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# Sobolev norms of eigenfunctions on a closed Riemannian manifold

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**Abstract**

Let $\chi_\lambda$ (cf (1.1)) be the unit spectral projection operator with respect to the Laplace-Beltrami operator $\Delta$ on a closed Riemannian manifold $M$. We generalize the ($L^2$, $L^\infty$) estimate of $\chi_\lambda$ by Hörmander [3] to those of covariant derivatives of $\chi_\lambda$ Moreover we extend the $(L^2, L^p)$ estimates of $\chi_\lambda$ by Sogge [7] [8] to ($L^2$, Sobolev $L^p$) estimates of $\chi_\lambda$.

## 1 Introduction

At first let us set the notation for the results of Hörmander and Sogge. Let $(M,\ g)$ be a smooth closed Riemannian manifold of dimension $\geq 2$ and $\Delta$ the positive Laplace-Beltrami operator on $M$. Let $L^2(M)$ be the space of square integrable functions on $M$ with respect to the Riemannian density $dv(M) := \sqrt{\det\ (g_{ij})}\ dx$. Recall that $L^2(M)$ admits a complete orthogonal direct sum decomposition with respect to the eigenspaces of $\Delta$. That is, one can write

$$L^2(M) = \sum_{j=0}^{\infty} E_j\ ,$$

where $E_j$ is the $j$th eigenspace corresponding to the eigenvalues $\lambda_j^2$. The eigenvalues are counted with multiplicity and are arranged in increasing order, i.e. $0 \leq \lambda_0^2 \leq \lambda_1^2 \leq \lambda_2^2 \leq \cdots$, where $\lambda_j$ are nonnegative real numbers. Also, $\mathbf{e}_j$ will be denote the projection onto the $j$th eigenspace $E_j$. Thus , an $L^2$ function $f$ can be written as

$$f = \sum_{j=0}^{\infty} \mathbf{e}_j(f),$$

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where the partial sum converges in the $L^2$ norm.

Let $e_j(x)$, a real eigenfunction of $\Delta$, be the base of the eigenspace $E_j$ with the normalized $L^2$ norm. That is, $\{e_j(x)\}_{j=0}^{\infty}$ becomes a complete orthonormal basis of $L^2(M)$. Let $\lambda$ be a postive real number $\geq 1$. We define the spectral function $e(x,y,\lambda)$ and the unit spectral projection operator (USPO) $\chi_\lambda$ as follows:

$$e(x,y,\lambda) := \sum_{\lambda_j \leq \lambda} e_j(x) e_j(y) \ ,$$

$$\chi_\lambda f := \sum_{\lambda_j \in [\lambda,\, \lambda+1]} \mathbf{e}_j(f) \ . \tag{1.1}$$

As a consequence of the sharp form of the Weyl formula (cf Theorem 4.4 in [3]), Hörmander proved the uniform estimate of eigenfunctions for $x \in M$

$$\sum_{\lambda_j \in [\lambda,\, \lambda+1]} |e_j(x)|^2 \leq C\, \lambda^{n-1} \ , \tag{1.2}$$

which implies

$$||\chi_\lambda f||_2 \leq C \lambda^{(n-1)/2} ||f||_1 \tag{1.3}$$

and

$$||\chi_\lambda f||_\infty \leq C \lambda^{(n-1)/2} ||f||_2 \ , \tag{1.4}$$

where $||f||_r$ $(1 \leq r \leq \infty)$ means the $L^r$ norm of the function $f$ on $M$. Let $\delta(r)$ be the critical exponent $\max(n \cdot |1/r - 1/2| - 1/2,\ 0)$ for Bochner Riesz means of the Laplacian on $L^r(\mathbf{R}^n)$. With the help of the oscillatory integral theorems of Carleson-Sjölin [1] and Stein [9], Sogge showed in [7] and [8]

$$||\chi_\lambda f||_2 \leq C \lambda^{\delta(p)} ||f||_q,\ q = 2(n+1)/(n+3) \tag{1.5}$$

by using the Hadamard parametrix for $\Delta - (\lambda + i)^2$ and the wave operator $(\partial/\partial t)^2 + \Delta$ respectively. By the duality and the above inequality, the following estimate holds:

$$||\chi_\lambda f||_p \leq C \lambda^{\delta(q)} ||f||_2,\ p = 2(n+1)/(n-1) \tag{1.6}$$

Interpolating (1.6) with (1.4) and the inequality

$$||\chi_\lambda f||_2 \leq ||f||_2 \tag{1.7}$$

from the orthogonal relation, Sogge proved the following

**Proposition 1.1.** (cf C. D. Sogge [7] and [8])

$$\begin{aligned} &||\chi_\lambda f||_r \leq C \lambda^{(n-1)(r-2)/4r} ||f||_2,\ 2 \leq r \leq 2(n+1)/(n-1) \ , \\ &||\chi_\lambda f||_r \leq C \lambda^{\delta(r)} ||f||_2, \qquad\qquad 2(n+1)/(n-1) \leq r \leq \infty \ . \end{aligned}$$

Then we set the notation for our results. For $k$ a nonnegative integer and $u \in C^\infty(M)$, $\nabla^k u$ denotes the $k$th covariant derivative of $u$ (with the convention $\nabla^0 u = u$). As an example, the components of $\nabla u$ in local coordinates are given by $(\nabla u)_i = \partial_i u$, while the components of $\nabla^2 u$ in local coordinates are given by

$$(\nabla^2 u)_{ij} = \partial_{ij} u - \Gamma_{ij}^k \partial_k u \ , \tag{1.8}$$

where $\Gamma_{ij}^k$ are the Christoffel symbols of the Levi-Civita connection of $(M, g)$ and the Einstein's summation convention is adopted. We define the length $|\nabla^k u|$ of the $(0, k)$ tensor $\nabla^k u$ by

$$|\nabla^k u|^2 := g^{i_1 j_1} \cdots g^{i_k j_k} (\nabla^k u)_{i_1 \cdots i_k} (\nabla^k u)_{j_1 \cdots j_k} \ ,$$

where $(g^{ij})$ denotes the inverse martrix of $(g_{ij})$.

**Definition 1.1.** The Sobolev space $H_k^r(M)$ is the completion of $C^\infty(M)$ with respect to the norm

$$||u||_{H_k^r} := (\sum_{j=0}^k \int_M |\nabla^j u|^r dv(g))^{1/r}, \ 1 \le r < \infty \ ,$$

$$||u||_{H_k^r} := \sum_{j=0}^k \sup_{x \in M} |\nabla^j u(x)|, \qquad\qquad r = \infty \ .$$

Sometime we also write $C^k, H_k$ instead of $H_k^\infty, H_k^2$.

**Proposition 1.2.** *$H_k^r(M)$ does not depend on the Riemannian metric. And $H_k(M)$ is a Hilbert space.*

We generalize the results of Hörmander and Sogge by considering the Sobolev spaces in the following

**Theorem 1.1.** *For $k = 0, 1, \cdots$, the following inequalities hold:*

$$||\chi_\lambda f||_{H_k^r} \le C\lambda^{k+(n-1)(r-2)/4r} ||f||_2, \ 2 \le r \le 2(n+1)/(n-1) \ ,$$

$$||\chi_\lambda f||_{H_k^r} \le C\lambda^{k+\delta(r)} ||f||_2, \qquad 2(n+1)/(n-1) \le r \le \infty \ .$$

**Remark 1.1.** Recently Xiangjin Xu [10] obtained by the maximum principle argument the same estimate to the $C^1$ norm of the unit spectral projection operator with respect to the Dirichlet Laplacian on a compact Riemannian manifold with boundary. However our proof is different from his even in $C^1$ case.

