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REISSNER-MINDLIN EXTENSIONS OF KIRCHHOFF ELEMENTS FOR PLATE BENDING

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Abstract

An extension method of various existing Kirchhoff plate bending elements to Reissner-Mindlin ones is developed and tested. The essential idea is to use independent transverse shear strains and a special mixed formulation. The Nedelec edge element is convenient for assuming the shear strains. Furthermore, the displacements are carefully constructed so that the strain-displacement relations are strictly satisfied for the transverse shear strains. We present our approach for displacement-based three-node triangular elements including both conforming and non-conforming ones as the base Kirchhoff elements. It is also possible to reduce the shear variables from the element degrees of freedom by means of a special technique called the beam element approximation. Numerical results are obtained for some fundamental test problems, and it is observed that the results are reasonable over wide range of plate thickness and the tested sample element is actually free from the transverse shear locking.

Keywords: plate bending, Kirchhoff elements, Reissner-Mindlin elements, assumed transverse shear strains, mixed FEM.

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1 Introduction

There have been developed a large number of finite elements for plate bending due to the importance in wide fields of structural engineering [1, 2]. Besides the fully 3-D elements, they can be essentially classified into two categories: Kirchhoff elements and Reissner-Mindlin ones. In earlier development stage of finite elements, the former ones were mainly devised with full refinement, though they are applicable to thin plates only. Then the latter have been almost exclusively considered since they can deal with moderately thick plates as well as thin ones.

Actually, usual Reissner-Mindlin elements have often suffered from poor accuracy in the thin plate range where Kirchhoff elements work well [1, 2]. Such a defect has been attributed to excessive transverse shear stiffness referred to the shear locking on one hand, and to numerical instability caused by the spurious zero-energy modes on the other hand. The latter phenomenon typically occurs when some tricks such as the reduced integration are introduced to reduce the excessive stiffness of such elements. Even when such difficulties are overcome, the accuracy of the Reissner-Mindlin elements is frequently inferior to that of the Kirchhoff elements with the same degrees of freedom especially in the thin plate range. This is probably because the Reissner-Mindlin elements usually use lower order polynomials in the lateral deflection and rotations than the Kirchhoff ones, since the Reissner-Mindlin deflection is not originally coupled with the rotations unlike the Kirchhoff one.

Thus a natural idea arises: to extend appropriate Kirchhoff elements to Reissner-Mindlin ones by using the highly refined deflection patterns of the base Kirchhoff-elements. In other words, we can then unify the Kirchhoff and Reissner-Mindlin elements. Such an idea appears to be quite attractive since we can rehabilitate classical plate elements, which behave nicely in the thin plate range but are no longer used widely as they are not valid for thick plates. The idea is in fact easy to implement to the beam element. More specifically, what we should do is to develop a nice Timoshenko beam element, whose thin-limit is the well-established Bernoulli-Euler one based on the cubic Hermite interpolation. Quite fortunately, we obtained a satisfactory Timoshenko element at fairly early stage in the history of FEM by considering the equilibrium equation of the Timoshenko beam [3, 4]. However, it has been quite difficult and remained a dream to realize it in the case of plate elements: the key is to find reasonable Reissner-Mindlin displacement patterns generalizing the Kirchhoff ones. Fairly recently, the idea was realized for the discrete Kirchhoff elements such as the DKT (3-node triangle) one [5, 6, 7]. So such generalization is desirable to be achieved for various other Kirchhoff elements.

In this paper, we will present an approach to extend various existing 3-node triangular Kirchhoff elements to the corresponding Reissner-Mindlin ones. At present, we can apply it to various conforming and non-conforming Kirchhoff ones besides the discrete Kirchhoff ones. The essence is to use a special mixed formulation and assume the transverse shear strains (or the shear forces) elementwise, whose tangential components are continuous on the interelement edges. Such shear fields are conveniently represented by the Nedelec edge element [8]. They are originally independent of the displacements but are finally coupled with them to satisfy the shear-displacement relations exactly. The derived displacement pattern appears to be desirable since it reduces to the base Kirchhoff pattern in the thin plate limit. It is also possible to determine the shears explicitly from the nodal displacements by using the technique employed in the extension of the DKT element [5, 7]. Such a reduction method appears to be closely related to the stabilization methods [9].

Our approach is also tested for some fundamental problems, and it is observed that the results converge to the Kirchhoff ones in the thin plate limit and reasonably represent the transverse shear effects in the moderately thick range. Finally, it is to be noticed that exact mathematical error analysis is possible for some typical finite elements, although the details will be published elsewhere.

2 Explanation of Idea by Timoshenko Beam

We will briefly explain our idea for the Timoshenko beam, i. e., a one-dimensional analog of the 2-D Reissner-Mindlin plate, since the plate cases may be slightly complicated. The resulting beam element itself is not new, and may be found in classical textbooks of matrix structural analysis [3, 4].

