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The degree of symmetry of certain compact smooth manifolds II

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Abstract

In this paper, we prove the sharp estimates for the degree of symmetry and the semi-simple degree of symmetry of the fiber bundles with certain 4-dimensional fibers by virtue of the rigidity theorem with respect to the harmonic map due to Schoen and Yau. As a corollary of this estimate, we compute the degree of symmetry and the semi-simple degree of symmetry of $\mathbf{C}P^2 \times V$, where V is compact and real analytic Riemannian manifold of nonpositive curvature. In addition, by the Albanese map, we obtain the sharp estimate of the degree of symmetry of a compact smooth manifold satisfying some restrictions from its first cohomology.

1 Introduction

Since this paper is a complemental continuation of [17], we begin with recalling the necessary part of the backgroud in [17]. Let M^n be a compact connected smooth n-manifold and $N(M^n)$ the *degree of symmetry* of M^n, that is, the maximum of the dimensions of the isometry groups of all possible Riemannian metrics on M^n. Of course, $N(M)$ is the maximum of the dimensions of the compact Lie groups which can act effectively and smoothly on M. The *semi-simple degree of symmetry* $N_s(M)$ is defined similarly, where we consider only actions of semi-simple compact Lie groups on M. The following is well known:

$$N(M^n) \leq n(n+1)/2. \tag{1}$$

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In addition, if the equality holds, then M^n is diffeomorphic to the standard sphere S^n or the real projective space $\mathbf{R}P^n$. In [10] H. T. Ku, L. N. Mann, J. L. Sicks and J. C. Su obtained similar results on a product manifold $M^n = M_1^{n_1} \times M_2^{n_2}$ $(n \geq 19)$ where M_i is a compact connected smooth manifold of dimension n_i: they showed that

$$N(M) \leq n_1(n_1+1)/2 + n_2(n_2+1)/2, \tag{2}$$

and that if the equality holds, then M^n is a product of two spheres, two real projective spaces or a sphere and a real projective space. A preliminary lemma for the proof of Ku-Mann-Sicks-Su's results claims that if M^n $(n \geq 19)$ is a compact connected smooth n-manifold which is not diffeomorphic to the complex projective space $\mathbf{C}P^m$ $(n = 2m)$, then

$$N(M^n) \leq k(k+1)/2 + (n-k)(n-k+1)/2 \tag{3}$$

holds for each $k \in \mathbf{N}$ such that the k-th Betti number b_k of M is nonzero.

Let V be a connected compact manifold which can admit a real analytic Riemannian metric of nonpositive curvature. It was noted in Remark 1.2 of [17] that by the results in [3] and [11] the following holds:

Fact N $N(V)$ equals rank of the center of $\pi_1(V)$ and the only connected compact Lie groups which can act effectively and smoothly on V are tori.

Let E be a compact smooth fiber bundle over V with connected fiber F. In Theorem 1.1 of [17] the author generalized partially Ku-Mann-Sicks-Su's result (2) by showing the corresponding sharp estimates of $N(E)$ and $N_s(E)$ by assuming that the fiber F satisfies various topological properties. In particular, part of the statements of Theorem 1.1 and Corollary 1.1 in [17] says:

Fact F Suppose that E is oriented and that F is an $4m$-manifold $(m \geq 5)$ of nonzero signature. Then the followings hold:

$$N(E) \leq N(V) + 4m(m+1), \quad N_s(E) \leq 4m(m+1)\ . \tag{4}$$

In particular, if V is oriented, then

$$N(\mathbf{C}P^{2m} \times V) = N(V) + 4m(m+1), \quad N_s(\mathbf{C}P^{2m} \times V) = 4m(m+1)\ . \tag{5}$$

The case of $1 \leq m \leq 4$ could not be covered in [17] because the author used Ku-Mann-Sick-Su's result (3), in which the dimension of the manifold is assumed to be ≥ 19. In this paper we shall show that Fact F also holds for $m = 1$. That is,