Now we sketch the proof of Theorem 1.1. Recall that the wave kernel $K(t, x, y)$ is the Schwarz kernel of the wave operator $\cos(t\sqrt{\Delta})$ associated with Laplace-Beltrami operator $\Delta$. For each $x, y \in M$, in the sense of the distribution in $t$, the following equality holds:

$$K(t, x, y) = \sum_{j=0}^\infty \cos(t\lambda_j) e_j(x) e_j(y) \ .$$

In Corollary 2.1 in Subsection 2.3, we prove that if $\epsilon > 0$ is a positive number sufficiently small and depending on the geometry of $M$, then $t^{2k+n}\nabla_x^k\nabla_y^k K(t,x,y)|_{x=y}$ is a smooth $(0,2k)$ tensor valued function with respect to $(t,x) \in [0,\ \epsilon) \times M$. By Corollary 2.1 and the wave kernel method (cf D. Grieser [2]), in Subsection 2.4 we obtain the following estimates of the covariant derivatives of the spectral function uniformly for $x \in M$::

$$|\nabla^k \sum_{\lambda_j \in [\lambda,\ \lambda+1]} e_j(x)|^2 \leq C\lambda^{n-1+2k},\ k = 0, 1, \cdots \tag{1.9}$$

which implies the following $(L^2, C^k)$ estimates for the USPO $\chi_\lambda$:

$$||\chi_\lambda f||_{C^k} \leq C\lambda^{k+(n-1)/2}||f||_2\ . \tag{1.10}$$

By the $L^2$ a priori estimate of the elliptic operator $\Delta$ and the orthogonal relation, in Lemma 3.1 of Subsection 3.1 we obtain

$$||\chi_\lambda f||_{H_k^2} \leq C\lambda^k||f||_2\ , \tag{1.11}$$

combining which with the $L^p$ a priori estimate and the interpolation method, also in this subsection we in Lemma 3.2 reduce Theorem 1.1 to the following estimates:

$$||\chi_\lambda f||_{H_1^p} \leq C\,\lambda^{1+\delta(p)}||f||_2,\ ||\Delta\chi_\lambda f||_p \leq C\,\lambda^{2+\delta(p)}||f||_2\ . \tag{1.12}$$

Then similarly to [8], we approximate the operators in Subection 3.2

$$\nabla\cos(t\sqrt{\Delta}),\ \Delta\cos(t\sqrt{\Delta}),\ j = 0, 1, \cdots$$

by certain Fourier integral operators (cf Lemma 3.3) and then in Subsection 3.3 we argue as in [7] to prove (1.12) by virtue of the oscillatory integral theorems of Carleson-Sjölin [1] and Stein [9].

As long as the organization of the rest of this paper is concerned, in Subection 2.1 we set the notations related to the covariant derivatives on $M$ and prove the spectral resolution of $\Delta$ as a self-adjoint operator. In Subsection 2.2 we give a quick review the Hadamard parametrix of $\cos(t\sqrt{\Delta})$ approximating the wave kernel $K(t,x,y)$ as well as we desire, which will be the crucial tool to the proof of Theorem 1.1.

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# 2 $C^k$ estimates

## 2.1 The spectral resolution of $\Delta$ and the wave kernel

Let $X$ be an open set of $M$ and $x : X \to \mathbf{R}^n$ a diffeomorphism of $X$ into $\mathbf{R}^n$, that is, a chart on $M$. Then assocaited to the chart are $n$ coordinates vector fields, written as $\partial/\partial x_j$ or as $\partial_j$, $j = 1, \cdots, n$. For the given Riemannian metric on $M$, define

$$g_{jk} = \langle \partial_j, \partial_k \rangle,\ G = (g_{jk}),\ g = \det G,\ G^{-1} = (g^{jk}),$$

where $j, k = 1, \cdots, n$ and det denotes the determinant. Let $\nabla$ be the Levi-Civita connection determined by the Riemannian metric. On $X$ Christoffel symbols $\Gamma_{ij}^k$ are defined by

$$\nabla_{\partial_i} \partial_j = \Gamma_{ij}^k \partial_k$$

and using a standard argument we deduces

$$\Gamma_{ij}^k = \frac{1}{2} \sum_l g^{kl} (\partial_i g_{lj} + \partial_j g_{il} - \partial_l g_{ij}) \ .$$

For $k$ an nonnegative integer and $u \in C^\infty(M)$, by the similar computation to the equalities (1.8), we can see that the component $(\nabla^k u)_{i_1 \cdots i_k}$ of $\nabla^k u$ is equal to the main term $\partial_{i_1 \cdots i_k} u := \partial_{i_1} \cdots \partial_{i_k} u$ plus the lower-order partial derivatives of $u$ with smooth coefficients coming from the Riemannian metric. Therefore by Definition 1.1 we have the following

**Lemma 2.1.** *For $k$ a nonnegative integer and $u \in C^\infty(M)$,*

$$||u||_{C^k(X)} \leq C \sum_{|\alpha| \leq k} ||\partial^\alpha u||_{L^\infty(X)} \ ,$$

*where $\alpha = (\alpha_1, \cdots, \alpha_n)$, with $\alpha_j$ nonpositive integers, is a multi-index of length $\leq k$ and*

$$\partial^\alpha = (\partial/\partial x_1)^{\alpha_1} \cdots (\partial/\partial x_n)^{\alpha_n} \ .$$

In the chart $X$ the Laplace-Beltrami operator takes the form

$$\Delta = -\sum \partial_j (g^{jk} \partial_k) + \sum b^j \partial_j \ ,$$

where $b^j = -\sum_k g^{jk} \partial_k (\log \sqrt{g})$. By the Green's formula, $\Delta$ is symmetric with respect to the Riemannian density $\sqrt{g} dx$ :

$$(\Delta u,\ u) \geq 0,\ (\Delta\, u, v) = (u, \Delta\, v);\ u, v \in C^\infty(M),$$

where

$$(u,v) := \int_M u\bar{v}\sqrt{g}dx \ .$$

Let $\mathscr{P}$ be the operator defined by $\Delta$ in $L^2(M)$ with $\mathscr{D}_{\mathscr{P}} = H^2(M)$. Again by the Green's formula, we have

$$||u||_{H^1} \le C\,||(\mathscr{P}+1)u||_2,\ \ u \in \mathscr{D}_{\mathscr{P}}. \tag{2.1.1}$$

The estimate (2.1.1) implies that $\mathscr{P}$ has a discrete spectrum so that the spectral resolution of $\Delta$ in the beginning of Section 1 holds. Let $E_\mu$ be the spectral family of $\mathscr{P}$ and the wave operator $\cos(t\sqrt{\mathscr{P}})$ associated with $\mathscr{P}$ defined by

$$\cos(t\sqrt{\mathscr{P}}) = \int_0^\infty \cos(t\sqrt{\mu})dE_\mu \ .$$

By direct computation (cf Section 17.5 of [5]), we obtain that the wave kernel $K(t,x,y) \in \mathscr{D}'(\mathbf{R}\times M\times M)$ of $\cos(t\sqrt{\mathscr{P}})$ is the Fourier transformation with respect to $\tau$ of the temperate measure $dm(x,y,\tau)$,

$$m(x,y,\tau) = \sqrt{g(y)}(\mathrm{sgn}\ \tau)e(x,y,|\tau|) \ . \tag{2.1.2}$$

We remark that $K(t,x,y) = \widehat{dm(t)}$ is an even function with respect to $t$.

## 2.2 The Hadamard parametrix of the wave operator

In this subsection we shall quickly review a remarkably simple and precise construction due to J. Hadamard, which gives the singularities of the wave kernel $K(t,x,y)$ with any desired precision.

Let the open subset $X$ (cf Subsection 2.1) of $M$ be sufficiently small so that for every point in it we can introduce the geodesic normal coordinates which vanish there and satisfy the condition

$$\sum_k g_{jk}(x)x_k = \sum_k g_{jk}(0)x_k \ . \tag{2.2.1}$$

By Lemma 17.4.1 in [5], there exist unique smooth functions $u_0,\cdots,u_\nu$ with $u_0(0)=1$ satisfying

$$2\nu u_\nu - hu_\nu + 2\langle x, \partial u/\partial x\rangle + 2\Delta u_{\nu-1} \ , \tag{2.2.2}$$

where $u_{-1}=0$ and

$$h(x) = \sum g_{jk}(0)b^j(x)x_k = \sum g_{jk}(x)b^j(x)x_k \ . \tag{2.2.3}$$

It follows from Corollary C.5.2 of [5] that there is a neighborhood $\mathscr{V}$ of the zero section $\{0\} \times M$ of the tangent bundle $TM$, a neighborhood $\mathscr{W}$ of the diagonal in $M \times M$, and a well-defined diffeomorphism

$$\mathscr{V} \ni (\tilde{x}, y) \mapsto (\exp_y \tilde{x},\, y) \in \mathscr{W} \ ,$$

where $\exp_y$ is the exponential map at $y$ with $\exp_y 0 = y$ and $(d \exp_y)|_{\tilde{x}=0}$ equal to the identity. The metric tensor in the $\tilde{x}$ coordinates

$$\sum \tilde{g}^{jk}(\tilde{x}, y)\xi_j\xi_k = p(\exp_y \tilde{x},\, {}^t(d \exp_y)^{-1}(\tilde{x})\,\xi)$$