Just as in the two-node Bernoulli-Euler beam element, the lateral deflection w is assumed to be cubic in each element:

$$w = \alpha_1 + \alpha_2 x + \alpha_3 x^2 + \alpha_4 x^3 \,. \tag{1}$$

As is well known, w can be expressed as the Hermite interpolation polynomial in terms of the nodal displacements $\{w_i, w_{xi}\}_{i=1,2}$ with $w_x = dw/dx$. On the other hand, the element transverse shear strain γ_x is assumed to be constant:

$$\gamma_x = \beta \,. \tag{2}$$

Then the element rotation θ_x in anticlockwise orientation is given by using the strain-displacement relation $\gamma_x = w_x - \theta_x$ as

$$\theta_x = w_x - \gamma_x = w_x - \beta\,. \tag{3}$$

Consequently, the nodal values of w_x are expressed by β and nodal values of θ_x as

$$w_{xi} = \theta_{xi} + \beta \quad (i = 1, 2)\,, \tag{4}$$

which are available to express w in (1) in terms of β and $\{w_i, \theta_{xi}\}_{i=1,2}$. Thus the final element DOF (= Degrees Of Freedom) are the nodal deflection $\{w_i\}_{i=1,2}$, the nodal rotations $\{\theta_{xi}\}_{i=1,2}$ and the constant element shear strain $\gamma_x = \beta$.

In summary, both w and θ_x of each element are determined from the above-mentioned DOF. It is to be noted that w and θ_x are forced to be continuous at nodes but w_x is not so unlike in the Bernoulli-Euler case. Then we can calculate the potential energy of each element to derive element stiffness matrices and load vectors as usual. Furthermore, the element parameter β may be eliminated by the static condensation or, equivalently in this case, the kinetic relation between the shear force and the moment. Thus β is subjected to the other DOF as

$$\beta = \frac{\phi}{1+\phi}\left(\frac{w_2 - w_1}{\ell} - \frac{\theta_{x1} + \theta_{x2}}{2}\right); \qquad \phi = \frac{12EI}{\kappa GA\ell^2}\,, \tag{5}$$

where G is the shear modulus of the beam, EI the bending rigidity, A the cross-sectional area, ℓ the element length, and κ the shear correction factor depending on the cross-sectional geometry. The identical result may be obtained by various other methods such as the mixed finite element method. As was already mentioned, the resulting element matrices and vectors are identical to the classical ones obtained early in the matrix structural analysis.

Although our idea appears to be promising for 2-D plates, it has been actually difficult to implement. The main difficulty is in finding appropriate element transverse shear strain (or shear force) distribution, and it is also not clear how to eliminate or condense element shear parameters. However, in the case of the discrete Kirchhoff elements, breakthrough was achieved fairly recently as was noted in Introduction. We will now combine various ideas to develop nice Reissner-Mindlin elements by reusing existing Kirchhoff elements.

3 Plate Theory

Let us summarize the Reissner-Mindlin bending theory of isotropic homogeneous plates of uniform thickness, which is applicable to moderately thick plates as well as thin ones, and reduces to the Kirchhoff plate bending theory in the thin plate limit [10]. We introduce Cartesian coordinates (x, y, z) to the 3-D space where the plate is located, and choose the xy-plane the middle plane of the plate.

The displacements u, v and w of the plate in the x-, y- and z-directions are respectively assumed as

$$u = -z\theta_x(x, y), \quad v = -z\theta_y(x, y), \quad w = w(x, y), \tag{6}$$

where θ_x and θ_y are the rotations, and w is the lateral deflection. Then the generalized strain-displacement relations for the plate are given by

$$\begin{aligned} &k_x = -\partial_x\theta_x, \quad k_y = -\partial_y\theta_y, \quad k_{xy} = -\partial_y\theta_x - \partial_x\theta_y, \\ &\gamma_x = w_x - \theta_x, \quad \gamma_y = w_y - \theta_y, \end{aligned} \tag{7}$$

where k_x, k_y and k_{xy} are the curvature or twist changes, $\gamma_x = \gamma_{xz}$ and $\gamma_y = \gamma_{yz}$ the transverse shear strains, $\partial_x = \partial/\partial x$ and $\partial_y = \partial/\partial y$ partial differentiations, and $w_x = \partial_x w$ and $w_y = \partial_y w$. In the Kirchhoff case, the transverse shear strains vanish so that the curvature changes reduce to second order derivatives of w [10].

The generalized stress-strain relations are expressed as

$$\begin{aligned} &M_x = D_b(k_x + \nu k_y), \quad M_y = D_b(k_y + \nu k_x), \quad M_{xy} = \frac{1-\nu}{2} D_b k_{xy}, \\ &Q_x = D_s\gamma_x, \quad Q_y = D_s\gamma_y, \end{aligned} \tag{8}$$

where M_x, M_y and M_{xy} are the moments, Q_x and Q_y the transverse shear forces, ν is Poisson's ratio, $D_b = Et^3/\{12(1-\nu^2)\}$ the bending rigidity, and $D_s = Gt\kappa$ the shear rigidity. Moreover, E is Young's modulus, t the plate thickness, $G = E/\{2(1+\nu)\}$ the shear modulus, and κ the shear correction factor which is taken here as $5/6$ after Reissner [10].