Theorem 1.1. *Let V be a connected compact manifold which can admit a real analytic Riemannian metric of nonpositive curvature and E be a compact smooth fiber bundle over V such that the fiber F of E is connected. Suppose that the fiber bundle E is oriented and that the fiber F has dimension 4 and has nonzero signature. Then the followings hold:*

$$N(E) \leq N(V) + 8, \quad N_s(E) \leq 8 \ . \tag{6}$$

In particular, if V is oriented, then

$$N(\mathbf{C}P^2 \times V) = N(\mathbf{C}P^2) + N(V) = N(V) + 8, \quad N_s(\mathbf{C}P^2 \times V) = N_s(\mathbf{C}P^2) = 8 \ . \tag{7}$$

In fact, the assumption before (7) that V is oriented can be removed by the following

Theorem 1.2. *Let V be a connected compact manifold which can admit a real analytic Riemannian metric of nonpositive curvature and E be a compact smooth fiber bundle over V such that the fiber F of E is connected. Suppose that the fiber F has dimension 4 and is not cobordant mod 2 to either 0 or $\mathbf{R}P^4$. Then both (6) and (7) hold.*

Remark 1.1. The assumption that F is oriented and has nonzero signature in Theorem 1.1 is independent of that F is not cobordant mod 2 to either 0 or $\mathbf{R}P^4$ in Theorem 1.2. For two examples: the oriented 4-manifold $\mathbf{C}P^2 \sharp \mathbf{C}P^2$ has signature 2 and is cobordant mod 2 to 0, $\mathbf{R}P^2 \times \mathbf{R}P^2$ is an oriented 4-manifold having zero signature and is not cobordant to either 0 or $\mathbf{R}P^4$.

By Remark 1.4 in [17] the connectedness of F is necessary for the validity of Theorems 1.1, 1.2.

D. Burghelea and R. Schultz [1] showed that $N_s(M) = 0$ if there exist $\alpha_1, \cdots, \alpha_n$ in $H^1(M; \mathbf{R})$ with $\alpha_1 \cup \cdots \alpha_n \neq 0$. In Theorem 1.2 of [17] Burghelea-Schultz's result was generalized to the following

Fact C Let M be an n-dimensional compact connected smooth manifold. If there exist $\alpha_1, \cdots, \alpha_k$ in $H^1(M; \mathbf{R})$ with $\alpha_1 \cup \cdots \cup \alpha_k \neq 0$, then the followings hold:

$$N(M) \leq (n-k+1)(n-k)/2 + k \ ,$$

$$N_s(M) \left\{ \begin{array}{rl} \leq (n-k+1)(n-k)/2 & \text{if } n-k > 1 \\ = 0 & \text{otherwise} \ . \end{array} \right.$$

Further assuming $b_1(M) > k$, we obtain the following

Theorem 1.3. *Let Let M be an n-dimensional compact connected smooth manifold and $k \geq 3$ be a positive integer. If the first Betti number $b_1(M)$ of M is greater than k and there exist $\alpha_1, \cdots, \alpha_k$ in $H^1(M; \mathbf{R})$ with $\alpha_1 \cup \cdots \cup \alpha_k \neq 0$ in $H^k(M; \mathbf{R})$, then*

$$N(M) \leq \begin{cases} (n-k+1)(n-k)/2 + k - 2 & \text{if } n \geq k+3 \\ n & \text{if } n = k+2 \; . \end{cases} \tag{8}$$

Remark 1.2. Let T^n be the n-dimensional torus. Since $N(T^n) = n$, in case that the dimension n of M equals k or $k+1$, we can not improve the estimate of Fact C by adding the assumption $b_1(M) > k$. Because only under the conditions $b_1(M) \geq 1, 2, 3$ (ii), (iii) of Theorem 1.2 in [17] give the similar sharp estimates of $N(M)$ as (8) respectively, we assume $k \geq 3$ here.