satisfies (2.2.1), where $p$ is the principal symbol of $\Delta$. If $(x, y) \in \mathscr{W}$ we have a well-defined Riemannian distance $s(x, y)$ which is realized by a unique geodesic between $x$ and $y$. We choose $\mathscr{V}$ such that $\{\tilde{x} : (\tilde{x}, y) \in \mathscr{V}\}$ is convex for every $y \in M$. Pulling the functions $u_\nu(\tilde{x}, y)$ defined by (2.2.2) back to $\mathscr{W}$ from $\mathscr{V}$, we obtain uniquely defined $U_\nu \in C^\infty(\mathscr{W})$. Since $\mathscr{W}$ is open, we further choose the open set $X$ so small that $X \times X \subset \mathscr{W}$. We can choose $c > 0$ such that

$$X^c \times X^c \subset \mathscr{W}, \tag{2.2.4}$$

where

$$X^c = \{y \in M : \inf_{x \in X} s(x, y) < c\} \ .$$

As Lemma 17.4.2 of [5], with notation (3.2.17) of [4] In $\mathbf{R}_t \times \mathbf{R}^n_x$ we define the homogeneous distributions $E_\nu$ $(k \in \mathbf{Z})$ of degree $2\nu + 1 - n$ with support in the forward light cone $\{(t, x) : t \geq |x|\}$ by

$$E_\nu = 2^{-2\nu-1}\pi^{(1-n)/2}\chi_+^{\nu+(1-n)/2}(t^2 - |x|^2),\ t > 0\ . \tag{2.2.5}$$

We have

$$(\partial^2/\partial t^2 - \sum \partial^2/\partial x_j^2)E_\nu = \nu E_{\nu-1}, \nu \neq 0;\ (\partial^2/\partial t^2 - \sum \partial^2/\partial x_j^2)E_0 = \delta_{0,0}; \tag{2.2.6}$$

$$-2\partial E_\nu/\partial x = x E_{\nu-1},\ \nu \in \mathbf{Z}\ . \tag{2.2.7}$$

With some abuse of the notation we shall write $E_\nu(t, |x|)$ instead of $E_\nu(t, x)$ in what follows; when $t = 0$ this should be interpreted as the limit when $t \to +0$. Moreover it follows from the proof of Lemma 17.4.2 in [5] with the notation (3.2.10)' of [4] that

$$\partial_t(E_\nu(t, 0) - \check{E}_\nu(t, 0))$$

$$= \begin{cases} 2^{-2\nu}\,\pi^{(1-n)/2}\,\underline{t}^{2\nu-n}/\Gamma(\nu + (1-n)/2), & \text{if n is even} \\ 2^{-2\nu-1}\,\pi^{(1-n)/2}\,|t|^{2\nu-n}/\Gamma(\nu + (1-n)/2), & \text{if } n \text{ is odd and } 2\nu > n \\ (-1)^k\,2^{-2\nu-k}\,\pi^{(1-n)/2}\,\delta^{(2k)}/(1 \times 3 \times \cdots \times (2k-1)), & \\ \text{if } n \text{ is odd and } n - 1 - 2\nu = 2k \geq 0\ , & \end{cases} \tag{2.2.8}$$

where $\check{E}_\nu$ is the reflection of $E_\nu$ with respect to the origin of $\mathbf{R}_t$. It follows from (2.2.1) and (2.2.6) (cf Proposition 17.4.3 in [5]) that we have in $(-\infty, c)\times X^c\times X$

$$\begin{aligned}(\partial^2/\partial t^2 \ &+ \ \Delta)\mathscr{E}(t,x,y)\\ &= \ (\partial^2/\partial t^2+\Delta)\sum_0^N U_\nu(x,y)E_\nu(t,s(x,y))\\ &= \ \delta_{0,y}/\sqrt{g(y)}+(P(x,D)U_N(x,y))E_N(t,s(x,y))\ . \qquad (2.2.9)\end{aligned}$$

When $s(x,y)\le c$ the coefficients $U_j$ are defined by integrating the equation (2.2.2) in geodesic coordinates, and when $s(x,y)>c$ their definition is irrelevant.

By the proof of Theorem 17.5.5 in [5], in $(-c,\ c)\times X^c\times X$, we have

$$K(t,x,y)-\partial_t(\mathscr{E}(t,x,y)-\check{\mathscr{E}}(t,x,y))/\sqrt{g(y)}\in C^{N-n-3} \qquad (2.2.10)$$

and

$$\left|\partial^\alpha_{t,x,y}\left(K(t,x,y)-\partial_t(\mathscr{E}(t,x,y)-\check{\mathscr{E}}(t,x,y))/\sqrt{g(y)}\right)\right|\le C|t|^{2N-n-|\alpha|}\ , \quad (2.2.11)$$

where the multi-index $\alpha$ has length $|\alpha|\le N-n-3$. By the definition of $E_\nu$ we know that $\mathscr{E}(t,x,y)$ has support in the forward light cone $\{t\ge s(x,y)\}$ and its reflection $\check{\mathscr{E}}(t,x,y))$ with respect to the origin of $\mathbf{R}_t$ has support in the backward light cone $\{t\le -s(x,y)\}$. Here all terms are continuous functions of $(x,y)$ with values in $\mathscr{D}'(\mathbf{R})$ by Lemma 17.4.2 in [5]. Multiplying with $t^n$, by (2.2.9) we can get rid of the singularity of the wave kernel $K(t,x,x)$ on the diagonal. More precisely, the product

$$t^n\,\partial_t(E_0(t,0)-\check{E}_0(t,0))$$

of the principal term $\partial_t(E_0(t,0)-\check{E}_0(t,0))$ of $K(t,x,x)$ with $t^n$ is smooth on $[0,\ c)\times X$, from which we can in fact immediately reach the estimate (1.2) by the wave kernel method in Subsection 2.4. We shall further apply (2.2.10) to investigating the singularities of the derivatives $\partial^\alpha_x\partial^\alpha_y K(t,x,y)|_{x=y}$ of the wave kernel on the diagonal in the following subsection.

## 2.3 The derivatives of the wave kernel on the diagonal

Let $\alpha=(\alpha_1,\cdots,\alpha_n)$ be a multi-index of length $|\alpha|=k\ge 0$. In the coordinate chart $(X\times X,(x,y))$ of $M\times M$, we shall consider the singularities of $\partial^\alpha_x\partial^\alpha_y K(t,x,y)|_{x=y}$ $(x\in X)$. From now on, we let the $N$ in (2.2.10) be as large as necessary. By (2.2.10), we know

$$\begin{aligned}\partial^\alpha_x\partial^\alpha_y K(t,x,y)|_{x=y} \ &= \ \partial^\alpha_x\partial^\alpha_y(\,\partial_t(\mathscr{E}(t,x,y)-\check{\mathscr{E}}(t,x,y))/\sqrt{g(y)}\,)|_{x=y}\\ &+ \ C^{N-n-2k-3}\,\text{term}\ . \qquad (2.3.1)\end{aligned}$$

By the above equality we know that $\partial_x^\alpha \partial_y^\alpha K(t,x,y)|_{x=y}$ is the sum of a continuous function of $(t,x) \in (-c,\, c) \times X$ and finite homogeneous distributions of $t$ with coefficients smooth functions of $x \in X$. We call the distribution summand of $\partial_x^\alpha \partial_y^\alpha K(t,x,y)|_{x=y}$ with the lowest homogeneous degree the *principal singular term* of $\partial_x^\alpha \partial_y^\alpha K(t,x,y)|_{x=y}$. We observe that the principal singular term of $\partial_x^\alpha \partial_y^\alpha K(t,x,y)|_{x=y}$ is equal to a smooth function of $x$ depending on the metric of $X$ times

$$\partial_x^\alpha \partial_y^\alpha \partial_t (E_0(t,s(x,y)) - \check{E}_0(t,s(x,y)))|_{x=y} \ . \tag{2.3.2}$$

Firstly we need the following

**Lemma 2.2.** *In the open set $X$ with the geodesic coordinates, we have the following Taylor expansion of the square distance function $s(x,y)^2$:*

$$s(x,y)^2 = |x-y|^2 + \text{higher} - \text{order terms}.$$

**Proof.** By [6] we know the square distance function $s(x,y)^2$ is a smooth on $X \times X$. Let $\eta(x,y) = s(x,y)^2$. Under the geodesic coordinates, the square distance function $\eta$ satisfies the following properties:
(i) $\eta(0,0) = \eta(x,x) = 0$,
(ii) $\eta(0,x) = \eta(0,-x) = |x|^2$,
(iii) $\eta(x,y) = \eta(y,x)$.
From (ii), we obtain

$$\partial_{x_j}\eta(0,0) = \partial_{y_j}\eta(0,0) = 0 \ , \ \partial_{x_j}^2\eta(0,0) = \partial_{y_j}^2\eta(0,0) = 2 \ , \tag{2.3.3}$$

where $j = 1, \cdots, n$. Let $z = (0, \cdots, 0, x_j, 0, \cdots, 0)$ have the $j$-th coordinate $x_j$ and others 0. The restriction on $(0,0)$ of the second derivative of $\eta(z,z)$ gives

$$\partial_{x_j}^2\eta(0,0) + 2\,\partial_{x_j y_j}^2\eta(0,0) + \partial_{y_j}^2\eta(0,0) = 0 \ ,$$

which combined with (2.3.3) implies $\partial_{x_j y_j}^2\eta(0,0) = -2$. q.e.d.

