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WEAK-TYPE L^∞-BMO ESTIMATE OF FIRST-ORDER SPACE DERIVATIVES OF STOKES FLOW IN A HALF SPACE

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Abstract

We prove the L^∞-BMO boundness of first-order space derivatives of Stokes flow in a half space. To show the estimate, we apply the solution formula of Stokes equation in a half space, which is a modified version of Ukai's formula.

Keywords

Partial differential equations, Fourier analysis, Fluid mechanics, Stokes equation, BMO-function, Hardy space Riesz transform.

§1 Introduction

We consider the Stokes equation in the half space
$\mathbb{R}^n_+ = \{x = (x_1, \cdots, x_n) \in \mathbb{R}^n : x_n > 0\}$ $(n \geq 2)$:

$$
\begin{aligned}
(1.1) \qquad & u_t - \Delta u + \nabla p = 0, \operatorname{div} u = 0 \text{ in } \mathbb{R}^n_+ \times (0, \infty), \\
& u = u_0 \text{ at } t = 0, \\
& u = 0 \text{ on } \partial\mathbb{R}^n_+ \times (0, \infty).
\end{aligned}
$$

Here $u = (u^1, \ldots, u^n)$ is the unknown velocity field and p is the unknown pressure field. The initial data u_0 is assumed to satisfy a *compatibility* *condition* : $\mathrm{div} u_0 = 0$ in $\mathbb{R}^n_+$ and the normal component of u_0 equals zero on $\partial\mathbb{R}^n_+ = \{x_n = 0\}$.

This system is a typical parabolic-like equation and it has several properties resembling the heat equation. It is known that the Stokes equation in the whole space $\mathbb{R}^n$ can be reduced to the heat equation with initial data u_0 and we have the regularity-decay estimate

$$\|\nabla u(t)\|_{L^p(\mathbb{R}^n)} \le Ct^{-1/2}\|u_0\|_{L^p(\mathbb{R}^n)} \text{ for } t > 0, \tag{1.2}$$

for all $1 \le p \le \infty$ with C independent of t and u_0, where ∇ denotes the gradient in space variables.

In [13], we have proved the L^∞-estimate of first-derivatives of Stokes flow with zero boundary condition in a half space:

$$\|\nabla u(t)\|_{L^\infty(\mathbb{R}^n_+)} \le Ct^{-1/2}\|u_0\|_{L^\infty(\mathbb{R}^n_+)} \text{ for } t > 0, \tag{1.3}$$

where u_0 is the initial data. In this paper, we improve (1.3) by replacing the right hand side by the norm of bounded mean oscillation space BMO.

Theorem 1.1. *There exists a function $U = U(t,x)$ such that $U(t) \in L^\infty(\mathbb{R}^n)$ for all $t > 0$, $U|_{\mathbb{R}^n_+}$ equals to the solution of the Stokes equation in $\mathbb{R}^n_+$ with initial data $u_0 \in \mathrm{BMO}(\mathbb{R}^n_+)$ and such that*

$$\sum_{j=1}^{n}\left|\int_{\mathbb{R}^n} \partial_j U(t,x)\cdot\phi(x)dx\right| \le Ct^{-1/2}[u_0]_{\mathrm{BMO}}\|\phi\|_{L^1(\mathbb{R}^n)}, \tag{1.4}$$

for all $t > 0$, where ϕ is in $C_0^\infty(\mathbb{R}^n)$ and C is a constant independent of ϕ andu_0.

The estimate (1.4) means that the first derivatives of solution of the Stokes equation is well-defined in sense of distribution when u_0 is in BMO.

Before explaining our problem, we recall the known results for the half space. First, Ukai [15] showed $\|\nabla u(t)\|_{L^p(\mathbb{R}^n)} \le Ct^{-1/2}\|u_0\|_{L^p(\mathbb{R}^n)}$ for the case $1 < p < \infty$ by estimating the representation of solutions in L^p. In the case $p = 1$ or $p = \infty$, the estimates do not follow from Ukai's method because the solution involves singular integral operators such as Riesz transforms which are not bounded in L^1 or L^∞. Instead of L^1, by using the formula in the Hardy space $\mathcal{H}^1$ of Fefferman and Stein (1.4) for $p=1$ was established by Giga-Matsui-Shimizu [8]. Moreover, Shimizu [13] showed (1.4) for $p=\infty$ by applying the modified version of Ukai's formula.

