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Abstract

New symmetries of the Painlevé differential equations, called folding transformations, are determined. These transformations are not birational but algebraic transformations of degree 2, 3, or 4. These are associated with quotients of the spaces of initial conditions of each Painlevé equation. We make the complete list of such transformations up to birational symmetries. We also discuss correspondences of special solutions of Painlevé equations.

1 Introduction

The Painlevé equations, except for the first one, admit classical solutions or algebraic solutions when parameters contained in the equations take special values. For example, the second Painlevé equation P_{II}:

$$\frac{\mathrm{d}^2 q}{\mathrm{d}t^2} = 2q^3 + tq + a, \tag{1.1}$$

has an algebraic solution

$$q \equiv 0, \tag{1.2}$$

when $a = 0$; and when $a = -1/2$, a solution of the Riccati equation:

$$\frac{\mathrm{d}q}{\mathrm{d}t} = -q^2 - \frac{t}{2},$$

solves P_{II}. On the other hand, P_{II} has the Bäcklund transformations generated by

$$\begin{aligned} \pi: \quad & q \mapsto -q, \\ s_1: \quad & q \mapsto q + \frac{a+\frac{1}{2}}{\frac{\mathrm{d}q}{\mathrm{d}t} + q^2 + \frac{t}{2}}; \end{aligned}$$

see the section nine below. It follows that P_{II} admits a classical solution if $a+\frac{1}{2}$ is an integer, while it has a rational solution if a is an integer; see [11].

Besides the Bäcklund transformations, P_{II} with the special value, $a=0$, can be converted to that with $a=-\frac{1}{2}$. In fact, considering the equation

$$\frac{\mathrm{d}^2 q}{\mathrm{d}t^2} = 2q^3 + tq, \tag{1.3}$$

put

$$Q = \frac{2}{q}\frac{\mathrm{d}q}{\mathrm{d}t}. \tag{1.4}$$

Then we can show that $Q = Q(t)$ satisfies the differential equation:

$$\frac{\mathrm{d}^2 Q}{\mathrm{d}t^2} = \frac{1}{2}Q^3 - 2tQ + 2, \tag{1.5}$$

which is reduced again to P_{II} by means of a suitable change of scales. This fact was already pointed out by Gambier [2] and has been noticed in recent investigations; see [5, 13, 15].

We firstly make a study of the transformation: $q \mapsto Q$, given by (1.4). It is obvious that the equation (1.3) remains invariant under the exchange,

$$\pi : q \mapsto -q, \tag{1.6}$$

of a sign of the variable. We are interested in a differential equation satisfied by the variable

$$x = q^2, \tag{1.7}$$

invariant under the action of π. We obtain in fact

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = \frac{1}{2x}\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + 4x^2 + 2tx. \tag{1.8}$$

Since the general solution of (1.3) is meromorphic on the whole complex plane $\mathbb{C}$, so is the general solution of (1.8). Thus the equation (1.8) should be reduced to one of the Painlevé equations through a certain transformation. We can show that such transformation is given as follows:

$$\begin{aligned} Q &= \frac{1}{x}\frac{\mathrm{d}x}{\mathrm{d}t}, \\ x &= \frac{1}{4}\frac{\mathrm{d}Q}{\mathrm{d}t} + \frac{Q^2}{8} - \frac{t}{2}, \end{aligned}$$

and then we obtain (1.5). Remark that transformation (1.4) is not canonical one, since it relays (1.3) to (1.5) via (1.7); particular solution (1.2) is nothing but the fixed point of π.

We can regard transformation (1.4) through (1.7) as folding Painlevé equation (1.3) along special solution (1.2); by folding the equation up, we obtain again Painlevé equation (1.5). The transformation considered above is an example of the *folding transformations* of the Painlevé equations; the definition is given below in the second section. Note that (1.3) has rational solution (1.2) and (1.5) admits a particular solution defined by

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = -\frac{1}{2}Q^2 + 2t. \tag{1.9}$$

Therefore we see that folding transformation (1.4) combines the equation having rational solution (1.2) to that having Riccati solution (1.9).

In the present article, we deal with the Painlevé systems, that is, the Hamiltonian systems associated with the Painlevé equations. The transformation considered in what follows is that between the canonical variables of the system. It is well-known that birational canonical transformations of each of the Painlevé systems are placed under control of the affine Weyl group; see [8, 11]. A folding transformation of the system is neither birational nor canonical, but is contact one; this keeps algebraically Hamiltonian systems invariant.

The aim of this paper is to determine the whole list of folding transformations of the Painlevé systems by means of a viewpoint of algebraic geometry. Recall that, to each of the Painlevé systems, we associate the pair of a rational surface and a divisor, as the space of initial conditions. Such a space has been investigated by the second author ([10]), and the third author has classified space of initial conditions, starting from a pair of a rational surface and a divisor satisfying some suitable conditions ([14]). The space obtained in [14] are corresponding not only to the Painlevé systems but also to the difference Painlevé systems, q-difference Painlevé systems, and elliptic Painlevé systems.

Among them, we take interest in the case of Painlevé differential equations. From the viewpoint of algebraic geometry, we see that there are *eight* types of the Painlevé equations. Precisely, we can classify space of initial conditions of second order differential equations in eight types, which can be represented by the use of the Dynkin diagrams. The sixth Painlevé equation can be represented by the extended Dynkin diagram $D_4^{(1)}$, the fifth one by $D_5^{(1)}$, the fourth one by $E_6^{(1)}$, the second one by $E_7^{(1)}$, and the first one by $E_8^{(1)}$. The third Painlevé equation consists of three types, $D_6^{(1)}$, $D_7^{(1)}$, and $D_8^{(1)}$. In this paper we adopt this classification of the Painlevé equations. The degeneration of the eight equations are as follows:

$$\begin{array}{ccccccccc}
D_4^{(1)} & \to & D_5^{(1)} & \to & D_6^{(1)} & \to & D_7^{(1)} & \to & D_8^{(1)} \\
 & & & \searrow & & \searrow & & & \\
 & & & & E_6^{(1)} & \to & E_7^{(1)} & \to & E_8^{(1)}
\end{array}$$

For the third Painlevé equation:

$$\frac{\mathrm{d}^2 q}{\mathrm{d}t^2} = \frac{1}{q}\left(\frac{\mathrm{d}q}{\mathrm{d}t}\right)^2 - \frac{1}{t}\frac{\mathrm{d}q}{\mathrm{d}t} + \frac{1}{t}(aq^2+b) + cq^3 + \frac{d}{q},$$

we give in the following table the correspondence between the Dynkin diagrams and the three types of the equation:

$D_6^{(1)}$	$cd \neq 0$
$D_7^{(1)}$	$c=0, d\neq 0$ or $c \neq 0, d=0$
$D_8^{(1)}$	$c=d=0$

In the present paper, we distinguish these equations and write as follows:

$$P_{\mathrm{III}}^{D_6^{(1)}}, \quad P_{\mathrm{III}}^{D_7^{(1)}}, \quad P_{\mathrm{III}}^{D_8^{(1)}}.$$

By means of a suitable change of scales, the number of parameters contained in these equations are respectively

$$2, \quad 1, \quad 0.$$

Researches on P_{III} precedent to ours have been concerned mainly with $P_{\mathrm{III}}^{D_6^{(1)}}$, the generic case of the third Painlevé equation; see [11]. From the viewpoint of algebraic geometry and of Hamiltonian structure, it is necessary and quite natural to study these three cases separately. These three equations will be investigated in the forthcoming paper [9].

Besides the case of P_{II}, several examples of folding transformations have been known; for example a transformation between P_{V} and $P_{\mathrm{III}}^{D_6^{(1)}}$ has been given already by P. Painlevé ([12]). This transformation is a particular case of that given below in the section four. We cite the recent result [5] by A. V. Kitaev; he has studied quadratic transformations of a 2×2-Schlesinger system, whose monodromy preserving deformation is governed by the sixth Painlevé equation. In [15] the third Painlevé equation has been considered. Moreover, Ramani et al. have studied such transformations for difference Painlevé equations and q-difference Painlevé equations as well as for the Painlevé equations; see [13].