We denote

$$E_\nu'(t,x) = \partial_t E_\nu'(t,x), \ \check{E}_\nu'(t,x) = \partial_t \check{E}_\nu(t,x)$$

and then

$$2\partial E_\nu'/\partial x = x E_{\nu-1}', \ 2\partial \check{E}_\nu'/\partial x = x \check{E}_{\nu-1}' \tag{2.3.4}$$

hold. We compute the principal singular term of (2.3.2) as follows.

With $\eta = s(x,y)^2, \ s = s(x,y)$, then

$$\partial_{x_j}(E_\nu'(t,s) - \check{E}_\nu'(t,s)) = -\frac{1}{4}\partial_{x_j}\eta(E_{\nu-1}'(t,s) - \check{E}_{\nu-1}'(t,s))$$

By Lemma 2.2, (2.3.4) and above equality, we have

$$\begin{aligned} & \partial_{x_j y_j}^2(E_\nu'(t,s) - \check{E}_\nu'(t,s)) \\ = \ & -\frac{1}{4}\partial_{x_j y_j}^2\eta(E_{\nu-1}'(t,s) - \check{E}_{\nu-1}'(t,s)) + \frac{1}{16}\partial_{x_j}\eta\partial_{y_j}\eta(E_{\nu-2}'(t,s) - \check{E}_{\nu-2}'(t,s)) \end{aligned}$$

In particualr,

$$
\begin{aligned}
& \partial_{x_j}\partial_{y_j}(E'_0(t,s)-\check{E}'_0(t,s))|s=0 \\
= \; & -\frac{1}{4}\partial^2_{x_jy_j}\eta(E'_{-1}(t,s)-\check{E}'_{-1}(t,s))|_{s=0}+\frac{1}{16}\partial_{x_j}\eta\partial_{y_j}\eta(E'_{-2}(t,s)-\check{E}'_{-2}(t,s))|_{s=0} \\
= \; & \frac{1}{2}(E'_{-1}(t,0)-\check{E}'_{-1}(t,0)) \; ; \qquad (2.3.5)
\end{aligned}
$$

and

$$
\begin{aligned}
& \partial^2_{x_j}\partial^2_{y_j}\big(E'_0(t,s)-\check{E}'_0(t,s)\big)|_{s=0} \\
= \; & -\frac{1}{4}\partial^2_{x_j}\partial^2_{y_j}\eta\big(E'_{-1}(t,s)-\check{E}'_{-1}(t,s)\big)|_{s=0} \\
+ \; & \Big(\frac{1}{16}(\partial^2_{x_jy_j}\eta)^2+\frac{1}{8}(\partial^2_{x_j}\partial_{y_j}\eta\partial_{y_j}\eta+\partial^2_{y_j}\partial_{x_j}\eta\partial_{x_j}\eta) \\
+ \; & \frac{1}{16}\partial^2_{x_jy_j}(\partial_{x_j}\eta\partial_{y_j}\eta)\Big)\big(E'_{-2}(t,s)-\check{E}'_{-2}(t,s)\big)|_{s=0} \\
+ \; & \Big(-\frac{3}{32}\partial^2_{x_jy_j}\eta\partial_{x_j}\eta\partial_{y_j}\eta-\frac{1}{32}\partial^2_{x_j}\eta(\partial_{y_j}\eta)^2-\frac{1}{32}\partial^2_{y_j}\eta(\partial_{x_j}\eta)^2\Big)\big(E'_{-3}(t,s)-\check{E}'_{-3}(t,s)\big)|_{s=0} \\
+ \; & \frac{1}{256}(\partial_{x_j}\eta)^2(\partial_{y_j}\eta)^2\big(E'_{-4}(t,s)-\check{E}'_{-4}(t,s)\big)|_{s=0} \\
= \; & \frac{3}{4}(E'_{-2}(t,0)-\check{E}'_{-2}(t,0))+(E'_{-1}(t,0)-\check{E}'_{-1}(t,0)) \text{ times} \\
& \text{a constant depending on the geometry of } M \text{ near } x. \qquad (2.3.6)
\end{aligned}
$$

The equalities (2.3.5) and (2.3.6) tell us that the principal singular term of

$$\partial^l_{x_j}\partial^l_{y_j}(E'_0(t,s)-\check{E}'_0(t,s)|_{s=0}, \; l=1,2$$

are certain constants times

$$E'_{-l}(t,0)-\check{E}'_{-l}(t,0), \; l=1,2 \; .$$

In general we can prove the following

**Lemma 2.3.** *Let $\alpha\in\mathbf{Z}^n_+$ be a multi-index of length $|\alpha|=k>0$ and $X$ a geodesic coordinate chart of $M$ satisfying (2.2.4). Let $(t,\, x)$ be in $(-c,\, c)\times X$. Then the principal singular term of*

$$\partial^\alpha_x\partial^\alpha_y(E'_0(t,s(x,y))-\check{E}'_0(t,s(x,y))|_{x=y}$$

*is a smooth function of $x$ times*

$$E'_{-k}(t,0)-\check{E}'_{-k}(t,0) \; ,$$

*where the smooth function depends on $\alpha$ and the geometry of $M$ on $X$.*

**Proof.** By Lemma 2.2 and the induction argument, we can show that

$$\partial_x^\alpha \partial_y^\alpha (E_0'(t, s(x,y)) - \check{E}_0'(t, s(x,y)) = \sum_1^{2k} P_m(E'_{-m}(t, s(x,y)) - \check{E}'_{-m}(t, s(x,y))) \ ,$$

where $P_m$ $(0 \le m \le 2k)$ is a polynomials in partial derivatives of $s(x,y)^2$. In particular, if $m = k + l$ $(1 \le l \le k)$, every summand of $P_m$ has a divisor which is a polynomial of order $2l$ in the first partial derivatives $\partial_{x_j} s(x,y)^2$, $\partial_{y_j} s(x,y)^2$ of $s(x,y)^2$. Since

$$\partial_{x_j} s(x,y)^2|_{x=y} = \partial_{y_j} s(x,y)^2|_{x=y} = 0 \ ,$$

$$\partial_x^\alpha \partial_y^\alpha (E_0'(t, s(x,y)) - \check{E}_0'(t, s(x,y))|_{x=y} = \sum_1^{k} P_m(x,x)(E'_{-m}(t, 0) - \check{E}'_{-m}(t, 0)) \ .$$

q.e.d.

**Proposition 2.1.** *Under the assumptions of* Lemma 2.3*, the principal singular term of* $\partial_x^\alpha \partial_y^\alpha K(t,x,y)|_{x=y}$ *is a smooth function of* $x$ *times*

$$E'_{-k}(t,0) - \check{E}'_{-k}(t,0)$$

*for* $t \in (-c,\, c)$*, where the smooth function depends on* $\alpha$ *and the geometry of* $M$ *on* $X$*. Moreover,*

$$\left(\partial_x^\alpha \partial_y^\alpha K(t,x,y) - \partial_x^\alpha \partial_y^\alpha \sum_{0 \le 2\nu < 2k+n} \Big((E'_\nu(t, s(x,y)) - \check{E}'_\nu(t, s(x,y)))U_\nu(x,y)/\sqrt{g(y)}\Big)\right)_{x=y}$$

*is in* $C^\infty((-c,\, c) \times X)$ *if* $n$ *is even, and in* $C^\infty((-c,\, c) \times X)$ *after division by* $|t|$ *if* $n$ *is odd. All derivatives are bounded in* $(-c,\, c) \times X$*.*

**Proof.** The first statement directly follows from the equalities (2.2.9), (2.3.1) and Lemma 2.3. Write the function in the second statement briefly in $R(t,x)$. Then

$$\begin{aligned} R(t,x) = &\Big(\partial_x^\alpha \partial_y^\alpha K(t,x,y) - \partial_x^\alpha \partial_y^\alpha \big(\, \partial_t(\mathscr{E}(t,x,y) - \check{\mathscr{E}}(t,x,y))/\sqrt{g(y)}\,\big)\Big)_{x=y} \\ + \ &\partial_x^\alpha \partial_y^\alpha \sum_{2k+n \le 2\nu \le 2N} \Big((E'_\nu(t, s(x,y)) - \check{E}'_\nu(t, s(x,y)))U_\nu(x,y)/\sqrt{g(y)}\Big)_{x=y} \end{aligned} \tag{2.3.7}$$