We have two motivations for the estimate (1.2). First, we want to apply the estimate to the integral equation which is formally equivalent to the Navier-Stokes equations

$$u(t,x) = (e^{-tA}u_0)(x) - \int_0^t (e^{(s-t)A}P\nabla\cdot u(s)\otimes u(s))(x)ds, \tag{1.5}$$

where e^{-tA} is a solution operator of the Stokes equation in the half space and P is a projection associated with the Helmholtz decomposition in the half space. P is constructed from some Riesz transforms which are not bounded in L^∞. However, Riesz transforms are bounded in BMO, so (1.4) may be useful to solve the problem (1.5).

Second motivation comes from the duality arguement. In fact in [8], we have proved (1.2) for $p=1$ by more strong estimate

$$\|\nabla u(t)\|_{\mathcal{H}^1} \le Ct^{-1/2}\|u_0\|_p \text{ for } t>0. \tag{1.6}$$

Since the dual space of $\mathcal{H}^1$ is BMO, (1.4) can be regarded as the dual estimate of (1.6) although (1.4) does not follow from (1.6) directly.

An idea of this paper is to apply the modified version of Ukai's formula for ∇u obtained by Shimizu [13]. We can extend the formula in [13] in such a way that the terms involving the square root of the tangential Laplacian $\Lambda := -\sum_{i=1}^n \partial^2/\partial x_i^2$ have no singularities on $\{x_n=0\}$. By the duality arguement, it is sufficient to estimate the corresponding integral kernels in $\mathcal{H}^1$ and we have established to estimate the terms involving Λ. There remain the terms without Λ which contain singularities on $\{x_n=0\}$. In [13], we treated the corresponding integral kernels in L^1. However, by deeper analysis, we are able to estimate these terms by investigating their integral kernels in $\mathcal{H}^1$.

The proof of our theorem is divided in three sections. In section 2, we refine the solution formula obtained by Ukai [15]. We can eliminate some of these singularities in Ukai's formula by extending solutions u to $\{x_n<0\}$ as the odd function, so that the terms involving Λ have no singularities on $x_n = 0$. In section 3, we define the Hardy space and the BMO-space in $\mathbb{R}^n_+$. We also recall the duality between $\mathcal{H}^1$ and BMO. It shall be noted that we do not use the definition of BMO directly in our proof. Finally, in section 4 we prove our thorem. By duality arguement, it is sufficient to estimate the corresponding integral kernels in the representation formula in $\mathcal{H}^1$. The $\mathcal{H}^1$-estimates of the kernels involving Λ are obtainded by Shimizu [13]. It remains to estimate the kernels without Λ. These kernels have singularities on $x_n = 0$. Moreover, the tangential parts of these kernels consist of the boundary integral on $\mathbb{R}^{n-1}$. However, we can handle these parts in $\mathcal{H}^1$ by a careful investigation. These estimates then provide $\mathcal{H}^1$-estimates for integral kernels which are needed in estimating the terms in L^∞.

§2 Solution formula

In this section, we recall a new solution formula of (1.1) by Shimizu [13] and construct the functional in (1.2).

First, we fix some notation. For an n-dimensional vector a, we denote the tangential component $(a_1, \ldots, a_{n-1})$ by $a' \in \mathbb{R}^{n-1}$, so that $a = (a', a_n)$. We set $\partial_j = \partial/\partial x_j$ and let $\nabla' = (\partial_1, \cdots, \partial_{n-1})$. Hereafter, C denotes a positive constant which may differ from one occasion to another.

Let $\mathcal{F}$ be the Fourier transform in $\mathbb{R}^n$:

$$\mathcal{F}f(\xi) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x)dx.$$

The Riesz operators R_j $(j = 1, \ldots, n)$, and the operator Λ are defined by

$$\mathcal{F}(R_j f)(\xi) = \frac{i\xi_j}{|\xi|}\mathcal{F}f(\xi),$$
$$\mathcal{F}(\Lambda f)(\xi) = |\xi'|\mathcal{F}f(\xi).$$

We set $R' = (R_1, \ldots, R_{n-1})$.

