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SCATTERING THEORY FOR THE COUPLED KLEIN-GORDON-SCHRÖDINGER EQUATIONS IN TWO SPACE DIMENSIONS II

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ABSTRACT. We study the scattering theory for the coupled Klein-Gordon-Schrödinger equation with the Yukawa type interaction in two space dimensions. The scattering problem for this equation belongs to the borderline between the short range case and the long range one. We show the existence of the wave operators to this equation without any size restriction on the Klein-Gordon component of the final state and any restriction on the support of the Fourier transform of the final state.

1. INTRODUCTION

We study the scattering theory for the coupled Klein-Gordon-Schrödinger equation with the Yukawa type interaction in two space dimensions:

$$\begin{cases} i\partial_t u + \dfrac{1}{2}\Delta u = uv, \\ \partial_t^2 v - \Delta v + v = -|u|^2. \end{cases} \tag{KGS}$$

Here u and v are complex and real valued unknown functions of $(t,x) \in \mathbb{R} \times \mathbb{R}^2$, respectively. This paper is a sequel to the previous paper [21]. In the present paper, we prove the existence of the wave operators to the equation (KGS) without any size restriction on the Klein-Gordon component of the final state and any restriction on the support of the Fourier transform of the final state.

A large amount of works has been devoted to the asymptotic behavior of solutions for the nonlinear Schrödinger equation and for the nonlinear Klein-Gordon equation. We consider the scattering theory for systems centering on the Schrödinger equation, in particular, the Klein-Gordon-Schrödinger, the Wave-Schrödinger and the Maxwell-Schrödinger equations. In the scattering theory for the linear Schrödinger equation, the (ordinary) wave operators are defined as follows. Assume that for a solution of the free Schrödinger equation with given initial data ϕ, there exists a unique time global solution u for the perturbed Schrödinger equation such that u behaves like the given free solution as $t \to \infty$. (This case is called the short range case, and otherwise we call the long range case). Then we define the wave operator W_+ by the mapping from ϕ to $u|_{t=0}$. In the long range case, ordinary wave operators do not exist and we have to construct modified wave operators including a suitable phase correction in their definition. For the nonlinear Schrödinger equation, the nonlinear wave equation and systems centering on the Schrödinger equation, we can define the wave operators and introduce the modified wave operators in the same way. According to linear scattering theory, it seems that the equation (KGS) in two space dimensions belongs to the borderline between the short range case and the long range one, because the equation (KGS) has quadratic nonlinearities, and the solutions of the free Schrödinger equation and the free Klein-Gordon equation decay as t^{-1} in L^∞ as $t \to \infty$ in two space dimensions. The Maxwell-Schrödinger equation and the Wave-Schrödinger equation in three space dimensions also belong to the same case.

There are some results of the long range scattering for nonlinear equations and systems. Ozawa [15] and Ginibre and Ozawa [4] proved the existence of modified wave operators in the borderline case for the nonlinear Schrödinger equation in one space dimension and in two and three space dimensions, respectively. Their methods applied to the Klein-Gordon-Schrödinger equation in two space dimensions by Ozawa and Tsutsumi [16] and to the Maxwell-Schrödinger equation under the Coulomb gauge condition in three space dimensions by Tsutsumi [23]. In all results mentioned above, the restriction on the size of the final state is assumed. Furthermore in [16], the support of the Fourier transform of the Schrödinger data is restricted outside the unit disk in order to use the difference between the propagation property of the Schrödinger wave and the Klein-Gordon wave and to obtain additional time decay estimates for the nonlinear term uv, because we can not apply the method of the phase correction mentioned above to this nonlinear term by the fact that all derivatives of the solution for the free Klein-Gordon equation decay as fast as itself. In [23], the Fourier transform of the Schrödinger data vanishes in a neighborhood of the unit sphere by the same reason.

Recently Ginibre and Velo [5, 6, 7] have proved the existence of the modified wave operators for the Hartree equations with long range potentials with no restriction on the size of the final state. They decomposed the unknown function u into the complex amplitude w and the real phase φ, and solved the system for w and φ. Constructing the modified wave operators for those equations such that the domain and the range of them are same space, Nakanishi [13, 14] extended their results. Using the methods in [5, 6, 7], Ginibre and Velo showed the existence of modified wave operators for the Wave-Schrödinger equation ([8]) and for the Maxwell-Schrödinger equation under the Coulomb gauge condition ([9]) in three space dimensions with no restriction on the size of the final state. (The restriction on the support of the Fourier transform of the final state mentioned above is assumed in [8], and the vanishing asymptotic magnetic field is considered in [9]).

On the other hand, recently, the author has proved the existence of wave operators for the two dimensional Klein-Gordon-Schrödinger equation in [18], and the modified wave operators to the three dimensional Wave-Schrödinger equation in [17] and to the three dimensional Maxwell-Schrödinger equations under the Coulomb and the Lorentz gauge conditions in [19] for small scattered states without any restrictions on the support of the Fourier transform of them. The proof for the Klein-Gordon-Schrödinger equation is mainly based on the construction of suitable second correction term $(\tilde{u}_1, v_1)$ of the solution to that equation so that $(i\partial_t + \frac{1}{2}\Delta)\tilde{u}_1 - u_0 v_0$ and $(\partial_t^2 - \Delta + 1)v_1 + |u_0|^2$ decay faster than $u_0 v_0$ and $-|u_0|^2$ as $t \to \infty$, respectively, and that the Cook-Kuroda method is applicable. Here u_0 and v_0 are the solutions of the free Schrödinger and the free Klein-Gordon equations, respectively. Furthermore combining idea of [8] with that of [17], Ginibre and Velo [10] have proved the existence of modified wave operators for the three dimensional Wave-Schrödinger equation with restrictions on neither size of the scattered states nor the support of the Fourier transform of them.

In the previous paper [21], the author proved the existence of the wave operators for the equation (KGS) without any size restriction on the Klein-Gordon component of the final state. But in [21], the smallness of the Schrödinger data is assumed, and the support of the Fourier transform of the Schrödinger data is restricted outside the unit disk as in [16]. In order to remove the size restriction on the Klein-Gordon data from Ozawa and Tsutsumi [16], the difference between the exact solution for that equation and the asymptotic profile have to decay more rapidly than the derivatives of it as in [20] (see Proposition 2.1). That difference decays as t^{-k} $(1 < k < 2)$ as $t \to \infty$ in L^2, though the decay rate of that difference is order t^{-1} in [16, 18]. Because of this difficulty, we assumed the restriction on the support of the Fourier transform of the Schrödinger data in order to obtain an improved time decay estimate for the nonlinear term uv in [21]. For the Schrödinger component, the method of the phase correction was applied in order to handle slowly decaying terms caused by the second correction terms.

In this paper, we prove the existence of the wave operators to the equation (KGS) without any size restriction on the Klein-Gordon component of the final state and any support restriction on the Fourier transform of the Schrödinger component of the final state. Namely we remove the support restriction on the Fourier transform of the Schrödinger data from the previous result [21]. We only assume the smallness of the Schrödinger data. The proof is mainly based on the choice of a suitable asymptotic profile which approximates the equation (KGS) better than that in [18] for large time. More precisely, we construct second correction terms and third ones of the asymptotic profile, while, as mentioned above, we only constructed the second correction terms in [18]. For the Schrödinger component, the method of the phase correction is also applied in order to handle slowly decaying terms caused by the second correction terms as in [21].

Before stating our main result, we introduce some notations.

Notations. We use the following symbols:

$$\partial_0 = \partial_t = \frac{\partial}{\partial t}, \quad \partial_j = \frac{\partial}{\partial x_j} \quad \text{for } j = 1, 2,$$
$$\partial^\alpha = \partial_x^\alpha = \partial_1^{\alpha_1}\partial_2^{\alpha_2} \quad \text{for a multi-index } \alpha = (\alpha_1, \alpha_2),$$
$$\nabla = (\partial_1, \partial_2), \quad \Delta = \partial_1^2 + \partial_2^2,$$

for $t \in \mathbb{R}$ and $x = (x_1, x_2) \in \mathbb{R}^2$.

Let

$$L^q \equiv L^q(\mathbb{R}^2) = \left\{ \psi \colon \|\psi\|_{L^q} = \left(\int_{\mathbb{R}^2} |\psi(x)|^q \, dx \right)^{1/q} < \infty \right\} \text{ for } 1 \le q < \infty,$$
$$L^\infty \equiv L^\infty(\mathbb{R}^2) = \left\{ \psi \colon \|\psi\|_{L^\infty} = \operatorname{ess.sup}_{x \in \mathbb{R}^2} |\psi(x)| < \infty \right\}.$$

We use the L^2-scalar product

$$(\varphi, \psi) \equiv \int_{\mathbb{R}^2} \varphi(x) \overline{\psi(x)} \, dx.$$

$\mathcal{S}$ denotes the Schwartz class, that is, the set of rapidly decreasing functions on $\mathbb{R}^2$. Let $\mathcal{S}'$ be the set of tempered distributions on $\mathbb{R}^2$. For $w \in \mathcal{S}'$, we denote the Fourier transform of w by $\hat{w}$. For $w \in L^1(\mathbb{R}^n)$, $\hat{w}$ is represented as

$$\hat{w}(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} w(x) e^{-ix\cdot\xi} \, dx.$$

For $s, m \in \mathbb{R}$, we introduce the weighted Sobolev spaces $H^{s,m}$ corresponding to the Lebesgue space L^2 as follows:

$$H^{s,m} \equiv \{\psi \in \mathcal{S}' \colon \|\psi\|_{H^{s,m}} \equiv \|(1 + |x|^2)^{m/2} (1 - \Delta)^{s/2} \psi\|_{L^2} < \infty\}.$$

H^s denotes $H^{s,0}$. For $1 \le p \le \infty$ and a positive integer k, we define the Sobolev space W_p^k corresponding to the Lebesgue space L^p by

$$W_p^k \equiv \left\{ \psi \in L^p \colon \|\psi\|_{W_p^k} \equiv \sum_{|\alpha| \le k} \|\partial^\alpha \psi\|_{L^p} < \infty \right\}.$$

Note that for a positive integer k, $H^k = W_2^k$ and the norms $\|\cdot\|_{H^k}$ and $\|\cdot\|_{W_2^k}$ are equivalent.