The first term in the right hand side (RHS) of (2.3.7) is in $C^{N-n-2k-3}((-c,\, c)\times X)$ by (2.3.1). Since it is even in $t$, its quotient by $|t|$ is in $C^{N-n-2k-4}((-c,\, c)\times X)$. As a similar result of the first statement, the principal singular term of the summand

$$\partial_x^\alpha \partial_y^\alpha \Big(E'_\nu(t, s(x,y)) - \check{E}'_\nu(t, s(x,y)))U_\nu(x,y)/\sqrt{g(y)}\Big)_{x=y}, \ 2k+n \le 2\nu \le 2N,$$

of the second term in the RHS of (2.3.7) is a smooth function times

$$E'_{\nu-k}(t,0) - \check{E}'_{\nu-k}(t,0),\ 2\nu \geq n+2k$$

which by (2.2.8) is in $C^\infty((-c,\,c)\times X)$ if $n$ is even, and in $C^\infty((-c,\,c)\times X)$ after division by $|t|$ if $n$ is odd. The same result holds for the second term of the LHS of (2.3.7). Letting $N \to \infty$, we complete the proof. q.e.d.

**Corollary 2.1.** *Let $(t,\,x)$ be in $(-c,\,c)\times X$ and $\alpha$ a multi-index of length $|\alpha| = k \geq 0$. If $n$ is even, $\partial_x^\alpha \partial_y^\alpha K(t,x,y)|_{x=y}$ is equal to*

$$\underline{t}^{-2k-n} \times \text{a smooth function } + \text{ a smooth function};$$

*if $n$ is odd, $\partial_x^\alpha \partial_y^\alpha K(t,x,y)|_{x=y}$ is equal to*

$$\delta_0^{(n-1+2k)}(t) \times \text{a smooth function } + |t| \times \text{a smooth function}.$$

*In particular, $t^{2k+n}\partial_x^\alpha \partial_y^\alpha K(t,x,y)|_{x=y}$ is in $C^\infty([0,\,c)\times X)$.*

**Proof.** Firstly let $n$ be even. By the equality (2.2.8) and the proof of Lemma 2.3 and Proposition 2.1,

$$\partial_x^\alpha \partial_y^\alpha K(t,x,y)|_{x=y} = \sum_{-k}^{(2k+n-2)/2} (E'_\nu(t,0) - \check{E}'_\nu(t,0))Q_\nu(x) \in C^\infty((-c,\,c)\times X\ ,$$

where $Q_\nu(x)$ are smooth functions of $x$. The statement follows from that for $-k \leq \nu \leq (2k+n-2)/2$

$$E'_\nu(t,0) - \check{E}'_\nu(t,0) = \text{const}\, \underline{t}^{2\nu-n} = \text{const}\, \underline{t}^{-2k-n} \times t^{2(\nu+k)}\ .$$

Then let $n$ be odd. By the equality (2.2.8) and the proof of Lemma 2.3 and Proposition 2.1, the following holds:

$$\partial_x^\alpha \partial_y^\alpha K(t,x,y)|_{x=y} = \sum_{-k}^{(n-1)/2} (E'_\nu(t,0) - \check{E}'_\nu(t,0))Q_\nu(x) = |t| \times \text{a smooth function}\ ,$$

where $Q_\nu$ are smooth function of $x$. The statement follows from the equality

$$E'_\nu(t,0) - \check{E}'_\nu(t,0) = \text{const}\, \delta^{(n-1-2\nu)}(t) = \text{const}\, \delta^{(n-1+2k)}(t) \times t^{2(\nu+k)}$$

for $-k \leq \nu \leq (n-1)/2$. q.e.d.

## 2.4 The wave kernel method and $L^\infty$ estimates

By Corollary 2.1 and the wave kernel method (cf D. Grieser [2]), in this subsection we shall prove the following pointwise estimates of the covariant derivatives of the spectral function uniformly for $x \in M$:

$$\sum_{\lambda_j \in [\lambda,\, \lambda+1]} |\nabla^k e_j(x)|^2 \leq C\lambda^{n-1+2k},\ k = 0, 1, \cdots \tag{2.4.1}$$

which implies the following $(L^2, C^k)$ estimates for the USPO $\chi_\lambda$:

$$||\chi_\lambda f||_{C^k} \leq C\lambda^{k+(n-1)/2}||f||_2\ . \tag{2.4.2}$$

In fact, by Lemma 2.1 in order to prove the above inequality, we have only to show

$$||\partial^\alpha \chi_\lambda f||_{L^\infty(X)} \leq C\lambda^{|\alpha|+(n-1)/2}||f||_2\ .$$

Withoud loss of generality, we assume that the smooth function $f$ takes real values on $M$ in what follows. Since

$$\chi_\lambda f(x) = \int_M \sum_{\lambda_j \in [\lambda,\, \lambda+1]} e_j(x)e_j(y)f(y)\sqrt{g(y)}dy\ ,$$

for any $x \in X$, by the Cauchy-Schwarz inequality and (2.4.1) we have

$$\begin{aligned} |\partial_x^\alpha \chi_\lambda f(x)|^2 &\leq \sum_{\lambda_j \in [\lambda,\, \lambda+1]} |\partial_x^\alpha e_j(x)|^2 \sum_{\lambda_j \in [\lambda,\, \lambda+1]} \left( \int_M e_j(y)f(y)\sqrt{g(y)}dy \right)^2 \\ &\leq C\lambda^{n-1+2|\alpha|}||f||_2^2\ . \end{aligned}$$

Proof of (2.4.1) Since $M$ is compact, by the proof of Lemma 2.1 and the known result (1.2), we have only to show that for any multi-index $\alpha \in \mathbf{Z}_+^n$ with length $|\alpha| = k > 0$ the following inequality holds:

$$|\partial_x^\alpha \sum_{\lambda_j \in [\lambda,\, \lambda+1]} e_j(x)|^2 \leq C\lambda^{n-1+2k},\ x \in M, \tag{2.4.3}$$

where $\lambda$ is positive number $\geq 1$. Let $\rho$ be a Schwarz function in $\mathscr{S}(\mathbf{R})$ satisfying

$$\rho \geq 0,\ \rho|_{[0,\,1]} \geq 1,\ \operatorname{supp} \hat{\rho} \subset (-c, c)\ . \tag{2.4.4}$$

Theorem 17.5.3 of [5] says that $|\partial_{x,y}^\alpha e(x, y, \lambda)| \leq C_k\lambda^{n+|\alpha|}$, which implies that the sum

$$\sum_{j=0}^{\infty} \rho(\lambda - \lambda_j)\partial_x^\alpha e_j(x)\partial_y^\alpha e_j(y)$$

convergences absolutely. By Fourier inversion formular,

$$\rho(\lambda-\lambda_j)=2(\hat\rho(t)\cos(t\lambda_j))^\vee(\lambda)-\rho(\lambda+\lambda_j)\ .$$

Multiplying $\partial_x^\alpha e_j(x)\partial_y^\alpha e_j(y)$ with above equality and taking sum, from the equality

$$\partial_x^\alpha\partial_y^\alpha K(t,x,y)=\sum_{j=0}^{\infty}\cos(t\lambda_j)\partial_x^\alpha e_j(x)\partial_y^\alpha e_j(y)\ ,$$

we obtain the following estimates

$$\begin{aligned}\sum_{j=0}^{\infty}\rho(\lambda-\lambda_j)\partial_x^\alpha e_j(x)\partial_y^\alpha e_j(y) &= 2(\hat\rho(t)\partial_x^\alpha\partial_y^\alpha K(t,x,y)|_{x=y})^\vee(\lambda)\\ &+ \mathrm{O}(\lambda^{-\infty})\ . \end{aligned}\tag{2.4.5}$$

Letting $x=y$ in above inequality, we have

$$\sum_{|\lambda_j-\lambda|\le 1}|\partial_x^\alpha e_j(x)|^2\le 2(\hat\rho(t)\partial_x^\alpha\partial_y^\alpha K(t,x,y)|_{x=y})^\vee(\lambda)+\mathrm{O}(\lambda^{-\infty})\ .\tag{2.4.6}$$