We also define the operator $E(t)$, $F(t)$, and $H(t)$ by

$$[E_t f](x) = \int_{\mathbb{R}^n_+} \{G_t(x-y) - G_t(x'-y', x_n+y_n)\} f(y)dy,$$

$$[F_t f](x) = \int_{\mathbb{R}^n_+} \{G_t(x-y) + G_t(x'-y', x_n+y_n)\} f(y)dy,$$

$$[H_t f](x) = \int_{\mathbb{R}^n} G_t(x-y) f(y)dy,$$

where G_t is the Gauss kernel $G_t(x) = (4\pi t)^{-n/2} e^{-|x|^2/4t}$. Furthermore, we define the operator E_{t+} by

$$[E_{t+} f](x) = \begin{cases} [E_t f](x) & \text{for } x_n > 0, \\ [E_t f](x', -x_n) & \text{for } x_n < 0. \end{cases}$$

Note that $z = E_t f$ (resp. $F_t f$) solves the heat equation in $\mathbb{R}^n_+$ with zero-Dirichlet (resp. zero-Neumann) boundary condition;

$$\begin{aligned} z_t - \Delta z &= 0 \text{ in } \mathbb{R}^n_+ \times (0,T), \\ z|_{t=0} &= f, \\ z|_{x_n=0} &\equiv 0. \text{ (resp. } \partial_n z|_{x_n=0} = 0.) \end{aligned}$$

Moreover note that the functions $[E_t f](x)$ and $[F_t f](x)$ can be defined for all x in $\mathbb{R}^n$.

Let $f(x)$ be a function defined in $\mathbb{R}^n_+$. Then we denote the odd (resp. even) extension of f by $\tilde{f}$ (resp. $\bar{f}$), i.e.

$$\tilde{f}(x) = \begin{cases} f(x) & \text{for } x_n > 0, \\ -f(x', -x_n) & \text{for } x_n < 0, \end{cases}$$

$$\bar{f}(x) = \begin{cases} f(x) & \text{for } x_n > 0, \\ f(x', -x_n) & \text{for } x_n < 0. \end{cases}$$

Finally, we denote the characteristic function of $\{x_n > 0\}$ (resp. $\{x_n < 0\}$) by χ_+ (resp. χ_-), i.e.

$$\chi_+(x_n) = \begin{cases} 1 & \text{for } x_n > 0, \\ 0 & \text{for } x_n < 0, \end{cases}$$

$$\chi_-(x_n) = 1 - \chi_+(x_n).$$

Now we are ready to show the modified Ukai's formula obtained by Shimizu [13]. In this paper, we recall the formula for the space derivatives of solutions.

Theorem 2.1. *Assume that* u_0 *is in* $L^p(\mathbb{R}^n_+)$, $1 \le p \le \infty$ *and satisfies* $\mathrm{div} u_0 = 0$. *Let*

(2.1a)
$$\begin{aligned} U^n(t) =& -\Lambda(-\Delta)^{-1}\nabla' \cdot E(t)u_0' - \partial_n(-\Delta)^{-1}\nabla' \cdot E_+(t)u_0' \\ & + (-\Delta)^{-1}\Delta' E_+(t)u_0^n - \partial_n(-\Delta)^{-1}\Lambda E(t)u_0^n, \end{aligned}$$

(2.1b)
$$\begin{aligned} U'(t) =& E(t)u_0' + \Lambda^{-1}\nabla' E(t)u_0^n \\ & + \nabla'(-\Delta)^{-1}\{\nabla' \cdot E_+(t)u_0'\} - \partial_n(-\Delta)^{-1}\Lambda^{-1}\nabla'\{\nabla' \cdot E(t)u_0'\} \\ & - \nabla'(-\Delta)^{-1}\{\Lambda E(t)u_0^n\} + \partial_n(-\Delta)^{-1}\nabla' E_+(t)u_0^n. \end{aligned}$$

Then U *is a function defined in* $\mathbb{R}^n$ *and* $U|_{\mathbb{R}^n_+}$ *satisfies* (1.1) *in* $\Omega = \mathbb{R}^n_+$.

