x-y)dy.
\end{aligned} \tag{4.9d}$$

To estimate (4.9a,b), we apply the following inequalities:

$$|x-y|\le|x-z|+1, \tag{4.10a}$$
$$|x'-y'|\le|x'-z'|+1, \tag{4.10b}$$
$$|x-y|^2\ge(|x-z|^2-2)/2, \tag{4.10c}$$
$$|x'-y'|^2\ge(|x'-z'|^2-2)/2. \tag{4.10d}$$

So we have

$$\begin{aligned}
&|I_{1,1}(1,x,z)| \\
\le& C\int_{|y'-z',-z_n|<1}\frac{1}{|(y'-z',-z_n)|^{n-2}}(1+|x_i-z_i||x_k-\delta_{kn}z_k|) \\
&e^{-|x'-z'|^2/8+1/4}e^{-x_n^2/4}dy' \\
\le& C|x'-z'|^{-\alpha}|x_n|^{-\beta},
\end{aligned} \tag{4.11a}$$

$$\begin{aligned}
&|I_{1,2}(1,x,z)| \\
\le& C\int_{\mathbb{R}^n_+\cap\{|y-z|<1\}}\frac{1}{|y-z|^{n-2}}(1+|x-z|+|x-z|^3) \\
&e^{-|x-z|^2/8+1/4}dy \\
\le& C|x-z|^{-\gamma}.
\end{aligned} \tag{4.11b}$$

Next we estimate (4.9c,d). For their first terms, we apply the same method on (4.9a,b). For second terms, we apply the following inequalities:

$$|x'-z'|^\alpha\le C_\alpha(|x'-y'|^\alpha+|y'-z'|^\alpha), \tag{4.12a}$$
$$|x-z|^\gamma\le C_\gamma(|x-y|^\gamma+|y-z|^\gamma) \tag{4.12b}$$

For $\alpha \geq 0$ and $\gamma \geq 0$. Assuming that $0 \leq \alpha \leq n$ and $0 \leq \gamma \leq n+1$, we have

(4.13a)
$$\begin{aligned}&|I_{2,1}(1,x,z)|\\ \leq &C|x'-z'|^{-\alpha}|x_n|^{-\beta}\\ &+C|x'-z'|^{-\alpha}\int_{1/2<|y'-z',-z_n|}(|y'-z'|^{-n+\alpha}+|y'-z'|^{-n}|x'-y'|^{\alpha})\\ &G_t(x'-y',x_n)dy'\\ \leq &C|x'-z'|^{-\alpha}|x_n|^{-\beta},\end{aligned}$$

(4.13b)
$$\begin{aligned}&|I_{2,2}(1,x,z)|\\ \leq &C|x-z|^{-\gamma}\\ &+C|x-z|^{-\gamma}\int_{\mathbb{R}^n_+\cap\{1/2<|y-z|\}}(|y-z|^{-n+\gamma}+|y-z|^{-n}|x-y|^{\gamma})\\ &G_t(x-y,x_n)dy\\ \leq &C|x-z|^{-\gamma}\end{aligned}$$

and complete the proof. $\square$

By Lemma 4.2, we have that there exists a function $K=K(t,x,z)$ such that

$$\begin{aligned}&\partial_n[H_t(\chi_+(-\Delta)^{-1}\partial_i\partial_k\phi)](x)\\ =&\int_{\mathbb{R}^n}\phi(x)K(t,x,z)dz\end{aligned} \tag{4.14}$$

and satisfiessatisfies

$$\begin{aligned}&|K(t,x,z)|\\ \leq &C_0t^{(\alpha+\beta-n-1)/2}|x'-z'|^{-\alpha}|x_n|^{-\beta}+C_1t^{(\gamma-n-1)/2}|x-z|^{-\gamma}\end{aligned} \tag{4.15}$$

for $0 \leq \alpha \leq n$, $\beta \geq 0$, and $0 \leq \gamma \leq n+1$.

Finally, we show the following lemma.

Lemma 4.3. *There exists a constant C depending only on n such that*

$$\|K(t,\cdot,z)\|_{\mathcal{H}^1} \leq Ct^{-1/2}. \tag{4.16}$$

Proof. By Lemma 4.2, we have

$$\begin{aligned}|G_s*K(t,x,z)| =&|K(s+t,x,z)|\\ \leq &C_0(s+t)^{(\alpha+\beta-n-1)/2}|x'-z'|^{-\alpha}|x_n|^{-\beta}\\ &+C_1t^{(\gamma-n-1)/2}|x-z|^{-\gamma}\\ \leq &Ct^{(\alpha+\beta-n-1)/2}|x'-z'|^{-\alpha}|x_n|^{-\beta}\\ &+C_1t^{(\gamma-n-1)/2}|x-z|^{-\gamma}.\end{aligned} \tag{4.17}$$

Note that $\alpha+\beta-n-1$ and $\gamma-n-1$ are non-positive. Therefore we obtain

$$\begin{aligned}
&\|K(t,\cdot,z)\|_{\mathcal{H}^1} \\
&\le \sum_{k=1}^{4} C_0 t^{(\alpha+\beta-n-1)/2} \int_{\Omega_k} |x'-z'|^{-\alpha}|x_n|^{-\beta}dx, \\
&\quad + \sum_{k=5}^{6} C_1 t^{(\gamma-n-1)/2} \int_{\Omega_k} |x-z|^{-\gamma}dx,
\end{aligned} \tag{4.18}$$

where

$$\begin{aligned}
\Omega_1 &= \{|x'-z'| \le t^{1/2}, |x_n| \le t^{1/2}\}, \\
\Omega_2 &= \{|x'-z'| > t^{1/2}, |x_n| \le t^{1/2}\}, \\
\Omega_3 &= \{|x'-z'| \le t^{1/2}, |x_n| > t^{1/2}\}, \\
\Omega_4 &= \{|x'-z'| > t^{1/2}, |x_n| > t^{1/2}\} \\
\Omega_5 &= \{|x-z| \le t^{1/2}\}, \\
\Omega_6 &= \{|x-z| > t^{1/2}\}.
\end{aligned}$$

We compute the integrals on the right-hand side of (4.18), taking

$$\begin{aligned}
\alpha = \beta = 0 &\text{ for } k=1, \\
\alpha = n, \beta = 0 &\text{ for } k=2, \\
\alpha = 0, \beta = 0 &\text{ for } k=3, \\
\alpha = n-1/2, \beta = 3/2 &\text{ for } k=4, \\
\gamma = 0 &\text{ for } k=5, \\
\gamma = n+1 &\text{ for } k=6,
\end{aligned}$$

to find that the integrals of (4.18) are all bounded above by a constant multiple of $t^{-1/2}$. This proves (4.16). $\square$

By Lemma 3.5 and Lemma 4.3, we obtain

$$\|\partial_n[H_t(\chi_+(-\Delta)^{-1}\partial_i\partial_k\phi)]\|_{\mathcal{H}^1} \le Ct^{-1/2}\|\phi\|_{L^1(\mathbb{R}^n)}. \tag{4.19}$$

Combining the estimates in (4.2), (4.3), and (4.19), we finally obtain the desired estimate.