For $s > 0$, we define the homogeneous Sobolev spaces $\dot{H}^s$ by the completion of $\mathcal{S}$ with respect to the norm

$$\|w\|_{\dot{H}^s} \equiv \|(-\Delta)^{s/2} w\|_{L^2}. \tag{1.1}$$

$\dot{H}^s$ is a Banach space with the norm (1.1) for $s > 0$.

We set for $t \in \mathbb{R}$,

$$U(t) \equiv e^{\frac{it}{2}\Delta}, \quad \Omega \equiv (1-\Delta)^{1/2}, \quad \omega \equiv (-\Delta)^{1/2}$$
$$K(t) \equiv \Omega^{-1} \sin \Omega t, \quad \dot{K}(t) \equiv \cos \Omega t,$$
$$\mathcal{L} \equiv i\partial_t + \frac{1}{2}\Delta, \quad \mathcal{K} \equiv \partial_t^2 - \Delta + 1, \quad \Box \equiv \partial_t^2 - \Delta.$$

C denotes various constants, and they may differ from line to line, when it does not cause any confusion.

Let $(u_+, v_+, \dot{v}_+)$ be a final state. u_+ and $(v_+, \dot{v}_+)$ are the Schrödinger and the Klein-Gordon components, respectively. We introduce the following asymptotic functions:

$$\begin{aligned} u_0(t,x) =&(U(t)e^{-i|\cdot|^2/2t} e^{-iS(t,-i\nabla)} u_+)(x) \\ =&\frac{1}{it} e^{i|x|^2/2t - iS(t,x/t)} \hat{u}_+\left(\frac{x}{t}\right), \end{aligned} \tag{1.2}$$

$$\begin{aligned} u_1(t,x) =& \left(U(t)e^{-i|\cdot|^2/2t} e^{-iS(t,-i\nabla)} \frac{i|\cdot|^2}{2t} u_+\right)(x) \\ =& -\frac{1}{2t^2} e^{i|x|^2/2t - iS(t,x/t)} \Delta \hat{u}_+\left(\frac{x}{t}\right), \end{aligned} \tag{1.3}$$

$$\begin{aligned} \tilde{u}_1(t,x) =& -\frac{1}{it^2} f_1\left(\frac{x}{t}\right) a_0\left(\frac{x}{t}\right) \hat{u}_+\left(\frac{x}{t}\right) e^{i|x|^2/2t - iS(t,x/t) + i\sqrt{t^2-|x|^2}} \\ &+ \frac{1}{it^2} g_1\left(\frac{x}{t}\right) \overline{a_0\left(\frac{x}{t}\right)} \hat{u}_+\left(\frac{x}{t}\right) e^{i|x|^2/2t - iS(t,x/t) - i\sqrt{t^2-|x|^2}}, \end{aligned} \tag{1.4}$$

$$v_0(t,x) = (\dot{K}(t)v_+)(x) + (K(t)\dot{v}_+)(x), \tag{1.5}$$

$$v_1(t,x) = -\frac{1}{t^2}\left|\hat{u}_+\left(\frac{x}{t}\right)\right|^2, \tag{1.6}$$

$$\begin{aligned} \tilde{v}_1(t,x) =& \frac{i}{2t^2}\left(1 - \frac{|x|^2}{t^2}\right)^{1/2} \left(f_1\left(\frac{x}{t}\right) - g_1\left(\frac{x}{t}\right)\right) \left|\hat{u}_+\left(\frac{x}{t}\right)\right|^2 \\ &\times \left(a_0\left(\frac{x}{t}\right) e^{i\sqrt{t^2-|x|^2}} - \overline{a_0\left(\frac{x}{t}\right)} e^{-i\sqrt{t^2-|x|^2}}\right) \end{aligned} \tag{1.7}$$

for $(t,x) \in [1,\infty) \times \mathbb{R}^2$, where

$$f_1(x) = \frac{2(1-|x|^2)}{2(1-|x|^2)^{3/2} + |x|^2} \quad \text{for } |x| < 1, \tag{1.8}$$

$$g_1(x) = \frac{2(1-|x|^2)}{2(1-|x|^2)^{3/2} - |x|^2} \quad \text{for } |x| < 1, \tag{1.9}$$

$$
a_0(x) = \begin{cases} \dfrac{i}{2(1-|x|^2)} \left[\hat{v}_+ \left(-\dfrac{x}{(1-|x|^2)^{1/2}} \right) \right. & \\ \quad \left. - i(1-|x|^2)^{1/2} \hat{\dot{v}}_+ \left(-\dfrac{x}{(1-|x|^2)^{1/2}} \right) \right] & \text{if } |x| < 1, \\ 0 & \text{if } |x| \geq 1, \end{cases} \tag{1.10}
$$

$$
S(t,x) = \frac{1}{t}(|\hat{u}_+(x)|^2 + |a_0(x)|^2 f_1(x) - |a_0(x)|^2 g_1(x)). \tag{1.11}
$$

The functions u_0 and v_0 are principal terms of the asymptotic profile. Note that u_0 is an approximate solution for the free Schrödinger equation and v_0 is the solution for the free Klein-Gordon equation. It is well-known that the function

$$
\frac{1}{t} a_0 \left(\frac{x}{t} \right) e^{i\sqrt{t^2-|x|^2}} + \frac{1}{t} \overline{a_0 \left(\frac{x}{t} \right)} e^{-i\sqrt{t^2-|x|^2}}
$$

is the leading term of the asymptotic expansion of the solution for the free Klein-Gordon equation in two space dimensions (see Lemma 3.2 below).

Let D denote the unit disk in $\mathbb{R}^2$. We define the functions $\tilde{g}_1$ and $\tilde{g}_2$ in the unit disk D:

$$
\tilde{g}_1(x) = \frac{1}{2(1-|x|^2)^{3/2} - |x|^2}, \quad \tilde{g}_2(x) = \frac{1}{(1-|x|^2)^{3/2} - |x|^2}. \tag{1.12}
$$

Throughout this paper, we assume that the space dimension is two.

The main result is as follows.

Theorem. *Let $u_+ \in H^{2,8}$, $\tilde{g}_1^{10} \partial^\alpha \hat{u}_+ \in L^2(D)$ for $|\alpha| \leq 6$ and $\tilde{g}_1^5 \tilde{g}_2^5 \partial^\alpha \hat{u}_+ \in L^2(D)$ for $|\alpha| \leq 4$, $v_+ \in \mathcal{S}$ and $\dot{v}_+ \in \mathcal{S}$. Assume that $\|u_+\|_{H^{2,2}}$ is sufficiently small. Let $1 < k < 2$. Then the equation* (KGS) *has a unique solution (u,v) satisfying*

$$
\begin{aligned}
& u \in C(\mathbb{R}; H^2), \quad v \in C(\mathbb{R}; H^2) \cap C^1(\mathbb{R}; H^1), \\
& \sup_{t \geq 1} (t^k \|u(t) - u_0(t) - (u_1(t) + \tilde{u}_1(t))\|_{L^2} \\
& \qquad + t\|u(t) - u_0(t) - (u_1(t) + \tilde{u}_1(t))\|_{\dot{H}^2}) < \infty, \\
& \sup_{t \geq 1} [t^k (\|v(t) - v_0(t) - (v_1(t) + \tilde{v}_1(t))\|_{H^1} \\
& \qquad + \|\partial_t (v(t) - v_0(t) - (v_1(t) + \tilde{v}_1(t)))\|_{L^2}) \\
& \qquad + t(\|v(t) - v_0(t) - (v_1(t) + \tilde{v}_1(t))\|_{\dot{H}^1 \cap \dot{H}^2} \\
& \qquad + \|\partial_t (v(t) - v_0(t) - (v_1(t) + \tilde{v}_1(t)))\|_{\dot{H}^1})] < \infty.
\end{aligned}
$$

In particular,

$$
\begin{aligned}
& \|u(t) - U(t)u_+\|_{H^2} + \|v(t) - v_0(t)\|_{H^2} \\
& \qquad + \|\partial_t v(t) - \partial_t v_0(t)\|_{H^1} \to 0,
\end{aligned}
$$

as $t \to +\infty$.

Furthermore for the equation (KGS), *the wave operator*

$$W_+ : (u_+, v_+, \dot{v}_+) \mapsto (u(0), v(0), \partial_t v(0))$$

is well-defined.

A similar result holds for negative time.

Remark 1.1. In Theorem, neither the size restriction on the Klein-Gordon component $(v_+, \dot{v}_+)$ of the final state nor the restriction on the support of the Fourier transform of the final state is assumed, while we restrict the restriction on the size of the Schrödinger component u_+ of the final state.

Remark 1.2. Since the function $\tilde{g}_1$ has a singularity on some circle contained in the unit disk D, the singular assumption $\tilde{g}_1^{10} \partial^\alpha \hat{u}_+ \in L^2(D)$ for $|\alpha| \leq 6$ in Theorem implies that $\partial^\alpha \hat{u}_+$ vanishes on that circle as in [18]. The assumption $\tilde{g}_1^5 \tilde{g}_2^5 \partial^\alpha \hat{u}_+ \in L^2(D)$ for $|\alpha| \leq 4$ also causes a similar phenomenon.

Remark 1.3. It is well-known that the equation (KGS) is globally well-posed in $C(\mathbb{R}; H^2) \oplus [C(\mathbb{R}; H^2) \cap C^1(\mathbb{R}; H^1)]$ (see Bachelot [1], Baillon and Chadam [2], Fukuda and Tsutsumi [3] and Hayashi and von Wahl [11]).

The outline of this paper is as follows. In Section 2, we solve the final value problem for the equation (KGS) for the asymptotic profile satisfying suitable conditions (see Proposition 2.1). In Section 3, we determine an asymptotic profile satisfying the assumptions of above final value problem.

