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On centralizers of parabolic subgroups in Coxeter groups

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Abstract

Let (W,S) be a Coxeter system (not necessarily of finite rank), and put $T = \bigcup_{w \in W} wSw^{-1}$. For any subset I of S, the centralizer $Z_W(W_I)$ of the parabolic subgroup W_I is decomposed as $\langle T \cap Z_W(W_I) \rangle \rtimes G_I$, where $\langle T \cap Z_W(W_I) \rangle$ is a Coxeter group, and G_I is a complementary subgroup whose structure is described by using a certain graph constructed from the information about finite parabolic subgroups only.

Key words: Coxeter group, parabolic subgroup, centralizer, root system, fundamental groupoid

1 Introduction

We analyze the structure of the centralizer of an arbitrary parabolic subgroup in an arbitrary Coxeter group (not necessarily of finite rank). Namely, let (W,S) be a Coxeter system. A parabolic subgroup of W is the subgroup

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generated by a subset I of S, and denoted by W_I. First we decompose the centralizer $Z_W(W_I)$ of W_I in W in the following manner:

$$Z_W(W_I) = W_{I^{\mathrm{iso}}} \times (W(\Phi_I^{\perp}) \rtimes G_I) \; (= (W_{I^{\mathrm{iso}}} \times W(\Phi_I^{\perp})) \rtimes G_I).$$

Here I^{iso} is the set of all elements of I isolated in the Coxeter graph of W_I, hence $W_{I^{\mathrm{iso}}}$ forms an elementary abelian 2-group. The remaining factors are defined in terms of the reflection representation V and the root system $\Phi = \Phi^+ \sqcup \Phi^- \subset V$ for W with simple system $\Pi = \{\alpha_s \mid s \in S\}$ (cf. Section 2). Let $\Phi_I^{\perp}$ be the set of all roots orthogonal to $\Pi_I = \{\alpha_s \mid s \in I\}$. Then $W(\Phi_I^{\perp})$ is the subgroup of W generated by all reflections with respect to the roots in $\Phi_I^{\perp}$, which is also a Coxeter group due to a result by Deodhar [7] or Dyer [8]. If Φ is well understood, the Coxeter generators of $W(\Phi_I^{\perp})$ can be determined using a result by Deodhar [7]. In a subsequent paper [14], we will give a new method to give the Coxeter generators of $W(\Phi_I^{\perp})$ concretely, even if Φ is difficult to describe. Note that the factor $\langle T \cap Z_W(W_I) \rangle$ in Abstract coincides with $W_{I^{\mathrm{iso}}} \times W(\Phi_I^{\perp})$.

In this paper, we focus on the final factor G_I, consisting of all $w \in W$ such that (1) $w \cdot \alpha_s = \alpha_s$ for all $s \in I^{\mathrm{iso}}$, (2) $w \cdot \alpha_s = \pm\alpha_s$ for all $s \in I$ and (3) $w \cdot (\Phi_I^{\perp} \cap \Phi^+) = \Phi_I^{\perp} \cap \Phi^+$. We determine G_I using a graph $\mathcal{G}$, which we call the *transition graph* in this paper. The construction of $\mathcal{G}$ is based on the knowledge of Coxeter systems of finite type only. Our result shows that G_I is a quotient of the fundamental group of $\mathcal{G}$ (which is a free group), and gives a presentation of G_I in terms of paths in $\mathcal{G}$ and automorphisms of a subgraph $\mathcal{H}$ of $\mathcal{G}$. Furthermore, it is shown that G_I acts on $W(\Phi_I^{\perp})$ as automorphisms of the Coxeter graph of $W(\Phi_I^{\perp})$.

In some cases, the structure of $Z_W(W_I)$ has been known. If $I = S$, then it is well known that $Z_W(W_I)$, namely the center of W, is generated by the longest elements $w_0(J)$ of W_J where J runs over all connected components of S of finite type such that $w_0(J)$ acts on Π_J as multiplication by -1, and is an elementary abelian 2-group. If I consists of a single element s, Brink [2] showed that $Z_W(W_I)$ $(= Z_W(s))$ is the semidirect product of a certain Coxeter group by a free group. Our results generalize these cases. Recently, we learned that Bahls and Mihalik [1] described $Z_W(W_I)$ if (W, S) is even (that is, every product ss' of two generators has even or infinite order) by a different approach.

There are some results on related topics; the normalizers and commensurators of parabolic subgroups in Coxeter groups have been examined by Brink and Howlett [3] and Paris [16] respectively. Further, the centralizers of parabolic subgroups in Artin groups of certain types have also been described by Paris [15].