Birational transformation for the Painlevé systems can be reduced to contiguity relations for classical functions such as the Gauß hypergeometric functions, when we consider classical solutions of the Painlevé systems. On the other hand, in addition to contiguity relations, the Gauß hypergeometric function $F(a,b,c;x)$ admits quadratic transformations, called the Goursat transformations; for example,

$$F\left(a,b,\frac{a+b+1}{2};x\right) = F\left(\frac{a}{2},\frac{b}{2},\frac{a+b+1}{2};4x(1-x)\right).$$

The folding transformations for the Painlevé systems can be regarded as generalization of the Goursat transformations. We have stated above some results on quadratic transformations of the second Painlevé system. We consider now the fourth Painlevé system associated to the fourth Painlevé equation:

$$\frac{\mathrm{d}^2 q}{\mathrm{d}t^2} = \frac{1}{2q}\left(\frac{\mathrm{d}q}{\mathrm{d}t}\right)^2 + \frac{3}{2}q^3 + 4tq^2 + 2(t^2 - a)q + \frac{b}{q}, \tag{1.10}$$

which has a cubic transformation when, for example,

$$a = 0, \quad b = -\frac{2}{9}. \tag{1.11}$$

Equation (1.10) is equivalent to the Hamiltonian system:

$$\frac{\mathrm{d}q}{\mathrm{d}t} = \frac{\partial H}{\partial p}, \quad \frac{\mathrm{d}p}{\mathrm{d}t} = -\frac{\partial H}{\partial q}, \tag{1.12}$$

with

$$H = (p - q - 2t)pq - 2\alpha_1 p - 2\alpha_2 q. \tag{1.13}$$

If $\alpha_1 = \alpha_2 = 1/3$, which is equivalent to (1.11), then (1.12) is invariant under the transformation

$$\pi : q \mapsto -p, \quad p \mapsto -p + q + 2t. \tag{1.14}$$

Note that the order of π is three and the fixed point of π is the rational solution of (1.12):

$$q = -\frac{2}{3}t \quad p = \frac{2}{3}t.$$

By considering the variable

$$x = (p - q - 2t)pq - \frac{8}{27}t^3, \tag{1.15}$$

invariant under (1.14), we obtain the differential equation:

$$\left(\frac{\mathrm{d}^2 x}{\mathrm{d}t^2}\right)^2 + 12\left(x - t\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + 6\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^3 = 0. \tag{1.16}$$

On the other hand, Hamiltonian (1.13) satisfies the equation:

$$\left(\frac{\mathrm{d}^2 H}{\mathrm{d}t^2}\right)^2 - 4\left(H - t\frac{\mathrm{d}H}{\mathrm{d}t}\right)^2 + 2\frac{\mathrm{d}H}{\mathrm{d}t}\left(\frac{\mathrm{d}H}{\mathrm{d}t} + 4\alpha_1\right)\left(\frac{\mathrm{d}H}{\mathrm{d}t} - 4\alpha_2\right) = 0; \tag{1.17}$$

see [11]. By putting

$$X = -(-3)^{\frac{3}{4}}x, \quad s = (-3)^{\frac{1}{4}}t, \tag{1.18}$$

we deduce from (1.16) the equation:

$$\left(\frac{\mathrm{d}^2 X}{\mathrm{d}s^2}\right)^2 - 4\left(X - s\frac{\mathrm{d}X}{\mathrm{d}s}\right)^2 + 2\left(\frac{\mathrm{d}X}{\mathrm{d}s}\right)^3 = 0,$$

that is equation (1.17) with $\alpha_1 = \alpha_2 = 0$. Since, for a solution H of (1.17), a pair of functions (q, p) given by

$$
\begin{aligned}
q &= \frac{\frac{\mathrm{d}^2 H}{\mathrm{d}t^2} + 2\left(H - t\frac{\mathrm{d}H}{\mathrm{d}t}\right)}{2\left(\frac{\mathrm{d}H}{\mathrm{d}t} - 4\alpha_2\right)}, \\
p &= \frac{\frac{\mathrm{d}^2 H}{\mathrm{d}t^2} - 2\left(H - t\frac{\mathrm{d}H}{\mathrm{d}t}\right)}{2\left(\frac{\mathrm{d}H}{\mathrm{d}t} - 4\alpha_1\right)},
\end{aligned}
\tag{1.19}
$$

solves Hamiltonian system (1.12), then we obtain the folding transformation of the fourth Painlevé system, by combining (1.15), (1.18), and (1.19). The transformation converts the case of $\alpha_1 = \alpha_2 = 1/3$ to that of $\alpha_1 = \alpha_2 = 0$; see the section eight.

In the next section, we will give the definition of folding transformations and make the whole list of the transformations. In the rest of this paper, we investigate each of the Painlevé systems. Section 3 is devoted to the sixth Painlevé system and Section 4 to the fifth one. The third Painlevé systems $P_{\mathrm{III}}^{D_6^{(1)}}$, $P_{\mathrm{III}}^{D_7^{(1)}}$, and $P_{\mathrm{III}}^{D_8^{(1)}}$ are studied respectively in Section 5, 6, and 7. Section 8 concerns the fourth Painlevé system and we sum up in Section 9 results on the second one. The correspondence between the system admitting an algebraic solution and that having a classical solution is a subject of Section 11.

2 Folding transformations of the Painlevé equations

The birational symmetry of each Painlevé equation is completely determined by that of the space of initial conditions. The space of initial conditions is a fiber of a fiber bundle $\mathcal{P} = (E, \rho, B)$ equipped with a foliation $\mathcal{F}$ in $\mathcal{P}$, satisfying the following properties ([10]):

a) Each leaf of $\mathcal{F}$ intersects with each fiber transversally;

b) Each path γ on B can be lifted to a leaf γ_p that runs through a given point $p \in \rho^{-1}(\gamma(0))$;

c) $\rho|_{\gamma_p} : \gamma_p \to B$ is surjective and γ_p is a covering space of B by ρ.

Here $\mathcal{F}$ is associated with the Painlevé equation and $B = \mathbb{P}^1 \setminus \{\text{the fixed singular points}\}$.

The compactification of the space of initial conditions is a rational surface X with a unique anti-canonical divisor $D = \sum m_i D_i$ of canonical type; we call X a generalized Halphen surface ([14]). For each Painlevé equation, irreducible components of anti-canonical divisor D have an intersection form of type $D_l^{(1)}$, $l = 4, \ldots, 8$, or $E_k^{(1)}$, $k = 6, 7, 8$.

On the other hand, affine Weyl group symmetries for each Painlevé equation is also well known ([11]). These symmetries are birational and are described by the use of the group of

Cremona isometries of X. An automorphism σ of $\mathrm{Pic}(X)$ is called a Cremona isometry if the following properties are satisfied:

(Cr1) σ preserves the intersection form in $\mathrm{Pic}(X)$,

(Cr2) σ fixes the canonical class of X,

(Cr3) σ leaves the semi-group of effective classes $\mathrm{Pic}^+(X)$ invariant.

Note that X is parameterized by the parameters of the equation and by the independent variable t of the equation. We denote the group of Cremona isometries as Cr, which is determined as follows ([14]):

Table 1. R, Cr, and the Painlevé equations

R	$D_4^{(1)}$	$D_5^{(1)}$	$D_6^{(1)}$	$D_7^{(1)}$	$D_8^{(1)}$	$E_6^{(1)}$	$E_7^{(1)}$	$E_8^{(1)}$
$R^\perp$	$D_4^{(1)}$	$A_3^{(1)}$	$(2A_1)^{(1)}$	$A_1^{(1)}$	-	$A_2^{(1)}$	$A_1^{(1)}$	-
P_J	P_{VI}	P_{V}	$P_{\mathrm{III}}^{D_6^{(1)}}$	$P_{\mathrm{III}}^{D_7^{(1)}}$	$P_{\mathrm{III}}^{D_8^{(1)}}$	P_{IV}	P_{II}	P_{I}
Cr	$\widetilde{W}(D_4^{(1)})_\Lambda$	$\widetilde{W}(A_3^{(1)})_\Lambda$	$\widetilde{W}((2A_1)^{(1)})_\Lambda$	$\widetilde{W}(A_1^{(1)})$	C_2	$\widetilde{W}(A_2^{(1)})_\Lambda$	$\widetilde{W}(A_1^{(1)})_\Lambda$	1

Here R is a type of surface X (the intersection form), and P_J represents the J-th differential equation of Painlevé. The symbol $\widetilde{W}(R^\perp)$ denotes the extended affine Weyl group of type $R^\perp$ which is the group extension of affine Weyl group $W(R^\perp)$ by the group of Dynkin automorphisms, i.e. $\widetilde{W}(R^\perp) = \mathrm{Aut}(R^\perp) \ltimes W(R^\perp)$. We have used above the notation $G_S = \{g \in G \mid g(S) = S\}$ and denoted by Λ the set of classes of nodal curves disjoint from the irreducible components of $|-\mathcal{K}_X|$. In generic case, we have $\Lambda = \phi$ and $G_\Lambda = G$. If $\Lambda \neq \phi$ then there exists a Riccati type solution. Note that $(2A_1)^{(1)}$ represents the following intersection forms:

$$(2A_1)^{(1)} = A_1^{(1)} + A_1 : -\begin{pmatrix} 2 & -2 & 0 \\ -2 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \sim A_1 + A_1^{(1)} : -\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & -2 \\ 0 & -2 & 2 \end{pmatrix};$$

we denote by $\widetilde{W}((2A_1)^{(1)})$ the extended Weyl group isomorphic to the automorphism group of the root lattice characterized by this intersection form.

For each type of R, we can construct the action of Cr on a family of surfaces, parameterized by the parameters and the independent variable t of the equation; this action is a birational symmetry of the equation.

For a special value of parameters, a Cremona isometry yields an automorphism of a space of initial conditions. Each automorphism appears on a fixed point of the parameter space with respect to the Cremona actions coming from $G = \mathrm{Cr}/\mathrm{Cr}(D) \subset \mathrm{Cr}$ except for $D_7^{(1)}$. Here $\mathrm{Cr}(D)$ is the normal subgroup of Cr which leaves each irreducible component D_i invariant. In the case $R = D_7^{(1)}$, G is not realized in Cr, and there exists no such automorphism.