2. The Final Value Problem

In this section, we solve the final value problem, that is, the Cauchy problem at infinity, for the equation (KGS) of general form. Namely, for an asymptotic profile (A, B) satisfying suitable assumptions, we construct a unique solution (u, v) for the equation (KGS) which approaches (A, B) as $t \to \infty$.

For a given asymptotic functions (A, B), we introduce the following functions.

$$R_1[A, B] = \mathcal{L}A - AB, \tag{2.1}$$

$$R_2[A, B] = \mathcal{K}B + |A|^2. \tag{2.2}$$

Proposition 2.1. *Assume that there exist positive constants δ, L_1, L_2 L_3 and L_4 such that for $t \geq 1$,*

$$\|A(t)\|_{W^2_\infty} \leq \delta t^{-1} + L_1 t^{-2}, \tag{2.3}$$

$$\|B(t)\|_{W^2_\infty} \leq L_2 t^{-1}, \tag{2.4}$$

$$\|R_1[A, B](t)\|_{H^2} \leq L_3 t^{-3}, \tag{2.5}$$

$$\|R_2[A, B](t)\|_{H^1} \leq L_4 t^{-3}, \tag{2.6}$$

and assume that $\delta > 0$ is sufficiently small. Let $1 < k < 2$. Then there exists a constant $T \geq 1$, depending only on δ, L_1, L_2, L_3 and L_4, such that the equation (KGS) *has a unique solution (u, v) satisfying*

$$u \in C([T, \infty); H^2), \quad v \in C([T, \infty); H^2) \cap C^1([T, \infty); H^1), \tag{2.7}$$

$$\sup_{t \geq T} (t^k \|u(t) - A(t)\|_{L^2} + t\|u(t) - A(t)\|_{\dot{H}^2}) < \infty, \tag{2.8}$$

$$\begin{aligned} &\sup_{t \geq T} [t^k (\|v(t) - B(t)\|_{H^1} + \|\partial_t v(t) - \partial_t B(t)\|_{L^2}) \\ &\qquad + t(\|v(t) - B(t)\|_{\dot{H}^1 \cap \dot{H}^2} + \|\partial_t v(t) - \partial_t B(t)\|_{\dot{H}^1})] < \infty. \end{aligned} \tag{2.9}$$

We can prove this proposition exactly in same way as in the proof of Proposition 2.1 in [21]. Therefore we omit the proof of this proposition.

Remark 2.1. In Proposition 2.1, the asymptotic profile (A, B) is not determined explicitly. In Section 3, we construct the asymptotic profile satisfying the assumptions of Proposition 2.1.

Remark 2.2. In Proposition 2.1, we do not restrict the size of the positive constants L_1, L_2 L_3 and L_4, though the smallness on the constant $\delta > 0$ is assumed.

Remark 2.3. By the global well-posedness of the equation (KGS), the solution (u, v) on the time interval $[T, \infty)$ for the equation (KGS) obtained in Proposition 2.1 can be extended all times.

3. Asymptotics and Proof of Theorem

In this section, by constructing an asymptotic profile (u_a, v_a) satisfying the assumptions of Proposition 2.1 under suitable conditions on the final state, we prove Theorem. Let $(u_+, v_+, \dot{v}_+)$ be a final state. Throughout this section, we assume that all the assumptions in Theorem are satisfied. Namely, we assume that $u_+ \in H^{2,8}$, $\tilde{g}_1^{10} \partial^\alpha \hat{u}_+ \in L^2(D)$ for $|\alpha| \leq 6$, $\tilde{g}_1^5 \tilde{g}_2^5 \partial^\alpha \hat{u}_+ \in L^2(D)$ for $|\alpha| \leq 4$, $v_+ \in \mathcal{S}$, $\dot{v}_+ \in \mathcal{S}$, and that $\|u_+\|_{H^{2,2}} \leq 1$ is sufficiently small. Let $C(u_+)$, $C(v_+, \dot{v}_+)$ and $C(u_+, v_+, \dot{v}_+)$ denote various positive finite constants depending on u_+, $(v_+, \dot{v}_+)$ and $(u_+, v_+, \dot{v}_+)$, respectively, and they may differ from line to line.

We find an asymptotic profile of the form

$$(u_a, v_a) = (u_0 + (u_1 + \tilde{u}_1) + u_2, v_0 + (v_1 + \tilde{v}_1) + v_2). \tag{3.1}$$

u_0 and v_0 are the principal terms of u_a and v_a, respectively. $u_1 + \tilde{u}_1$ and $v_1 + \tilde{v}_1$ are the second correction terms, and u_2 and v_2 are the third correction ones of u_a and v_a. ($u_0 \gg u_1, \tilde{u}_1 \gg u_2$, $v_0 \gg v_1, \tilde{v}_1 \gg v_2$). It is natural to expect that (u_0, v_0) is the free profile or the modified free profile.

As in [21], we set

$$v_0(t, x) = (\dot{K}(t)v_+)(x) + (K(t)\dot{v}_+)(x).$$

v_0 is a solution of the free Klein-Gordon equation with initial data $(v_+, \dot{v}_+)$. The time decay estimates of v_0 are well-known. (See, e.g., Lemmas 2.2 and 2.3 in Ozawa and Tsutsumi [16]).

Lemma 3.1. *There exists a constant $C(v_+, \dot{v}_+) > 0$ such that for $t \geq 1$,*

$$\|v_0(t)\|_{H^2} \leq C(v_+, \dot{v}_+),$$
$$\|v_0(t)\|_{W^2_\infty} \leq C(v_+, \dot{v}_+)t^{-1}.$$

We recall the asymptotic expansion of the free profile v_0 for the Klein-Gordon equation. The following lemma is well-known (see, e.g., Section 7.2 in Hörmander [12] and Lemma 2.1 in Sunagawa [22]).

Lemma 3.2. *For any positive integer N, and any multi-index $\alpha \in \mathbb{Z}_+^2$, there exists a constant $C_{N,\alpha}(v_+, \dot{v}_+) > 0$ such that*

$$\left| \partial_x^\alpha \left\{ v_0(t,x) - 2\sum_{j=0}^{N-1} \mathrm{Re}\left(\frac{1}{t^{j+1}} a_j \left(\frac{x}{t}\right) e^{i\sqrt{t^2-|x|^2}} \right) \right\} \right| \leq C_{N,\alpha}(v_+, \dot{v}_+) t^{-1-N}$$

for $(t,x) \in [1,\infty) \times \mathbb{R}^2$, where the functions $a_j \in C^\infty(\mathbb{R}^2;\mathbb{C})$, $j = 0,1,2,\ldots,$ satisfy the following:

- *$a_j(x) = 0$ if $|x| \geq 1$.*
- *For any positive integer m and any multi-index $\alpha \in \mathbb{Z}_+^2$, there exists a constant $C_{j,\alpha,m}(v_+, \dot{v}_+) > 0$ such that*

$$|\partial^\alpha a_j(x)| \leq C_{j,\alpha,m}(v_+, \dot{v}_+)(1-|x|^2)^m \quad \text{for } |x| < 1.$$

In particular, a_0 is given by (1.10).

Remark 3.1. According to Section 7.2 in Hörmander [12], the function a_j in above lemma has the following form

$$a_j(x) = \begin{cases} \tilde{a}_j \left(-\dfrac{x}{(1-|x|^2)^{1/2}} \right) & \text{if } |x| < 1, \\ 0 & \text{if } |x| \geq 1 \end{cases}$$

with a suitable function $\tilde{a}_j \in \mathcal{S}$.

We use the asymptotic expansion of v_0 in Lemma 3.2 for $N = 1$ and $N = 2$. We introduce the following functions:

$$V^{(0)}(t,x) = \frac{1}{t} a_0 \left(\frac{x}{t}\right) e^{i\sqrt{t^2-|x|^2}} + \frac{1}{t}\overline{a_0\left(\frac{x}{t}\right)} e^{-i\sqrt{t^2-|x|^2}},$$
$$V^{(1)}(t,x) = \frac{1}{t^2} a_1 \left(\frac{x}{t}\right) e^{i\sqrt{t^2-|x|^2}} + \frac{1}{t^2}\overline{a_1\left(\frac{x}{t}\right)} e^{-i\sqrt{t^2-|x|^2}},$$

where the functions a_0 and a_1 appears in Lemma 3.2. According to Lemma 3.2, the functions $V^{(0)}$ and $V^{(0)} + V^{(1)}$ are asymptotic forms of v_0 for large time.

Let R_1 and R_2 be defined by (2.1) and (2.2), respectively. We consider the asymptotic profile (u_a, v_a), which has the form (3.1). Then we see that

$$\begin{aligned}
R_1[u_a, v_a] =& \mathcal{L}u_a - u_a v_a \\
=& \mathcal{L}(u_0 + u_1) + \mathcal{L}u_2 + (\mathcal{L}\tilde{u}_1 - u_0 V^{(0)}) \\
& - (u_1 + \tilde{u}_1)V^{(0)} - u_0 v_1 - u_0 V^{(1)} - u_0 \tilde{v}_1 \\
& - u_0(v_0 - (V^{(0)} + V^{(1)})) - (u_1 + \tilde{u}_1)(v_0 - V^{(0)}) \\
& - ((u_1 + \tilde{u}_1) + u_2)(v_1 + \tilde{v}_1) - u_2 v_0 - u_a v_2,
\end{aligned} \tag{3.2}$$

$$\begin{aligned}
R_2[u_a, v_a] =& \mathcal{K}v_a + |u_a| \\
=& (\mathcal{K}v_1 + |u_0|^2) + (\mathcal{K}\tilde{v}_1 + 2\,\mathrm{Re}(\bar{u}_0 \tilde{u}_1)) \\
& + (\mathcal{K}v_2 + 2\,\mathrm{Re}(\bar{u}_0 u_1)) + 2\,\mathrm{Re}(\bar{u}_0 u_2) \\
& + |u_1 + \tilde{u}_1 + u_2|^2.
\end{aligned} \tag{3.3}$$

In the second equality of (3.3), we have used the fact $\mathcal{K}v_0 = 0$.