Remark 1.3. By the definition of degree of symmetry, it is easy to see that for a product manifold $M_1 \times M_2$, where M_i is a compact connected smooth manifold, the following holds:

$$N(M_1 \times M_2) \geq N(M_1) + N(M_2) \; . \tag{9}$$

Let Σ_g be the oriented closed surface of genus g and $M^n = S^{n-k} \times T^{k-2} \times \Sigma_g$ ($n \geq k+3,\ g \geq 2$). Then M^n satisfies the assumption of Theorem 1.3. Since by Fact N $N(T^{k-2} \times \Sigma_g) = k-2$, by (9) and Theorem 1.3 we obtain the equality

$$N(S^{n-k} \times T^{k-2} \times \Sigma_g) = (n-k+1)(n-k)/2 + k - 2 \; ,$$

combining which with the equality $N(T^{k+2}) = k+2$, we show that the estimate (8) is best possible.

This paper is organized as follows. In Section 2, we prepare for the following sections. In particular, we cite some results in [17] and prove a key lemma (cf Lemma 2.2) for the proofs of Theorems 1.1, 1.2. In Section 3, we prove Theorem 1.1 (1.2) with the help of this key lemma and the oriented (unoriented) cobordism theory. In Section 4, we prove Theorem 1.3 by virtue of the unique continuation property of harmonic maps.

2 Preliminaries

For a compact Riemannian manifold M let $I^0(M)$ be the identity component of the isometry group of M. The following proposition will provide the frame for the proof of Theorems 1.1, 1.2.

Proposition 2.1. (cf [16] Theorem 4) *Suppose that M, N are compact real analytic Riemannian manifolds and N has nonpositive curvatures. Suppose that $h : M \to N$ is a surjective harmonic map and its induced map $h_* : \pi_1(M) \to \pi_1(N)$ is also surjective. Then the space of surjective harmonic maps homotopic to h is represented by $\{\beta \circ h | \beta \in I^0(N)\}$, where $I^0(N)$ is a torus group of dimension = rank of the center of $\pi_1(N)$ = the degree of symmetry of V.*

We cite a topological result from [17].

Proposition 2.2. (cf [17] Proposition 3.1) *Let $p_0 : E \to B$ be a fiber bundle over a compact connected smooth manifold B such that the fiber of E is also connected. Then any continuous map homotopic to $p_0 : E \to B$ is surjective.*

We cite a lemma in [17], which is also necessary for Theorems 1.1, 1.2.

Lemma 2.1. (cf [17] Lemma 2.1) *Let M^m be a connected Riemannian manifold and f a smooth map from it to a smooth manifold N^n. Suppose $y \in N$ is a regular point of f and F is a connected component of the submanifold $f^{-1}(y)$. If an isometry α of M satisfies that $h \circ \alpha = h$ and that*

$$\alpha(x) = x \text{ for any } x \in F,$$

then α is the identity map of M.

Lemma 2.2. (Key lemma of Theorems 1.1, 1.2) *Let Y be a compact connected smooth 4-manifold not diffeomorphic to either S^4 or $\mathbf{R}P^4$. Then $N(Y) \leq 8$. The equality $N(Y) = 8$ holds if and only if Y is diffeomorphic to $\mathbf{C}P^2$. Moreover, $N_s(\mathbf{C}P^2) = 8$.*

Proof. By (1) $N(Y) \leq 9$. Then $N(Y) \leq 8$ follows from Theorem A′ in Ishihara [8] which claims that there exists no 4-dimensional Riemannian manifold having a 9-dimensional isometry group. If Y is a Riemannian manifold whose isometry group has dimension 8, then by Theorem 5 in Ishihara [8] Y is a Kählerian space with positive constant holomorphic sectional curvatures. Since the holomorphic section curvature of a Kähler manifold determines completely its Riemannian curvature tensor (cf [18] Lemma 7.19.), Y has positive sectional curvature and then by the theorem of Synge (cf [2] Theorem 5.9.) Y is simply connected. By the theorem of Cartan-Ambrose-Hicks (cf [2] Theorem 1.36.), Y is isometric to the Kähler manifold $\mathbf{C}P^2$ with the Fubini-Study metric. Since the compact Lie group SU(3) acting isometrically on $\mathbf{C}P^2$ is semi-simple, $N_s(\mathbf{C}P^2) = N(\mathbf{C}P^2) = 8$. q.e.d.