We have to consider only the subgroup G_t of G, which leave the independent variable t invariant. In fact, we can prove that each surface has automorphisms only from such constructions:

Theorem 2.1 *Let X be a generalized Halphen surface with* $\dim|-\mathcal{K}_X| = 0$ *of type $D_l^{(1)}$, $l = 4, \ldots, 8$, and $E_k^{(1)}$, $k = 6, 7, 8$. Then the group of automorphisms of X is a subgroup of G_t.*

Before prove the theorem, we give the list of G_t:

Table 2. $R^{\perp}$, G, and G_t

	P_{VI}	P_{V}	$P_{\mathrm{III}}^{D_6^{(1)}}$	$P_{\mathrm{III}}^{D_7^{(1)}}$	$P_{\mathrm{III}}^{D_8^{(1)}}$	P_{IV}	P_{II}	P_{I}
$R^{\perp}$	$D_4^{(1)}$	$A_3^{(1)}$	$(2A_1)^{(1)}$	$A_1^{(1)}$	-	$A_2^{(1)}$	$A_1^{(1)}$	-
G	$\mathfrak{S}_4$	$\mathfrak{D}_8$	$\mathfrak{D}_8$	$\mathfrak{D}_8$	C_2	$\mathfrak{S}_3$	C_2	1
G_t	$C_2 \times C_2$	C_4	$C_2 \times C_2$	C_4	C_2	C_3	C_2	1

Here $R^{\perp}$ denotes the type of symmetries; $\mathfrak{D}_{2n}$ is the dihedral group of order $2n$, $\mathfrak{S}_n$ is the symmetric group of order $n!$, and C_n is the cyclic group of order n.

Proof of Theorem 2.1. Let σ be an automorphism of X, then σ induces a Cremona isometry. Thus we obtain a homomorphism $\varphi : \mathrm{Aut}(X) \to \mathrm{Cr}$. The kernel of φ is 1. In fact, X has a blowing-down to $\mathbb{P}^2$ and if σ belongs to the kernel then σ leaves this blowing-down structure invariant. Here σ induces an automorphism of $\mathbb{P}^2$; on the other hand, an automorphism which fixes the center of blowing-up in $\mathbb{P}^2$ is only the identity; see [14]. Now we can conclude that $\mathrm{Aut}(X)$ is included in Cr.

We show next that $\sigma \in \mathrm{Cr}(D)$ coming from an automorphism of X is only the identity. Let X be not a $D_7^{(1)}$-surface. A $D_8^{(1)}$-surface and a $E_8^{(1)}$-surface have 1 as $\mathrm{Cr}(D)$. Aside from these, the group $\mathrm{Cr}(D)$ is described as $W(R^{\perp})_{\Lambda}$. The parameter space neglecting t (t is the independent variable of P_J) is identified with $H := \mathfrak{h}_1/\mathbb{C}K$, where $\mathfrak{h}_1 = \{h \in \mathfrak{h}_{\mathbb{C}} | \langle \delta, h \rangle = 1\}$ and K is the canonical central element. The intersection with the fundamental chamber $C = \{h \in \mathfrak{h}_{\mathbb{C}} | \langle \alpha_i, h \rangle \leq 0 \text{ for } \alpha_i : \text{simple root}\}$ is a fundamental domain for the action of $W(R^{\perp})$ on H. The theory of affine Weyl groups shows us that, for $h \in C \cap H$, $W(R^{\perp})_h = \{w \in W(R^{\perp}) | w(h) = h\}$ is generated by the simple reflections; see [4]. Therefore, for our purpose we have only to check that a simple reflection which fixes parameterization is the identity.

If a simple reflection s_i fixes the parameterization then the corresponding parameter α_i equals to 0. But in this case s_i works as the identity.

For a $D_7^{(1)}$-surface, the subgroup of Cr which fixes the parameter t is expressed as translations on the other parameter. So a $D_7^{(1)}$-surface has no other automorphism except the identity. $\square$

Now we consider the quotient of the space of initial conditions with respect to such an automorphism. After resolution of singularities, the quotient space may be again a space of initial conditions of a certain Painlevé equation. We will show that it occurs certainly and have a new class of symmetries of the Painlevé equations.

The rational function field of the quotient space coincides with that of the invariants with respect to subgroup S of automorphisms:

$$L = F(q,p), \quad L^S = F(P,Q),$$

where F is the coefficient field attached with the parameters and the independent variable t of the Painlevé equation considered; (q,p) and (Q,P) are the dependent variables.

Definition 2.2 *An algebraic transformation of the Painlevé systems is called a folding transformation, if it gives rise to a non-trivial quotient map of the space of initial conditions.*

We denote by $\psi^{[n]}_{J\to K}$ the folding transformation from P_J to P_K with degree n. An explicit form of each transformation will be given in the following sections.

Theorem 2.3 *The whole list of folding transformations of the Painlevé equations, up to birational transformations, is given as follows (see Figure 1):*

$$\psi^{[2]}_{\mathrm{VI}}, \quad \psi^{[4]}_{\mathrm{VI}}, \quad \psi^{[2]}_{\mathrm{V}\to\mathrm{III}(D_6^{(1)})}, \quad \psi^{[4]}_{\mathrm{V}}, \quad \psi^{[2]}_{\mathrm{III}(D_6^{(1)})\to\mathrm{V}},$$
$$\psi^{[2]}_{\mathrm{III}(D_6^{(1)}\to D_8^{(1)})}, \quad \psi^{[4]}_{\mathrm{III}(D_6^{(1)})}, \quad \psi^{[2]}_{\mathrm{III}(D_8^{(1)}\to D_6^{(1)})}, \quad \psi^{[3]}_{\mathrm{IV}}, \quad \psi^{[2]}_{\mathrm{II}}.$$

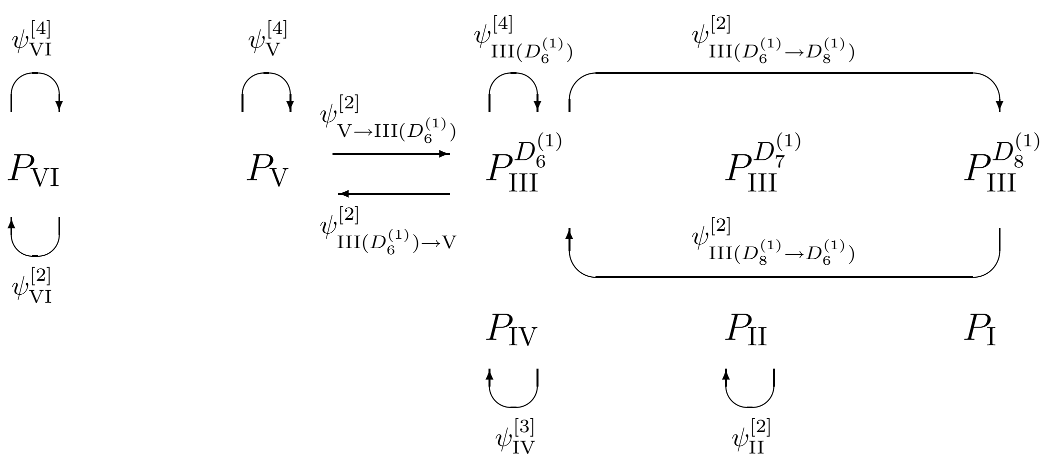

Figure 1: Folding transformations

Remark 2.4 In the case when a group is of the form: $S = S_1 \ltimes S_2$, we can consider quotients of surfaces twice successively. Then we can describe a folding transformation in the form of composition of transformations. Here is the table of all the possible cases:

$$\psi_{\mathrm{VI}}^{[4]} = \sigma_3 \circ \psi_{\mathrm{VI}}^{[2]}\Big|_{\epsilon_0=0,\epsilon_1=2\epsilon} \circ \sigma_3 \circ \psi_{\mathrm{VI}}^{[2]}\Big|_{\epsilon_0=\epsilon_1=\epsilon}, \tag{2.1}$$

$$\psi_{\mathrm{V}}^{[4]} = \psi_{\mathrm{III}(D_6^{(1)})\to\mathrm{V}}^{[2]}\Big|_{\epsilon=0} \circ \psi_{\mathrm{V}\to\mathrm{III}(D_6^{(1)})}^{[2]}\Big|_{\epsilon=1/4}, \tag{2.2}$$

$$\begin{aligned}\psi_{\mathrm{III}(D_6^{(1)})}^{[4]} &= \psi_{\mathrm{V}\to\mathrm{III}(D_6^{(1)})}^{[2]}\Big|_{\epsilon=0} \circ \psi_{\mathrm{III}(D_6^{(1)})\to\mathrm{V}}^{[2]}\Big|_{\epsilon=1/2} \\ &= \psi_{\mathrm{III}(D_8^{(1)}\to D_6^{(1)})}^{[2]} \circ \psi_{\mathrm{III}(D_6^{(1)}\to D_8^{(1)})}^{[2]},\end{aligned} \tag{2.3}$$

where ϵ, ϵ_0, and ϵ_1 denote parameters of the Painlevé equations.