By Corollary 2.1, in order to estimate $(\rho(t)\partial_x^\alpha\partial_y^\alpha K(t,x,y)|_{x=y})^\vee(\lambda)$, instead of $\partial_x^\alpha\partial_y^\alpha K(t,x,y)|_{x=y}$ we may only consider

$$\underline{t}^{-2k-n}\ ,\qquad \text{if } n \text{ is even ;}$$

and

$$\delta^{(n-1+2k)}(t),\quad \text{if } n \text{ is odd .}$$

By Example 7.1.17 of [4], the Fourier transformation of $\underline{t}^{-l-1}$ is const $\times$ $(\operatorname{sgn}\xi)\xi^l$ if $l\ge 0$ and const $\times$ $\delta^{-l-1}$ if $l<0$. For $n$=even, as follows:

$$\begin{aligned}|(\hat\rho(t)\underline{t}^{-2k-n})^\vee(\lambda)| &= |(\rho*(\underline{t}^{-2k-n})^\vee(\lambda)|\\ &\le C\,(\rho*(|\xi|^{n+2k-1}))(\lambda)\\ &\le C\int_{\mathbf{R}}(1+|\xi|)^{-(n+2k+1)}|\lambda-\xi|^{n+2k-1}d\xi\\ &\le C\sum_{0}^{n+2k-1}\int_{\mathbf{R}}(1+|\xi|)^{j-(n+2k+1)}\lambda^j d\xi\\ &\le C\lambda^{n+2k-1}\ .\end{aligned}$$

For $n$=odd, we have also the above estimate. q.e.d.

# 3 Sobolev $L^p$ estimate

## 3.1 $L^p$ a priori estimates and Sobolev $L^p$ norms of USPO

In this subsection we shall reduce Theorem 1.1 to the estimates of certain Sobolev norms of USPO $\chi_\lambda$. Firstly we show some elliptic estimates related to Sobolev $L^p$ norms.

**Proposition 3.1.** *Let $u$ be a smooth function on $M$, $1 \le r < \infty$ and $k$ a positive integer. Then the followings hold:*

$$||u||_{H^r_{2k}} \le C \sum_{j=0}^{k} ||\Delta^j u||_r, \ ||u||_{H^r_{2k+1}} \le C \sum_{j=0}^{k} ||\Delta^j u||_{H^r_1}. \tag{3.1.1}$$

**Proof.** Let $l$ be a positive integer $\ge 2$. By the well known elliptic a priori estimate

$$||u||_{H^r_2} \le C\left(||u||_r + ||\Delta u||_r\right), \tag{3.1.2}$$

and that the commutator operator $[\Delta, \partial^\alpha]$ has order $< 2 + |\alpha|$, the following inequality holds:

$$\begin{aligned} ||u||_{H^r_l} &\le C \sum_{|\alpha| \le l-2} ||\partial^\alpha u||_{H^r_2} \le C \sum_{|\alpha| \le l-2} \left(||\partial^\alpha u||_r + ||\Delta \partial^\alpha u||_r\right) \\ &\le C \left(||u||_{H^r_{l-2}} + \sum_{|\alpha| \le l-2} ||\, [\Delta, \partial^\alpha]\, u\, ||_r + \sum_{|\alpha| \le l-2} ||\partial^\alpha \Delta u||_r\right) \\ &\le C\left(||u||_{H^r_{l-1}} + ||\Delta u||_{H^r_{l-2}}\right). \end{aligned} \tag{3.1.3}$$

Letting $l = 3$ in above inequality gives, by (3.1.2) we have

$$||u||_{H^r_3} \le C\left(||u||_r + ||\Delta u||_{H^r_1}\right) \le C\left(||u||_{H^r_1} + ||\Delta u||_{H^r_1}\right). \tag{3.1.4}$$

In the following we show the inequalities (3.1.1) by induction argument. Firstly let (3.1.1) hold for $\le 2k$. Then by (3.1.3) we have

$$\begin{aligned} ||u||_{H^r_{2k+1}} &\le C\left(||u||_{H^r_{2k}} + ||\Delta u||_{H^r_{2k-1}}\right) \le C\left(\sum_{j=0}^{k} ||\Delta^j u||_r + \sum_{j=1}^{k} ||\Delta^j u||_{H^r_1}\right) \\ &\le C \sum_{j=0}^{k} ||\Delta^j u||_{H^r_1}. \end{aligned}$$

That is, (3.1.1) holds for $2k + 1$.

Finally letting (3.1.1) hold for $\leq 2k+1$. Then by this assumption and (3.1.2), we show (3.1.1) holds for $2k+2$ as following:

$$\begin{aligned} ||u||_{H^r_{2k+2}} &\leq C\left(||u||_{H^r_{2k+1}} + ||\Delta u||_{H^r_{2k}}\right) \leq C\left(\sum_{j=0}^{k} ||\Delta^j u||_{H^r_1} + \sum_{j=1}^{k} ||\Delta^j u||_r\right) \\ &\leq C\sum_{j=0}^{k+1} ||\Delta^j u||_r \ . \end{aligned}$$

By above two inequalities and the inequalities (3.1.2) and (3.1.4), we complete the induction argument. q.e.d.

Let $u$ be a real valued smooth function on the Riemannian manifold $M$. The gradient $\mathbf{grad}\, u$ of $u$ is defined to be the dual vector field of one form $du = \nabla u$ by

$$g(\mathbf{grad}\, u,\, V) = du(V)$$

for arbitrary smooth vector field $V$ on $M$. We shall identity $\nabla u$ with $\mathbf{grad}\, u$ and only consider the latter in what follows. Since in the coordinate chart $(X, x)$

$$|\mathbf{grad}\, u| = |\nabla u| = \sum g^{jk}\partial_j u \partial_k u \ , \tag{3.1.5}$$

by the Green's formula we have

$$||u||^2_{H^1} = ||u||^2_2 + \int_M |\mathbf{grad}\, u|^2 \, dv(M) = ||u||^2_2 + \int_M u\, \Delta u dv(M). \tag{3.1.6}$$

Adding up to (3.1.6) the equality $\Delta\chi_\lambda f = \sum_{|\lambda-\lambda_j|\leq 1} \lambda_j^2\, \mathbf{e_j}(f)$ and the orthogonal relation, we obtain

$$||\chi_\lambda f||_{H^1} \leq C\lambda ||f||_2 \ , \tag{3.1.7}$$

combining which with (3.1.1) and the equalities

$$\chi_\lambda^2 = \chi_\lambda \text{ on } L^2(M),\ \Delta\,\chi_\lambda = \chi_\lambda\,\Delta \text{ on } C^\infty(M),$$

we have

$$\begin{aligned} ||\chi_\lambda f||_{H^{2k+1}} &\leq C\sum_{j=0}^{k} ||\Delta^j \chi_\lambda f||_{H^1} = C\sum_{j=0}^{k} ||\chi_\lambda(\Delta^j \chi_\lambda f)||_{H^1} \\ &\leq C\lambda\sum_{j=0}^{k} ||\Delta^j \chi_\lambda f||_2 \leq C\lambda^{1+2k}||f||_2 \ . \end{aligned} \tag{3.1.8}$$

It follows also from (3.1.1) that

$$||\chi_\lambda f||_{H^{2k}} \leq C\sum_{j=0}^{k} ||\Delta^j \chi_\lambda f||_2 \leq C\lambda^{2k}||f||_2 \ . \tag{3.1.9}$$

Summing up the inequalities (3.1.6), (3.1.8) and (3.1.9), we obtain the following

**Lemma 3.1.** *Let $k$ be a nonnegative integer. The following $(L^2, H^k)$ estimates for the USPO $\chi_\lambda$ hold:*

$$||\chi_\lambda f||_{H^k} \leq C\,\lambda^k ||f||_2\ , k = 0, 1, \cdots . \tag{3.1.10}$$

Now we can do reduction of Theorem 1.1 in the following

**Lemma 3.2.** *Suppose that the following estimates hold for $p = 2(n+1)/(n-1)$*

$$||\mathbf{grad}\,\chi_\lambda f||_p \leq C\,\lambda^{1+\delta(p)}||f||_2, \tag{3.1.11}$$

$$||\Delta\chi_\lambda f||_p \leq C\,\lambda^{2+\delta(p)}||f||_2\ . \tag{3.1.12}$$

*In* (3.1.11) *the $L^p$ norm $||\mathbf{grad}\,u||_p = \Big(\displaystyle\int_M |\mathbf{grad}\,u|^p\,dv(M)\Big)^{1/p}$ for a smooth function $u$ on $M$. Then* Theorem 1.1 *holds.*

**Proof.** By the interpolation method and the estimates (2.4.2), (3.1.10), we have only to prove $(*)$ : $\;||\chi_\lambda f||_{H^p_k} \leq C\,\lambda^{k+\delta(p)}||f||_2$, for any nonnegative integer $k$. Firstly let $k$ be even. By (3.1.12) and Proposition 1.1, for any nonnegative integer $j$, the following inequality holds by induction:

$$||\Delta^j\chi_\lambda f||_p \leq C\,\lambda^{2j+\delta(p)}||f||_2\ .$$

Then $(*)$ follows from Proposition 1.1, Proposition 3.1 and above inequality.