Theorem 2.2. *Let* U *be a function in Theorem 2.1. Then*

(2.1a)
$$\begin{aligned} \partial_j U^n =& -R_j R' \cdot \Lambda E(t)u_0' + R_j R_n \nabla' \cdot E_+(t)u_0' \\ & - R_j R' \cdot \nabla' E_+(t)u_0^n - R_j R_n \Lambda E(t)u_0^n, \end{aligned}$$

(2.1b)
$$\begin{aligned} \partial_j U' =& \partial_j E(t)u_0' + w_j \\ & + R_j R'\{\nabla' \cdot E_+(t)u_0'\} - R_j R_n \Lambda^{-1}\nabla'\{\nabla' \cdot E(t)u_0'\} \\ & - R_j R'\{\Lambda E(t)u_0^n\} + R_j R_n \nabla' E_+(t)u_0^n, \end{aligned}$$

where

(2.1c)
$$w_j = \begin{cases} \Lambda^{-1}\partial_j \nabla' E(t)u_0^n & \text{for } j < n, \\ \Lambda^{-1}\nabla'\{\nabla' \cdot F(t)u_0'\} & \text{for } j = n. \end{cases}$$

Note that the terms containing Λ do not contain E_+ (which has singularities at $x_n = 0$).

By duality arguement, we have the following theorem:

Theorem 2.3. *Let U be the function in Theorem 2.1 and let ϕ be in $C_0^\infty(\mathbb{R}^n)$. Then*

$$
\begin{aligned}
&\sum_{j=1}^{n}\int_{\mathbb{R}^n}\partial_j U(t,x)\cdot\phi(x)dx\\
&=-\sum_{j=1}^{n}\int_{\mathbb{R}^n}\Big\{\tilde{u_0'}(x)\cdot R'R_j\Lambda[H_t\phi^n](x)\\
&+\tilde{u_0'}(x)\cdot[H_t(\chi_+\nabla'R_jR_n\phi^n)](x)\\
&-\tilde{u_0'}(x)\cdot[H_t(\chi_-\nabla'R_jR_n\phi^n)](x)\\
&+\tilde{u_0^n}(x)\cdot[H_t(\chi_+\nabla'\cdot R'R_j\phi^n)](x)\\
&-\tilde{u_0^n}(x)\cdot[H_t(\chi_-\nabla'\cdot R'R_j\phi^n)](x)\\
&+\tilde{u_0^n}(x)\Lambda R_jR_n[H_t\phi^n](x)\Big\}dx\\
&-\sum_{j=1}^{n}\int_{\mathbb{R}^n}\tilde{u_0'}(x)\partial_j[H_t\phi'](x)dx\\
&+\sum_{j=1}^{n-1}\int_{\mathbb{R}^n}\tilde{u_0'}(x)\cdot\nabla'\partial_j\Lambda^{-1}[H_t\phi^n](x)dx\\
&+\int_{\mathbb{R}^n}\bar{u_0'}(x)\cdot\nabla'(\nabla'\cdot\Lambda^{-1}[H_t\phi'])(x)dx\\
&-\sum_{j=1}^{n}\int_{\mathbb{R}^n}\Big\{\tilde{u_0'}(x)\cdot[H_t\{\chi_+\nabla'(R_jR'\cdot\phi')\}](x)\\
&-\tilde{u_0'}(x)\cdot[H_t\{\chi_-\nabla'(R_jR'\cdot\phi')\}](x)\\
&+\tilde{u_0'}(x)\cdot\nabla'(R_jR_n\Lambda^{-1}\nabla'\cdot[H_t\phi'](x)\\
&+\tilde{u_0^n}(x)\Lambda R_jR'\cdot[H_t\phi'](x)\\
&+\tilde{u_0^n}(x)[H_t\{\chi_+\nabla'\cdot(R_jR_n\phi')\}](x)\\
&-\tilde{u_0^n}(x)[H_t\{\chi_-\nabla'\cdot(R_jR_n\phi')\}](x)\Big\}dx.
\end{aligned}
$$

§3 Study of bouded mean oscillation spaces

In this section, we introduce two function space that appear in our theorem and proof. First, we introduce the Hardy space $\mathcal{H}^1$ that is a subspace of L^1.