§5 Proof of Theorem 1.2

In this section, we show Theorem 1.2, the estimates of linear-part. By Proposition 3.3, it is sufficient to show that the terms with ϕ in (2.7) are in $\mathcal{H}^1$.

First we show the pointwise estimates of terms with $\chi_\pm$, that is,

$$I_{\alpha,i,j,k}(t,x) = \Lambda_{\alpha,j}H_t(\chi_\pm\partial_i\partial_k(-\Delta)^{-1}\phi)(x), \tag{5.1}$$

where ϕ is in $C^\infty(\mathbb{R}^n)$.

Lemma 5.1. *Let $0 < \alpha < 1$, $1 \le i,j \le n-1$, and $1 \le k \le n$. Assume that $0 \le \lambda \le n-\alpha+1$, $0 \le \mu \le n-\alpha$, and $\lambda \ge 0$. Then*

$$
\begin{aligned}
&|I_{\alpha,i,j,k}(t,x)| \\
\le & C\int_{\mathbb{R}^n} |\phi(w)|(t^{(\lambda-n+\alpha-1)/2}|x-w|^{-\lambda} \\
&+ t^{(\mu+\nu-n+\alpha-1)/2}|x'-w'|^{-\mu}|x_n-w_n|^{\nu})dw
\end{aligned}
\tag{5.2}
$$

holds for any $(t,x) \in (0,\infty)\times\mathbb{R}^n_+$, where C is independ of ϕ.

Before starting the proof of lemma, we define some functions. Let ψ_l, $l = 1,2$ be a smooth function defined in section 4. And for $l = 1,2$, define $\psi_l'(x') = \psi_l(x',0)$.

Proof. By scaling arguement, we may fix $t = 1$. Applying the integral presentation of operators, we have

$$
\begin{aligned}
& I(x) := I_{\alpha,i,j,k}(1,x) \\
=& \partial_{x_j}\int_{\mathbb{R}^{n-1}} \frac{1}{|z'-x'|^{n-1-\alpha}} \\
& \Big\{\int_{\mathbb{R}^n_+} G_1(z'-y',x_n-y_n) \\
& \Big(\partial_{y_i}\partial_{y_k}\int_{\mathbb{R}^n}\frac{1}{|w-y|^{n-2}}\phi(w)\bar{d}w\Big)dy\Big\}\bar{d}z' \\
=& \sum_{l,m=1}^{2}\partial_{x_j}\int_{\mathbb{R}^{n-1}}\frac{\psi_l'(z'-x')}{|z'-x'|^{n-1-\alpha}} \\
& \Big\{\int_{\mathbb{R}^n_+} G_1(z'-y',x_n-y_n) \\
& \Big(\partial_{y_i}\partial_{y_k}\int_{\mathbb{R}^n}\frac{\psi_m(w-y)}{|w-y|^{n-2}}\phi(w)\bar{d}w\Big)dy\Big\}\bar{d}z', \\
=& \sum_{l,m=1}^{2} I^{l,m}(x)
\end{aligned}
\tag{5.3}
$$

where $\bar{d}w$ denotes the integral with a constant of operator $(-\Delta)^{-1}$.

First, we show the estimate of $I^{1,1}$. Repeating partial integrals, we have

$$
\begin{aligned}
& I^{1,1}(x) \\
=& -\int_{\mathbb{R}^n}\phi(w)\Big\{\int_{\mathbb{R}^{n-1}}\frac{\psi_1'(z'-x')}{|z'-x'|^{n-1-\alpha}} \\
& \Big(\int_{\mathbb{R}^n_+}\partial_{y_i}\partial_{z_j}(G_1(z'-y',x_n-y_n))\partial_{y_k}\Big(\frac{\psi_1(w-y)}{|w-y|^{n-2}}\Big)dy\Big)\bar{d}z'\Big\}\bar{d}w.
\end{aligned}
\tag{5.4}
$$

So we have

$$
\begin{aligned}
&|I^{1,1}(x)| \\
\le & C\int_{\mathbb{R}^n}|\phi(w)|\Big\{\int_{|z'-x'|\le 1}\frac{1}{|z'-x'|^{n-1-\alpha}} \\
&\Big(\int_{\{|w-y|\le 1\}_+}(1+|z'-y'|^2)G_1(z'-y',x_n-y_n) \\
&\Big(\frac{1}{|w-y|^{n-2}}+\frac{1}{|w-y|^{n-1}}\Big)dy\Big)\bar{d}z'\Big\}\bar{d}w,
\end{aligned}
\tag{5.5}
$$

where $\{|w-y|<1\}_+$ denotes $\{|w-y|<1\}\cap\mathbb{R}^n_+$. Since $|z'-x'|\le 1$ and $|w-y|\le 1$, the following inequalities holds:

$$|z'-y'|\le |x-w|+2, \tag{5.6a}$$

$$|(z'-y',x_n-y_n)|^2\ge \frac{1}{4}|x-w|^2-\frac{3}{2}. \tag{5.6b}$$

Applying (5.6) to (5.5), we have

$$
\begin{aligned}
&|I^{1,1}(x)| \\
\le & C\int_{\mathbb{R}^n}|\phi(w)|\Big\{\int_{|z'-x'|\le 1}\frac{1}{|z'-x'|^{n-1-\alpha}} \\
&\Big(\int_{\{|w-y|\le 1\}_+}(3+|x-w|^2)e^{-|x-w|^2/16}e^{3/8} \\
&\Big(\frac{1}{|w-y|^{n-2}}+\frac{1}{|w-y|^{n-1}}\Big)dy\Big)\bar{d}z'\Big\}\bar{d}w \\
\le & C\int_{\mathbb{R}^n}|\phi(w)||x-w|^{-\lambda}\bar{d}w
\end{aligned}
\tag{5.7}
$$

for $\lambda\ge 0$.