Hereafter we construct functions u_0, u_1, $\tilde{u}_1$, u_2, v_1, $\tilde{v}_1$ and v_2 such that the asymptotic profile (u_a, v_a) of the form (3.1) and the functions $R_1[u_a, v_a]$ and $R_2[u_a, v_a]$ satisfy the assumptions in Proposition 2.1.

Recall that the function

$$\frac{1}{it} e^{i|x|^2/2t} \hat{\phi}\left(\frac{x}{t}\right) - \frac{1}{2t^2} e^{i|x|^2/2t} \Delta\hat{\phi}\left(\frac{x}{t}\right)$$

is an asymptotics of the free profile $U(t)\phi$ for the Schrödinger equation. In view of this, as in [21] we define

$$\begin{aligned}
u_0(t,x) =& (U(t)e^{-i|\cdot|^2/2t} e^{-iS(t,-i\nabla)} u_+)(x) \\
=& \frac{1}{it} \hat{u}_+\left(\frac{x}{t}\right) e^{i|x|^2/2t - iS(t,x/t)}, \\
u_1(t,x) =& \left(U(t)e^{-i|\cdot|^2/2t} e^{-iS(t,-i\nabla)} \frac{i|\cdot|^2}{2t} u_+\right)(x) \\
=& -\frac{1}{2t^2} \Delta\hat{u}_+\left(\frac{x}{t}\right) e^{i|x|^2/2t - iS(t,x/t)}.
\end{aligned}$$

We will determine a real phase function S satisfying the following estimate later: If $|\alpha| \le 4$ and $j = 0, 1$, then there exist constants $C(u_+, v_+, \dot{v}_+) > 0$ and $C(u_+) > 0$ such that

$$|\partial_x^\alpha \partial_t^j S(t,x)| \le \begin{cases} C(u_+, v_+, \dot{v}_+)|\tilde{g}_1(x)| t^{-1-j} & \text{if } |x| < 1, \\ C(u_+) t^{-1-j} & \text{if } |x| \ge 1 \end{cases} \tag{3.4}$$

for $t \ge 1$, where the function $\tilde{g}_1$ is defined by (1.12).

We consider the first term $\mathcal{K}v_1 + |u_0|^2$ in the right hand side of the equation (3.3). Because $|u_0|^2 = t^{-2}|\hat{u}_+(x/t)|^2$, $\||u_0(t)|^2\|_{L^2}$ decays as $O(t^{-1})$. $|u_0|^2$ does not satisfy the assumption (2.6) on R_2 in Proposition 2.1. In order to obtain improved time decay estimates of R_2, we choose the second correction term v_1 of v_a such that $\mathcal{K}v_1+|u_0|^2$ decays faster than $|u_0|^2=t^{-2}|\hat{u}_+(x/t)|^2$ as in [16, 18, 21]. We put

$$v_1(t,x)=-|u_0(t,x)|^2=-\frac{1}{t^2}\left|\hat{u}_+\left(\frac{x}{t}\right)\right|^2.$$

This function coincides with the right hand side of (1.6). Note that $|u_0|^2$ is independent of a choice of the real phase function S though it has not yet determined explicitly. (As mentioned above, S will be determined later). Then

$$\mathcal{K}v_1(t,x)+|u_0(t,x)|^2=-\Box\left(\frac{1}{t^2}\left|\hat{u}_+\left(\frac{x}{t}\right)\right|^2\right).$$

By a direct calculation, we have the following lemma.

Lemma 3.3. *Let $k=0,1,2$. There exists a constant $C(u_+)>0$ such that for $t\geq 1$,*

$$\|\omega^k v_1(t)\|_{L^2}\leq C(u_+)t^{-k-1},$$
$$\sum_{|\alpha|=k}\|\partial^\alpha v_1(t)\|_{L^\infty}\leq C(u_+)t^{-k-2},$$
$$\|\mathcal{K}v_1(t)+|u_0(t)|^2\|_{H^1}\leq C(u_+)t^{-3}.$$

We consider the third term $\mathcal{L}\tilde{u}_1-u_0V^{(0)}$ in right hand side of the equality (3.2). Because $u_0V^{(0)}$ decays as t^{-1} in L^2 as $t\to\infty$, it does not satisfy the assumption (2.5) on R_1 of Proposition 2.1. Since all derivatives of $V^{(0)}$ decay as fast as itself ($=O(t^{-1})$ in L^∞), we can not apply the method of the phase correction to the slowly decaying term $u_0V^{(0)}$. In order to overcome this difficulty, we find a second correction term $\tilde{u}_1$ such that $\mathcal{L}\tilde{u}_1-u_0V^{(0)}$ decays faster than $u_0V^{(0)}$ as $t\to\infty$ as in [18].

Let $b\geq 1$, $m\in\mathbb{R}\setminus\{0\}$, let P be a function on $\mathbb{R}^2$ supported in the unit disk $\{x\in\mathbb{R}^2;|x|<1\}$, and let

$$F(t,x)=\frac{1}{it}P\left(\frac{x}{t}\right)e^{i|x|^2/2t-iS(t,x/t)}.$$

By a direct calculation, we have

$$\begin{aligned}
&\mathcal{L}\left(\frac{1}{t^b}e^{im\sqrt{t^2-|x|^2}}F(t,x)\right)\\
&\quad=-\frac{1}{t^b}\left(m\left(1-\frac{|x|^2}{t^2}\right)^{1/2}+m^2\frac{|x|^2/t^2}{2(1-(|x|^2/t^2))}\right)e^{im\sqrt{t^2-|x|^2}}F(t,x)\\
&\qquad-\frac{i}{t^{b+1}}\left(b+m\frac{2-(|x|^2/t^2)}{2(1-(|x|^2/t^2))^{3/2}}\right)e^{im\sqrt{t^2-|x|^2}}F(t,x)\\
&\qquad-\frac{im}{t^{b+1}}\frac{x/t}{(1-(|x|^2/t^2))^{1/2}}e^{im\sqrt{t^2-|x|^2}}\cdot\tilde{F}(t,x)
\end{aligned}$$

$$
\begin{aligned}
&- \frac{m}{t^{b+1}} \frac{x/t}{(1-(|x|^2/t^2))^{1/2}} e^{im\sqrt{t^2-|x|^2}} \cdot \nabla S\left(t, \frac{x}{t}\right) F(t,x) \\
&+ \frac{1}{t^b} e^{im\sqrt{t^2-|x|^2}} \mathcal{L}F(t,x),
\end{aligned} \tag{3.5}
$$

where

$$
\tilde{F}(t,x) = \frac{1}{it} \nabla P\left(\frac{x}{t}\right) e^{i|x|^2/2t - iS(t,x/t)}.
$$

Remark 3.2. Noting the equality

$$
\begin{aligned}
\mathcal{L}F(t,x) =& \frac{1}{it} e^{i|x|^2/2t} \Bigg(\frac{1}{2t^2} \Delta_y (e^{-iS(t,y)} P(y))|_{y=x/t} \\
&\qquad + (\partial_0 S)\left(t, \frac{x}{t}\right) e^{-iS(t,x/t)} P\left(\frac{x}{t}\right) \Bigg),
\end{aligned}
$$

we see that if the phase function S satisfies (3.4) and P is a polynomial of $\hat{u}_+$ and its derivative, then the first term in the right hand side of (3.5) decays as t^{-b}, the second and the third ones decay as t^{-b-1}, and the last two terms decay as t^{-b-2} as $t \to \infty$ in H^2. Indeed, the first term in the right hand side of (3.5) is the principal part, and the other terms are the remainder ones.

By the definitions of u_0 and $V^{(0)}$, we see that

$$
\begin{aligned}
u_0(t,x)V^{(0)}(t,x) =& \frac{1}{t} e^{i\sqrt{t^2-|x|^2}} \left(\frac{1}{it} a_0\left(\frac{x}{t}\right) \hat{u}_+\left(\frac{x}{t}\right) e^{i|x|^2/2t - iS(t,x/t)} \right) \\
&+ \frac{1}{t} e^{-i\sqrt{t^2-|x|^2}} \left(\frac{1}{it} \overline{a_0\left(\frac{x}{t}\right)} \hat{u}_+\left(\frac{x}{t}\right) e^{i|x|^2/2t - iS(t,x/t)} \right).
\end{aligned}
$$

In view of this, we construct a second correction term $\tilde{u}_1$ of the form

$$
\tilde{u}_1(t,x) = \frac{1}{t} e^{i\sqrt{t^2-|x|^2}} F_1(t,x) + \frac{1}{t} e^{-i\sqrt{t^2-|x|^2}} F_2(t,x),
$$

where

$$
\begin{aligned}
F_1(t,x) &= \frac{1}{it} P_1\left(\frac{x}{t}\right) e^{i|x|^2/2t - iS(t,x/t)}, \quad \operatorname{supp} P_1 \subset \{x \in \mathbb{R}^2; |x| < 1\}, \\
G_1(t,x) &= \frac{1}{it} Q_1\left(\frac{x}{t}\right) e^{i|x|^2/2t - iS(t,x/t)}, \quad \operatorname{supp} Q_1 \subset \{x \in \mathbb{R}^2; |x| < 1\}.
\end{aligned}
$$