3 The sixth Painlevé equation: P_{VI}

The Hamiltonian considered in what follows is of the form:

$$\begin{aligned} t(t-1)H_{\mathrm{VI}} &= q(q-1)(q-t)p^2 - \{\alpha_4(q-1)(q-t) + \alpha_3 q(q-t) + (\alpha_0-1)q(q-1)\}p \\ &\quad +\alpha_2(\alpha_1+\alpha_2)(q-t). \end{aligned} \tag{3.1}$$

The equation for $y = q$ is the sixth Painlevé equation P_{VI}:

$$\begin{aligned}\frac{\mathrm{d}^2y}{\mathrm{d}t^2} &= \frac{1}{2}\left(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-t}\right)\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{y-t}\right)\frac{\mathrm{d}y}{\mathrm{d}t} \\ &\quad + \frac{y(y-1)(y-t)}{t^2(t-1)^2}\left\{a + b\frac{t}{y^2} + c\frac{t-1}{(y-1)^2} + d\frac{t(t-1)}{(y-t)^2}\right\},\end{aligned}$$

where

$$a = \frac{{\alpha_1}^2}{2}, \quad b = -\frac{{\alpha_4}^2}{2}, \quad c = \frac{{\alpha_3}^2}{2}, \quad d = -\frac{{\alpha_0}^2-1}{2}, \tag{3.2}$$

and $\alpha_0 + \alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4 = 1$. The symmetry of the equation is described as follows:

$$\begin{aligned} &\mathrm{Cr} = \widetilde{W}(D_4^{(1)}) = G \ltimes W(D_4^{(1)}), \\ &W(D_4^{(1)}) = \langle s_0, s_1, s_2, s_3, s_4\rangle, \\ &G = \mathrm{Aut}(D_4^{(1)}) = \mathfrak{S}_4 = \langle \sigma_1, \sigma_2, \sigma_3\rangle, \\ &G_t = C_2 \times C_2 = \langle \pi_1, \pi_2\rangle. \end{aligned}$$

The list of birational transformations are given by the following table:

x	α_0	α_1	α_2	α_3	α_4	q	p	t
$s_0(x)$	$-\alpha_0$	α_1	$\alpha_2+\alpha_0$	α_3	α_4	q	$p-\frac{\alpha_0}{q-t}$	t
$s_1(x)$	α_0	$-\alpha_1$	$\alpha_2+\alpha_1$	α_3	α_4	q	p	t
$s_2(x)$	$\alpha_0+\alpha_2$	$\alpha_1+\alpha_2$	$-\alpha_2$	$\alpha_3+\alpha_2$	$\alpha_4+\alpha_2$	$q+\frac{\alpha_2}{p}$	p	t
$s_3(x)$	α_0	α_1	$\alpha_2+\alpha_3$	$-\alpha_3$	α_4	q	$p-\frac{\alpha_3}{q-1}$	t
$s_4(x)$	α_0	α_1	$\alpha_2+\alpha_4$	α_3	$-\alpha_4$	q	$p-\frac{\alpha_4}{q}$	t
$\pi_1(x)$	α_3	α_4	α_2	α_0	α_1	$\frac{t}{q}$	$-\frac{q(qp+\alpha_2)}{t}$	t
$\pi_2(x)$	α_1	α_0	α_2	α_4	α_3	$\frac{(q-1)t}{(q-t)}$	$-\frac{p(q-t)^2+\alpha_2(q-t)}{t(t-1)}$	t
$\sigma_1(x)$	α_0	α_1	α_2	α_4	α_3	$1-q$	$-p$	$1-t$
$\sigma_2(x)$	α_0	α_4	α_2	α_3	α_1	$\frac{1}{q}$	$-q(qp+\alpha_2)$	$\frac{1}{t}$
$\sigma_3(x)$	α_4	α_1	α_2	α_3	α_0	$\frac{t-q}{t-1}$	$-(t-1)p$	$\frac{t}{t-1}$

(3.3)

Here $\pi_1=\sigma_2\sigma_1\sigma_3\sigma_1$ and $\pi_2=\sigma_1\sigma_2\sigma_3\sigma_2$.

3.1. The fixed value with respect to the action of $S_1=\langle\pi_1\rangle\subset G_t$ is

$$\alpha_0=\alpha_3=\epsilon_0,\quad \alpha_1=\alpha_4=\epsilon_1.$$

We have the folding transformation $\psi=\psi_{\mathrm{VI}}^{[2]}$ given by

$$\begin{aligned}
&F=\mathbb{C}(\epsilon_0,\epsilon_1,t),\quad L=F(q,p),\\
&\widetilde{F}=\mathbb{C}(\epsilon_0,\epsilon_1,\sqrt{t}),\quad \widetilde{L}=\widetilde{F}(q,p),\\
&\widetilde{L}^{S_1}=\widetilde{F}(Q,P),\\
\psi:&(q,p,t,H)\mapsto(Q,P,s,K),\\
&Q=\frac{1}{2}+\frac{1}{4}\left(\frac{q}{\sqrt{t}}+\frac{\sqrt{t}}{q}\right),\\
&P=4\sqrt{t}\frac{q}{q^2-t}\left(2qp-\left(\epsilon_0+\epsilon_1-\frac{1}{2}\right)\right),\\
&s=\frac{1}{2}+\frac{1}{4}\left(\sqrt{t}+\frac{1}{\sqrt{t}}\right),\\
&s(s-1)K=\frac{1}{\sqrt{t}}\left\{t(t-1)H-\frac{t-1}{2}\left(qp-\left(\epsilon_0-\frac{1}{2}\right)\left(\epsilon_0+\epsilon_1-\frac{1}{2}\right)\right)\right\},
\end{aligned}$$

where

$$\begin{aligned}
H&=H_{\mathrm{VI}}(\alpha_0,\alpha_1,\alpha_2,\alpha_3,\alpha_4;t;q,p),\quad \alpha_0=\alpha_3=\epsilon_0,\alpha_1=\alpha_4=\epsilon_1,\\
K&=H_{\mathrm{VI}}(\alpha_0,\alpha_1,\alpha_2,\alpha_3,\alpha_4;s;Q,P),\quad \alpha_0=2\epsilon_0,\alpha_1=2\epsilon_1,\alpha_3=\alpha_4=0.
\end{aligned}$$

We can verify by straightforward computation that ψ is exactly the transformation of P_{VI}; we do not enter into details of computation.

Remark 3.1 The transformation ψ is not canonical but a contact transformation; we have in fact

$$\mathrm{d}P \wedge \mathrm{d}Q - \mathrm{d}K \wedge \mathrm{d}s = 2\left(\mathrm{d}p \wedge \mathrm{d}q - \mathrm{d}H \wedge \mathrm{d}t\right).$$

Remark 3.2 Y. I. Manin [6] considered the same transformation in different formalization. Through the change of dependent and independent variables, P_{VI} is rewritten in the following form:

$$\frac{\mathrm{d}^2 z}{\mathrm{d}\tau^2} = \frac{1}{(2\pi\sqrt{-1})^2} \sum_{j=0}^{3} a_j \wp_z\left(z + \frac{T_j}{2}, \tau\right), \tag{3.4}$$

with $(T_0, \ldots, T_3) = (0, 1, \tau, 1+\tau)$ and $(a_0, a_1, a_2, a_3) = ({\alpha_1}^2/2, {\alpha_4}^2/2, {\alpha_3}^2/2, {\alpha_0}^2/2)$. Here $\wp = \wp(z, \tau)$ is the Weierstrass elliptic $\wp$-function with the primitive periods 2 and 2τ; and $\wp_z(z, \tau)$ denotes the partial derivative of $\wp$-function with respect to z.

Consider the case when $a_0 = a_1 = {\epsilon_1}^2/2$ and $a_2 = a_3 = {\epsilon_0}^2/2$. By applying the Landen transform:

$$\wp_z\left(z, \frac{\tau}{2}\right) = \wp_z(z, \tau) + \wp_z\left(z + \frac{\tau}{2}, \tau\right),$$

we see that equation (3.4) is rewritten as

$$\begin{aligned}
(2\pi\sqrt{-1})^2 \frac{\mathrm{d}^2 z}{\mathrm{d}\tau^2} &= \frac{\epsilon_1^2}{2}\left\{\wp_z(z, \tau) + \wp_z\left(z + \frac{\tau}{2}, \tau\right)\right\} + \frac{\epsilon_0^2}{2}\left\{\wp_z\left(z + \frac{1}{2}, \tau\right) + \wp_z\left(z + \frac{1+\tau}{2}, \tau\right)\right\} \\
&= \frac{\epsilon_1^2}{2}\wp_z\left(z, \frac{\tau}{2}\right) + \frac{\epsilon_0^2}{2}\wp_z\left(z + \frac{1}{2}, \frac{\tau}{2}\right);
\end{aligned}$$

thus we have

$$\frac{\mathrm{d}^2 z}{\mathrm{d}(\tau/2)^2} = \frac{1}{(2\pi\sqrt{-1})^2}\left\{2\epsilon_1^2 \wp_z\left(z, \frac{\tau}{2}\right) + 2\epsilon_0^2 \wp_z\left(z + \frac{1}{2}, \frac{\tau}{2}\right)\right\}.$$

This gives a transformation of solutions of (3.4). That is, if $z(\tau)$ is any solution of (3.4) with $(\alpha_0, \alpha_1, \alpha_3, \alpha_4) = (\epsilon_0, \epsilon_1, \epsilon_1, \epsilon_0)$ then $z(2\tau)$ solves (3.4) with $(\alpha_0, \alpha_1, \alpha_3, \alpha_4) = (0, 2\epsilon_1, 0, 2\epsilon_0)$.