Definition 3.1. A function $f \in L^1(\mathbb{R}^n)$ belongs to the Hardy space $\mathcal{H}^1 = \mathcal{H}^1(\mathbb{R}^n)$ if

$$f^+(x) = \sup_{s>0} |G_s * f(x)| \in L^1(\mathbb{R}^n), \tag{3.1}$$

where the symbol $*$ denotes the convolution with respect to the space variable x. The norm of $f \in \mathcal{H}^1(\mathbb{R}^n)$ is defined by

$$\|f\|_{\mathcal{H}^1} := \|f^+\|_{L^1(\mathbb{R}^n)} \tag{3.2}$$

Next, we define the space of "Bounded Mean Oscillation" BMO.

Definition 3.2. A function g belongs to the space of bounded mean oscillation BMO if $g \in L^1_{loc}(\mathbb{R}^n)$ and

$$[g]_{\mathrm{BMO}} = \sup_{Q\text{: cube}} \frac{1}{|Q|} \int_Q |g(x) - g_Q| dx < \infty, \tag{3.3}$$

where $|Q|$ means the lebesgue measure of Q and where g_Q means the mean of g on Q such that

$$g_Q = \frac{1}{|Q|} \int_Q g(x) dx. \tag{3.4}$$

Note that $[\cdot]_{\mathrm{BMO}}$ is semi-norm because $[C]_{\mathrm{BMO}} = 0$ for any constant function C. So we usually consider the quotient space $\mathrm{BMO}/\mathbb{R}$.

The definition of BMO seems to be complicated to apply. We do not use this definition but the duality characterization.

Proposition 3.3(Fefferman-Stein[6]. *Assume that $f \in \mathcal{H}^1$ and $g \in \mathrm{BMO}$. Then*

$$\Big| \int_{\mathbb{R}^n} f(x) g(x) dx \Big| \le C \|f\|_{\mathcal{H}^1} [g]_{\mathrm{BMO}}. \tag{3.5}$$

Corollary 3.4. *Assume that* $f \in \mathcal{H}^1$ *and* $g \in \mathrm{BMO}$*. Then the convolution function* $f * g$ *is in* L^∞ *and*

$$\|f * g\|_{L^\infty} \le C\|f\|_{\mathcal{H}^1}[g]_{\mathrm{BMO}}. \tag{3.6}$$

Proof. By the definition of convolution,

$$f * g(x) = \int_{\mathbb{R}^n} f(x-y)g(y)dy.$$

Let $f_x(y) = f(x-y)$. Then

$$\begin{aligned} G_s * f_x(y) &= \int_{\mathbb{R}^n} G_s(y-z)f_x(z)dz \\ &= \int_{\mathbb{R}^n} G_s(y-z)f(x-z)dz \\ &= \int_{\mathbb{R}^n} G_s(x+y-w)f(w)dw \\ &= G_s * f(x+y). \end{aligned}$$

So we have $f_x^+(y) = f^+(x+y)$ and $\|f_x\|_{\mathcal{H}^1} = \|f\|_{\mathcal{H}^1}$. Applying Proposition 3.3, we obtain (3.6). $\square$

We note that the Riesz operators R_j are bounded in $\mathcal{H}^1$ and BMO, i.e.

$$\begin{aligned} \|R_j f\|_{\mathcal{H}^1} &\le C_1\|f\|_{\mathcal{H}^1}, \\ [R_j g]_{\mathrm{BMO}} &\le C_2[g]_{\mathrm{BMO}}. \end{aligned}$$

Remark. We remark the BMO space in the half space. Assume that g is a function defined in the half space. A function g belongs to BMO if there exists an extension function over the whole space which is equal to g in the half space and belongs to BMO. The norm of g is defined as

$$[g]_{\mathrm{BMO}(\mathbb{R}^n_+)} := \inf_{G:\text{extension}} [G]_{\mathrm{BMO}}.$$

§4 Proof of theorem

Now we are ready to prove our theorem. By Proposition 3.3, it is sufficient to show that the integral kernels in Theorem 2.3 are in $\mathcal{H}^1$.