Applying the equality (3.5) to the cases of $(b,m) = (1,1)$ and $(b,m) = (1,-1)$, we see that

$$
\begin{aligned}
\mathcal{L}\tilde{u}_1(t,x) =& -\frac{1}{t} e^{i\sqrt{t^2-|x|^2}} \frac{1}{f_1(x/t)} F_1(t,x) + \frac{1}{t} e^{-i\sqrt{t^2-|x|^2}} \frac{1}{g_1(x/t)} G_1(t,x) \\
&- \frac{i}{t^2} \left(1 + \frac{2 - (|x|^2/t^2)}{2(1-(|x|^2/t^2))^{3/2}} \right) e^{i\sqrt{t^2-|x|^2}} F_1(t,x)
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{i}{t^2}\left(1-\frac{2-(|x|^2/t^2)}{2(1-(|x|^2/t^2))^{3/2}}\right)e^{-i\sqrt{t^2-|x|^2}}G_1(t,x)\\
&- \frac{i}{t^2}\frac{x/t}{(1-(|x|^2/t^2))^{1/2}}e^{i\sqrt{t^2-|x|^2}}\cdot\tilde{F}_1(t,x)\\
&+ \frac{i}{t^2}\frac{x/t}{(1-(|x|^2/t^2))^{1/2}}e^{-i\sqrt{t^2-|x|^2}}\cdot\tilde{G}_1(t,x)\\
&- \frac{1}{t^2}\frac{x/t}{(1-(|x|^2/t^2))^{1/2}}e^{i\sqrt{t^2-|x|^2}}\cdot\nabla S\left(t,\frac{x}{t}\right)F_1(t,x)\\
&+ \frac{1}{t^2}\frac{x/t}{(1-(|x|^2/t^2))^{1/2}}e^{-i\sqrt{t^2-|x|^2}}\cdot\nabla S\left(t,\frac{x}{t}\right)G_1(t,x)\\
&+ \frac{1}{t}e^{i\sqrt{t^2-|x|^2}}\mathcal{L}F_1(t,x)+\frac{1}{t}e^{-i\sqrt{t^2-|x|^2}}\mathcal{L}G_1(t,x),
\end{aligned} \tag{3.6}
$$

where

$$
\begin{aligned}
\tilde{F}_1(t,x) &= \frac{1}{it}\nabla P_1\left(\frac{x}{t}\right)e^{i|x|^2/2t-iS(t,x/t)},\\
\tilde{G}_1(t,x) &= \frac{1}{it}\nabla Q_1\left(\frac{x}{t}\right)e^{i|x|^2/2t-iS(t,x/t)},
\end{aligned}
$$

and the functions f_1 and g_1 are defined by (1.8) and (1.9), respectively. We note that the first and the second terms in the right hand side of (3.6) decay most slowly if the phase function S satisfies the estimate (3.4) (see Remark 3.2). If we choose the functions P_1 and Q_1 such that

$$
\begin{aligned}
u_0(t,x)V^{(0)}(t,x) = &-\frac{1}{t}e^{i\sqrt{t^2-|x|^2}}\frac{1}{f_1(x/t)}F_1(t,x)\\
&+\frac{1}{t}e^{-i\sqrt{t^2-|x|^2}}\frac{1}{g_1(x/t)}G_1(t,x)
\end{aligned} \tag{3.7}
$$

holds, then $\mathcal{L}\tilde{u}_1 - u_0V^{(0)}$ decays faster than $u_0V^{(0)}$. In fact, if we set

$$
P_1(x) = -f_1(x)a_0(x)\hat{u}_+(x), \quad Q_1(x) = g_1(x)\overline{a_0(x)}\hat{u}_+(x), \tag{3.8}
$$

then the equality (3.7) is satisfied. It follows from definitions (3.8) of P_1 and Q_1 that

$$
\begin{aligned}
\tilde{u}_1(t,x) =& \frac{1}{t}e^{i\sqrt{t^2-|x|^2}}F_1(t,x)+\frac{1}{t}e^{-i\sqrt{t^2-|x|^2}}F_2(t,x)\\
=& -\frac{1}{it^2}f_1\left(\frac{x}{t}\right)a_0\left(\frac{x}{t}\right)\hat{u}_+\left(\frac{x}{t}\right)e^{i|x|^2/2t-iS(t,x/t)+i\sqrt{t^2-|x|^2}}\\
&+\frac{1}{it^2}g_1\left(\frac{x}{t}\right)\overline{a_0\left(\frac{x}{t}\right)}\hat{u}_+\left(\frac{x}{t}\right)e^{i|x|^2/2t-iS(t,x/t)-i\sqrt{t^2-|x|^2}}.
\end{aligned}
$$

By the equality (3.6), we have

$$
\begin{aligned}
&\mathcal{L}\tilde{u}_1(t,x) - u_0(t,x)V^{(0)}(t,x)\\
&= -\frac{i}{t^2}\left(1 + \frac{2-(|x|^2/t^2)}{2(1-(|x|^2/t^2))^{3/2}}\right)e^{i\sqrt{t^2-|x|^2}}F_1(t,x)\\
&\quad -\frac{i}{t^2}\left(1 - \frac{2-(|x|^2/t^2)}{2(1-(|x|^2/t^2))^{3/2}}\right)e^{-i\sqrt{t^2-|x|^2}}G_1(t,x)\\
&\quad -\frac{i}{t^2}\frac{x/t}{(1-(|x|^2/t^2))^{1/2}}e^{i\sqrt{t^2-|x|^2}}\cdot\tilde{F}_1(t,x)\\
&\quad +\frac{i}{t^2}\frac{x/t}{(1-(|x|^2/t^2))^{1/2}}e^{-i\sqrt{t^2-|x|^2}}\cdot\tilde{G}_1(t,x)\\
&\quad -\frac{1}{t^2}\frac{x/t}{(1-(|x|^2/t^2))^{1/2}}e^{i\sqrt{t^2-|x|^2}}\cdot\nabla S\left(t,\frac{x}{t}\right)F_1(t,x)\\
&\quad +\frac{1}{t^2}\frac{x/t}{(1-(|x|^2/t^2))^{1/2}}e^{-i\sqrt{t^2-|x|^2}}\cdot\nabla S\left(t,\frac{x}{t}\right)G_1(t,x)\\
&\quad +\frac{1}{t}e^{i\sqrt{t^2-|x|^2}}\mathcal{L}F_1(t,x) + \frac{1}{t}e^{-i\sqrt{t^2-|x|^2}}\mathcal{L}G_1(t,x).
\end{aligned}
\tag{3.9}
$$

From this equality and Remark 3.2, we see that $\mathcal{L}\tilde{u}_1 - u_0V^{(0)}$ decays as t^{-2} in H^2 if the phase function S satisfies the estimate (3.4). This term does not satisfy the assumption (2.5) on R_1 in Proposition 2.1 though it decays faster than $u_0V^{(0)}$, which decays as t^{-1} in L^2. To overcome difficulty, we have to construct a third correction term u_2 for the Schrödinger component. This will be done at the end of the choice of an asymptotic profile (u_a, v_a) in the same manner.

Before constructing a third correction term u_2 for the Schrödinger component, we have to find another second correction term $\tilde{v}_1$ for the Klein-Gordon component in order to obtain an improved time decay estimate for the second term $\mathcal{K}\tilde{v}_1 + 2\,\mathrm{Re}(\bar{u}_0\tilde{u}_1)$ in the right hand side of the equality (3.3). From the definitions of u_0 and $\tilde{u}_1$, we see

$$
\begin{aligned}
&2\,\mathrm{Re}(\overline{u_0(t,x)}\tilde{u}_1(t,x))\\
&=2\,\mathrm{Re}\left[\frac{1}{t^3}\overline{\hat{u}_+\left(\frac{x}{t}\right)}P_1\left(\frac{x}{t}\right)e^{i\sqrt{t^2-|x|^2}}\right.\\
&\qquad\left. + \frac{1}{t^3}\overline{\hat{u}_+\left(\frac{x}{t}\right)}Q_1\left(\frac{x}{t}\right)e^{-i\sqrt{t^2-|x|^2}}\right]\\
&=2\,\mathrm{Re}\left[\frac{1}{t^3}\left(\overline{\hat{u}_+\left(\frac{x}{t}\right)}P_1\left(\frac{x}{t}\right) + \hat{u}_+\left(\frac{x}{t}\right)\overline{Q_1\left(\frac{x}{t}\right)}\right)e^{i\sqrt{t^2-|x|^2}}\right],
\end{aligned}
\tag{3.10}
$$

where the functions P_1 and Q_1 are defined by (3.8). Here we note that the function $2\,\mathrm{Re}(\bar{u}_0\tilde{u}_1)$ is independent of a choice of the phase function S though it has not yet determined explicitly. It follows from above equality that the function $2\,\mathrm{Re}(\bar{u}_0\tilde{u}_1)$ decays as t^{-2} in L^2 and that this term does not satisfy the assumption (2.6) on R_2 in Proposition 2.1. We construct a second correction term $\tilde{v}_1$ for the Klein-Gordon component such that $\mathcal{K}\tilde{v}_1 + 2\,\mathrm{Re}(\bar{u}_0\tilde{u}_1)$ decays faster than $2\,\mathrm{Re}(\bar{u}_0\tilde{u}_1)$. In view of the equality (3.10), we find a second correction term $\tilde{v}_1$ of the form

$$v_1(t,x) = 2\,\mathrm{Re}\left(\frac{1}{t^b}Y\left(\frac{x}{t}\right)e^{i\sqrt{t^2-|x|^2}}\right),$$

where $b \geq 1$ and Y is a function on $\mathbb{R}^2$ supported in the unit disk $\{x \in \mathbb{R}^2; |x| < 1\}$. We determine the constant b and the function Y.