Next, we show the estimate of $I^{1,2}$. We have

$$
\begin{aligned}
&I^{1,2}(x) \\
=&\int_{\mathbb{R}^n}\phi(w)\Big\{\int_{\mathbb{R}^{n-1}}\frac{\psi_1'(z'-x')}{|z'-x'|^{n-1-\alpha}} \\
&\Big(\int_{\mathbb{R}^n_+}G_1(z'-y',x_n-y_n) \\
&\partial_{y_i}\partial_{y_j}\partial_{y_k}\Big(\frac{\psi_2(w-y)}{|w-y|^{n-2}}\Big)dy\Big)\bar{d}z'\Big\}\bar{d}w
\end{aligned}
\tag{5.8}
$$

and

$$
\begin{aligned}
&|I^{1,2}(x)| \\
\le& C_1 \int_{\mathbb{R}^n} |\phi(w)| \Big\{ \int_{|z'-x'|\le 1} \frac{1}{|z'-x'|^{n-1-\alpha}} \\
&\Big(\int_{\{1/2\le|w-y|\le 1\}_+} G_1(z'-y', x_n-y_n) \\
&\Big(\frac{1}{|w-y|^{n-2}} + \frac{1}{|w-y|^{n-1}} \Big) dy \Big) \bar{d}z' \Big\} \bar{d}w \\
&+ C_2 \int_{\mathbb{R}^n} |\phi(w)| \Big\{ \int_{|z'-x'|\le 1} \frac{1}{|z'-x'|^{n-1-\alpha}} \\
&\Big(\int_{\{|w-y|\ge 1/2\}_+} G_1(z'-y', x_n-y_n) \frac{1}{|w-y|^{n+1}} dy \Big) \bar{d}z' \Big\} \bar{d}w \\
=:& C_1 I^{1,2,1} + C_2 I^{1,2,2}.
\end{aligned} \tag{5.9}
$$

Since the integral domains are bounded, we can estimate $I^{1,2,1}$ same as $I^{1,1}$, so we have

$$
\begin{aligned}
&|I^{1,2,1}(x)| \\
\le& C \int_{\mathbb{R}^n} |\phi(w)||x-w|^{-\lambda} \bar{d}w
\end{aligned} \tag{5.10}
$$

for $\lambda \ge 0$. To estimate $I^{1,2,2}$, we apply that

$$
|x-w|^\lambda \le C_\lambda (|z'-x'|^\lambda + |w-y|^\lambda + |(z'-y', x_n-y_n)|^\lambda)
$$

holds for $\lambda \ge 0$. Now we assume that $0 \le \lambda \le n+1$. Then we have

$$
\begin{aligned}
&|I^{1,2,2}(x)| \\
\le& \int_{\mathbb{R}^n} |\phi(w)| \frac{C_\lambda}{|x-w|^\lambda} \Big\{ \int_{|z'-x'|\le 1} \frac{1}{|z'-x'|^{n-1-\alpha}} \\
&\Big(\int_{\{|w-y|\ge 1/2\}_+} G_1(z'-y', x_n-y_n) \\
&\frac{|z'-x'|^\lambda + |w-y|^\lambda + |(z'-y', x_n-y_n)|^\lambda}{|w-y|^{n+1}} dy \Big) \bar{d}z' \Big\} \bar{d}w \\
\le& C \int_{\mathbb{R}^n} |\phi(w)||x-w|^{-\lambda} \Big\{ \int_{|z'-x'|\le 1} \frac{1}{|z'-x'|^{n-1-\alpha}} \\
&\Big(\int_{\{|w-y|\ge 1/2\}_+} G_1(z'-y', x_n-y_n) \\
&(2^{n+1}|z'-x'|^\lambda + 2^{n+1-\lambda}|w-y|^\lambda \\
&+ 2^{n+1}|(z'-y', x_n-y_n)|^\lambda) dy \Big) \bar{d}z' \Big\} \bar{d}w
\end{aligned} \tag{5.11}
$$

$$\le C\int_{\mathbb{R}^n}|\phi(w)||x-w|^{-\lambda}\Big(\int_{|z'-x'|\le 1}\frac{1+|z'-x'|^{\lambda}}{|z'-x'|^{n-1-\alpha}}\bar{d}z'\Big)\bar{d}w$$
$$\le C\int_{\mathbb{R}^n}|\phi(w)||x-w|^{-\lambda}\bar{d}w$$

for $0\le\lambda\le n+1$.

Next, we show the estimate of $I^{2,1}$. We have

$$\begin{aligned}&I^{2,1}(x)\\ =&-\int_{\mathbb{R}^n}\phi(w)\Big\{\int_{\mathbb{R}^n_+}\partial_{y_k}\frac{\psi_1(w-y)}{|w-y|^{n-2}}\\ &\Big(\int_{\mathbb{R}^{n-1}}G_1(z'-y',x_n-y_n)\partial_{z_i}\partial_{x_j}\Big(\frac{\psi_2'(z'-x')}{|z'-x'|^{n-1-\alpha}}\Big)\bar{d}z'\Big)dy\Big\}\bar{d}w\end{aligned}\tag{5.12}$$

and

$$\begin{aligned}&|I^{2,1}(x)|\\ \le&C_1\int_{\mathbb{R}^n}|\phi(w)|\\ &\Big\{\int_{\{|w-y|\le 1\}_+}\Big(\frac{1}{|w-y|^{n-2}}+\frac{1}{|w-y|^{n-2}}\Big)\\ &\Big(\int_{1/2\le|z'-x'|\le 1}G_1(z'-y',x_n-y_n)\\ &\Big(\frac{1}{|z'-x'|^{n-1-\alpha}}+\frac{1}{|z'-x'|^{n-\alpha}}\Big)\bar{d}z'\Big)dy\Big\}\bar{d}w\\ &+C_2\int_{\mathbb{R}^n}|\phi(w)|\\ &\Big\{\int_{\{|w-y|\le 1\}_+}\Big(\frac{1}{|w-y|^{n-2}}+\frac{1}{|w-y|^{n-2}}\Big)\\ &\Big(\int_{1/2\le|z'-x'|\le 1}G_1(z'-y',x_n-y_n)\frac{1}{|z'-x'|^{n+1-\alpha}}\bar{d}z'\Big)dy\Big\}\bar{d}w\\ =:&C_1I^{2,1,1}+C_2I^{2,1,2}.\end{aligned}\tag{5.13}$$

By similar arguement on $I^{1,1}$, we have

$$\begin{aligned}&|I^{2,1,1}(x)|\\ \le&C\int_{\mathbb{R}^n}|\phi(w)||x-w|^{-\lambda}\bar{d}w\end{aligned}\tag{5.14}$$