First we estimate the kernels without $\chi_\pm$. Note that the estimates of Gauss kernel in Hardy space has been obtained by Giga-Matsui-Shimizu [8].

Lemma 4.1. *Let G_t be the Gauss kernel. Then*

$$\|\partial_i G_t\|_{\mathcal{H}^1} \le Ct^{-1/2} \text{ for } 1 \le i \le n, \tag{4.1a}$$

$$\|\Lambda G_t\|_{\mathcal{H}^1} \le Ct^{-1/2}, \tag{4.1b}$$

$$\|\partial_j\partial_k\Lambda^{-1} G_t\|_{\mathcal{H}^1} \le Ct^{-1/2} \text{ for } 1 \le j,k \le n-1. \tag{4.1c}$$

By Lemma 4.1 and Corollary 3.4, we have

$$\begin{aligned} \|\partial_j H_t\phi\|_{\mathcal{H}^1} &\le \|\phi\|_{L^1(\mathbb{R}^n)}\|\partial_j G_t\|_{\mathcal{H}^1} \\ &\le Ct^{-1/2}\|\phi\|_{L^1(\mathbb{R}^n)}, \end{aligned} \tag{4.2a}$$

$$\|\partial_j\partial_k\Lambda^{-1}E(t)\phi\|_{\mathcal{H}^1} \le Ct^{-1/2}\|\phi\|_{L^1(\mathbb{R}^n)}, \tag{4.2b}$$

$$\|\partial_j\partial_k\Lambda^{-1}f(t)\phi\|_{\mathcal{H}^1} \le Ct^{-1/2}\|\phi\|_{L^1(\mathbb{R}^n)}. \tag{4.2c}$$

By the boundedness of the Riesz operators in the Hardy space, we have

$$\begin{aligned} \|R_jR_k\Lambda H_t u_0\|_{L^\infty} &\le \|\phi\|_{L^1(\mathbb{R}^n)}\|R_jR_k\Lambda G_t\|_{\mathcal{H}^1} \\ &\le C\|\phi\|_{L^1(\mathbb{R}^n)}\|\Lambda G_t\|_{\mathcal{H}^1} \\ &\le Ct^{-1/2}\|\phi\|_{L^1(\mathbb{R}^n)}. \end{aligned} \tag{4.3}$$

Next, we estimate the terms containing $\chi_\pm$, i.e. $[H_t(\chi_\pm\partial_i R_j R_k\phi)](x)$, where $1 \le i \le n-1$ and $1 \le j,k \le n$. We may assume $j \ne n$, because if $j = k = n$, then we can reduce to $j \ne n$ by using the property of the Riesz kernels, i.e.

$$\sum_{\alpha=1}^{n} R_\alpha^2 = -I.$$

Since R_jR_k equals to $\partial_j\partial_k(-\Delta)^{-1}$ and the integral kernel of $(-\Delta)^{-1}$ is $c_n|x|^{-n+2}$, we have

(4.4)
$$\begin{aligned}
&[H_t(\chi_+\partial_i R_j R_k\phi)](x)\\
=&-\delta_{kn}\int_{\mathbb{R}^n}\phi(z)\int_{\mathbb{R}^{n-1}}\frac{C_{n-2}}{|(z'-y',z_n)|^{n-2}}(\partial_i\partial_j G_t)(x'-y',x_n)dy'dz\\
&-\int_{\mathbb{R}^n}\phi(z)\int_{\mathbb{R}^n_+}\frac{C_{n-2}}{|z-y|^{n-2}}(\partial_i\partial_j\partial_k G_t)(x-y)dydz\\
=&-\delta_{kn}\int_{\mathbb{R}^n}\phi(z)I_{1,t}(x,z)dz-\int_{\mathbb{R}^n}\phi(z)I_{2,t}(x,z)dz
\end{aligned}$$

where δ_{kn} is Kronecker's delta.

The term I_2 is essentially same as the recent case, so we can estimate the second term such as Lemma 4.1.

Now we show the estimate of the first term.