By a direct calculation, we have

$$\begin{aligned}&\mathcal{K}\left(\frac{1}{t^b}Y\left(\frac{x}{t}\right)e^{i\sqrt{t^2-|x|^2}}\right)\\ =&\left[-\frac{1}{t^{b+1}}\frac{2(b-1)i}{(1-(|x|^2/t^2))^{1/2}}Y\left(\frac{x}{t}\right) + \Box\left(\frac{1}{t^b}Y\left(\frac{x}{t}\right)\right)\right]e^{i\sqrt{t^2-|x|^2}}.\end{aligned} \tag{3.11}$$

It is easyly seen that the first term in the right hand side of (3.11) decays as t^{-b} in L^2 and the second one decays as t^{-b-1} in the same space. Namely the first term is the leading term. Now we put

$$b = 2, \tag{3.12}$$

$$\begin{aligned}Y(x) =& -\frac{i}{2}(1-|x|^2)^{1/2}(\overline{\hat{u}_+(x)}P_1(x) + \hat{u}_+(x)\overline{Q_1(x)})\\ =&\frac{i}{2}(1-|x|^2)^{1/2}(f_1(x)-g_1(x))a_0(x)|\hat{u}_+(x)|^2\end{aligned} \tag{3.13}$$

so that the equality

$$\begin{aligned}&-2\,\mathrm{Re}(\overline{u_0(t,x)}\tilde{u}_1(t,x))\\ &\quad=2\,\mathrm{Re}\left(-\frac{1}{t^{b+1}}\frac{2(b-1)i}{(1-(|x|^2/t^2))^{1/2}}Y\left(\frac{x}{t}\right)e^{i\sqrt{t^2-|x|^2}}\right)\end{aligned} \tag{3.14}$$

holds. Therefore we determine the second correction term $\tilde{v}_1$ for the Klein-Gordon component by

$$\tilde{v}_1(t,x) = 2\,\mathrm{Re}\left(\frac{1}{t^2}Y\left(\frac{x}{t}\right)e^{i\sqrt{t^2-|x|^2}}\right), \tag{3.15}$$

where the function Y is defined by (3.13). Note that this function coincides with the right hand side of (1.7). It follows from the equalities (3.10), (3.11), (3.14) and (3.15) that

$$\begin{aligned}&\mathcal{K}\tilde{v}_1(t,x) + 2\,\mathrm{Re}(\overline{u_0(t,x)}\tilde{u}_1(t,x))\\ &\quad=2\,\mathrm{Re}\left[\left\{\Box\left(\frac{1}{t^2}Y\left(\frac{x}{t}\right)\right)\right\}e^{i\sqrt{t^2-|x|^2}}\right].\end{aligned} \tag{3.16}$$

By the equalities (3.15) and (3.16), we obtain the following lemma.

Lemma 3.4. *There exists a constant $C(u_+, v_+, \dot{v}_+) > 0$ such that for $t \geq 1$,*

$$\|\tilde{v}_1(t)\|_{H^2} \leq C(u_+, v_+, \dot{v}_+)t^{-1},$$
$$\|\tilde{v}_1(t)\|_{W^2_\infty} \leq C(u_+, v_+, \dot{v}_+)t^{-2},$$
$$\|\mathcal{K}\tilde{v}_1(t) + 2\,\mathrm{Re}(\overline{u_0(t)}\tilde{u}_1(t))\|_{H^1} \leq C(u_+, v_+, \dot{v}_+)t^{-3}.$$

Remark 3.3. In this lemma, we have used the assumption $\tilde{g}_1^{10}\partial^\alpha \hat{u}_+ \in L^2(D)$ for $|\alpha| \leq 6$, where $\tilde{g}_1$ is defined by (1.12) and D is the unit disk in $\mathbb{R}^2$, because the function g_1 appears in the definition of the function $\tilde{v}_1$.

We next consider the third term $\mathcal{K}v_2 + 2\,\mathrm{Re}(\bar{u}_0 u_1)$ in the right hand side of the equation (3.3). By the definitions of u_0 and u_1, $\|\bar{u}_0 u_1\|_{L^2}$ decays as $O(t^{-2})$. This is not sufficient to satisfy the assumption (2.6) on R_2 of Proposition 2.1. In order to obtain improved time decay estimates of R_2, we choose the third correction term v_2 of v_a such that $\mathcal{K}v_2 + 2\,\mathrm{Re}(\bar{u}_0 u_1)$ decays faster than $2\,\mathrm{Re}(\bar{u}_0 u_1)$ in the same manner as in [21]. We put

$$v_2(t, x) = -2\,\mathrm{Re}(\bar{u}_0 u_1) = -\frac{1}{t^3}\,\mathrm{Im}\left(\overline{\hat{u}_+\left(\frac{x}{t}\right)}\Delta\hat{u}_+\left(\frac{x}{t}\right)\right).$$

Here we note that the function $2\,\mathrm{Re}(\bar{u}_0 u_1)$ is independent of a choice of the phase function S though it has not yet determined explicitly. Then

$$\mathcal{K}v_2(t, x) + 2\,\mathrm{Re}(\bar{u}_0 u_1) = -\Box\left[\frac{1}{t^3}\,\mathrm{Im}\left(\overline{\hat{u}_+\left(\frac{x}{t}\right)}\Delta\hat{u}_+\left(\frac{x}{t}\right)\right)\right].$$

By a direct calculation, we have the following lemma.

Lemma 3.5. *Let $k = 0, 1, 2$. There exists a constant $C(u_+) > 0$ such that for $t \geq 1$,*

$$\|\omega^k v_2(t)\|_{L^2} \leq C(u_+)t^{-k-2},$$
$$\sum_{|\alpha|=k} \|\partial^\alpha v_2(t)\|_{L^\infty} \leq C(u_+)t^{-k-3}$$
$$\|\mathcal{K}v_2(t) + 2\,\mathrm{Re}(\overline{u_0(t)}u_1(t))\|_{H^1} \leq C(u_+)t^{-3}.$$

Finally we construct a real phase function S, which appears in the definitions of u_0 and u_1, and a third correction term u_2 of the Schrödinger component u_a.

By the definitions of u_0, $V^{(0)}$ and $\tilde{u}_1$, we see that

$$
\begin{aligned}
&\tilde{u}_1(t,x)V^{(0)}(t,x) \\
&\quad = -\frac{1}{t^2}\left(f_1\left(\frac{x}{t}\right) - g_1\left(\frac{x}{t}\right)\right)\left|a_0\left(\frac{x}{t}\right)\right|^2 u_0(t,x) \\
&\quad\quad + \frac{1}{t^2}e^{2i\sqrt{t^2-|x|^2}}\frac{1}{it}a_0\left(\frac{x}{t}\right)P_1\left(\frac{x}{t}\right)e^{i|x|^2/2t-iS(t,x/t)} \\
&\quad\quad + \frac{1}{t^2}e^{-2i\sqrt{t^2-|x|^2}}\frac{1}{it}\overline{a_0\left(\frac{x}{t}\right)}Q_1\left(\frac{x}{t}\right)e^{i|x|^2/2t-iS(t,x/t)},
\end{aligned}
\tag{3.17}
$$

where P_1 and Q_1 are defined by (3.8). By the equalities (3.2), (3.9) and (3.17), we decompose $R_1[u_a, v_a]$ into three parts:

$$
R_1[u_a, v_a] = q_1 + q_2 + q_3, \tag{3.18}
$$

where

$$
\begin{aligned}
q_1 =& \mathcal{L}(u_0+u_1) - \left[v_1 - \frac{1}{t^2}\left(f_1\left(\frac{x}{t}\right) - g_1\left(\frac{x}{t}\right)\right)\left|a_0\left(\frac{x}{t}\right)\right|^2\right]u_0 \\
=& \mathcal{L}(u_0+u_1) \\
&+ \frac{1}{t^2}\left[\left|\hat{u}_+\left(\frac{x}{t}\right)\right|^2 + \left(f_1\left(\frac{x}{t}\right) - g_1\left(\frac{x}{t}\right)\right)\left|a_0\left(\frac{x}{t}\right)\right|^2\right]u_0,
\end{aligned}
\tag{3.19}
$$

$$
\begin{aligned}
q_2 =& \mathcal{L}u_2 - (u_1V^{(0)} + u_0V^{(1)} + u_0\tilde{v}_1) \\
&- \frac{i}{t^2}\left(1 + \frac{2-(|x|^2/t^2)}{2(1-(|x|^2/t^2))^{3/2}}\right)e^{i\sqrt{t^2-|x|^2}}F_1(t,x) \\
&- \frac{i}{t^2}\left(1 - \frac{2-(|x|^2/t^2)}{2(1-(|x|^2/t^2))^{3/2}}\right)e^{-i\sqrt{t^2-|x|^2}}G_1(t,x) \\
&- \frac{i}{t^2}\frac{x/t}{(1-(|x|^2/t^2))^{1/2}}e^{i\sqrt{t^2-|x|^2}}\cdot\tilde{F}_1(t,x) \\
&+ \frac{i}{t^2}\frac{x/t}{(1-(|x|^2/t^2))^{1/2}}e^{-i\sqrt{t^2-|x|^2}}\cdot\tilde{G}_1(t,x) \\
&- \left[\frac{1}{t^2}e^{2i\sqrt{t^2-|x|^2}}\frac{1}{it}a_0\left(\frac{x}{t}\right)P_1\left(\frac{x}{t}\right)e^{i|x|^2/2t-iS(t,x/t)}\right. \\
&\left.+ \frac{1}{t^2}e^{-2i\sqrt{t^2-|x|^2}}\frac{1}{it}\overline{a_0\left(\frac{x}{t}\right)}Q_1\left(\frac{x}{t}\right)e^{i|x|^2/2t-iS(t,x/t)}\right],
\end{aligned}
\tag{3.20}
$$

$$
\begin{aligned}
q_3 =& -u_0(v_0 - (V^{(0)} + V^{(1)})) - (u_1 + \tilde{u}_1)(v_0 - V^{(0)}) \\
&- ((u_1+\tilde{u}_1) + u_2)(v_1 + \tilde{v}_1) - u_2v_0 - u_av_2 \\
&- \frac{1}{t^2}\frac{x/t}{(1-(|x|^2/t^2))^{1/2}}e^{i\sqrt{t^2-|x|^2}}\cdot\nabla S\left(t,\frac{x}{t}\right)F_1(t,x) \\
&+ \frac{1}{t^2}\frac{x/t}{(1-(|x|^2/t^2))^{1/2}}e^{-i\sqrt{t^2-|x|^2}}\cdot\nabla S\left(t,\frac{x}{t}\right)G_1(t,x) \\
&+ \frac{1}{t}e^{i\sqrt{t^2-|x|^2}}\mathcal{L}F_1(t,x) + \frac{1}{t}e^{-i\sqrt{t^2-|x|^2}}\mathcal{L}G_1(t,x).
\end{aligned}
\tag{3.21}
$$