We do some preparations for the proof of Theorem 1.3 in the following.

For a compact oriented Riemannian manifold M with nonzero first Betti number $b_1(M)$, let $\mathcal{H}$ be the real vector space of all harmonic 1-forms on M and ν the natural projecion from the universal covering $\tilde{M}$ of M. For $x_0 \in \tilde{M}$, set $p_0 = \nu(x_0)$. We define a smooth map $\tilde{a} : \tilde{M} \to \mathcal{H}^*$ from $\tilde{M}$ to the dual space $\mathcal{H}^*$ of $\mathcal{H}$ by a line integral

$$\tilde{a}(x)(\omega) = \int_{x_0}^{x} \nu^*\omega.$$

For $\sigma \in \pi_1(M)$

$$\tilde{a}(\sigma x) = \tilde{a}(x) + \psi(\sigma)$$

holds, where $\psi(\sigma)(\omega) = \int_{x_0}^{\sigma x_0} \nu^*\omega$, so that ψ is a homomorphism from $\pi_1(M)$ into $\mathcal{H}^*$ as an additive group. It is a fact that $\Delta = \psi(\pi_1(M))$ is a lattice in the vector space $\mathcal{H}^*$, and clearly this vector space has a natural Euclidean metric from the global inner prduct of forms on M. With the quotient metric, we call the torus $A(M) = \mathcal{H}^*/\Delta$ the *Albanese torus* of Riemannian manifold M. By the above relation between $\tilde{a}$ and ψ, we obtain a map $a : M \to A(M)$ satisfying $\tilde{a}(x) \in a \circ \nu(x)$ for any $x \in \tilde{M}$. We call the map a the *Albanese map*. From the very construction of a, we see that the map it induces on fundamental groups

$$a_* : \pi_1(M) \to \pi_1(A(M))$$

is surjective and that a^* maps the space of harmonic 1-forms on $A(M)$ isomorphically onto $\mathcal{H}$. By Corollary 1 in [14], the Albanese map is harmonic. Set

$$r_a := \max\{\operatorname{rank} da(p) | p \in M\} .$$

Lemma 2.3. (cf [17] Lemma 4.3) *Let M be an n-dimensional oriented compact Riemannian manifold with nonzero first Betti number b_1. Let $a : M \to A(M)$ be the Albanese map. Suppose there exist $\alpha_1, \cdots, \alpha_k$ in $H^1(M;\mathbf{R})$ with $\alpha_1 \cup \cdots \cup \alpha_k \neq 0$ in $H^k(M;\mathbf{R})$. Then $r_a \geq k$ holds.*

Lemma 2.4. (cf [17] Lemmas 4.1, 4.2) *Let M be a non-orientable compact manifold and $\pi : M' \to M$ be its orientable double covering. Then the followings hold :*
(i) $N(M) \leq N(M')$
(ii) $b_1(M) \leq b_1(M')$
(iii) *If M has the property that there exist k one dimensional real cohomology classes $\alpha_1, \cdots, \alpha_k$ of M such that $\alpha_1 \cup \cdots \cup \alpha_k$ is nonzero in $H^k(M;\mathbf{R})$, then so does M'.*

3 Proof of Theorems 1.1, 1.2

Proof of Theorem 1.1 For the proof of (6), by Corollary A.1 we have only to show that for any real analytic Riemannian metric on E and any compact semi-simple subgroup G of $I^0(E)$, the inequalities hold:

$$\dim I^0(E) \le N(V) + 8,\ \dim G \le 8\ . \tag{10}$$

Since the fiber F is connected, the fiber bundle projection $p : E \to V$ induces a surjective map $p_* : \pi_1(E) \to \pi_1(V)$. Using a well known result by Eells-Sampson [5], we see that there exist harmonic maps homotopic to $p : E \to V$. By Proposition 2.2, each of them is surjective and then satisfies the assumptions of Proposition 2.1. Taking a harmonic map $h : E \to V$ homotopic to $p : E \to V$, by Proposition 2.1, for any $\alpha \in I^0(E)$ we can find a unique $\rho(\alpha) \in I^0(V)$ with $h \circ \alpha = \rho(\alpha) \circ h$. We see that $\rho : I^0(E) \to I^0(V)$ is a Lie group homomorphism. Since G is contained in Ker ρ, the proof of (10) is completed if we can show that Ker ρ, which acts isometrically on E, has dimension ≤ 8.