Now the potential energy of the plate can be evaluated as

$$\Pi[w, \theta_x, \theta_y] = \iint \{\frac{D_b}{2}(k_x^2 + k_y^2 + 2\nu k_x k_y + \frac{1-\nu}{2}k_{xy}^2) + \frac{D_s}{2}(\gamma_x^2 + \gamma_y^2) - pw\} dx\, dy\,, \tag{9}$$

where the integration should be performed over the middle plane of the plate, the strains are calculated from the displacements, the displacements must a priori satisfy appropriate kinematic boundary conditions, and only the distributed lateral force p is considered as external forces for simplicity. As usual, the present energy functional can be used to derive various finite element models for Reissner-Mindlin plates. Of course the transverse shear energy is neglected in the Kirchhoff plates.

Originally, the displacements w, θ_x and θ_y are primary unknown functions. Actually, some couplings may be introduced to them to reduce the excessive stiffness as commonly observed in the Reissner-Mindlin formulation, but still they are basically independent of each other [2, 11, 12, 13]. We will employ here a slightly different approach based on the use of assumed shear strains and a mixed formulation.

Another variational formulation is based on the Hellinger-Reissner functional [10], which is given in a little simplified form as

$$\Pi_1^*[w,\theta_x,\theta_y,Q_x,Q_y] = \iint \{ \frac{D_b}{2}(k_x^2 + k_y^2 + 2\nu k_x k_y + \frac{1-\nu}{2}k_{xy}^2) + Q_x(w_x - \theta_x) + Q_y(w_y - \theta_y) - \frac{1}{2D_s}(Q_x^2 + Q_y^2) - pw\} dx\, dy\,, \qquad (10)$$

where the curvatures are calculated from the rotations, and the shear forces are now independent of the displacements. This functional is used as the basis of mixed finite element methods for the Reissner-Mindlin plates. For the Kirchhoff plates, the quadratic terms $\frac{1}{2D_s}(Q_x^2 + Q_y^2)$ are omitted in the above functional.

We can also use the shear strains as the argument functions instead of the shear forces:

$$\Pi_2^*[w,\theta_x,\theta_y,\gamma_x,\gamma_y] = \iint \{ \frac{D_b}{2}(k_x^2 + k_y^2 + 2\nu k_x k_y + \frac{1-\nu}{2}k_{xy}^2) + D_s\gamma_x(w_x - \theta_x) + D_s\gamma_y(w_y - \theta_y) - \frac{D_s}{2}(\gamma_x^2 + \gamma_y^2) - pw\} dx\, dy\,. \qquad (11)$$

This functional is, however, difficult to use in the Kirchhoff case since $\gamma_x = \gamma_y = 0$.

Our approach to be proposed may be considered a kind of mixed method in which independent shear strains or shear forces play important roles.

4 Basic Procedure for Triangular Elements

To fix the idea, we only consider 3-node triangular elements. As nodal displacements of usual Reissner-Mindlin triangular elements, we employ the lateral deflection w and two rotations θ_x and θ_y at each vertex node: $\{w_i, \theta_{xi}, \theta_{yi}\}_{i=1,2,3}$ (i = node number). However, we also use some unknown parameters for the transverse shear strains in our approach.

4.1 Base Kirchhoff element

Our approach starts from an appropriate Kirchhoff element. As a base 3-node triangular Kirchhoff element, we take either a conforming one such as the HCT element or a nonconforming one in implementation, although a conforming one may be convenient from purely theoretical standpoint [1, 14].

In the derivation of such a base Kirchhoff element, we only need the deflection pattern for w, which is usually expressed in terms of nodal displacements $\{w_i, w_{xi}, w_{y_i}\}_{i=1,2,3}$ (i = node number). On the other hand, $\{w_i, \theta_{xi}, \theta_{yi}\}_{i=1,2,3}$ are employed as the nodal displacements of a Reissner-Mindlin element. Thus we will later eliminate $\{w_{xi}, w_{yi}\}_{i=1,2,3}$ by introducing a special technique based on the shear-displacement relations in (7).

4.2 Assumed shear strains

Before assuming the rotations, we consider the shear strains $\boldsymbol{\gamma} = \{\gamma_x, \gamma_y\}$ and the shear forces in each element. We employ for $\boldsymbol{\gamma}$ the simplest Nedelec element [8]. More specifically, γ_x and γ_y are incomplete linear polynomials in each element:

$$\gamma_x = \beta_1 + \beta_3 y, \quad \gamma_y = \beta_2 - \beta_3 x, \tag{12}$$

where β_i's are coefficients. Such a shear vector function has a constant tangential value on each edge, and β_i's are uniquely determinable from these three edge values. Thus it is easy to make the tangential component of $\boldsymbol{\gamma}$ continuous across the interelement edges, so that the present choice is mathematically natural [15].

The shear forces $\boldsymbol{Q} = \{Q_x, Q_y\}$ take the same form as that of $\boldsymbol{\gamma}$, and is determined from $\boldsymbol{\gamma}$ by (8) in the Reissner-Mindlin element. For the Kirchhoff element, $\boldsymbol{\gamma}$ must be specified as zero, while $\boldsymbol{Q}$ can be assumed independently.