We choose a real phase function S such that q_1 defined by (3.19) decays as t^{-3} in H^2 as $t \to \infty$ (see assumption (2.5) in Proposition 2.1). We set

$$W = W_0 + W_1, \quad W_0(t,x) = \hat{u}_+(x), \quad W_1(t,x) = -\frac{i}{2t}\Delta \hat{u}_+(x).$$

Then by a direct calculation, we have

$$\begin{aligned} q_1 =& MDe^{-iS}\Bigg[i\partial_t W_0 + \left(i\partial_t W_1 + \frac{1}{2t^2}\Delta W_0 \right) \\ &+ \left(\partial_t S + \frac{1}{t^2}(|\hat{u}_+|^2 + |a_0|^2 f_1 - |a_0|^2 g_1) \right) W \\ &+ \frac{1}{2t^2}\Delta W_1 - \frac{i}{2t^2}(2\nabla S \cdot \nabla W + W\Delta S) - \frac{1}{2t^2}|\nabla S|^2 W \Bigg] \\ =& MDe^{-iS}\Bigg[\left(\partial_t S + \frac{1}{t^2}(|\hat{u}_+|^2 + |a_0|^2 f_1 - |a_0|^2 g_1) \right) W \\ &+ \frac{1}{2t^2}\Delta W_1 - \frac{i}{2t^2}(2\nabla S \cdot \nabla W + W\Delta S) - \frac{1}{2t^2}|\nabla S|^2 W \Bigg], \end{aligned}$$

where M and D are the following operators:

$$(Mf)(t,x) = e^{i|x|^2/2t} f(x), \quad (Dg)(t,x) = \frac{1}{it} g\left(t, \frac{x}{t}\right).$$

In the same way as in [21], we determine

$$S(t,x) = \frac{1}{t}(|\hat{u}_+(x)|^2 + |a_0(x)|^2 f_1(x) - |a_0(x)|^2 g_1(x))$$

so that

$$\partial_t S(t,x) = -\frac{1}{t^2}(|\hat{u}_+(x)|^2 + |a_0(x)|^2 f_1(x) - |a_0(x)|^2 g_1(x))$$

for $(t,x) \in [1,\infty) \times \mathbb{R}^2$. Then

$$\begin{aligned} q_1 =& MDe^{-iS}\Bigg[\frac{1}{2t^2}\Delta W_1 - \frac{i}{2t^2}(2\nabla S \cdot \nabla W + W\Delta S) \\ &- \frac{1}{2t^2}|\nabla S|^2 W \Bigg], \end{aligned} \tag{3.22}$$

We note that the function S defined above coincides with the right hand side of (1.11) and that it satisfies the time decay estimate (3.4). Therefore u_0, u_1 and $\tilde{u}_1$ are determined completely, and they are equal to the right hand sides of (1.2), (1.3) and (1.4), respectively.

By using Lemma 3.2 and noting the equation (3.22), we have the following lemma exactly in the same way as in the derivation of Lemma 3.4 in [21].

Lemma 3.6. *There exist constants $C > 0$ and $C(u_+, v_+, \dot{v}_+)$ such that for $t \geq 1$,*

$$
\begin{aligned}
\|u_0(t)\|_{H^2} &\leq C\|u_+\|_{H^2} + C(u_+, v_+, \dot{v}_+)t^{-1}, \\
\|u_0(t)\|_{W^2_\infty} &\leq C\|u_+\|_{H^{2,2}}t^{-1} + C(u_+, v_+, \dot{v}_+)t^{-2}, \\
\|u_1(t)\|_{H^2} &\leq C(u_+, v_+, \dot{v}_+)t^{-1}, \\
\|u_1(t)\|_{W^2_\infty} &\leq C(u_+, v_+, \dot{v}_+)t^{-2}, \\
\|\tilde{u}_1(t)\|_{H^2} &\leq C(u_+, v_+, \dot{v}_+)t^{-1}, \\
\|\tilde{u}_1(t)\|_{W^2_\infty} &\leq C(u_+, v_+, \dot{v}_+)t^{-2}, \\
\|q_1(t)\|_{H^2} &\leq C(u_+, v_+, \dot{v}_+)t^{-3}.
\end{aligned}
$$

Here the constant C is independent of $(u_+, v_+, \dot{v}_+)$, and the constant $C(u_+, v_+, \dot{v}_+)$ depends on them.

Remark 3.4. In this lemma, we have used the assumption $\tilde{g}_1^{10}\partial^\alpha \hat{u}_+ \in L^2(D)$ for $|\alpha| \leq 6$, where $\tilde{g}_1$ is defined by (1.12) and D is the unit disk in $\mathbb{R}^2$, because the function g_1 appears in the definitions of the functions $\tilde{u}_1$ and S.

We next construct a third correction term u_2 for the Schrödinger component such that q_2 decays as t^{-3} in H^2 as $t \to \infty$ (see assumption (2.5) in Proposition 2.1) exactly in the same way as in the construction of the second correction term $\tilde{u}_1$. Recalling the definitions of u_0, u_1, $V^{(0)}$, $V^{(1)}$ and $\tilde{v}_1$, we rewrite q_2 defined in (3.20) as

$$
\begin{aligned}
q_2 =& \mathcal{L}u_2 - \left[\frac{1}{t^2}e^{i\sqrt{t^2-|x|^2}}\frac{1}{it}\Phi^{(1)}\left(\frac{x}{t}\right)e^{i|x|^2/2t-iS(t,x/t)}\right. \\
&+ \frac{1}{t^2}e^{-i\sqrt{t^2-|x|^2}}\frac{1}{it}\Psi^{(1)}\left(\frac{x}{t}\right)e^{i|x|^2/2t-iS(t,x/t)} \\
&+ \frac{1}{t^2}e^{2i\sqrt{t^2-|x|^2}}\frac{1}{it}\Phi^{(2)}\left(\frac{x}{t}\right)e^{i|x|^2/2t-iS(t,x/t)} \\
&\left.+ \frac{1}{t^2}e^{-2i\sqrt{t^2-|x|^2}}\frac{1}{it}\Psi^{(2)}\left(\frac{x}{t}\right)e^{i|x|^2/2t-iS(t,x/t)}\right],
\end{aligned}
\tag{3.23}
$$

where

$$
\begin{aligned}
\Phi^{(1)}(x) =& i\left(1 + \frac{2-|x|^2}{2(1-|x|^2))^{3/2}}\right)P_1(x) \\
&+ \frac{ix}{(1-|x|^2)^{1/2}} \cdot \nabla P_1(x) - \frac{i}{2}\Delta\hat{u}_+(x)a_0(x) \\
&+ \hat{u}_+(x)a_1(x) + \hat{u}_+(x)Y(x),
\end{aligned}
$$

$$\begin{aligned}\Psi^{(1)}(x) =& i\left(1-\frac{2-|x|^2}{2(1-|x|^2))^{3/2}}\right)Q_1(x)\\ &-\frac{ix}{(1-|x|^2)^{1/2}}\cdot\nabla Q_1(x)-\frac{i}{2}\Delta\hat{u}_+(x)\overline{a_0(x)}\\ &+\hat{u}_+(x)\overline{a_1(x)}+\hat{u}_+(x)\overline{Y(x)},\\ \Phi^{(2)}(x) &= a_0(x)P_1(x),\\ \Psi^{(2)}(x) &= \overline{a_0(x)}Q_1(x).\end{aligned}$$

We construct a third correction term u_2 such that q_2 decays faster than $[\dots]$ in the right hand side of (3.23), which decays as t^{-2} in L^2. We find u_2 of the form

$$\begin{aligned}u_2(t,x) =& \frac{1}{t^2}e^{i\sqrt{t^2-|x|^2}}F_{2,1}(t,x)+\frac{1}{t^2}e^{-i\sqrt{t^2-|x|^2}}G_{2,1}(t,x)\\ &+\frac{1}{t^2}e^{2i\sqrt{t^2-|x|^2}}F_{2,2}(t,x)+\frac{1}{t^2}e^{-2i\sqrt{t^2-|x|^2}}G_{2,2}(t,x),\end{aligned}$$

where

$$\begin{aligned}F_{2,j}(t,x) &= \frac{1}{it}P_{2,j}\left(\frac{x}{t}\right)e^{i|x|^2/2t-iS(t,x/t)}, \quad \operatorname{supp} P_{2,j}\subset\{x\in\mathbb{R}^2; |x|<1\},\\ G_{2,j}(t,x) &= \frac{1}{it}Q_{2,j}\left(\frac{x}{t}\right)e^{i|x|^2/2t-iS(t,x/t)}, \quad \operatorname{supp} Q_{2,j}\subset\{x\in\mathbb{R}^2; |x|<1\}\end{aligned}$$

for $j=1,2$. We determine the functions $P_{2,j}$ and $Q_{2,j}$ $(j=1,2)$.

Applying the equality (3.5) to the cases of $(b,m)=(2,1)$, $(2,-1)$, $(2,2)$, $(2,-2)$ and noting Remark 3.2, we see that the principal part of $\mathcal{L}u_2$ is

$$\begin{aligned}&-\frac{1}{t^2}e^{i\sqrt{t^2-|x|^2}}\frac{1}{f_1(x/t)}F_{2,1}(t,x)+\frac{1}{t^2}e^{-i\sqrt{t^2-|x|^2}}\frac{1}{g_1(x/t)}G_{2,1}(t,x)\\ &\quad-\frac{2}{t^2}e^{2i\sqrt{t^2-|x|^2}}\frac{1}{f_2(x/t)}F_{2,2}(t,x)+\frac{2}{t^2}e^{-2i\sqrt{t^2-|x|^2}}\frac{1}{g_2(x/t)}G_{2,2}(t,x),\end{aligned}$$

where the functions f_1 and g_1 are defined by (1.8) and (1.9), respectively, and

$$\begin{aligned}f_2(x) &= \frac{1-|x|^2}{2((1-|x|^2)^{3/2}+|x|^2)} \quad \text{for } |x|<1,\\ g_2(x) &= \frac{1-|x|^2}{2((1-|x|^2)^{3/2}-|x|^2)} \quad \text{for } |x|<1.\end{aligned}$$

As in the construction of $\tilde{u}_1$, we put

$$\begin{aligned}P_{2,j}(x) &= -f_j(x)\Phi^{(j)}(x), \quad (j=1,2),\\ Q_{2,j}(x) &= g_j(x)\Psi^{(j)}(x), \quad (j=1,2)\end{aligned}$$

so that the principal term of $\mathcal{L}u_2$ mentioned above coincides with $[\dots]$ in the right hand side of (3.23). Then we see that q_2 decays faster than $[\dots]$ in the right hand side of (3.23).