The explicit formula of Manin's transformation can be written in terms of the folding transformation and birational transformations as $\sigma_3 \circ \psi_{\mathrm{VI}}^{[2]} \circ \sigma_1$, that is nothing but the folding transformation arising from the quotient with respect to the action of $\langle \pi_1 \circ \pi_2 \rangle$.

3.2. The fixed value with respect to the action of $S_2 = \langle \pi_1, \pi_2 \rangle = G_t$ is

$$\alpha_0 = \alpha_1 = \alpha_3 = \alpha_4 = \epsilon.$$

The folding transformation $\psi = \psi_{\rm VI}^{[4]}$ is described as follows:

$$
\begin{aligned}
&F = \mathbb{C}(\epsilon, t), \quad L = F(q,p),\\
&L^{S_2} = F(Q,P),\\
\psi &: (q,p,t,H) \mapsto (Q,P,s,K),\\
&Q = \frac{(q^2-t)^2}{4q(q-1)(q-t)},\\
&P = \frac{4q(q-1)(q-t)}{(q^2-t)(q^2-2q+t)(q^2-2qt+t)} \times\\
&\qquad \times \left\{ 4q(q-1)(q-t)p - (2\epsilon - \frac{1}{2})(q(q-1)+q(q-t)+(q-1)(q-t)) \right\},\\
&s = t,\\
&s(s-1)K = 4t(t-1)H - 2t(t-1)\,\frac{2q(q-1)p - (2\epsilon - \frac{1}{2})(2q-1)}{q^2-2qt+t}\\
&\qquad\qquad + (2\epsilon - \frac{1}{2})(2\epsilon - \frac{3}{2})(2t-1),
\end{aligned}
$$

where

$$
\begin{aligned}
H &= H_{\rm VI}(\alpha_0,\alpha_1,\alpha_2,\alpha_3,\alpha_4;t;q,p), \quad \alpha_0=\alpha_1=\alpha_3=\alpha_4=\epsilon,\\
K &= H_{\rm VI}(\alpha_0,\alpha_1,\alpha_2,\alpha_3,\alpha_4;s;Q,P), \quad \alpha_0=\alpha_3=\alpha_4=0, \alpha_1=4\epsilon.
\end{aligned}
$$

We have

$$
{\rm d}P \wedge {\rm d}Q - {\rm d}K \wedge {\rm d}s = 4\left({\rm d}p \wedge {\rm d}q - {\rm d}H \wedge {\rm d}t\right).
$$

3.3. By using the above folding transformations, we can construct another symmetry of $P_{\rm VI}$, not birational nor folding transformations. We show some examples of these transformations. In what follows, we consider $P_{\rm VI}$ with $\alpha_i = 0$ $(i \neq 2)$ and $\alpha_2 = 1/2$.

(i) We can iterate the folding transformation $\psi = \psi_{\rm VI}^{[2]}$. The transformation ψ^n $(n = 1, 2, \ldots)$ gives a transformation of the solutions of $P_{\rm VI}$. Note that ψ^n is an algebraic transformation of degree 2^n.

(ii) Let $\pi_1^{(n)} = (\psi^{-1})^n \circ \pi_1 \circ \psi^n$ $(n = 1, 2, \ldots)$. Then $\pi_1^{(n)}$ also is a transformation of $P_{\rm VI}$. For $n = 1$, the explicit form of this transformation is given as follows:

$$
\begin{aligned}
&\pi_1^{(1)} : (q,p,t,H) \to (Q,P,s,K),\\
&\quad \left(q + \frac{t}{q} + 2\sqrt{t}\right)\left(Q + \frac{t}{Q} + 2\sqrt{t}\right) = 4\sqrt{t}(1+\sqrt{t})^2,\\
&\quad \frac{q+\sqrt{t}}{q-\sqrt{t}}\left(2qp + \frac{1}{2}\right) + \frac{Q+\sqrt{t}}{Q-\sqrt{t}}\left(2QP + \frac{1}{2}\right) + \frac{1}{2} = 0,\\
&\quad s = t,
\end{aligned}
$$

where $H = H_{\rm VI}(t;q,p)$ and $K = H_{\rm VI}(s;Q,P)$. Note that

$$
{\rm d}P \wedge {\rm d}Q - {\rm d}K \wedge {\rm d}s = {\rm d}p \wedge {\rm d}q - {\rm d}H \wedge {\rm d}t.
$$

4 The fifth Painlevé equation: $P_{\rm V}$

Consider the Hamiltonian:

$$tH_{\rm V} = p(p+t)q(q-1) + \alpha_2 qt - \alpha_3 pq - \alpha_1 p(q-1). \tag{4.1}$$

By putting $y = 1 - 1/q$, we have the fifth Painlevé equation $P_{\rm V}$:

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} = \left(\frac{1}{2y} + \frac{1}{y-1}\right)\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 - \frac{1}{t}\frac{\mathrm{d}y}{\mathrm{d}t} + \frac{(y-1)^2}{t^2}\left(ay + \frac{b}{y}\right) + c\frac{y}{t} + d\frac{y(y+1)}{y-1}, \tag{4.2}$$

where

$$a = \frac{{\alpha_1}^2}{2}, \quad b = -\frac{{\alpha_3}^2}{2}, \quad c = \alpha_0 - \alpha_2, \quad d = -\frac{1}{2}, \tag{4.3}$$

and $\alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 = 1$. The symmetry of the equation is

$$\begin{aligned}
&\mathrm{Cr} = \widetilde{W}(A_3^{(1)}) = G \ltimes W(A_3^{(1)}),\\
&W(A_3^{(1)}) = \langle s_0, s_1, s_2, s_3\rangle,\\
&G = \mathrm{Aut}(D_5^{(1)}) = \mathrm{Aut}(A_3^{(1)}) = \mathfrak{D}_8 = \langle \sigma, \pi\rangle,\\
&G_t = C_4 = \langle \pi \rangle.
\end{aligned}$$

The birational transformations are given as follows:

x	α_0	α_1	α_2	α_3	q	p	t
$s_0(x)$	$-\alpha_0$	$\alpha_1+\alpha_0$	α_2	$\alpha_3+\alpha_0$	$q+\frac{\alpha_0}{p+t}$	p	t
$s_1(x)$	$\alpha_0+\alpha_1$	$-\alpha_1$	$\alpha_2+\alpha_1$	α_3	q	$p-\frac{\alpha_1}{q}$	t
$s_2(x)$	α_0	$\alpha_1+\alpha_2$	$-\alpha_2$	$\alpha_3+\alpha_2$	$q+\frac{\alpha_2}{p}$	p	t
$s_3(x)$	$\alpha_0+\alpha_3$	α_1	$\alpha_2+\alpha_3$	$-\alpha_3$	q	$p-\frac{\alpha_3}{q-1}$	t
$\pi(x)$	α_1	α_2	α_3	α_0	$-\frac{p}{t}$	$t(q-1)$	t
$\sigma(x)$	α_0	α_3	α_2	α_1	$1-q$	$-p$	$-t$

(4.4)

4.1. The fixed value with respect to the action of $S_1 = \langle \pi^2 \rangle \subset G_t$ is

$$\alpha_1 = \alpha_3 = \epsilon, \quad \alpha_0 = \alpha_2 = \frac{1}{2} - \epsilon.$$

The folding transformation $\psi = \psi^{[2]}_{\mathrm{V}\to\mathrm{III}(D_6^{(1)})}$ is given as follows:

$$\begin{aligned}
&F = \mathbb{C}(\epsilon, t), \quad L = F(q,p),\\
&L^{S_1} = F(Q,P),\\
\psi :\ &(q,p,t,H) \mapsto (Q,P,s,K),\\
&Q = \frac{2p+t}{4(2q-1)},\\
&P = 4q(1-q),\\
&s = \frac{t^2}{16},\\
&sK = tH + \frac{(\epsilon - q)t}{2},
\end{aligned}$$

where

$$\begin{aligned}
&H = H_{\mathrm{V}}(\alpha_0,\alpha_1,\alpha_2,\alpha_3;t;q,p), \quad \alpha_1=\alpha_3=\epsilon, \alpha_0=\alpha_2=\frac{1}{2}-\epsilon,\\
&K = H_{\mathrm{III}}^{D_6^{(1)}}(\alpha_0,\alpha_1,\beta_0,\beta_1;s;Q,P), \quad \alpha_1=2\epsilon, \alpha_0=1-2\epsilon, \beta_1=0, \beta_0=1.
\end{aligned}$$