4.3 Specification of displacements

As was already noted, we need nodal values $\{w_{xi}, w_{yi}\}_{i=1,2,3}$ besides $\{w_i\}_{i=1,2,3}$ to specify w of the base Kirchhoff element. This may be done by utilizing the following relations based on (7) at nodes:

$$w_{xi} = \theta_{xi} + \gamma_{xi}\,, \quad w_{yi} = \theta_{yi} + \gamma_{yi}\,; \quad 1 \le i \le 3\,, \tag{13}$$

where γ_{xi}'s and γ_{yi}'s are the nodal values of $\boldsymbol{\gamma}$ which are determinable from the three edge values.

The next step is to assume two rotations $\boldsymbol{\theta} = \{\theta_x, \theta_y\}$. This may be essentially done by using relations (7) in each element, that is,

$$\theta_x = w_x - \gamma_x\,, \quad \theta_y = w_y - \gamma_y\,, \tag{14}$$

which are consistent with (13) at nodes.

Now we can completely determine all the displacements in each element from their nodal values and the edge values of $\boldsymbol{\gamma}$. The boundary conditions for displacements and $\boldsymbol{\gamma}$ are easy to specify at least in typical cases such as the clamped and simply supported ones.

4.4 Continuity of displacements

To check the continuity of w and $\boldsymbol{\theta}$ on element edges, let us consider an edge i facing to the vertex i and connecting vertices j and k. For simplicity, we restrict to the cases where vertices i, j and k are placed anticlockwise. Then the tangential component γ_s of $\boldsymbol{\gamma}$ is given by

$$\gamma_s = -n_y \gamma_x + n_x \gamma_y = \text{constant}, \tag{15}$$

where $\boldsymbol{n} = \{n_x, n_y\}$ is the unit outward normal vector of the considered edge. We denote the above value by γ_{si} for edge i $(1 \leq i \leq 3)$.

In usual 3-node elements, w is a cubic polynomial on each element edge. Then the tangential component θ_s of $\boldsymbol{\theta}$ is a quadratic one there as may be seen from (14). They, together with $w_s = \partial_s w$ ($\partial_s = \partial/\partial s$; $s =$ the line coordinate on the considered edge), are equal to their nodal values at both ends j and k of edge i, and the strain-displacement relation $w_s - \theta_s = \gamma_s$ on the edge holds. Consequently w and θ_s on edge i are uniquely determinable from their end values and the constant value of γ_s, so that they are continuous there.

On the other hand, the normal derivative $\partial_n w$ of w for usual conforming elements is linear on each edge, so that the normal component θ_n of $\vec{\theta}$ is linear due to (12) and (14). The continuity of θ_n follows from its linearity and continuity at both ends of each edge. However, the continuity of $\partial_n w$ does not necessarily hold except in the thin plate limit. Of course, the present arguments for θ_n hold only in the case where the base Kirchhoff element is conforming.

4.5 Mixed method

We can now decide the curvature changes as well as the shear strains in each element, which are necessary for calculating the stiffness matrices. Substituting these into the functional Π_2^* in (11) and making it stationary, we can obtain a mixed finite element scheme. Notice here that this is a special mixed method since the displacements and the shear strains are strongly coupled to satisfy (7) or (14) identically. Thus our approach may be also considered a kind of displacement-based method, so that we may call the arising element matrices "stiffness" ones unlike in the standard mixed methods. That is, we can directly use Π in (9) by regarding the shear strains as the enriching terms for the displacements. However, in the Kirchhoff plate case, we should use a purely mixed method based on Π_1^*, where the shear forces are assumed independently in the same form as the general shear strains, but the shear strains themselves are set to zero to match the Kirchhoff conditions.

4.6 Reduction of shear strains

It is also possible to determine $\boldsymbol{\gamma}$ from the nodal displacements so that we can eliminate γ_s from element DOF. That is, using a kinetic relation for a Timoshenko beam (5) or rather analogous one to a strip plate, we can specify γ_s on edge i in terms of nodal displacements of the two endpoints j and k [5, 7]:

$$\gamma_s = \frac{\phi_i}{1+\phi_i}\left(\frac{w_k - w_j}{\ell_i} - \frac{\theta_{sj}+\theta_{sk}}{2}\right); \qquad \phi_i = 12D_b/\{\ell_i^2 D_s\}\,, \tag{16}$$

where ℓ_i is the length of edge i, and θ_{sj} and θ_{sk} are values of $\theta_s = -n_y\theta_x + n_x\theta_y$ at node j and k, respectively. Using these values, we can uniquely decide γ_x and γ_y of (12) in each element. It is to be noted that $\phi_i/\{1+\phi_i\} \to 0$ $(t/\ell_i \to 0)$ and $\to 1$ $(t/\ell_i \to \infty)$. That is, γ_s vanishes in the thin plate limit, while it reduces to a kind of finite difference approximation of $\gamma_s = \partial_s w - \theta_s$ in the thick plate limit. However, it appears to be difficult for the present approach to determine the shear forces in the Kirchhoff case. On the other hand, they can be calculated as the Lagrange multipliers of the original mixed (i.e., non-reduced) formulation even in such a case.