Since the critical value set of h is compact and has Lebesgue measure zero in V, there exists an open set U of V such that any point in U is a regular value of h. Choosing a smooth homotopy $P : E \times [0,\ 1] \to V$ between p and h, and taking a regular value y of P from U, we have the following

Claim 1 $P^{-1}(y)$ is a oriented submanifold in $E \times [0,\ 1]$ with boundary $p^{-1}(y) + h^{-1}(y)$. That is, there exists an oriented cobordism in E for the submanifolds F and $h^{-1}(y)$.

Proof of Claim 1 Since y is the regular value of P and $P^{-1}(y)$ is non-empty, it is easy to see that $P^{-1}(y)$ is a submanifold of $E \times [0,\ 1]$ with boundary

$$\partial P^{-1}(y) \cap E \times 0 + \partial P^{-1}(y) \cap E \times 1 = p^{-1}(y) + h^{-1}(y) \cong F + h^{-1}(y).$$

Let $\tilde{F} = h^{-1}(y)$ and $\iota : \tilde{F} \to E$ be the embedding of $\tilde{F}$ in E. Since E is oriented and

$$TE|_{\tilde{F}} = T\tilde{F} \oplus h^*(T_yV) = T\tilde{F} \oplus \text{ a trivial vector bundle },$$

we have the following relation of first Stiefel-Whitney classes:

$$w_1(T\tilde{F}) = w_1(TE|_{\tilde{F}}) = \iota^* w_1(TE) = 0\ .$$

That is, the fiber $\tilde{F}$ can be induced a natural orientation in E by h. Similarly, we can show that both F and $P^{-1}(y)$ have their natural orientations in E and $E \times [0,\ 1]$ respectively.

By Hirzebruch's signature theorem (cf [7] Theorem 8.2.2) and Claim 1, up to the difference of " $\pm$ " the signature of $\tilde{F}$ equals that of F so that there exists a connected component F^* of $\tilde{F}$ having nonzero signature. Hence F^* is not diffeomorphic to either S^4 or $\mathbf{R}P^4$. Since by Lemma 2.1 Ker ρ acts effectively on F^*, Lemma 2.2 tells us that

Ker ρ has dimension ≤ 8. We complete the proof of (6). Since $\mathbf{C}P^2$ has signature 1, (7) follows from (6) and (9). q.e.d.

PROOF OF THEOREM 1.2 Repeating the part of the proof of Theorem 1.1 before Claim 1, we can see that $P^{-1}(y)$ is a submanifold of $E \times [0, 1]$ with boundary $p^{-1}(y) + h^{-1}(y)$. That is, $h^{-1}(y)$ is cobordant mod 2 to F. Since F is not cobordant mod 2 to either 0 or $\mathbf{R}P^4$, there exists a connected component F^* of $h^{-1}(y)$ such that F^* is not diffeomorphic to either S^4 or $\mathbf{R}P^4$. Since Ker ρ acts effectively on F^*, by Lemma 2.2 Ker ρ has dimension ≤ 8. Therefore the inequalities in (6) hold. To show (7), we only need to show $\mathbf{C}P^2$ is not cobordant mod 2 to zero or $\mathbf{R}P^4$, which follows from that $w_2^2[\mathbf{R}P^4] = 0$ and $w_2^2[\mathbf{C}P^2] \neq 0$. q.e.d.