Then let $k$ be odd. It follows from Proposition 1.1, (3.1.5) and (3.1.11) that

$$||\chi_\lambda f||_{H^p_1} \leq C\left(||\chi_l f||_p + ||\mathbf{grad}\,\chi_l f||_p\right) \leq C\lambda^{1+\delta(p)}||f||_2\ .$$

In particular, the following holds for any nonnegative integer $j$:

$$||\Delta^j\chi_l f||_{H^p_1} = ||\chi_l(\Delta^j\chi_l f)||_{H^p_1} \leq C\,\lambda^{1+\delta(p)}||\Delta^j\chi_l f||_2 \leq C\,\lambda^{1+2j+\delta(p)}||f||_2\ .$$

The $(*)$ follows from Proposition 3.1 and above inequality. q.e.d.

## 3.2 Fourier Integral operators

In this subsection, we shall approximate operators $\Delta\cos(t\sqrt{\mathscr{P}})$, $\mathbf{grad}\,\cos(t\sqrt{\mathscr{P}})$ by Fourier integral operators (FIOs) and reduce the estimates (3.1.11) and (3.1.12) to the corresponding estimates of the FIOs. In subsection 2.2, in order to write the covariant derivatives of the wave kernel explicitly, we constructed the Hadamard parametrix in the coordinate chart $X$. Here we consider the parametrix on $M$. Using the same argument in Subsection 2.2 , there exists a positive number $c$ depending on $M$ such that in $(-c,\ c) \times M \times M$, we have

$$K(t,x,y) - \partial_t(\mathscr{E}(t,x,y) - \check{\mathscr{E}}(t,x,y))/\sqrt{g(y)} \in C^{N-n-3} \tag{3.2.1}$$

where

$$\mathscr{E}(t,x,y)=\sum_0^N U_\nu(x,y)E_\nu(t,s(x,y))\ , \tag{3.2.2}$$

$U_\nu(x,y)$ are $C^\infty$ function and can be assumed to be supported in a neighbourhood

$$\mathscr{C}=\{(x,y)\in M\times M\ :\ s(x,y)<c\}$$

of the diagonal, and $U_0(x,x)=1$. By Lemma 17.4.2 of [5], we have

$$E_0'(t,x)-\check{E}_0'(t,x)=(2\pi)^{-n}\int_{\mathbf{R}^n}\cos(t|\xi|)e^{i\langle x\,,\,\xi\rangle}d\xi\ .$$

Therefore, the followings holds:

$$\partial(E_0'(t,x)-\check{E}_0'(t,x))/\partial x=(2\pi)^{-n}\int_{\mathbf{R}^n}\cos(t|\xi|)(i\xi)e^{i\langle x\,,\,\xi\rangle}d\xi\ , \tag{3.2.3}$$

$$\sum_{j=1}^n(\partial/\partial x_j)^2(E_0'(t,x)-\check{E}_0'(t,x))=(2\pi)^{-n}\int_{\mathbf{R}^n}\cos(t|\xi|)(-|\xi|^2)e^{i\langle x\,,\,\xi\rangle}d\xi\ \ . \tag{3.2.4}$$

Firstly we consider the $L^p$ gradient estimate (3.1.11) and do some preparations for it. The Riemannian metric on $M$ induces naturally the the inner products on the space of smooth vector fields of $M$, and on $\Lambda^1 M$, the space of smooth one forms of $M$. Also the spaces $L^2(TM)$, $L^2(\Lambda^1 M)$ consisting of square integrable vector fields, square integrable one forms can be defined respectively. We denote the inner products in $L^2(TM)$, $L^2(\Lambda^1 M)$ simutaneously by $(\ ,\ )$ if there is no confusion. Then $(\mathbf{grad}\,u,\mathbf{grad}\,u)=(du\,,\,du)$ holds for $u\in C^\infty(M)$. In particular, by Green's formula, we know

$$(de_j\,,\,de_k)=(\mathbf{grad}\,e_j,\mathbf{grad}\,e_k)=(\Delta e_j\,,\,e_k)=\delta_{jk}\lambda_j^2\ . \tag{3.2.5}$$

That is, $\{\mathbf{grad}\ e_j\}$, $\{de_j\}$ are orthogonal basis of $L^2(TM)$, $L^2(\Lambda^1 M)$ respectively.

Let $\rho$ be the Schwarz function in $\mathscr{S}(\mathbf{R})$ satisfying (2.4.4). To prove (3.1.11), by the dual argument and (3.2.5), it is enough for us to show

$$||\tilde{\chi}_{g,\lambda}f||_2\le C\,\lambda^{1+\delta(q)}||f||_q\ , \tag{3.2.6}$$

where $\tilde{\chi}_{g,\lambda}f=\sum\rho(\lambda-\lambda_j)\,\mathbf{grad}\,\mathbf{e}_j(f)$ and $q=2(n+1)/(n+3)$. Without loss of generality, we assume $f$ to be in $C^\infty(M)$ in what follows. And $\sum$ will mean $\sum\limits_0^\infty$. We can write $\tilde{\chi}_{g,\lambda}f$ as

$$\tilde{\chi}_{g,\lambda}f=(2\pi)^{-1}\int_{\mathbf{R}}\hat{\rho}(\tau)e^{-i\tau\lambda}\sum e^{i\tau\lambda_j}\ \mathbf{grad}\ \mathbf{e}_j(f)d\tau\ .$$

We define $\tilde{\tilde{\chi}}_{g,\lambda}$ as

$$\tilde{\tilde{\chi}}_{g,\lambda} = (2\pi)^{-1} \int_{\mathbf{R}} \hat{\rho}(\tau) e^{-i\tau t} \sum \cos(\tau\lambda_j)\, \mathbf{grad}\ \mathbf{e}_j(f) d\tau \ , \tag{3.2.7}$$

so that

$$2\,\tilde{\tilde{\chi}}_{g,\lambda} f = \tilde{\chi}_{g,\lambda} f + \sum \rho(\lambda+\lambda_j)\, \mathbf{grad}\, \mathbf{e}_j f \ .$$

By the $L^\infty$ gradient estimate $|\mathbf{grad} \sum_{|\lambda_j-\lambda|\le 1} e_j| \le C\,\lambda^{(n+1)/2}$ (cf (2.4.1)) and the Young's inequality, we have

$$||\sum \rho(\lambda+\lambda_j)\, \mathbf{grad}\, \mathbf{e}_j||_{(L^1(M),\, L^2(TM))} = \mathrm{O}(\lambda^{-\infty}) \ ,$$

interpolating which with

$$||\sum \rho(\lambda+\lambda_j)\, \mathbf{grad}\, \mathbf{e}_j||_{(L^2(M),\, L^2(TM))} = \mathrm{O}(1)$$

followed from (3.2.5), we obtain

$$||\sum \rho(\lambda+\lambda_j)\, \mathbf{grad}\, \mathbf{e}_j||_{(L^q(M),\, L^2(TM))} = \mathrm{O}(\lambda^{-\infty}) \ .$$

Thus it suffices to show that the operator $\tilde{\tilde{\chi}}_{g,\lambda}$ satisfies (3.2.6).