The present technique may be also related to a kind of stabilization method [9]. Some arbitrariness, however, remains in the amplitudes of the stabilization parameters corresponding to ϕ_i above.

4.7 Extension of present approach

The present approach is available in various conforming and non-conforming 3-node elements. Moreover, it may be extended to some other elements such as discrete Kirchhoff and hybrid ones. In the discrete Kirchhoff elements, the first-order derivatives of w are not directly used as above: instead, special approximations $\partial_{hx}w$ and $\partial_{hy}w$ are used for the first-order derivatives $w_x = \partial_x w$ and $w_y = \partial_y w$ of w. In the case of DKT, they are exactly the Kirchhoff rotations determined from w by the discrete Kirchhoff assumptions, and extension to the Reissner-Mindlin element is now well-established [5, 7]. Our approach also appears to be promising in some hybrid Kirchhoff elements by determining the curvature changes or the moments appropriately. We will formulate and test such cases in due course.

4.8 Comments on error analysis

Based on the abstract error analysis of mixed FEM [15], we can evaluate errors of the present mixed (non-reduced) finite element solutions in the case of the purely conforming base Kirchhoff displacement fields. In particular, the inf-sup condition is essential, and can be established after carefully checking the approximate displacements with special coupling (14). Interpolation error analysis is also complicated due to the special form of displacements. Under some assumptions on the plate domain shape, we have derived error estimates in terms of the mesh size, which are uniform in the plate thickness after appropriate normalization on the exact plate solutions. In particular, error estimates for the shear forces in some weak topologies are also obtainable in the case of the mixed (i.e., non-reduced) formulation. We will publish the details of such analysis separately since they are rather lengthy and may be outside the interests from purely computational viewpoints.

For the reduced-shear element, the analysis is slightly more complicated but is expected to be accomplished by means of the techniques of the stabilization methods [9], and the coupled interpolation [11, 12, 13]. Analysis of more general cases including non-conforming base K-elements and DKT one needs further preparations.

5 Implementation

Here we give some comments on implementation that are specific to our approach. In other respects, our method can be implemented as the standard displacement-based finite element method.

5.1 General case

We have presented essentially two finite element formulations in the preceding section, both of which can be implemented according to the standard finite element procedures. That is, all the quantities are determinable from the DOF for the displacements and $\boldsymbol{\gamma}$ (or $\boldsymbol{Q} = \{Q_x, Q_y\}$ in the Kirchhoff case), where those for $\boldsymbol{\gamma}$ can be eliminated by (16) in the second approach. In particular, we require the curvature changes as well as the shear strains in each element. As usual, the necessary quantities can be finally arranged in the form of the element stiffness matrices and the element load vectors [1].

In general, the element stiffness matrix $[K]$ may be expressed in the form of sum:

$$[K] = [K_b] + [K_s]\,, \tag{17}$$

where $[K_b]$ and $[K_s]$ are respectively the bending and shearing parts. More specifically, the former is the part due to the bending energy, while the latter is to the transverse shear energy. In the calculations of $[K_b]$, it is to be noted that, for the present assumed shear strains (12), the curvature changes simply become

$$k_x = -\partial_x\theta_x = -\partial_x^2 w,\;\; k_y = -\partial_y\theta_y = -\partial_y^2 w,\;\; k_{xy} = -\partial_y\theta_x - \partial_x\theta_y = -2\partial_x\partial_y w, \tag{18}$$

which are formally the same as those for Kirchhoff plates. Actually, $[K_b]$ does not necessarily coincide with that of the base Kirchhoff element, since the nodal values of $w_x = \partial_x w$ and $w_y = \partial_y w$ are different from those of the Kirchhoff one as may be seen from (13). However, it can be easily obtained from $[K_b]$ of the original Kirchhoff one by applying the linear transformation (13). More specifically, first calculating a transformation matrix $[T]$ expressing (13) and then multiplying it to the original Kirchhoff bending matrix $[K_b^{(K)}]$, we can obtain $[K_b]$ as

$$[K_b] = [T]^t[K_b^{(K)}][T] \qquad (t\text{: transpose of matrix}). \tag{19}$$

On the other hand, $[K_s]$ is calculated directly from the assumed $\boldsymbol{\gamma}$.

As was already noted, $[K_b]$ does not coincide with the corresponding Kirchhoff matrix for finite plate thickness, but we can see that the components of $[K_b]$ converge to the corresponding Kirchhoff values in the thin plate limit. The shear part $[K_s]$ of the element stiffness matrix for the reduced shear formulation based on (16), obtained by evaluating the transverse shear energy in (9) or (11), is seen to vanish in the thin plate limit.