4 Proof of Theorem 1.3

PROOF OF THEOREM 1.3 By Lemma 2.4, we may assume M is an oriented Riemannian manifold. Let $a : M \to a(M)$ be the Albanese map and b_1 the first Betti numer of M. By Corollary A.1 we have only to consider the analytic Riemannian metric on M. For any $\gamma \in I^0(M)$, $a \circ \gamma$ is also a harmonic map from M to the Albanese torus $A(M)$ and homotopic to a. By Lemma 3 in [14] there is a unique translation $\rho(\gamma)$ of the torus $A(M)$ such that

$$a \circ \gamma = \rho(\gamma) \circ a \ .$$

Then we have a Lie group homomorphism $\rho : I^0(M) \to T^{b_1}$, where the torus group T^{b_1} is the translation group of the Albanese torus $A(M)$. By the proof of Lemma 2.3 in [17], we have

$$\dim \ \mathrm{Ker}\ \rho \leq \frac{1}{2}(n - r_a + 1)(n - r_a), \quad \dim \ \mathrm{Im}\ \rho \leq r_a. \tag{11}$$

We see from Lemma 2.3 that $r_a \geq k$. If $r_a \geq k + 1$, then from (11)

$$\dim \ I(M) = \dim \ \mathrm{Ker}\ \rho + \dim \ \mathrm{Im}\ \rho \leq \frac{1}{2}(n - k - 1)(n - k) + k + 1.$$

Suppose $r_a = k$ in what follows. We claim that dim Im ρ will be less than $k - 1$ so that by (11)

$$\dim \ I^0(M) \leq \frac{1}{2}(n - k + 1)(n - k) + k - 2.$$

Otherwise, suppose dim Im $\rho \geq k - 1$. By the definition of ρ, the Lie group Im ρ acting on $A(M)$ in fact acts on the image $a(M)$ of a. Hence we can assume that there exists a subgroup T^{k-1} of the translation group T^{b_1} which acts freely and isometrically on $a(M)$.

Since both M and $A(M)$ are real analytic, a theorem of Morrey [13] shows that the harmonic mapping a is in fact real analytic. By well-known theorems in real analytic geometry [12] we know that both M and $A(M)$ can be triangulated so that $a(M)$ is a k-dimensional compact connected simplicial subcomplex of $A(M)$. We write the orbit space of the free and isometric T^{k-1} actions on $A(M)$ and $a(M)$ by $A(M)/T^{k-1}$ and $a(M)/T^{k-1}$ respectively, in which the former is in fact also a flat torus of dimension $b_1 - k + 1$. Since the natural projection map $\pi : A(M) \to A(M)/T^{k-1}$ is totally geodesic, we see that by a result in [4] the composition map $\pi \circ a : M \to A(M)/T^{k-1}$ is a harmonic map, whose image is $a(M)/T^{k-1}$, the orbit space of the free T^{k-1} action on the k-dimensional simplicial subcomplex $a(M)$ of $A(M)$. Hence $a(M)/T^{k-1}$, the image of $\pi \circ a$ in $A(M)/T^{k-1}$ has dimensional 1 so that the differential of harmonic map $\pi \circ a$ has rank ≤ 1 at any point of M. By the unique continuation property of harmonic maps (cf [15] Theorem 3), we see that $\pi \circ a$ maps M onto a closed geodesic of $A(M)/T^{k-1}$, which means that $a(M)$ is a principal T^{k-1}-bundle over S^1. Since S^1 is connected, there exists a section on this bundle so that $a(M)$ is a trivial T^{k-1}-bundle, i.e. a k-dimensional torus. This contradicts the surjectivity of the homomorphism $a_* : \pi_1(M) \to \pi_1(A(M)) \cong \mathbf{Z}^{b_1}$ $(b_1 > k)$. Hence we obtain dim $I^0(M)$ is not greater than the maximum of $(n-k-1)(n-k)/2 + k + 1$ and $(n-k)(n-k+1)/2 + k - 2$. q.e.d.

Appendix

A Real Analytic Group Action

In this appendix, we will prove the following theorem.