Lemma 4.2. *Assume that a real parameter α and β satisfies $0\le\alpha\le n$ and $\beta\ge 0$. Then there exists a constant $C=C_{n,\alpha,\beta}$ independent of $x\in\mathbb{R}^n$ and $t\ge 0$ such that*

$$|I_{1,t}(x,z)|\le Ct^{(\alpha+\beta-n-1)/2}|x'-z'|^{-\alpha}|x_n|^{-\beta} \tag{4.5}$$

Proof. By the parameter arguement, it suffices to show (4.5) for $t=1$.

Let ψ_1 be a smooth function with supp$\psi_1\subset B(0,1)$, $0\le\psi_1\le 1$ and $\psi_1|_{B(0,1/2)}=1$. Let ψ_2 be $\psi_2=1-\psi_1$. Then we have

(4.6)
$$\begin{aligned}
I_{1,1}&=\sum_{l=1}^{2}\int_{\mathbb{R}^{n-1}}\frac{C\psi_l(z'-y',z_n)}{|(z'-y',z_n)|^{n-2}}(\partial_{y_i}\partial_{y_j}G_1)(x'-y',x_n)dy'\\
&=J_1+J_2.
\end{aligned}$$

First we estimate the term J_1. We have

(4.7)
$$\begin{aligned}
&|J_1|\\
\le&\int_{|(z'-y',z_n)|\le 1}\frac{C}{|(z'-y',z_n)|^{n-2}}\Big(\frac12+\frac14|x'-y'|^2\Big)e^{-|x'-y'|^2/4}e^{-x_n^2/4}dy'.
\end{aligned}$$

Since $|z'-y'| \le |(z'-y',z_n)| \le 1$, we have $|x'-y'| \le |x'-z'|+1$ and $|x'-y'|^2 \ge (|x'-z'|^2-2)/2$. Therefore we have

$$\begin{aligned} |J_1| &\le \int_{|(z'-y',z_n)|\le 1} \frac{C}{|(z'-y',z_n)|^{n-2}} \Big\{ \frac{1}{2} + \frac{1}{4}(|x'-z'|^2+1) \Big\} \\ &\qquad e^{-|x'-z'|^2/8+1/4} e^{-x_n^2/4} dy' \\ &\le C|x'-z'|^{-\alpha}|x_n|^{-\beta} \end{aligned} \tag{4.8}$$

for $\alpha \ge 0$ and $\beta \ge 0$.

Now we show the estimate of the term J_2. Integrating partially, we have

$$\begin{aligned} J_2 &= C\int_{\mathbb{R}^{n-1}} \partial_{w_i}\partial_{w_j}\psi_2(z'-x'+w',z_n)|(z'-x'+w',z_n)|^{-n+2} \\ &\qquad G_1(w',x_n)dw' \\ &+ C\int_{\mathbb{R}^{n-1}} \big\{\partial_{w_i}\psi_2(z'-x'+w',z_n)\partial_{w_j}|(z'-x'+w',z_n)|^{-n+2} \\ &\qquad +\partial_{w_j}\psi_2(z'-x'+w',z_n)\partial_{w_i}|(z'-x'+w',z_n)|^{-n+2}\big\} \\ &\qquad G_1(w',x_n)dw' \\ &+ C\int_{\mathbb{R}^{n-1}} \psi_2(z'-x'+w',z_n)\big\{\partial_{w_i}\partial_{w_j}|(z'-x'+w',z_n)|^{-n+2}\big\} \\ &\qquad G_1(w',x_n)dw' \\ &= J_{21}+J_{22}+J_{23}. \end{aligned} \tag{4.9}$$

Since the support of $\nabla\psi_2$ is compact, We can obtain the estimate of J_{21} and J_{22} by the same method on J_1. So we have

$$|J_{2l}| \le C|x'-z'|^{-\alpha}|x_n|^{-\beta} \tag{4.10}$$

for $l=1,2$, $\alpha \ge 0$ and $\beta \ge 0$. Finally, we estimate the term J_{23}. We have

$$|J_{23}| \le C\int_{|(z'-x'+w',z_n)|\ge 1/2} |(z'-x'+w',z_n)|^{-n} G_1(w',x_n)dw'. \tag{4.11}$$