for $\lambda\ge 0$. To estimate $I^{2,1,2}$, we apply that

$$|x'-w'|^{\mu}\le C_\mu(|x'-z'|^{\mu}+|z'-y'|^{\mu}+|w-y|^{\mu})$$

holds for $\mu \geq 0$. Now we assume $0 \leq \mu \leq n-\alpha+1$. Then we have

$$
\begin{aligned}
(5.15)\qquad & |I^{2,1,2}(x)| \\
\leq & \int_{\mathbb{R}^n} |\phi(w)| \frac{C_\mu}{|x'-w'|^\mu} \\
& \Big\{ \int_{\{|w-y|\leq 1\}_+} \Big(\frac{1}{|w-y|^{n-2}} + \frac{1}{|w-y|^{n-2}} \Big) \\
& \Big(\int_{1/2\leq |z'-x'|\leq 1} G_1(z'-y', x_n-y_n) \\
& \frac{|z'-x'|^\mu + |z'-y'|^\mu + |w-y|^\mu}{|z'-x'|^{n+1-\alpha}} \bar{d}z' \Big) dy \Big\} \bar{d}w \\
\leq & C \int_{\mathbb{R}^n} |\phi(w)||x'-w'|^{-\mu} \\
& \Big\{ \int_{\{|w-y|\leq 1\}_+} e^{-|x_n-y_n|^2/4} \Big(\frac{1}{|w-y|^{n-2}} + \frac{1}{|w-y|^{n-2}} \Big) \\
& \Big(\int_{1/2\leq |z'-x'|\leq 1} e^{-|z'-y'|^2/4} \\
& (2^{n-\alpha+1-\mu} + 2^{n-\alpha+1}|z'-y'|^\mu + 2^{n-\alpha+1}|w-y|^\mu) \bar{d}z' \Big) dy \Big\} \bar{d}w \\
\leq & C \int_{\mathbb{R}^n} |\phi(w)||x'-w'|^{-\mu} \\
& \Big\{ \int_{\{|w-y|\leq 1\}_+} e^{-|x_n-y_n|^2/4} \Big(\frac{1}{|w-y|^{n-2}} + \frac{1}{|w-y|^{n-2}} \Big) \\
& (1+|w-y|^\mu) dy \Big\} \bar{d}w.
\end{aligned}
$$

Since $|w_n - y_n| \leq |w-y| \leq 1$, we have

$$
|x_n - y_n| \geq \frac{1}{2}|x_n - w_n|^2 - 1. \tag{5.16}
$$

Applying (5.16) to (5.15), we have

$$
\begin{aligned}
(5.17)\qquad & |I^{2,1,2}(x)| \\
\leq & C \int_{\mathbb{R}^n} |\phi(w)||x'-w'|^{-\mu} \\
& \Big\{ \int_{\{|w-y|\leq 1\}_+} e^{-|x_n-w_n|^2/8+1/4} \\
& \Big(\frac{1}{|w-y|^{n-2}} + \frac{1}{|w-y|^{n-2}} \Big) (1+|w-y|^\mu) dy \Big\} \bar{d}w \\
\leq & C \int_{\mathbb{R}^n} |\phi(w)||x'-w'|^{-\mu}|x_n-w_n|^{-\nu} \bar{d}w
\end{aligned}
$$

for $0 \le \mu \le n-\alpha+1$ and $\nu \ge 0$.

Finally we show the estimate of $I^{2,2}$. We have

$$
\begin{aligned}
(5.18)\qquad & I^{2,2}(x) \\
&= \int_{\mathbb{R}^n} \phi(w)\Big\{\int_{\mathbb{R}^{n-1}} \partial_{x_j} \frac{\psi_1'(z'-x')}{|z'-x'|^{n-1-\alpha}} \\
&\Big(\int_{\mathbb{R}^n_+} G_1(z'-y', x_n-y_n)\partial_{y_i}\partial_{y_k}\Big(\frac{\psi_2(w-y)}{|w-y|^{n-2}}\Big)dy\Big)\bar{d}z'\Big\}\bar{d}w.
\end{aligned}
$$

Now let $z'-y'=\xi'$ and $x_n-y_n=\xi_n$. Then we have

$$
\begin{aligned}
(5.19)\qquad & \int_{\mathbb{R}^{n-1}} \partial_{x_j} \frac{\psi_1'(z'-x')}{|z'-x'|^{n-1-\alpha}} \\
&\Big(\int_{\mathbb{R}^n_+} G_1(z'-y', x_n-y_n)\partial_{y_i}\partial_{y_k}\Big(\frac{\psi_2(w-y)}{|w-y|^{n-2}}\Big)dy\Big)\bar{d}z' \\
&= \int_{\mathbb{R}^n_+} G_1(\xi)\Big(\int_{\mathbb{R}^{n-1}} \partial_{x_j} \frac{\psi_1'(z'-x')}{|z'-x'|^{n-1-\alpha}} \\
&\partial_{\xi_i}\partial_{\xi_k}\Big(\frac{\psi_2(w'-z'+\xi', w_n-x_n+\xi_n)}{|(w'-z'+\xi', w_n-x_n+\xi_n)|^{n-2}}\Big)\bar{d}z'\Big)d\xi.
\end{aligned}
$$

Applying (5.19), we have

$$
\begin{aligned}
(5.20)\qquad & \\
& |I^{2,2}(x)| \\
&\le C_1 \int_{\mathbb{R}^n} |\phi(w)|\Big\{\int_{1/2\le|z'-x'|\le 1} \frac{1}{|z'-x'|^{n-1-\alpha}} \\
&\Big(\int_{\{1/2\le|w-y|\le 1\}_+} G_1(z'-y', x_n-y_n) \\
&\Big(\frac{1}{|w-y|^{n-2}} + \frac{1}{|w-y|^{n-1}}\Big)dy\Big)\bar{d}z'\Big\}\bar{d}w \\
&+ C_2 \int_{\mathbb{R}^n} |\phi(w)|\Big\{\int_{\mathbb{R}^n_+} G_1(\xi) \\
&\Big(\int_{D_2(x,w,\xi)} \frac{1}{|z'-x'|^{n-1-\alpha}} \frac{1}{|(w'-z'+\xi', w_n-x_n+\xi_n)|^{n}}\bar{d}z'\Big)dy\Big\}\bar{d}w \\
&+ C_3 \int_{\mathbb{R}^n} |\phi(w)| \\
&\Big\{\int_{\{1/2\le|w-y|\le 1\}_+} \Big(\frac{1}{|w-y|^{n-2}} + \frac{1}{|w-y|^{n-1}}\Big) \\
&\Big(\int_{|z'-x'|\ge 1/2} G_1(z'-y', x_n-y_n)\frac{1}{|z'-x'|^{n-\alpha}}dy\Big)\bar{d}z'\Big\}\bar{d}w
\end{aligned}
$$

$$
\begin{aligned}
&+ C_4 \int_{\mathbb{R}^n} |\phi(w)| \Big\{ \int_{\mathbb{R}^n_+} G_1(\xi) \\
&\Big(\int_{D_4(x,w,\xi)} \frac{1}{|z'-x'|^{n-\alpha}} \frac{1}{|(w'-z'+\xi', w_n - x_n + \xi_n)|^n} \bar{d}z' \Big) dy \Big\} \bar{d}w \\
=: &\sum_{h=1}^4 C_h I^{2,2,h},
\end{aligned}
$$

where

$$
\begin{aligned}
D_2(x,w,\xi) =& \{1/2 \le |z'-x'| \le 1\} \\
& \cap \{|(w'-z'+\xi', w_n - x_n + \xi_n)| \ge 1/2\}, \\
D_4(x,w,\xi) =& \{|z'-x'| \ge 1/2\} \\
& \cap \{|(w'-z'+\xi', w_n - x_n + \xi_n)| \ge 1/2\}.
\end{aligned}
$$