Since the consistent element load vectors are slightly complicated to compute due to the coupling of w with $\boldsymbol{\theta}$ and $\boldsymbol{\gamma}$, we here use the lumped one. That is, in the case of uniform p, we have as the load vector $\lfloor\, pS/3\ pS/3\ pS/3\ 0\ 0\ \cdots\cdots 0\ \ 0 \,\rfloor$, where S is the area of the triangular element, and the nodal variables are arranged in the order $w_1, w_2, w_3, \theta_{x1}, \cdots, \theta_{y3}, \gamma_{s1}, \gamma_{s2}, \gamma_{s3}$. Here the last three terms are for the transverse shear and their signs must be adjusted with those of the neighboring triangles occupying common edges. Of course, the last three terms are absent in the final stage of the reduced formulation.

5.2 Felippa-Bergan element

As a concrete base triangular Kirchhoff element, let us employ here the Felippa-Bergan nonconforming one based on the energy-orthogonal concept [16, 17], since it is easier to implement than various existing conforming ones and still has nice numerical performance.

The base deflection pattern w in each triangular element is of the form

$$w = \sum_{i=1}^{3} \{\alpha_i L_i + \alpha_{i+3} L_j L_k + \alpha_{i+6}(L_j - L_k)^3\} \ , \tag{20}$$

where L_i $(i = 1, 2, 3)$ are the area coordinates, α_m $(1 \leq m \leq 9)$ are coefficients, and $\{i, j, k\}$ is each cyclic permutation of $\{1, 2, 3\}$. The cubic terms are so chosen that they are energy-orthogonal to the lower terms. We can express the above coefficients uniquely in terms of the nodal displacements $\{w_i, w_{xi}, w_{yi}\}_{i=1,2,3}$.

Then we modify w as

$$w^\star = \sum_{i=1}^{3} \{\alpha_i^\star L_i + \alpha_{i+3}^\star L_j L_k + \alpha_{i+6}(L_j - L_k)^3\} \ , \tag{21}$$

where the modified quadratic part $\sum_{i=1}^{3}\{\alpha_i^\star L_i + \alpha_{i+3}^\star L_j L_k\}$ is nothing but the quadratic interpolation of w by using the vertices and midpoints of edges as 6 nodes. Such a modification is made so that the element passes the constant curvature patch test as the well-known convergence criterion [1]. Note that $w^\star$ is no longer a usual interpolation function in general, but still reduces to w when w is an arbitrary quadratic polynomial.

Using the above $w^\star$, we can obtain the stiffness matrix in the standard fashion. As was shown both numerically and theoretically, the present Kirchhoff element gives generally reasonable accuracy [16, 17]. Moreover, generalization of this element has been also made to optimize the numerical performance [16, 18]. It is straightforward to implement our approach if we use $w^\star$ in (21) as w in the preceding general formulation.

6 Numerical Results

We numerically tested our approach by a few sample problems. The base Kirchhoff element tested here is only the afore-mentioned Felippa-Bergan element since we focus on the point that our approach actually gives an expected Reissner-Mindlin extension. Due to the same reason, we here tested plates of square shape only. But we are planning to test various other elements and plates of various shape in due course. It is to be noted here that the DKT base element gives the Reissner-Mindlin extension identical to that of Katili [5], and Soh et al.[7] so long as the reduced shear variable formulation is employed.

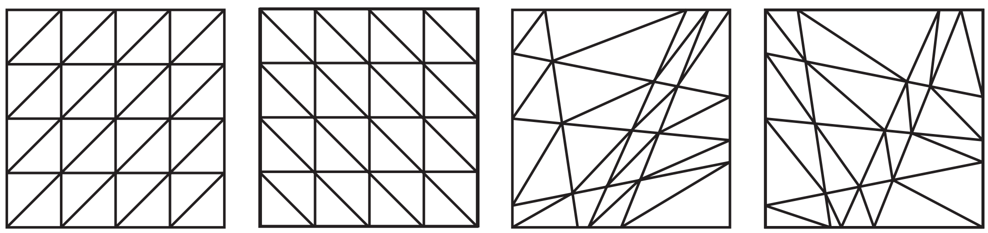

Figure 1: Examples of meshes (left to right: 4×4 A, 4×4 B, irregular A, irregular B)

Here we show some results for a simply supported or clamped square plate subjected to uniform lateral loading. Due to the double symmetry of the problems, only a quarter portion of the plate is divided into uniform $N \times N$ ($N = 2, 4, 8$) square meshes, each small square of which is composed of two triangles generated by a diagonal of a common direction, see Fig. 1. By A we denote meshes with diagonals running toward the plate center, and by B those with diagonals running perpendicular to those of A. Moreover, irregular meshes A and B included in Fig. 1 are also tested. In every case, the plate center is located at the upper-right corner of the mesh. Poisson's ratio is taken as 0 to compare with the results of Felippa-Bergan for Kirchhoff plates [16].