Theorem A.1. *Let G be a compact Lie group acting smoothly and effectively on a compact smooth manifold M. Then there exists a real analytic manifold M' on which G acts real analytically such that there exists a G-isomorphism between (M, G) and (M', G). That is, there is a diffeomorphism $f : M \to M'$ which satisfies for any $x \in M$ and $g \in G$*

$$f(gx) = gf(x) \ .$$

We put the proof of Theorem A.1 afterward. Firstly, from it we have the following

Corollary A.1. *The degree of symmetry of M equals the maximum of the dimensions of the isometry groups of all the real analytic Riemannian metrics on M.*

Proof. Let G be a compact Lie group acting smoothly and effectively on a compact smooth manifold M. By Theorem A.1 there exists another real analytical G action on the unique real analytic structure of M compatible to the existed smooth structure o M. Moreover, the new G action is equivariant to the old one on the smooth structure of M. Thus the new one is also effective. Taking a real analytic Riemannian metric on M, by the invariant integration on G we can construct a new real analytic one on which G acts isometrically. q.e.d.

Corollary A.2. *Any element g of a compact subgroup G in the diffeomorphism group* Diff(M) *of a compact smooth manifold M is real analytic with respect to the unique real analytic structure of M compatible to the existed smooth structure on M.*

Proof. Since G is a compact subgroup of Diff (M), G has a Lie group structure. That is, G is a compact Lie group acting smoothly and effectively on M. The statement follows from the proof of Corollary A.1.

Let $f : M \to N$ be a smooth map between smooth manifolds M and N. It is *transverse* to a submanifold $A \subset N$ if and only if whenever $f(x) = y \in A$, then the tangent space to N at y is spanned by the tangent space to A at y and the image of the tangent space to M at x. That is,

$$T_yA + df(T_xM) = T_yN.$$

Lemma A.1. (cf Theorem 1.3.3. in [6]) *Let $f : M \to N$ be a smooth map and $A \subset N$ a submanifold of codimension l. If f is transverse to A, then $f^{-1}(A)$ is a submanifold of M of codimension l.*

Lemma A.2. (cf Theorem 4.12. in [9]) *Let G be a compact Lie group and M a compact manifold on which G acts smoothly. Then there exists a representation space (V, μ) of G and a smooth G-embedding $\iota : M \to V$. That is, for any $x \in M$ and $g \in G$,*

$$\iota(gx) = \mu(g)\iota(x)\ .$$

Moreover, if the G-action on M is effective, then the representation (V, μ) of G is faithful.

Let G be a compact group acting smoothly on two manifolds M and N. A smooth map $f : M \to N$ is a *G-map* if and only if for any $x \in M$ and $g \in G$ the following holds:

$$gf(x)) = f(gx)\ .$$

Lemma A.3. (An equivariant version of Theorem 2.5.2. in [6]) *Let G be a compact Lie group acting isometrically on Euclidean spaces $\mathbf{R}^q$ and $\mathbf{R}^s$. Let $M \subset \mathbf{R}^q$ be a G-invariant compact submanifold of codimension > 0 and E a G-invariant tubular neighborhood of M in $\mathbf{R}^q$. Let $f : E \to W$ be a smooth G-map into a G-invariant open set W of $\mathbf{R}^s$. Let $v : \mathbf{R}^q \to \mathbf{R}$ be a smooth G-invariant function with support in E, equal to 1 on a G-invariant compact neighborhood K of M. Set $h(x) = v(x)f(x) = v(x)(f_1(x), \cdots, f_s(x))$. Let $\delta : \mathbf{R}^q \to \mathbf{R}$, $\delta(x) = \exp(-|x|^2)$. Let $C = 1/\int_{\mathbf{R}^q} \delta$. Let $\epsilon > 0$. Then for $k > 0$ sufficiently large,*

$$\psi(x) = (\psi_1(x), \cdots, \psi_s(x)) := (h_1(x), \cdots, h_s(x)) * (Ck^q\delta(kx))$$

is an analytic G-map and satisfies $||\psi - f||_{C^1,K} < \epsilon$.