We have

$$\mathrm{d}P\wedge\mathrm{d}Q - \mathrm{d}K\wedge\mathrm{d}s = 2\left(\mathrm{d}p\wedge\mathrm{d}q - \mathrm{d}H\wedge\mathrm{d}t\right).$$

4.2. The fixed value with respect to the action of $S_2 = \langle\pi\rangle = G_t$ is

$$\alpha_0=\alpha_1=\alpha_2=\alpha_3=\frac{1}{4}.$$

The folding transformation $\psi = \psi^{[4]}_{\mathrm{V}}$ is given as follows:

$$\begin{aligned}
&F = \mathbb{C}(t), \quad L = F(q,p),\\
&L^{S_2} = F(Q,P),\\
\psi :\ &(q,p,t,H) \mapsto (Q,P,s,K),\\
&Q = \frac{1}{2} + \frac{\sqrt{-1}(p+tq)(p+t-tq)}{t(2q-1)(2p+t)},\\
&P = 2\sqrt{-1}\frac{t(2q-1)^2(2p+t)^2}{(2p+t)^2+t^2(2q-1)^2},\\
&s = -2\sqrt{-1}t,\\
&sK = 4\left(tH + \frac{qp}{2} - \frac{p}{4} - \frac{tq}{4} - \frac{t^2}{16}\right),
\end{aligned}$$

where

$$\begin{aligned}
&H = H_{\mathrm{V}}(\alpha_0,\alpha_1,\alpha_2,\alpha_3;t;q,p), \quad \alpha_0=\alpha_1=\alpha_2=\alpha_3=\frac{1}{4},\\
&K = H_{\mathrm{V}}(\alpha_0,\alpha_1,\alpha_2,\alpha_3;s;Q,P), \quad \alpha_0=1, \alpha_1=\alpha_2=\alpha_3=0.
\end{aligned}$$

We have

$$\mathrm{d}P\wedge\mathrm{d}Q - \mathrm{d}K\wedge\mathrm{d}s = 4\left(\mathrm{d}p\wedge\mathrm{d}q - \mathrm{d}H\wedge\mathrm{d}t\right).$$

5 The third Painlevé equation: $P_{\mathrm{III}}^{D_6^{(1)}}$

The Hamiltonian is

$$tH_{\mathrm{III}}^{D_6^{(1)}} = q^2p^2 - (q^2 - (\alpha_1+\beta_1)q - t)p - \alpha_1 q. \tag{5.1}$$

The equation for $y = q/\tau$, $t = \tau^2$ is the third Painlevé equation P_{III}:

$$\frac{\mathrm{d}^2 y}{\mathrm{d}\tau^2} = \frac{1}{y}\left(\frac{\mathrm{d}y}{\mathrm{d}\tau}\right)^2 - \frac{1}{\tau}\frac{\mathrm{d}y}{\mathrm{d}\tau} + \frac{1}{\tau}(ay^2+b) + cy^3 + \frac{d}{y}, \tag{5.2}$$

with

$$a = 4(\alpha_1 - \beta_1), \quad b = -4(\alpha_1+\beta_1-1), \quad c = 4, \quad d = -4. \tag{5.3}$$

The symmetry of the equation is

$$\begin{aligned}
&\mathrm{Cr} = \widetilde{W}((2A_1)^{(1)}) = G \ltimes W((2A_1)^{(1)}),\\
&W((2A_1)^{(1)}) = \langle s_0, s_1, s_0', s_1'\rangle,\\
&G = \mathrm{Aut}(D_6^{(1)}) = \mathrm{Aut}((2A_1)^{(1)}) = \mathfrak{D}_8 = \langle \pi_1, \pi_2, \sigma_1\rangle,\\
&G_t = C_2 \times C_2 = \langle \pi_1, \pi_2\rangle.
\end{aligned}$$

The birational transformations are given as follows:

x	α_0	α_1	β_0	β_1	q	p	t
$s_0(x)$	$-\alpha_0$	$\alpha_1+2\alpha_0$	β_0	β_1	$q+\frac{\alpha_0}{(p-1)+\frac{\alpha_1+\beta_1-1}{q}+\frac{t}{q^2}}$	$p-\frac{\alpha_0(2q(p-1)+\alpha_1+\beta_1-1)}{f_1}-\frac{\alpha_0^2\,t}{f_1^2}$	t
$s_1(x)$	$\alpha_0+2\alpha_1$	$-\alpha_1$	β_0	β_1	$q+\frac{\alpha_1}{p}$	p	t
$s_0'(x)$	α_0	α_1	$-\beta_0$	$\beta_1+2\beta_0$	$q+\frac{\beta_0}{p+\frac{\alpha_1+\beta_1-1}{q}+\frac{t}{q^2}}$	$p-\frac{\beta_0(2qp+\alpha_1+\beta_1-1)}{f_2}-\frac{\beta_0^2\,t}{f_2^2}$	t
$s_1'(x)$	α_0	α_1	$\beta_0+2\beta_1$	$-\beta_1$	$q+\frac{\beta_1}{p-1}$	p	t
$\pi_1(x)$	α_1	α_0	β_0	β_1	$-\frac{t}{q}$	$\frac{q}{t}(q(p-1)+\beta_1)+1$	t
$\pi_2(x)$	α_0	α_1	β_1	β_0	$\frac{t}{q}$	$-\frac{q}{t}(qp+\alpha_1)$	t
$\sigma_1(x)$	β_0	β_1	α_0	α_1	$-q$	$1-p$	$-t$
$\sigma_2(x)$	β_1	β_0	α_1	α_0	q	$p+\frac{\alpha_1+\beta_1-1}{q}+\frac{t}{q^2}$	$-t$

(5.4)

Here $\alpha_0 = 1-\alpha_1$, $\beta_0 = 1-\beta_1$, $f_1 = \beta_1 q + (p-1)q^2 + t$, and $f_2 = \alpha_1 q + pq^2 + t$.

5.1. The fixed value with respect to the action of $S_1 = \langle \pi_1\rangle \subset G_t$ is

$$\alpha_1 = \alpha_0 = \frac{1}{2}, \quad \beta_1 = \epsilon, \quad \beta_0 = 1-\epsilon.$$

The folding transformation $\psi = \psi^{[2]}_{\mathrm{III}(D_6^{(1)})\to \mathrm{V}}$ is given as follows:

$$
\begin{aligned}
&F = \mathbb{C}(\epsilon, t), \quad L = F(q,p),\\
&\widetilde{F} = \mathbb{C}(\epsilon, \sqrt{t}), \quad \widetilde{L} = \widetilde{F}(q,p),\\
&\widetilde{L}^{S_1} = \widetilde{F}(Q,P),\\
\psi &: (q,p,t,H) \mapsto (Q,P,s,K),\\
&Q = \frac{1}{2} + \frac{\sqrt{-1}}{4}\left(\frac{q}{\sqrt{t}} - \frac{\sqrt{t}}{q}\right),\\
&P = -4\sqrt{-1}\,\frac{2(p-1)q+\epsilon}{\dfrac{q}{\sqrt{t}} + \dfrac{\sqrt{t}}{q}},\\
&s = -8\sqrt{-t},\\
&sK = 4tH - 2q(p-1) + \left(2\sqrt{-t}-\epsilon\right)^2,
\end{aligned}
$$

where

$$
\begin{aligned}
H &= H_{\mathrm{III}}^{D_6^{(1)}}(\alpha_0,\alpha_1,\beta_0,\beta_1;t;q,p), \quad \alpha_1=\alpha_0=\frac{1}{2}, \beta_1=\epsilon, \beta_0 = 1-\epsilon,\\
K &= H_{\mathrm{V}}(\alpha_0,\alpha_1,\alpha_2,\alpha_3;s;Q,P), \quad \alpha_1=\alpha_3=0, \alpha_0=1-\epsilon, \alpha_2=\epsilon.
\end{aligned}
$$

We have

$$
\mathrm{d}P\wedge\mathrm{d}Q - dK\wedge\mathrm{d}s = 2\left(\mathrm{d}p\wedge\mathrm{d}q - \mathrm{d}H\wedge\mathrm{d}t\right).
$$

5.2. The fixed value with respect to the action of $S_2 = \langle \pi_1\circ\pi_2\rangle \subset G_t$ is

$$
\alpha_1 = \alpha_0 = \beta_1 = \beta_0 = \frac{1}{2}.
$$

The folding transformation $\psi = \psi^{[2]}_{\mathrm{III}(D_6^{(1)}\to D_8^{(1)})}$ is given by

$$
\begin{aligned}
&F = \mathbb{C}(t), \quad L = F(q,p),\\
&L^{S_2} = F(Q,P),\\
\psi &: (q,p,t,H) \mapsto (Q,P,s,K),\\
&Q = \frac{q^2}{2},\\
&P = \frac{2p-1}{q} + \frac{t}{q^3},\\
&s = \frac{t^2}{4},\\
&sK = tH + \left(2qp - q - p + \frac{1}{2q}\right)t,
\end{aligned}
$$

where

$$H = H_{\mathrm{III}}^{D_6^{(1)}}(\alpha_0, \alpha_1, \beta_0, \beta_1; t; q, p), \quad \alpha_1 = \alpha_0 = \beta_1 = \beta_0 = \frac{1}{2},$$
$$K = H_{\mathrm{III}}^{D_8^{(1)}}(s; Q, P).$$