Table 1 summarizes the results of approximate central deflections w normalized by the exact Kirchhoff solutions. We take various values of t/L ($L =$side length of the plate). The case $t/L = 0$ is the Kirchhoff one, and the associated results are obtained by setting ϕ_i in (16) to 0. Although a plate with $t/L = 1$ is physically meaningless, such a case is considered here to see whether the transverse shear effects are reasonably represented. Here the simply-supported edge conditions are the so-called "hard" simply supported ones, that is, $w = \theta_s = 0$ on the edges. In the simple-support case, we also include the exact Reissner-Mindlin results based on the double Fourier series solutions.

We can see that the results are generally reasonable and the shear effects are also well represented as far as the central deflections are concerned. In particular, we checked that the results completely coincide with those based on the Felippa-Bergan

Table 1: Calculated central deflections normalized by the exact Kirchhoff solutions

nr : non-reduced shear variable formulation
r : reduced shear variable formulation (omitted when the difference is negligible in the displayed digits)

mesh		nr/r	t/L				
division	type		(0)	0.001	0.01	0.1	1.0
hard simply supported plate							
2×2	A	nr	1.0338	1.0338	1.0342	1.0726	4.8800
		r				1.0686	4.8616
4×4	A	nr	1.0078	1.0078	1.0082	1.0448	4.7162
		r			1.0081	1.0424	4.7115
8×8	A	nr	1.0019	1.0019	1.0022	1.0383	4.6562
		r				1.0374	4.6550
irregular	A	nr	0.9521	0.9521	0.9526	0.9934	4.8118
		r			0.9523	0.9787	4.6737
2×2	B	nr	0.9080	0.9080	0.9083	0.9374	3.9801
		r			0.9082	0.9359	3.9741
4×4	B	nr	0.9780	0.9780	0.9784	1.0122	4.4350
		r				1.0110	4.4330
8×8	B	nr	0.9946	0.9946	0.9949	1.0302	4.5719
		r				1.0298	4.5713
irregular	B	nr	0.9489	0.9489	0.9494	0.9880	4.5591
		r			0.9492	0.9802	4.5591
exact			1.0000	1.0000	1.0004	1.0363	4.6270
clamped plate							
2×2	A	nr	1.2803	1.2803	1.2822	1.4518	13.8460
		r			1.2820	1.4380	13.7877
4×4	A	nr	1.0731	1.0732	1.0751	1.2291	13.0938
		r			1.0748	1.2216	13.0790
8×8	A	nr	1.0185	1.0186	1.0205	1.1595	12.8367
		r		1.0185	1.0202	1.1568	12.8331
irregular	A	nr	0.9971	0.9972	0.9993	1.1583	13.4739
		r			0.9985	1.1144	13.0310
2×2	B	nr	1.0128	1.0128	1.0137	1.1083	10.8786
		r				1.1033	10.8592
4×4	B	nr	1.0121	1.0121	1.0134	1.1336	12.1594
		r			1.0133	1.1289	12.1526
8×8	B	nr	1.0040	1.0040	1.0054	1.1340	12.5575
		r			1.0053	1.1318	12.5556
irregular	B	nr	0.9803	0.9803	0.9818	1.1178	12.6192
		r			0.9814	1.0903	12.4664

element when the plate is sufficiently thin, so that the present element is completely free from the shear locking. The results obtained by the reduced shear formulation are also included, but the difference from the non-reduced one is fairly small even when the plate is thick. The results for A type meshes and various plate thickness by the non-reduced shear formulation are also illustrated in Fig. 2 in the case of the simply supported plate. We can thus observe both the convergence behavior for mesh refinement and the robustness on the plate thickness. The evaluation of the shear strains and forces is also important, and related numerical results will be reported in due course.

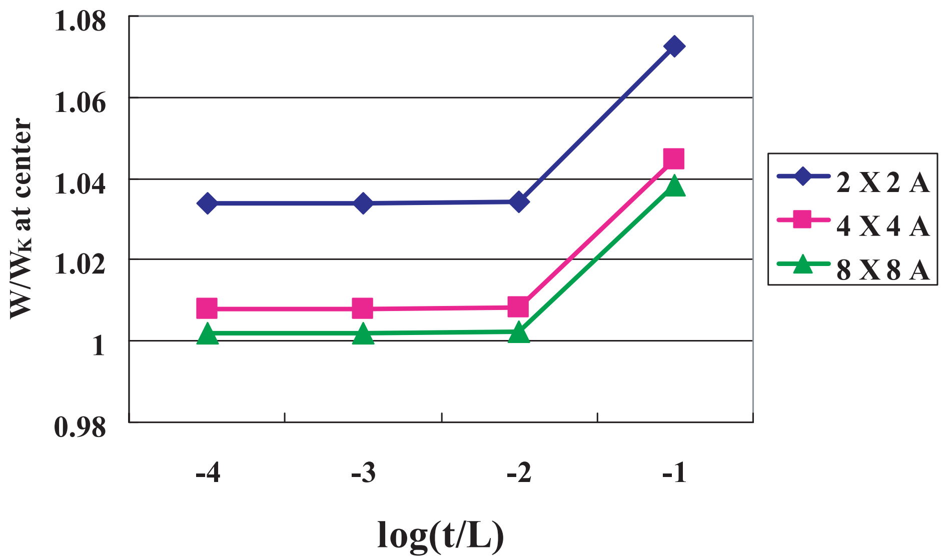

Figure 2: w/w_K at center vs t/L (w_K = exact w by Kirchhoff theory)