Proof. The proof of the analytic property of ψ and the estimate $||\psi - f||_{C^1,K} < \epsilon$ for $k > 0$ large enough is straightforward. We have only to show that $h * \delta$ is G-invariant. Since $v(gx) = v(x)$, $h = vf : E \to \mathbf{R}^s$ is G-map. For any $g \in G$, since g acts on $\mathbf{R}^q$ isometrically and δ is a radial function on $\mathbf{R}^q$,

$$\begin{aligned}
(h * \delta)(gx) &= \int_{\mathbf{R}^q} h(y)\delta(gx - y)\, dy = \int_{\mathbf{R}^q} h(gz)\delta(g(x - z))\, dz \\
&= \int_{\mathbf{R}^q} h(gz)\delta(x - z)\, dz = g\Big(\int_{\mathbf{R}^q} h(z)\delta(x - z)\, dz\Big) \\
&= g\Big(h * \delta(x)\Big) .
\end{aligned}$$

q.e.d.

Proof of Theorem A.1 By Lemma A.2, there exists a faithful representation space V of G and a G-embedding $\iota : M \to \mathbf{R}^q$. By the invariant integration on G, we can induce a G-invariant inner product $(\ ,\ )$ on V, equipped with which V becomes a Euclidean space $\mathbf{R}^q$ and G becomes a subgroup of $\mathrm{O}\,(q)$. Let k be the codimension of M in $\mathbf{R}^q$. Since M is a G-invariant submanifold, by Theorem 4.8 in [9] there exists a G-invariant normal tubular neighborhood E of M, which can be identified with a G-invariant neighborhood of the zero section of the normal bundle of M. Let $p : E \to M$ be the restriction of the bundle projection, which is a G-map.

Let $G_{q,k}$ be the Grassmann manifold of k-dimensional linear subspaces of $\mathbf{R}^q$ and $E_{q,k} \to G_{q,k}$ be the Grassmann bundle, the fiber of $E_{q,k}$ over the k-plane $P \subset \mathbf{R}^q$ is the set of pairs (P, x) where $x \in P$. Then the G-action on $\mathbf{R}^q$ induces the natural real analytic actions on $G_{q,k}$ and $E_{q,k}$ respectively such that the bundle projection $E_{q,k} \to G_{q,k}$ is a G-map. Let $h : M \to G_{q,k}$ be the map sending $x \in M$ to the k-plane normal to M at x and $f : E \to E_{q,k}$ be the natural map covering h; thus

$$f(y) = (h \circ p(y), y) \in E_{q,k} \subset G_{q,k} \times \mathbf{R}^q .$$

Since G acts isometrically on M, as a linear map on $\mathbf{R}^q$ $dg = g$ maps the k-plane normal to M at x to the one normal to M at gx. Therefore both h and f are G-maps. Moreover, f is transverse to the zero section $G_{q,k} \subset E_{q,k}$ and

$$f^{-1}(G_{q,k}) = M \ .$$

Now we embed $E_{q,k}$ analytically in $\mathbf{R}^s$ with $s = q^2 + q$. For this it suffices to embed $G_{q,k}$ in $\mathbf{R}^{q^2}$. This is done by mapping a k-plane $P \in G_{q,k}$ to the linear map $\mathbf{R}^q \to \mathbf{R}^q$ given by the orthogonal projection on P. There exists a natural isometric G-action on $\mathbf{R}^s$ such that the embedding $E_{q,k} \subset \mathbf{R}^s$ is a G-map. Then we can find a G-invariant normal tubular neighborhood W of $E_{q,k}$. Let $\Pi : W \to E_{q,k}$ be the real analytic G-invariant projection.

Let $f' : E \to W$ be the extension of the map $f : E \to E_{q,k}$ to W. It follows from Lemma A.3 that f' can be approximated near M by an analytic G-map $\psi : E \to W$. Then $\phi = \Pi \circ \psi$ is an analytic G-invariant approximation of $f : E \to E_{q,k}$. Put $M' = \phi^{-1}(G_{q,k})$. If ϕ is sufficiently C^1 close to f, then ϕ is also transverse to $G_{q,k}$ and the restriction of G-map $p : E \to M$ to M' is a G-isomorphism from M to M'. q.e.d.

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