By similar arguement on $I^{1,1}$, we have

$$
\begin{aligned}
&|I^{2,2,1}(x)| \\
\le & C \int_{\mathbb{R}^n} |\phi(w)| |x-w|^{-\lambda} \bar{d}w
\end{aligned}
\tag{5.21}
$$

for $\lambda \ge 0$, and by similar arguement on $I^{2,1,2}$, we have

$$
\begin{aligned}
&|I^{2,2,3}(x)| \\
\le & C \int_{\mathbb{R}^n} |\phi(w)| |x'-w'|^{-\mu} |x_n - w_n|^{-\nu} \bar{d}w
\end{aligned}
\tag{5.22}
$$

for $0 \le \mu \le n - \alpha + 1$ and $\nu \ge 0$. To estimate $I^{2,2,2}$ and $I^{2,2,4}$, we apply that

$$
|x-w|^\lambda \le C_\lambda (|z'-x'|^\lambda + |(w'-z'+\xi', w_n - x_n + \xi)|^\lambda + |\xi|^\lambda)
$$

holds for $\lambda \ge 0$. Now we assume that $0 \le \lambda \le n - \alpha + 1$. Moreover, we denote that

$$
\begin{aligned}
D_{2,1} &= D_2 \cap \{|z'-x'| \le |(w'-z'+\xi', w_n - x_n + \xi_n)|\}, \\
D_{2,2} &= D_2 \cap \{|z'-x'| \ge |(w'-z'+\xi', w_n - x_n + \xi_n)|\}, \\
D_{4,1} &= D_4 \cap \{|z'-x'| \le |(w'-z'+\xi', w_n - x_n + \xi_n)|\}, \\
D_{4,2} &= D_4 \cap \{|z'-x'| \ge |(w'-z'+\xi', w_n - x_n + \xi_n)|\}.
\end{aligned}
$$

Now we have

(5.23)

$$
\begin{aligned}
&|I^{2,2,2}| \\
\le & \int_{\mathbb{R}^n} |\phi(w)| \frac{C_\lambda}{|x-w|^{-\lambda}} \Big\{ \int_{\mathbb{R}^n_+} G_1(\xi) \\
&\Big(\int_{D_2(x,w,\xi)} \frac{1}{|z'-x'|^{n-1-\alpha-\lambda}} \frac{1}{|(w'-z'+\xi', w_n - x_n + \xi_n)|^n} \bar{d}z'
\end{aligned}
$$

$$
\begin{aligned}
&+\int_{D_{2,1}(x,w,\xi)} \frac{1}{|z'-x'|^{n-1-\alpha}} \frac{1}{|(w'-z'+\xi', w_n-x_n+\xi_n)|^{n-\lambda}} \bar{d}z' \\
&+\int_{D_{2,2}(x,w,\xi)} \frac{1}{|z'-x'|^{n-1-\alpha}} \frac{1}{|(w'-z'+\xi', w_n-x_n+\xi_n)|^{n-\lambda}} \Big) \bar{d}z' \\
&+|\xi|^{\lambda} \int_{D_2(x,w,\xi)} \frac{1}{|z'-x'|^{n-1-\alpha}} \\
&\frac{1}{|(w'-z'+\xi', w_n-x_n+\xi_n)|^{n}} \Big) \bar{d}z' \Big) dy \Big\} \bar{d}w \\
=: &\int_{\mathbb{R}^n} |\phi(w)| \frac{C_\lambda}{|x-w|^{-\lambda}} \Big\{ \int_{\mathbb{R}^n_+} G_1(\xi)(J_1+J_2+J_3+|\xi| J_4) dy \Big\} \bar{d}w.
\end{aligned}
$$

Since $|z'-x'|^{-n-1-\alpha-\lambda}$ is integrable for any $\lambda \in \mathbb{R}$ and $|(w'-z'+\xi', w_n-x_n+\xi_n)|^n \geq 2^n$, we have

$$
\begin{aligned}
&J_1 \\
\leq &\int_{D_2} \frac{2^n}{|z'-x'|^{n-1-\alpha-\lambda}} \bar{d}z' \\
\leq &C < \infty
\end{aligned}
\tag{5.24}
$$

for $0 \leq \lambda \leq n-\alpha+1$. By the properties of the domains $D_{2,1}$ and $D_{2,2}$, we have

$$
\begin{aligned}
&J_2 \\
\leq &\int_{D_{2,1}} \frac{1}{|z'-x'|^{n-1-\alpha+n-\lambda}} \bar{d}z' \\
\leq &C < \infty,
\end{aligned}
\tag{5.25}
$$

$$
\begin{aligned}
&J_3 \\
\leq &\int_{D_{2,2}} \frac{|z'-x'|}{|(w'-z'+\xi', w_n-x_n+\xi_n)|^{n-\alpha-\lambda+n}} \bar{d}z' \\
\leq &\int_{D_{2,2}} \frac{1}{|(w'-z'+\xi', w_n-x_n+\xi_n)|^{n-\alpha-\lambda+n}} \bar{d}z' \\
\leq &C < \infty
\end{aligned}
\tag{5.26}
$$

for $0 \leq \lambda \leq n-\alpha+1$. Since $n-1-\alpha > 0$ and $n > 0$, we have

$$
\begin{aligned}
&J_4 \\
\leq &\int_{D_2} 2^{n-1-\alpha} 2^n \bar{d}z' \\
\leq &C < \infty
\end{aligned}
\tag{5.27}
$$

for $0 \leq \lambda \leq n-\alpha+1$. Combining (5.24-27), we have

$$
\begin{aligned}
&|I^{2,2,2}| \\
\leq & C\int_{\mathbb{R}^n}|\phi(w)|\frac{C_\lambda}{|x-w|^{-\lambda}}\Big\{\int_{\mathbb{R}^n_+}G_1(\xi)(1+|\xi|)dy\Big\}\bar{d}w \\
\leq & C\int_{\mathbb{R}^n}|\phi(w)||x-w|^{-\lambda}\bar{d}w
\end{aligned}
\tag{5.28}
$$

for $0 \leq \lambda \leq n-\alpha+1$. The estimate of $I^{2,2,4}$ is similar to $I^{2,2,2}$. We have