We have

$$\mathrm{d}P \wedge \mathrm{d}Q - \mathrm{d}K \wedge \mathrm{d}s = 2\left(\mathrm{d}p \wedge \mathrm{d}q - \mathrm{d}H \wedge \mathrm{d}t\right).$$

5.3. The fixed value with respect to the action of $S_3 = \langle \pi_1, \pi_2 \rangle = G_t$ is

$$\alpha_1 = \alpha_0 = \beta_1 = \beta_0 = \frac{1}{2}.$$

The folding transformation $\psi = \psi^{[4]}_{\mathrm{III}(D_6^{(1)})}$ is given by

$$\begin{aligned}
&F = \mathbb{C}(t), \quad L = F(q,p),\\
&L^{S_3} = F(Q,P),\\
&\psi : (q,p,t,H) \mapsto (Q,P,s,K),\\
&Q = -4\frac{2pq - q + \frac{1}{2} + \frac{t}{q}}{\frac{q^2}{t} - \frac{t}{q^2}},\\
&P = \frac{1}{2} + \frac{1}{4}\left(\frac{q^2}{t} + \frac{t}{q^2}\right),\\
&s = -4t,\\
&sK = 4tH - 2qp + q + \frac{t}{q} - 4t + \frac{1}{4},
\end{aligned}$$

where

$$H = H_{\mathrm{III}}^{D_6^{(1)}}(\alpha_0, \alpha_1, \beta_0, \beta_1; t; q, p), \quad \alpha_1 = \alpha_0 = \beta_1 = \beta_0 = \frac{1}{2},$$
$$K = H_{\mathrm{III}}^{D_6^{(1)}}(\alpha_0, \alpha_1, \beta_0, \beta_1; s; Q, P), \quad \alpha_1 = 0, \alpha_0 = 1, \beta_1 = 0, \beta_0 = 1.$$

We have

$$\mathrm{d}P \wedge \mathrm{d}Q - \mathrm{d}K \wedge \mathrm{d}s = 4\left(\mathrm{d}p \wedge \mathrm{d}q - \mathrm{d}H \wedge \mathrm{d}t\right).$$

6 The third Painlevé equation: $P_{\mathrm{III}}^{D_7^{(1)}}$

The Hamiltonian is

$$tH_{\mathrm{III}}^{D_7^{(1)}} = q^2p^2 + \alpha_1 qp + tp + q. \tag{6.1}$$

The equation for $y = q/\tau$, $t = \tau^2$ is the special case of P_{III}:

$$\frac{\mathrm{d}^2 y}{\mathrm{d}\tau^2} = \frac{1}{y}\left(\frac{\mathrm{d}y}{\mathrm{d}\tau}\right)^2 - \frac{1}{\tau}\frac{\mathrm{d}y}{\mathrm{d}\tau} + \frac{1}{\tau}(ay^2 + b) + cy^3 + \frac{d}{y}, \tag{6.2}$$

with

$$a = -8, \quad b = 4(1-\alpha_1), \quad c = 0, \quad d = -4. \tag{6.3}$$

The symmetry of the equation is

$$\mathrm{Cr} = \widetilde{W}(A_1^{(1)}) = \langle s_1, \sigma \rangle,$$
$$\mathrm{Cr}_t = \mathbb{Z} = \langle \pi \rangle.$$

Here Cr_t denotes the subgroup of Cr which leaves the independent variable t invariant, and $\pi = \sigma \circ s_1$.

The birational transformations are given by

x	α_0	α_1	q	p	t
$s_0(x)$	$-\alpha_0$	$\alpha_1 + 2\alpha_0$	q	$p + \frac{\alpha_0}{q} - \frac{t}{q^2}$	$-t$
$s_1(x)$	$\alpha_0 + 2\alpha_1$	$-\alpha_1$	$-q + \frac{\alpha_1}{p} + \frac{1}{p^2}$	$-p$	$-t$
$\sigma(x)$	α_1	α_0	tp	$-\frac{q}{t}$	$-t$

(6.4)

where $\alpha_0 = 1 - \alpha_1$.

Any element of Cr_t has no fixed value of parameters; therefore $P_{\mathrm{III}}^{D_7^{(1)}}$ has no folding transformation.

7 The third Painlevé equation: $P_{\mathrm{III}}^{D_8^{(1)}}$

The Hamiltonian is

$$tH_{\mathrm{III}}^{D_8^{(1)}} = q^2p^2 + qp - \frac{1}{2}\left(q + \frac{t}{q}\right). \tag{7.1}$$

The equation for $y = q/\tau$, $t = \tau^2$ is the special case of P_{III}:

$$\frac{\mathrm{d}^2 y}{\mathrm{d}\tau^2} = \frac{1}{y}\left(\frac{\mathrm{d}y}{\mathrm{d}\tau}\right)^2 - \frac{1}{\tau}\frac{\mathrm{d}y}{\mathrm{d}\tau} + \frac{1}{\tau}(ay^2 + b) + cy^3 + \frac{d}{y}, \tag{7.2}$$

with

$$a = 4, \quad b = -4, \quad c = 0, \quad d = 0. \tag{7.3}$$

The symmetry of the equation is

$$\mathrm{Cr} = G = G_t = C_2 = \langle \pi \rangle.$$

The birational transformations are given by

x	q	p	t
$\pi(x)$	$\frac{t}{q}$	$-\frac{q(2qp+1)}{2t}$	t

(7.4)

The folding transformation $\psi = \psi^{[2]}_{\mathrm{III}(D_8^{(1)} \to D_6^{(1)})}$ is given as follows:

$$\begin{aligned}
&F = \mathbb{C}(\epsilon, t), \quad L = F(q,p),\\
&L^S = F(Q,P), \quad S = \langle \pi \rangle,\\
\psi :\ &(q,p,t,H) \mapsto (Q,P,s,K),\\
&Q = -2\frac{4qp+1}{\frac{q}{\sqrt{t}} - \frac{\sqrt{t}}{q}},\\
&P = \frac{1}{2} + \frac{1}{4}\left(\frac{q}{\sqrt{t}} + \frac{\sqrt{t}}{q}\right),\\
&s = -8\sqrt{t},\\
&sK = 4tH - 2qp - 4\sqrt{t} + \frac{1}{4},
\end{aligned}$$

where

$$\begin{aligned}
H &= H_{\mathrm{III}}^{D_8^{(1)}}(t;q,p),\\
K &= H_{\mathrm{III}}^{D_6^{(1)}}(\alpha_1,\alpha_0,\beta_1,\beta_0;s;Q,P), \quad \alpha_1 = \beta_1 = 0,\ \alpha_0 = \beta_0 = 1.
\end{aligned}$$

We have

$$\mathrm{d}P \wedge \mathrm{d}Q - \mathrm{d}K \wedge \mathrm{d}s = 2\left(\mathrm{d}p \wedge \mathrm{d}q - \mathrm{d}H \wedge \mathrm{d}t\right).$$

8 The fourth Painlevé equation: P_{IV}

The Hamiltonian is

$$H_{\mathrm{IV}} = (p - q - 2t)pq - 2\alpha_1 p - 2\alpha_2 q. \tag{8.1}$$

The equation for $y = q$ is the fourth Painlevé equation P_{IV}:

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} = \frac{1}{2y}\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 + \frac{3y^3}{2} + 4ty^2 + 2(t^2 - a)y + \frac{b}{y}, \tag{8.2}$$

where

$$a = \alpha_0 - \alpha_2, \quad b = -2{\alpha_1}^2. \tag{8.3}$$

The symmetry of the equation is

$$\begin{aligned}
&\mathrm{Cr} = \widetilde{W}(A_2^{(1)}) = G \ltimes W(A_2^{(1)}),\\
&W(A_2^{(1)}) = \langle s_0, s_1, s_2 \rangle,\\
&G = \mathrm{Aut}(E_6^{(1)}) = \mathrm{Aut}(A_2^{(1)}) = \mathfrak{S}_3 = \langle \sigma_1, \sigma_2 \rangle,\\
&G_t = C_3 = \langle \pi \rangle.
\end{aligned}$$

The birational transformations are

x	α_0	α_1	α_2	q	p	t
$s_0(x)$	$-\alpha_0$	$\alpha_1+\alpha_0$	$\alpha_2+\alpha_0$	$q+\frac{2\alpha_0}{f}$	$p+\frac{2\alpha_0}{f}$	t
$s_1(x)$	$\alpha_0+\alpha_1$	$-\alpha_1$	$\alpha_2+\alpha_1$	q	$p-\frac{2\alpha_1}{q}$	t
$s_2(x)$	$\alpha_0+\alpha_2$	$\alpha_1+\alpha_2$	$-\alpha_2$	$q+\frac{2\alpha_2}{p}$	p	t
$\pi(x)$	α_1	α_2	α_0	$-p$	$-f$	t
$\sigma_1(x)$	α_0	α_2	α_1	$-\sqrt{-1}p$	$-\sqrt{-1}q$	$\sqrt{-1}t$
$\sigma_2(x)$	α_2	α_1	α_0	$\sqrt{-1}f$	$\sqrt{-1}p$	$\sqrt{-1}t$

(8.4)

where $\alpha_0 = 1-\alpha_1-\alpha_2$ and $f = p-q-2t$.