Therefore, recalling Lemma 3.2, we have the following estimates.

Lemma 3.7. *There exists a constant $C(u_+, v_+, \dot{v}_+)$ such that for $t \geq 1$,*

$$\|u_2(t)\|_{H^2} \leq C(u_+, v_+, \dot{v}_+)t^{-2},$$
$$\|u_2(t)\|_{W^2_\infty} \leq C(u_+, v_+, \dot{v}_+)t^{-3},$$
$$\|q_2(t)\|_{H^2} \leq C(u_+, v_+, \dot{v}_+)t^{-3}.$$

Remark 3.5. In this lemma, we have used the assumptions $\tilde{g}_1^{10}\partial^\alpha \hat{u}_+ \in L^2(D)$ for $|\alpha| \leq 6$ and $\tilde{g}_1^5 \tilde{g}_2^5 \partial^\alpha \hat{u}_+ \in L^2(D)$ for $|\alpha| \leq 4$, where $\tilde{g}_1$ and $\tilde{g}_2$ are defined by (1.12) and D is the unit disk in $\mathbb{R}^2$, as in Lemma 3.6. (Note that the function g_2 appears in the definition of the function u_2).

Now the asymptotic profile (u_a, v_a) of the form (3.1) is determined explicitly. Noting Lemmas 3.1–3.7 and Remark 3.2 and using the Hölder inequality and the Sobolev embedding theorem, we have the following lemma.

Lemma 3.8. *There exists a constant $C(u_+, v_+, \dot{v}_+)$ such that for $t \geq 1$,*

$$\|q_3(t)\|_{H^2} \leq C(u_+, v_+, \dot{v}_+)t^{-3}.$$

Recalling the definitions of the functions (u_a, v_a), $R_1[u_a, v_a]$ and $R_2[u_a, v_a]$ and using Lemmas 3.1–3.8, the Hölder inequality and the Sobolev embedding theorem, we obtain the following time decay estimates for u_a, v_a, $R_1[u_a, v_a]$ and $R_2[u_a, v_a]$.

Lemma 3.9. *There exist constants $C > 0$ and $C(u_+, v_+, \dot{v}_+)$ such that for $t \geq 1$,*

$$\|u_a(t)\|_{W^2_\infty} \leq C\|u_+\|_{H^{2,2}}t^{-1} + C(u_+, v_+, \dot{v}_+)t^{-2},$$
$$\|v_a(t)\|_{W^2_\infty} \leq C(u_+, v_+, \dot{v}_+)t^{-1},$$
$$\|R_1[u_a, v_a](t)\|_{H^2} \leq C(u_+, v_+, \dot{v}_+)t^{-3},$$
$$\|R_2[u_a, v_a](t)\|_{H^1} \leq C(u_+, v_+, \dot{v}_+)t^{-3}.$$

Here the constant C is independent of $(u_+, v_+, \dot{v}_+)$, and the constant $C(u_+, v_+, \dot{v}_+)$ depends on them.

Proof of Theorem. We assume that all the assumptions of Theorem are satisfied. If we put

$$(A, B) = (u_a, v_a),$$
$$\delta = C\|u_+\|_{H^{2,2}},$$
$$L_1 = L_2 = L_3 = L_4 = C(u_+, v_+, \dot{v}_+),$$

where $C > 0$ and $C(u_+, v_+, \dot{v}_+)$ are the constants which appear in Lemma 3.9, then the assumptions in Proposition 2.1 are satisfied. By

Proposition 2.1, if $\|u_+\|_{H^{2,2}}$ is sufficiently small and if $T \geq 1$, which depends on $\|u_+\|_{H^{2,2}}$ and $C(u_+, v_+, \dot{v}_+)$, is sufficiently large, then there exists a unique solution (u, v) satisfying

$$u \in C([T,\infty); H^2), \quad v \in C([T,\infty); H^2) \cap C^1([T,\infty); H^1), \tag{3.24}$$

$$\sup_{t\geq T}(t^k\|u(t) - u_a(t)\|_{L^2} + t\|u(t) - u_a(t)\|_{\dot{H}^2}) < \infty, \tag{3.25}$$

$$\begin{aligned}\sup_{t\geq T}[t^k(&\|v(t) - v_a(t)\|_{H^1} + \|\partial_t v(t) - \partial_t v_a(t)\|_{L^2}) \\ &+ t(\|v(t) - v_a(t)\|_{\dot{H}^1\cap\dot{H}^2} + \|\partial_t v(t) - \partial_t v_a(t)\|_{\dot{H}^1})] < \infty.\end{aligned} \tag{3.26}$$

Since the equation (KGS) is globally well-posed in $C(\mathbb{R}; H^2)\oplus[C(\mathbb{R}; H^2)\cap C^1(\mathbb{R}; H^1)]$ (see Bachelot [1], Baillon and Chadam [2], Fukuda and Tsutsumi [3] and Hayashi and von Wahl [11]), the unique solution (u, v) on the time interval $[T, \infty)$ obtained above can be extended to all times. Since $\|u_2(t)\|_{H^2} = O(t^{-2})$ and $\|v_2(t)\|_{H^2} = O(t^{-2})$ (see Lemmas 3.5 and 3.7) and since $1 < k < 2$, the third correction terms u_2 and v_2 are negligible in the estimates (3.25) and (3.26), respectively. This completes the proof of Theorem. □

References

[1] A. Bachelot, *Probléme de Cauchy pour des systèms hyperboliques semi-linéaires*, Ann. Inst. H. Poincaré, Anal. Non Linéaire, **1** (1984), 453–478.

[2] J. B. Baillon and J. M. Chadam, *The Cauchy problem for the coupled Schrödinger-Klein-Gordon equations*, in "Contemporary Developments in Continuum Mechanics and Partial Differential Equations", (G. M. de La Penha and L. A. Medeiros Eds), North-Holland, Amsterdam (1989), 37–44.

[3] I. Fukuda and M. Tsutsumi, *On coupled Klein-Gordon-Schrödinger equations II*, J. Math. Anal. Appl., **66** (1978), 358–378.

[4] J. Ginibre and T. Ozawa, *Long range scattering for nonlinear Schrödinger and Hartree equations in space dimension $n \geq 2$*, Comm. Math. Phys., **151** (1993), 619–645.

[5] J. Ginibre and G. Velo, *Long range scattering and modified wave operators for some Hartree type equations I*, Rev. Math. Phys, **12** (2000), 361–429.

[6] J. Ginibre and G. Velo, *Long range scattering and modified wave operators for some Hartree type equations II*, Ann. Henri Poincaré, **1** (2000), 753–800.

[7] J. Ginibre and G. Velo, *Long range scattering and modified wave operators for some Hartree type equations III, Gevrey spaces and low dimensions*, J. Differential Equations, **175** (2001), 415–501.

[8] J. Ginibre and G. Velo, *Long range scattering and modified wave operators for the Wave-Schrödinger system*, Ann. Henri Poincaré, **3** (2002), 537–612.

[9] J. Ginibre and G. Velo, *Long range scattering and modified wave operators for the Maxwell-Schrödinger system I. The case of vanishing asymptotic magnetic field*, Comm. Math. Phys., **236** (2003), 395–448.

[10] J. Ginibre and G. Velo, *Long range scattering and modified wave operators for the Wave-Schrödinger system II*, to appear in Ann. Henri Poincaré.

[11] N. Hayashi and W. von Wahl, *On the global strong solutions of coupled Klein-Gordon-Schrödinger equations*, J. Math. Soc. Japan, **39** (1987), 489–497.

[12] L. Hörmander, *Lectures on Nonlinear Hyperbolic Differential Equations*, Mathématiques & Applications **26**, Springer-Verlag (1997).

[13] K. Nakanishi, *Modified wave operators for the Hartree equation with data, image and convergence in the same space*, Commun. Pure Appl. Anal., **1** (2002), 237–252.

[14] K. Nakanishi, *Modified wave operators for the Hartree equation with data, image and convergence in the same space, II*, Ann. Henri Poincaré, **3** (2002), 503–535.

[15] T. Ozawa, *Long range scattering for nonlinear Schrödinger equations in one space dimension*, Comm. Math. Phys., **139** (1991), 479–493.

[16] T. Ozawa and Y. Tsutsumi, *Asymptotic behavior of solutions for the coupled Klein-Gordon-Schrödinger equations*, preprint, RIMS-775 (1991); Adv. Stud. Pure Math., **23** (1994), 295–305.

[17] A. Shimomura, *Modified wave operators for the coupled Wave-Schrödinger equations in three space dimensions*, to appear in Discrete Contin. Dyn. Syst.

[18] A. Shimomura, *Wave operators for the coupled Klein-Gordon-Schrödinger equations in two space dimensions*, to appear in Funkcial. Ekvac.

[19] A. Shimomura, *Modified wave operators for Maxwell-Schrödinger equations in three space dimensions*, to appear in Ann. Henri Poincaré.

[20] A. Shimomura, *Scattering theory for Zakharov equations in three space dimensions with large data*, to appear in Commun. Contemp. Math.

[21] A. Shimomura, *Scattering theory for the coupled Klein-Gordon-Schrödinger equations in two space dimensions*, Preprint, Preprint Series, UTMS 2003-32.

[22] H. Sunagawa, *A note on the large time asymptotics for a system of Klein-Gordon systems*, to appear in Hokkaido Math. J.

[23] Y. Tsutsumi, *Global existence and asymptotic behavior of solutions for the Maxwell-Schrödinger equations in three space dimensions*, Comm. Math. Phys., **151** (1993), 543–576.

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