$$
\begin{aligned}
&|I^{2,2,4}| \\
\leq & \int_{\mathbb{R}^n}|\phi(w)|\frac{C_\lambda}{|x-w|^{-\lambda}}\Big\{\int_{\mathbb{R}^n_+}G_1(\xi) \\
&\Big(\int_{D_{4,1}(x,w,\xi)}\frac{1}{|z'-x'|^{n-\alpha-\lambda}}\frac{1}{|(w'-z'+\xi',w_n-x_n+\xi_n)|^n}\bar{d}z' \\
&+\int_{D_{4,2}(x,w,\xi)}\frac{1}{|z'-x'|^{n-\alpha-\lambda}}\frac{1}{|(w'-z'+\xi',w_n-x_n+\xi_n)|^n}\bar{d}z' \\
&+\int_{D_{4,1}(x,w,\xi)}\frac{1}{|z'-x'|^{n-\alpha}}\frac{1}{|(w'-z'+\xi',w_n-x_n+\xi_n)|^{n-\lambda}}\bar{d}z' \\
&+\int_{D_{4,2}(x,w,\xi)}\frac{1}{|z'-x'|^{n-\alpha}}\frac{1}{|(w'-z'+\xi',w_n-x_n+\xi_n)|^{n-\lambda}}\Big)\bar{d}z' \\
&+|\xi|^\lambda\int_{D_{4,1}(x,w,\xi)}\frac{1}{|z'-x'|^{n-\alpha}} \\
&\frac{1}{|(w'-z'+\xi',w_n-x_n+\xi_n)|^n}\Big)\bar{d}z'\Big)dy\Big\}\bar{d}w \\
&+|\xi|^\lambda\int_{D_{4,2}(x,w,\xi)}\frac{1}{|z'-x'|^{n-\alpha}} \\
&\frac{1}{|(w'-z'+\xi',w_n-x_n+\xi_n)|^n}\Big)\bar{d}z'\Big)dy\Big\}\bar{d}w \\
=: & \int_{\mathbb{R}^n}|\phi(w)|\frac{C_\lambda}{|x-w|^{-\lambda}}\Big\{\int_{\mathbb{R}^n_+}G_1(\xi) \\
&\Big(K_1+K_2+K_3+K_4+|\xi|K_5+|\xi|K_6)\Big)dy\Big\}\bar{d}w.
\end{aligned}
\tag{5.29}
$$

By similar arguement on J_2 and J_3, we have

$$
\begin{aligned}
&K_1 \\
\leq & \int_{D_{4,1}}\frac{1}{|z'-x'|^{n-\alpha-\lambda+n}}\bar{d}z' \\
\leq & C<\infty,
\end{aligned}
\tag{5.30}
$$

$$
\begin{aligned}
&K_2 \\
\le&\int_{D_{4,2}} \frac{1}{|(w'-z'+\xi', w_n-x_n+\xi_n)|^{n-\alpha-\lambda+n}}\bar{d}z' \\
\le&C<\infty,
\end{aligned} \tag{5.31}
$$

$$
\begin{aligned}
&K_3 \\
\le&\int_{D_{4,1}} \frac{1}{|z'-x'|^{n-\alpha-\lambda+n}}\bar{d}z' \\
\le&C<\infty,
\end{aligned} \tag{5.32}
$$

$$
\begin{aligned}
&K_4 \\
\le&\int_{D_{4,2}} \frac{1}{|(w'-z'+\xi', w_n-x_n+\xi_n)|^{n-\alpha-\lambda+n}}\bar{d}z' \\
\le&C<\infty,
\end{aligned} \tag{5.33}
$$

$$
\begin{aligned}
&K_5 \\
\le&\int_{D_{4,1}} \frac{1}{|z'-x'|^{n-\alpha+n}}\bar{d}z' \\
\le&C<\infty,
\end{aligned} \tag{5.34}
$$

$$
\begin{aligned}
&K_6 \\
\le&\int_{D_{4,2}} \frac{1}{|(w'-z'+\xi', w_n-x_n+\xi_n)|^{n-\alpha+n}}\bar{d}z' \\
\le&C<\infty
\end{aligned} \tag{5.35}
$$

for $0 \le \lambda \le n-\alpha+1$. Combining (4.30-35), we have

$$
\begin{aligned}
&|I^{2,2,4}| \\
\le&C\int_{\mathbb{R}^n} |\phi(w)|\frac{C_\lambda}{|x-w|^{-\lambda}}\Big\{\int_{\mathbb{R}^n_+} G_1(\xi)(1+|\xi|)dy\Big\}\bar{d}w \\
\le&C\int_{\mathbb{R}^n} |\phi(w)||x-w|^{-\lambda}\bar{d}w
\end{aligned} \tag{5.36}
$$

for $0 \le \lambda \le n-\alpha+1$. Finally, we combine the estimates of $I^{l,m}$. Then we have

$$
\begin{aligned}
&I(x) \\
\le&C\int_{\mathbb{R}^n} |\phi(w)|(|x-w|^{-\lambda} \\
&+|x'-w'|^{-\mu}|x_n-w_n|^{\nu})dw
\end{aligned} \tag{5.37}
$$

for $0 \le \lambda \le n-\alpha+1$, $0 \le \mu \le n-\alpha$, and $\lambda \ge 0$, which is our desired estimate. □

Using Lemma 5.1, we can obtain $\mathcal{H}^1$-estimate of $I_{\alpha,i,j,k}$.

Lemma 5.2. *There exists a constant C depending only on n such that*

$$\|I_{\alpha,i,j,k}\|_{\mathcal{H}^1} \leq Ct^{(\alpha-1)/2}\|\phi\|_{L^1(\mathbb{R}^n)}. \tag{5.38}$$

Proof. The proof of this lemma is similar to Lemma 4.3. By Lemma 5.1, we have

$$\begin{aligned}|G_s * I_{\alpha,i,j,k}(t,x)| =& |I_{\alpha,i,j,k}(s+t,x)|\\ \leq& C\int_{\mathbb{R}^n} |\phi(w)|((s+t)^{(\lambda-n+\alpha-1)/2}|x-w|^{-\lambda}\\ &+ (s+t)^{(\mu+\nu-n+\alpha-1)/2}|x'-w'|^{-\mu}|x_n-w_n|^{\nu})dw\\ \leq& C\int_{\mathbb{R}^n} |\phi(w)|(t^{(\lambda-n+\alpha-1)/2}|x-w|^{-\lambda}\\ &+ (t^{(\mu+\nu-n+\alpha-1)/2}|x'-w'|^{-\mu}|x_n-w_n|^{\nu})dw\end{aligned} \tag{5.39}$$