The fixed value with respect to the action of $S = \langle\pi\rangle = G_t$ is

$$\alpha_0 = \alpha_1 = \alpha_2 = \frac{1}{3}.$$

The folding transformation $\psi = \psi_{\mathrm{IV}}^{[3]}$ is described as follows:

$$\begin{aligned}
&F = \mathbb{C}(t), \quad L = F(q,p),\\
&L^S = F(Q,P),\\
\psi &: (q,p,t,H) \mapsto (Q,P,s,K),\\
&Q = \frac{(3p-3q-4t)(3p-2t)(3q+2t) - (-3)^{1/2}3(p+q)(2p-q-2t)(p-2q-2t)}{2(-3)^{3/4}(3(p-q-2t)(p-q)+3pq+4t^2)},\\
&P = \frac{-(3p-3q-4t)(3p-2t)(3q+2t) - (-3)^{1/2}3(p+q)(2p-q-2t)(p-2q-2t)}{2(-3)^{3/4}(3(p-q-2t)(p-q)+3pq+4t^2)},\\
&s = (-3)^{1/4}t,\\
&K = -(-3)^{3/4}\left(H + \frac{2}{3}q + \frac{2}{3}p - \frac{8}{27}t^3\right),
\end{aligned}$$

where

$$\begin{aligned}
H &= H_{\mathrm{IV}}(\alpha_0,\alpha_1,\alpha_2;t;q,p), \quad \alpha_0=\alpha_1=\alpha_2=\frac{1}{3},\\
K &= H_{\mathrm{IV}}(\alpha_0,\alpha_1,\alpha_2;s;Q,P), \quad \alpha_0 = 1, \quad \alpha_1=\alpha_2=0.
\end{aligned}$$

We have

$$\mathrm{d}P\wedge\mathrm{d}Q - \mathrm{d}K\wedge\mathrm{d}s = 3(\mathrm{d}p\wedge\mathrm{d}q - \mathrm{d}H\wedge\mathrm{d}t).$$

9 The second Painlevé equation: P_{II}

The Hamiltonian is

$$H_{\mathrm{II}} = \frac{1}{2}p^2 - \left(q^2+\frac{t}{2}\right)p - \alpha_1 q. \tag{9.1}$$

The equation for $y = q$ is the second Painlevé equation P_{II}:

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} = 2y^3 + ty + a, \tag{9.2}$$

where $a = \alpha_1 - \frac{1}{2}$. The symmetry of the equation is

$$\begin{aligned}
&\mathrm{Cr} = \widetilde{W}(A_1^{(1)}) = G \ltimes W(A_1^{(1)}), \\
&W(A_1^{(1)}) = \langle s_0, s_1 \rangle, \\
&G = \mathrm{Aut}(E_7^{(1)}) = \mathrm{Aut}(A_1^{(1)}) = C_2 = \langle \pi \rangle = G_t.
\end{aligned}$$

The birational transformations are given by

x	α_0	α_1	q	p	t
$s_0(x)$	$-\alpha_0$	$\alpha_1 + 2\alpha_0$	$q + \frac{\alpha_0}{f}$	$p + \frac{4\alpha_0 q}{f} + \frac{2\alpha_0{}^2}{f^2}$	t
$s_1(x)$	$\alpha_0 + 2\alpha_1$	$-\alpha_1$	$q + \frac{\alpha_1}{p}$	p	t
$\pi(x)$	α_1	α_0	$-q$	$-f$	t

(9.3)

where $\alpha_0 = 1 - \alpha_1$ and $f = p - 2q^2 - t$.

The fixed value with respect to the action of $S = \langle \pi \rangle = G_t$ is

$$\alpha_0 = \alpha_1 = \frac{1}{2}.$$

The folding transformation $\psi = \psi_{\mathrm{II}}^{[2]}$ is given by the following:

$$\begin{aligned}
&F = \mathbb{C}(t), \quad L = F(q,p), \\
&L^S = F(Q,P), \\
&\psi : (q,p,t,H) \mapsto (Q,P,s,K), \\
&Q = -2^{-\frac{1}{3}} \frac{p - q^2 - t/2}{q}, \\
&P = 2^{\frac{1}{3}} q^2, \\
&s = -2^{\frac{1}{3}} t, \\
&K = -2^{\frac{2}{3}} \left(H + \frac{q}{2} + \frac{t^2}{8} \right),
\end{aligned}$$

where

$$\begin{aligned}
&H = H_{\mathrm{II}}(\alpha_0, \alpha_1; t; q, p), \quad \alpha_0 = \alpha_1 = \frac{1}{2}, \\
&K = H_{\mathrm{II}}(\alpha_0, \alpha_1; s; Q, P), \quad \alpha_1 = 0, \alpha_0 = 1.
\end{aligned}$$

We have

$$\mathrm{d}P \wedge \mathrm{d}Q - \mathrm{d}K \wedge \mathrm{d}s = 2(\mathrm{d}p \wedge \mathrm{d}q - \mathrm{d}H \wedge \mathrm{d}t).$$

10 The first Painlevé equation: P_{I}

The Hamiltonian is

$$H_{\mathrm{I}} = p^2 - q^3 - \frac{t}{2}q. \tag{10.1}$$

The equation for $y = q$ is the first Painlevé equation P_{I}:

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} = 6y^2 + t. \tag{10.2}$$

This equation has no symmetry and thus no folding transformation.

11 Correspondences of special solutions

Consider the case when the Painlevé system admits algebraic solutions. These are two types of such solutions; the first type of them appears as a special case of Riccati solutions and the other one is written by means of special polynomials: Yablonskii-Vorob'ev polynomials for P_{II}, Okamoto polynomials for P_{IV}, Umemura polynomials for P_{III}, P_{V}, and P_{VI}; see for example [7].

It is known that the latter occurs as the fixed point of a Dynkin automorphism. We see that folding transformations is a correspondence from such an algebraic solution to Riccati solutions. In fact, an algebraic solution is related to *a point* of the space of initial conditions and Riccati solutions can be regarded as to be raid on a projective line $\mathbb{P}^1$ in the space of initial conditions.

If we regard a Dynkin automorphism as a transformation of the space of initial conditions, the fixed point of the automorphism defines an algebraic solution. By considering the quotient space with respect to the automorphism, we obtain from the fixed point the quotient singularity, and by means of blowing-up procedure we have $\mathbb{P}^1$, which is corresponding to Riccati solutions. A folding transformation describes this correspondence; see Figure 2.

For example, consider the rational solution of P_{II}:

$$(\alpha_0, \alpha_1; q, p) = \left(\frac{1}{2}, \frac{1}{2}; 0, \frac{t}{2}\right), \tag{11.1}$$

that is the fixed point with respect to the Dynkin automorphism π given by (9.3). We see that the folding transformation $\psi_{\mathrm{II}}^{[2]}$, given above in Section 9, maps (11.1) to

$$(\alpha_0, \alpha_1; P, Q) = (1, 0; 0, Q).$$

Here Q satisfies the Riccati equation:

$$\frac{\mathrm{d}Q}{\mathrm{d}s} = -Q^2 - \frac{s}{2}, \quad s = -2^{1/3}t.$$

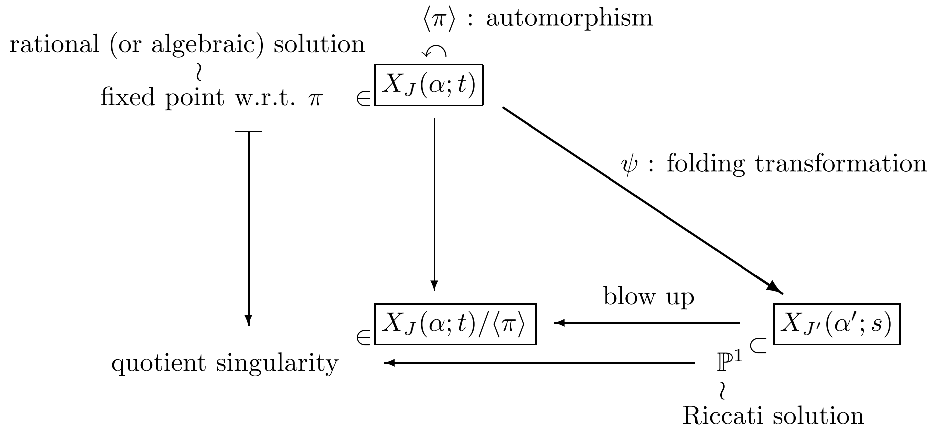

Figure 2: Geometric picture of a folding transformation

Note that all of rational solutions of $P_{\rm II}$ are obtained from (11.1) through birational symmetries.

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