7 Concluding Remarks

We have presented a special mixed finite element method for Reissner-Mindlin plates by reusing classical Kirchhoff elements. We carefully choose the displacements and the transverse shear strains so that they can exactly satisfy the Kirchhoff condition in the thin plate limit as well as the shear-displacement relations in the finite thickness case. In particular, the derived element reduces to the base Kirchhoff element in the thin plate limit. Thus the element is free from the shear locking and also from the spurious zero-energy mode instability. Such desirable performance is in fact observed in the numerical tests. Theoretical error analysis is also available at least for a few conforming elements since the approximate displacements pass the inf-sup condition, a common criterion in the mixed finite element method. It is also possible to eliminate the transverse shear variables from DOF by using the beam element approximation, and the associated numerical results are shown to be reasonable.

Extension to 4-node quadrilateral elements is straightforward in some special cases such as the DKQ element [6], and may be available in some other cases. However, in the case of general convex quadrilateral shape, it is not necessarily easy to find a nice base Kirchhoff element and a reliable shear strain field. We will continue our study to cover more general cases including some non-conforming base Kirchhoff elements, DKT and DKQ ones, and hybrid ones. Anyway, our approach appears to be effective to develop efficient Reissner-Mindlin elements by rehabilitating classical Kirchhoff elements.

References

[1] O. C. Zienkiewicz and R. L. Taylor, *The Finite Element Method, Volume II, Solid Mechanics*, 5th ed. (Butterworth-Heinemann, Oxford, 2000).

[2] T. J. R. Hughes, *Finite Element Method: Linear Static and Dynamic Finite Element Analysis* (Dover Publications, New York, 2000).

[3] H. C. Martin, *Introduction to Matrix Methods of Structural Analysis* (McGraw-Hill, New York, 1966).

[4] J. S. Przemieniecki, *Theory of Matrix Structural Analysis* (McGraw-Hill, New York, 1967).

[5] I. Katili, "A new discrete Kirchhoff-Mindlin element based on Mindlin-Reissner plate theory and assumed shear strain fields– Part I: An extended DKT element for thick-plate bending analysis", *International Journal for Numerical Methods in Engineering* **36**, 1859-1883 (1993).

[6] I. Katili, "A new discrete Kirchhoff-Mindlin element based on Mindlin-Reissner plate theory and assumed shear strain fields– Part II: An extended DKQ element for thick-plate bending analysis", *International Journal for Numerical Methods in Engineering* **36**, 1885-1908 (1993).

[7] A. K. Soh, Z. F. Long and S. Cen, "A new nine DOF triangular element for analysis of thick and thin plates", *Computational Mechanics* **24**, 408-417 (1999).

[8] J.-C. Nedelec, "Mixed finite elements in $\mathbf{R}^3$", *Numerische Mathematik* **35**, 315-341 (1980).

[9] M. Lyly and R. Stenberg, "A survey of stabilized plate elements", in *Mathematical Modeling and Numerical Simulation in Continuum Mechanics — Proceedings of the International Symposium on Mathematical Modeling and Numerical Simulation in Continuum Mechanics*, ed. I. Babuska, P. G. Ciarlet and T. Miyoshi, pp. 11-21 (Springer-Verlag, Berlin-Heidelberg, 2002)

[10] K. Washizu, *Variational Methods in Elasticity and Plasticity*, 2nd ed. (Pergamon, Oxford, 1975).

[11] F. Kikuchi and Y. Ando, "Triangular finite element for plate bending analysis with transverse shear deformation", *Journal of Nuclear Science and Technology* **8**, 715-717 (1971).

[12] F. Kikuchi and K. Ishii, "An improved 4-node quadrilateral plate bending element of the Reissner-Mindlin type", *Computational Mechanics* **23**, 240-249 (1999).

[13] A. Tessler and T. J. R. Hughes, "An improved treatment of transverse shear in the Mindlin-type four-node quadrilateral element", *Computer Methods in Applied Mechanics and Engineering* **39**, 311-335 (1983).

[14] P. G. Ciarlet, *The Finite Element Method for Elliptic Problems* (SIAM, Philadelphia, 2002).

[15] F. Brezzi and M. Fortin, *Mixed and Hybrid Finite Element Methods* (Springer-Verlag, New York, 1991).

[16] C. A. Felippa and P. G.Bergan, "A triangular bending element based on an energy-orthogonal free formulation", *Computer Methods in Applied Mechanics and Engineering* **61**, 129-160 (1987).

[17] Z.-C. Shi, S.-C. Chen and F. Zhang, "Convergence analysis of Bergan's energy-orthogonal plate element", *Mathematical Models and Methods in Applied Sciences* **4**, 489-507 (1994).

[18] C. A. Felippa and C. Militello, "Construction of optimal 3-node plate bending triangles by templates", *Computational Mechanics* **24**, 1-13 (1999).

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