Note that $\lambda-n+\alpha-1$ and $\mu+\nu-n+\alpha-1$ is non-positive. Therefore we have

$$\begin{aligned}&\|I_{\alpha,i,j,k}\|_{\mathcal{H}^1}\\ &\leq \sum_{k=1}^{2} C_1 t^{(\lambda-n-1)/2}\int_{\mathbb{R}^n}|\phi(w)|\int_{\Omega_k}|x-w|^{-\lambda}dxdw\\ &\quad+\sum_{k=3}^{6} C_0 t^{(\mu+\nu-n+\alpha-1)/2}\int_{\mathbb{R}^n}|\phi(w)|\int_{\Omega_k}|x'-w'|^{-\mu}|x_n-w_n|^{-\nu}dxdw\end{aligned} \tag{5.40}$$

where

$$\begin{aligned}\Omega_1 &= \{|x-w| \leq t^{1/2}\},\\ \Omega_2 &= \{|x-w| > t^{1/2},\\ \Omega_3 &= \{|x'-w'| \leq t^{1/2}, |x_n-w_n| \leq t^{1/2}\},\\ \Omega_4 &= \{|x'-w'| > t^{1/2}, |x_n-w_n| \leq t^{1/2}\},\\ \Omega_5 &= \{|x'-w'| \leq t^{1/2}, |x_n-w_n| > t^{1/2}\},\\ \Omega_6 &= \{|x'-w'| > t^{1/2}, |x_n-w_n| > t^{1/2}\}.\end{aligned}$$

We compute the integrals on the right-hand side of (4.18), taking

$$\begin{aligned}\lambda = 0 \text{ for } k = 1,\\ \lambda = n+1-\alpha \text{ for } k = 2,\end{aligned}$$

$$\mu = \nu = 0 \text{ for } k = 3,$$
$$\mu = n + 1 - \alpha, \beta = 0 \text{ for } k = 4,$$
$$\mu = 0, \beta = n + 1 - \alpha \text{ for } k = 5,$$
$$\alpha = n - (1 - \alpha)/2, \beta = (3 - \alpha)/2 \text{ for } k = 6,$$

to find that the integrals of (5.40) are all bounded above by a constant multiple of $t^{(\alpha-1)/2}$. This proves (5.38). □

Next we show the estimate of terms without $\chi_\pm$. By similar arguement in Theorem 5.1, we have the following lemma:

Lemma 5.3. *Let $0 < \alpha < 1$, $1 \le i, j \le n - 1$, and $1 \le k \le n$. Assume that $0 \le \lambda \le n - \alpha + 1$, $0 \le \mu \le n - \alpha$, and $\lambda \ge 0$. Then*

$$|\Lambda_{\alpha,j} G_t(x)| \le C t^{(\mu+\nu-n+\alpha-1)/2} |x|^{-\mu} |x_n|^{\nu}, \tag{5.41a}$$

$$|\nabla' \Lambda_{\alpha-1,j} G_t(x)| \le C t^{(\mu+\nu-n+\alpha-1)/2} |x|^{-\mu} |x_n|^{\nu}, \tag{5.41b}$$

$$|\Lambda^{1-\alpha} G_t(x)| \le C t^{(\mu+\nu-n+\alpha-1)/2} |x|^{-\mu} |x_n|^{\nu} \tag{5.41c}$$

holds for any $(t, x) \in (0, \infty) \times \mathbb{R}^n_+$, where C is independ of ϕ.

Applying Lemma 5.3 and Lemma 3.5, we obtain the $\mathcal{H}^1$-estimates of terms without $\chi_\pm$, that is,

$$\|\Lambda_{\alpha,j}[H_t \phi]\|_{\mathcal{H}^1} \le C t^{(\alpha-1)/2} \|\phi\|_{L^1(\mathbb{R}^n)}, \tag{5.42a}$$

$$\|\nabla \Lambda_{\alpha+1,j}[H_t \phi]\|_{\mathcal{H}^1} \le C t^{(\alpha-1)/2} \|\phi\|_{L^1(\mathbb{R}^n)}, \tag{5.42b}$$

$$\|R_i R_j \Lambda^{1-\alpha}[H_t \phi]\|_{\mathcal{H}^1} \le C t^{(\alpha-1)/2} \|\phi\|_{L^1(\mathbb{R}^n)}. \tag{5.42c}$$

Combining the estimates in (5.38) and (5.42), we finally obtain our desired estimate and conclusion of Theorem 1.2.

§6 Application: Local-solution of Navier-Stokes equation

In this section, we show Theorem 1.3, the existence of local solution of Navier-Stokes equation in a half space, by applying Theorem 1.1 and Theorem 1.2. We recall the integral equation provided by Navier-Stokes equation:

$$u(t, x) = [e^{-tA} u_0](x) - \sum_{i=1}^{n} \int_0^t e^{-(t-s)A} Q_i(u^i u)(s, x) ds,$$

where e^{-tA} is the solution operator defined in Theorem 2.1 and P_i is the combination of the projection operator and space derivative defined in (2.2).

Before begining to prove theorem, we recall a function space $BC_{q,T}$, $0<q<\infty$;

$$\begin{aligned} &BC_{q,T}\\ =&\{f\in C((0,T);L^\infty(\mathbb{R}^n_+));\|f\|_{BC_{q,T}}=\sup_{0<t<T}t^q\|f\|_{L^\infty(\mathbb{R}^n_+)}<\infty\}.\end{aligned}$$

Now we begin to prove our theorem.

Proof of Theorem 1.3. We define the sequence $\{u_k\}_{k=1}^\infty$ by

$$u_1(t)=e^{-tA}\Lambda_{\alpha,j}v_0, \tag{6.1a}$$

$$\begin{aligned} u_{k+1}(t)=[e^{-tA}\Lambda_{\alpha,j}v_0](x)-\sum_{i=1}^n\int_0^t e^{-(t-s)A}Q_i(u_k^iu_k)(s)ds\\ \text{for } k\geq 1.\end{aligned} \tag{6.1b}$$

First we show that $\{u_k\}$ is bounded in $BC_{(1-\alpha)/2,T}$. By Theorem 1.1 and Theorem 1.2, we have

$$\|u_1(t)\|_{L^\infty}\leq C_0t^{-(1-\alpha)/2}[v_0]_{\mathrm{BMO}}, \tag{6.2a}$$

$$\begin{aligned}\|u_{k+1}(t)\|_{L^\infty}&\leq C_0t^{-(1-\alpha)/2}[v_0]_{\mathrm{BMO}}\\ &\quad+C_1\sum_{i=1}^n\int_0^t(t-s)^{-1/2}[u_k^iu_k(s)]_{\mathrm{BMO}}ds\\ &\leq C_0t^{-(1-\alpha)/2}[v_0]_{\mathrm{BMO}}\\ &\quad+C_1\int_0^t(t-s)^{-1/2}\|u_k(s)\|_{L^\infty}^2ds\\ &\text{for } k\geq 1.\end{aligned} \tag{6.2b}$$