To do this, we use the Hadamard parametric (3.2.1) and (3.2.3), which for $t \in (-c\,,\,c)$ allows us to write the gradient of $\cos(t\sqrt{\mathscr{P}})f$ as follows

$$\mathbf{grad}_x \cos(t\sqrt{\mathscr{P}})f(x) = \int_M \int_{\mathbf{R}^n} \mathbf{grad}_x\, e^{i\Phi(x,y,\xi)} \cos(t|\xi|)\, U_0(x,y) f(y) d\xi dy + R_t f(x) \ . \tag{3.2.8}$$

Here $R_t$ is also a Fourier integral operator, but it is of one order lower. Also,

$$\Phi(x,y,\xi) = \langle x - \tilde{y}\,,\,\xi \rangle \ , \tag{3.2.9}$$

where, for a given $x$, $\tilde{y}$ denotes the geodesic normal coordinates of $y$. This phase function is always well defined in $\mathscr{C}$. It follows from (3.2.7) and (3.2.8) that, modulo an operator which has an $(L^q\,,\,L^2)$ norm that is $\mathrm{O}(\lambda^{-1})$ better by the argument in Subsection 3.3, $\tilde{\tilde{\chi}}_{g,\lambda}$ has kernel

$$\mathcal{K}_\lambda(x,y) = U_0(x,y) \int\int \mathbf{grad}_x\, e^{i\Phi(x,y,\xi)} \cos(\lambda|\xi|)\, \hat{\rho}(\tau)\, e^{-i\tau\lambda}\, d\tau d\xi \ .$$

However, it is easy to check that the kernel

$$U_0(x,y) \int\int \mathbf{grad}_x\, e^{i\Phi(x,y,\xi)-\tau|\xi|}\, \hat{\rho}(\tau)\, e^{-i\tau\lambda}\, d\tau d\xi \ = \ U_0(x,y) \int \mathbf{grad}_x\, e^{i\Phi(x,y,\xi)} \rho(\lambda+|\xi|) d\xi$$

give rise to a operator with rapidly decreasing $(L^q\,,\,L^2)$ norm, which, in turn, implies that we need only show that the operator with kernel

$$\begin{aligned}\mathscr{K}_\lambda(x,y) &= U_0(x,y)\int\int \mathbf{grad}_x\, e^{i\Phi(x,y,\xi)+\tau|\xi|}\,\hat{\rho}(\tau)\,e^{-i\tau\lambda}\,d\tau d\xi \\ &= U_0(x,y)\int \mathbf{grad}_x\, e^{i\Phi(x,y,\xi)}\rho(\lambda-|\xi|)d\xi \qquad (3.2.10)\end{aligned}$$

satisfies (3.2.6)

Using (3.2.4) and the parallel argument as above, we can prove the similar reduction of the $(L^2, L^p)$ estimate (3.1.12). Therefore we have the following

**Lemma 3.3.** *Let the two operators* $\Upsilon_{g,\lambda} : L^q(M) \to L^2(TM)$ *and* $\Upsilon_{\Delta,\lambda} : L^q(M) \to L^2(M)$ *have kernels of* $\mathscr{K}_\lambda(x,y)$ *in* (3.2.10) *and* $\tilde{\mathscr{K}_\lambda}(x,y)$ *defined by*

$$\tilde{\mathscr{K}_\lambda}(x,y) = U_0(x,y)\int\sum g^{jk}(x)\xi_j\xi_k\, e^{i\Phi(x,y,\xi)}\rho(\lambda-|\xi|)d\xi \qquad (3.2.11)$$

*respectively. Suppose that the following estimates hold for* $q = 2(n+1)/(n+3)$ :

$$||\Upsilon_{g,\lambda}f||_{L^2(TM)} \leq C\,\lambda^{1+\delta(q)}||f||_q, \qquad (3.2.12)$$

$$||\Upsilon_{\Delta,\lambda}f||_{L^2(M)} \leq C\,\lambda^{2+\delta(q)}||f||_q\ . \qquad (3.2.13)$$

*Then the estimates in* (3.1.11) *and* (3.1.12) *hold.*

This Lemma will be proved in the following subsection.

## 3.3 Oscillatory integrals

In this subsection we shall apply the Carleson-Sjölin method and oscillatory integral theorems of Carleson-Sjölin [1] and Stein [9] to proving Lemma 3.3, combining which with Lemma 3.2, we immediately obtain the proof of Theorem 1.1. We firstly take computations involving stationary phase in the following lemma to find the essential parts in kernels $\mathscr{K}_\lambda$, $\tilde{\mathscr{K}_\lambda}$ as $\lambda \to +\infty$, which determine the $(L^q,\ ,L^2)$ norms of $\Upsilon_{g,\lambda}$, $\Upsilon_{\Delta,\lambda}$.

**Lemma 3.4.** *Let* $(x,y)$ *be in the neighbourhood* $\mathscr{C}$ *of the diagonal in* $M\times M$ *and* $x \neq y$. *Then every component of* $\mathscr{K}_\lambda(x,y)$ *is essentially a* $C^\infty$ *function times*

$$\lambda^{(n+1)/2}e^{i\lambda s(x,y)}\,s(x,y)^{-(n-1)/2}$$

*and* $\tilde{\mathscr{K}_\lambda}(x,y)$ *is essentially a* $C^\infty$ *function times*

$$\lambda^{(n+3)/2}e^{i\lambda s(x,y)}\,s(x,y)^{-(n-1)/2}\ .$$

**Proof.** Firstly we prove the following

**Claim 1** The integral $\int_{\mathbf{R}^n} e^{iz\cdot\xi}\rho(\lambda-\xi)d\xi$ is essentially a $C^\infty$ function times

$$\lambda^{(n-1)/2}e^{i\lambda|z|}|z|^{-(n-1)/2}$$

for $0\neq z$ and $\lambda\to+\infty$.

We prove Claim 1 by stationary phase argument. Let $J_m(r)$, $m\geq 0$ an integer or a half-integral, be the Bessel function defined by

$$J_m(r)=\frac{r^m}{\pi^{1/2}2^m\Gamma(m+1/2)}\cdot\int_{-1}^{1}e^{irt}(1-t^2)^{m-1/2}\,dt\ ,$$

whose asympototic form as $r\to+\infty$ can be written by

$$\sqrt{r}e^{ir}\times\text{a smooth function }+\sqrt{r}e^{-ir}\times\text{a smooth function}\ .$$

Without loss of generality, we shall only consider the first term of above when using the Bessel function in what follows. By [9], the Fourier transform $\widehat{d\sigma}(\xi)$ of the Lebesgue measure on the unit sphere $\mathbf{S}^{n-1}\subset\mathbf{R}^n$ satisfies

$$\widehat{d\sigma}(\xi)=\int_{\mathbf{S}^{n-1}}e^{-2\pi i\theta\cdot\xi}\,d\sigma(\theta)=2\pi|\xi|^{(2-n)/2}J_{(n-1)/2}(2\pi|\xi|)\ .$$

By the above equalities, we have

$$\begin{aligned}\int_{\mathbf{R}^n}e^{iz\cdot\xi}\rho(\lambda-\xi)d\xi &= \text{const }\times\int_0^\infty\rho(\lambda-r)r^{n-1}\widehat{d\sigma}(\frac{r|z|}{2\pi})dr\\ &= |z|^{(1-n)/2}\int_0^\infty e^{ir|z|}r^{(n-1)/2}\rho(\lambda-r)dr\ \times\text{ a smooth function}\\ &= \lambda^{(n-1)/2}e^{i\lambda|z|}|z|^{-(n-1)/2}\ \times\text{ a smooth function}\end{aligned}$$

Since $\mathbf{grad}_x\, e^{i\Phi(x,y,\xi)}$ has the components

$$\big(\mathbf{grad}_x\,\Phi(x,y,\xi)\big)_j=\sum_{k=1}^n g^{jk}(x)(i\xi_k)e^{i\Phi(x,y,\xi)}\ ,$$

by the similar computation as above the proof for $\mathscr{K}_\lambda$ is completed. By (3.2.11) we can use the similar argument in the proof for $\tilde{\mathscr{K}_\lambda}$. q.e.d.

Proof of Lemma 3.3 By Lemma 3.4, to prove Lemma 3.3, we have only to show the operator $\Upsilon_\lambda: L^q(M)\to L^2(M)$ with kernel

$$\lambda^{(n-1)/2}e^{i\lambda\, s(x,y)}\,s(x,y)^{-(n-1)/2}\,U_0(x,y)$$

satisfies the estimate

$$||\Upsilon f||_{L^q(M)} \leq C\,\lambda^{\delta(q)}||f||_{L^2(M)}\ .$$

By using the compactness of $M$ and the local coordinates, it is clear from the dual argument that above estimate would be a consequence of the following estimate:

$$\left|\left|\int_{\mathbf{R}^n} \eta(x,y)\lambda^{(n-1)/2}\, e^{i\lambda s(x,y)}\, s(x,y)^{-(n-1)/2}\, f(y)dy\right|\right|_{L^p(\mathbf{R}^n)} \leq C\,\lambda^{\delta(p)}||f||_{L^2(\mathbf{R}^n)}\ , \tag{3.3.1}$$

where $p = 2(n+1)/(n-1)$ and $\eta \in C_0^\infty(\mathbf{R}^n \times \mathbf{R}^n)$ has support $\subset \{(x,y)|s(x,y) \leq c\}$. On the other hand, Sogge (cf (4.6) of [7]) proved (3.3.1) by using the Carleson-Sjölin method and oscillatory integral theorems of Carleson-Sjölin [1] and Stein [9]. q.e.d.

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