Since $|x'-z'|^\alpha \le C(|x'-z'-w'|^\alpha + |w'|^\alpha) \le C(|(z'-x'+w', z_n)|^\alpha + |w'|^\alpha)$ for $\alpha \ge 0$, we have

$$
\begin{aligned}
&|J_{23}| \\
&\le C|x'-z'|^{-\alpha} \int_{|(z'-x'+w',z_n)|\ge 1/2} (|(z'-x'+w', z_n)|^{-n+\alpha} \\
&\qquad + |(z'-x'+w', z_n)|^{-n}|w'|^\alpha) G_1(w', x_n) dw' \\
&\le C|x'-z'|^{-\alpha} \int_{|(z'-x'+w',z_n)|\ge 1/2} (1 + |w'|^\alpha) G_1(w', x_n) dw' \\
&\le C|x'-z'|^{-\alpha} e^{-|x_n|^2/4} \\
&\le C|x'-z'|^{-\alpha} |x_n|^{-\beta}
\end{aligned}
\tag{4.12}
$$

for $0 \le \alpha \le n$ and $\beta \ge 0$.

Combining the estimate J_{21}, J_{22}, J_{23}, and J_1, we finally obtain

$$
|I_{1,1}| \le C|x'-z'|^{-\alpha}|x_n|^{-\beta} \tag{4.13}
$$

for $0 \le \alpha \le n$ and $\beta \ge 0$. $\square$

Finally we show the key lemma for the main theorem.

Lemma 4.3. *There exists a constant C depending only on n such that*

$$
\|I_{1,t}(\cdot, z)\|_{\mathcal{H}^1} \le Ct^{-1/2}. \tag{4.14}
$$

Proof. By Lemma 4.2, we have

$$
\begin{aligned}
|G_s * I_{1,t}(x,z)| =& |I_{1,s+t}(x,z)| \\
\le& C(s+t)^{(\alpha+\beta-n-1)/2} |x'-z'|^{-\alpha}|x_n|^{-\beta} \\
\le& Ct^{(\alpha+\beta-n-1)/2} |x'-z'|^{-\alpha}|x_n|^{-\beta},
\end{aligned}
\tag{4.15}
$$

where α and β satisfies the assumption in Lemma 4.2. Therefore we obtain

$$
\begin{aligned}
&\|I_{1,t}(\cdot, z)\|_{\mathcal{H}^1} \\
&\le \sum_{k=1}^4 C_{n,l} t^{(\alpha+\beta-n-1)/2} \int_{\Omega_k} |x'-z'|^{-\alpha}|x_n|^{-\beta} dx,
\end{aligned}
\tag{4.16}
$$

where

$$\Omega_1 = \{|x' - z'| \le t^{1/2}, |x_n| \le t^{1/2}\},$$
$$\Omega_2 = \{|x' - z'| > t^{1/2}, |x_n| \le t^{1/2}\},$$
$$\Omega_1 = \{|x' - z'| \le t^{1/2}, |x_n| > t^{1/2}\},$$
$$\Omega_2 = \{|x' - z'| > t^{1/2}, |x_n| > t^{1/2}\}.$$

We estimate the integrals on the right-hand side of (4.16), taking $\alpha = 0$ and $\beta = 0$ for $k = 1$, $\alpha = n$ and $\beta = 0$ for $k = 2$, $\alpha = 0$ and $\beta = n$ for $k = 3$, and $\alpha = n - 1/2$ and $\beta = 3/2$ for $k = 4$, to find that the integrals of (4.16) are all bounded above by a constant multiple of $t^{-1/2}$. This proves (4.14). □

By Lemma 4.4 and Corollary 3.4, we obtain

$$\|[H_t(\chi_+ \partial_i R_j R_k \phi)]\|_{\mathcal{H}^1} \le Ct^{-1/2} \|\phi\|_{L^1(\mathbb{R}^n)}. \tag{4.17}$$

Combining the estimates in (4.2), (4.3), and (4.17), we finally obtain the desired estimate

$$\sum_{j=1}^{n} \Big| \int_{\mathbb{R}^n} \partial_j U(t,x) \cdot \phi(x) dx \Big| \le Ct^{-1/2} [u_0]_{\mathrm{BMO}} \|\phi\|_{L^1(\mathbb{R}^n)}.$$

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