Here we set $X_k=\|u_k\|_{BC_{(1-\alpha)/2,T}}$. Then

$$X_1\leq C_0[v_0]_{\mathrm{BMO}}, \tag{6.3a}$$

$$\begin{aligned} t^{(1-\alpha)/2}\|u_{k+1}(t)\|_{L^\infty}&\leq C_0[v_0]_{\mathrm{BMO}}\\ &\quad+C_1t^{(1-\alpha)/2}X_k^2\int_0^t(t-s)^{-1/2}s^{-(1-\alpha)}ds\\ &\leq C_0[v_0]_{\mathrm{BMO}}+C_2t^{\alpha/2}X_k^2\\ &\text{for } k\geq 1.\end{aligned} \tag{6.3b}$$

Taking supremum for $0<t<T$ in (6.3b), we have

$$X_{k+1}\leq C_0[v_0]_{\mathrm{BMO}}+C_2T^{\alpha/2}X_k^2 \tag{6.4}$$

for $k \geq 1$. Now we assume that $D = 1 - 4C_0C_2T^{\alpha/2}[v_0]_{\mathrm{BMO}}$ satisfies $0 < D < 1$. Then we have

$$X_k \leq \frac{1 - D^{1/2}}{2C_2T^{\alpha/2}} = X_\infty \tag{6.5}$$

for $k \geq 1$. (6.5) shows that $\{u_k\}$ is bounded in $C_{(1-\alpha)/2,T}$.

Next, we show that $\{u_k\}$ is a Cauchy sequence in $BC_{(1-\alpha)/2,T}$. By (6.1), we have

$$(u_2 - u_1)(t) = -\sum_{i=1}^{n} \int_0^t e^{-(t-s)A} Q_i(u_1^i u_1)(s)ds, \tag{6.6a}$$

$$(u_{k+1} - u_k)(t) = -\sum_{i=1}^{n} \int_0^t e^{-(t-s)A} Q_i(u_k^i u_k - u_{k-1}^i u_{k-1})(s)ds \quad \text{for } k \geq 2. \tag{6.6b}$$

By same arguement for boundedness, we have

$$\|(u_2 - u_1)(t)\|_{L^\infty} \leq C_1 \int_0^t (t-s)^{-1/2} \|u_1(s)\|_{L^\infty}^2 ds, \tag{6.7a}$$

(6.7b)

$$\|(u_{k+1} - u_k)(t)\|_{L^\infty} \leq C_1 \int_0^t (t-s)^{-1/2} \|(u_k - u_{k-1})(s)\|_{L^\infty} (\|u_k(s)\|_{L^\infty} + \|u_{k-1}(s)\|_{L^\infty}) ds \quad \text{for } k \geq 2.$$

Here we set $Y_k = \|u_{k+1} - u_k\|_{BC_{(1-\alpha)/2,T}}$. Then we have

$$\begin{aligned} t^{(1-\alpha)/2} \|(u_2 - u_1)(t)\|_{L^\infty} &\leq C_1 t^{(1-\alpha)/2} X_1^2 \int_0^t (t-s)^{-1/2} s^{(1-\alpha)} ds \\ &\leq C_1 t^{\alpha/2} X_\infty^2 \end{aligned} \tag{6.8a}$$

(6.8b)

$$\begin{aligned} t^{(1-\alpha)/2} \|(u_{k+1} - u_k)(t)\|_{L^\infty} &\leq C_1 t^{(1-\alpha)/2} Y_{k-1} (X_k + X_{k-1}) \\ &\int_0^t (t-s)^{-1/2} s^{1-\alpha} ds \\ &\leq 2C_1 t^{\alpha/2} X_\infty Y_{k-1} \\ &\text{for } k \geq 2. \end{aligned}$$

Taking supremum for $0 < t < T$ in (6.8), we have

$$Y_1 \leq C_2 T^{\alpha/2} X_\infty^2 \tag{6.9a}$$

$$
\begin{aligned}
Y_k &\le C_1 t^{(1-\alpha)/2} Y_{k-1}(X_k + X_{k-1}) \int_0^t (t-s)^{-1/2} s^{1-\alpha} ds \\
&\le 2C_2 T^{\alpha/2} X_\infty Y_{k-1} \\
&\le (1 - D^{1/2}) Y_{k-1} \text{ for } k \ge 2.
\end{aligned} \tag{6.9b}
$$

It is easy to see Y_1 is bounded. By the assumption of D, there exists a constant λ such that $0 < \lambda < 1$ and satisfies

$$
\begin{aligned}
Y_k &\le \lambda Y_{k-1} \\
&\le \lambda^{k-1} Y_1 \text{ for } k \ge 1.
\end{aligned} \tag{6.10}
$$

(6.10) shows that $\{u_k\}$ is the Cauchy sequence in $BC_{(1-\alpha)/2,T}$. By the completeness of the space of continuous functions, we finally obtain the existence of the solution for (1.8).

Finally, we show the uniqueness of the solution for (1.8). Assume that the function v also satisfies (1.8). Let $w = u - v$. Then w satisfies

$$
w(t) = -\sum_{i=1}^n \int_0^t e^{-(t-s)A} Q_i(u^i w + w^i v)(s) ds. \tag{6.11}
$$

Now we show that $\|w\|_{BC_{0,T}} = 0$. By Theorem 1.1, we have

$$
\|w(t)\|_{L^\infty} \le C_1 \int_0^t (t-s)^{-1/2} \|w(s)\|_{L^\infty} (\|u(s)\|_{L^\infty} + \|v(s)\|_{L^\infty}) ds. \tag{6.12}
$$

By (6.5) and the assumption of $\{v_k\}$, we have

$$
\|u\|_{BC_{(1-\alpha)/2,T}} = \|v\|_{BC_{(1-\alpha)/2,T}} < X_\infty.
$$

Now we take a non-negative number t_* in $(0, T)$ and set $W = \|w\|_{BC_{0,t_*}}$. Then we have

$$
\begin{aligned}
\|w(t)\|_{L^\infty} &\le 2C_1 \int_0^t (t-s)^{-1/2} W s^{-(1-\alpha)/2} X_\infty ds \\
&\le 2C_1 B' t^{\alpha/2} X_\infty W.
\end{aligned} \tag{6.13}
$$

Taking supremum for $0 < t < t_*$ in (6.13), we have

$$
W \le 2C_1 B' t_*^{\alpha/2} X_\infty W. \tag{6.14}
$$

Now we assume that $t_* < (2C_1 B' X_\infty)^{-2/\alpha}$. Then we have $W = 0$ and $w(t,x) = 0$ for almost every $(t,x) \in (0,t_*) \times \mathbb{R}^n_+$. Since X_∞ depends only on T and $[v_0]_{\mathrm{BMO}}$, we can extend the function $w(s)$ from BC_{0,t_*} onto $BC_{0,T}$, by repeating the procedure (6.11-6.14) for T/t^* times. So we finally obtain $\|w\|_{BC_{0,T}} = 0$ and complete the proof. $\square$

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