This paper is organized as follows. In Section 2, we recall some terminology and basic properties of Coxeter groups and groupoids, which we use in the analysis of G_I, and show some lemmas used in the following sections. In Section 3, we show the decomposition of $Z_W(W_I)$ as described above, together with some remarks on the first two factors. Note that $W_{I^{\mathrm{iso}}}$, $W(\Phi_I^{\perp})$ and G_I are denoted by $W_{[x]^{\mathrm{iso}}}$, $W(\Phi_{[x]}^{\perp})$ and $G_{x,x}$ respectively in the text, by taking x as in the previous paragraph. In Section 4, we define the transition graph $\mathcal{G}$ and its subgraph $\mathcal{H}$. In Section 5, we examine the group G_I. To do this, we define a groupoid anti-homomorphism g from the fundamental groupoid of $\mathcal{G}$ to a certain subgroupoid G' of G having the same vertex groups as G. Then we show that g is surjective, and give a generating set of the kernel of g as a normal subgroupoid, in terms of paths of $\mathcal{G}$ and automorphisms of $\mathcal{H}$. This yields a presentation of G_I. Finally, Section 6 deals with an example in full; we compute $Z_W(W_I)$ for an affine Coxeter group using the results of previous sections. Moreover, we also recover the preceding results on special cases mentioned above.

2 Background material

A group W is called a *Coxeter group* (or a pair (W,S) is called a *Coxeter system*) if W is presented as

$$W = \langle S \mid (ss')^{m_{s,s'}} = 1 \ (s,s' \in S, m_{s,s'} < \infty) \rangle$$

for certain $(m_{s,s'})_{s,s' \in S}$ such that $m_{s,s} = 1$ for all $s \in S$ and $m_{s,s'} = m_{s',s} \in \{2,3,\dots\} \sqcup \{\infty\}$ for all $s,s' \in S$, $s \neq s'$ (cf. [11]). The *Coxeter graph Γ* of (W,S) is the simple, undirected graph on S which has an edge between s and s' labeled $m_{s,s'}$ if and only if $m_{s,s'} \geqslant 3$ (these labels are usually omitted for the case $m_{s,s'} = 3$). For $I \subset S$, a "connected component of I" means the vertex set of a connected component of Γ_I, where Γ_I is the restriction of Γ to I.

Let V be a real vector space with a symmetric bilinear form $\langle\,,\,\rangle$, and let $\Pi = \{\alpha_s \mid s \in S\} \subset V$. Then, as in [10], Π is called a *root basis* if

$$\langle \alpha_s, \alpha_{s'} \rangle = -\cos(\pi/m_{s,s'}) \text{ if } m_{s,s'} < \infty,\ \langle \alpha_s, \alpha_{s'} \rangle \leqslant -1 \text{ if } m_{s,s'} = \infty$$

and $0 \in V$ cannot be written as a nontrivial positive linear combination of elements of Π (note that S is assumed in [10] to be a finite set, but the following properties also hold for the case $|S| = \infty$). For any root basis Π, an action of W on V is well defined by $s \cdot v = v - 2\langle \alpha_s, v \rangle \alpha_s$, $s \in S$, $v \in V$. Note that this action preserves the bilinear form. Let $\Phi = W \cdot \Pi$. Then we have $\Phi = \Phi^+ \cup \Phi^-$, $\Phi^+ \cap \Phi^- = \emptyset$, where Φ^+ is the set of all $\gamma \in \Phi$ which is written as a positive linear combination of elements of Π and $\Phi^- = -\Phi^+$. Φ is called a *root system* of (W, S), and every element of Φ^+, Φ^- is called a *positive, negative root* respectively. Any element of Π is called a *simple root*. For $\Psi \subset \Phi$, define $\Psi^\pm = \Psi \cap \Phi^\pm$. For a root γ, we write $\gamma > 0$, $\gamma < 0$ instead of $\gamma \in \Phi^+$, $\gamma \in \Phi^-$ respectively. Further, for $w \in W$, let $\Phi_w^+ = \{\gamma \in \Phi^+ \mid w \cdot \gamma < 0\}$. Then it is shown in [10] that $|\Phi_w^+| = \ell(w)$, where $\ell(w)$, the *length* of w, is the minimal number k such that $w = s_1 s_2 \cdots s_k$ for some $s_i \in S$. This implies the following:

Lemma 2.1. **(i)** *Let $w_1, w_2 \in W$. Then*

$$\ell(w_1 w_2) = \ell(w_1) + \ell(w_2) - 2\left|\Phi_{w_1}^+ \cap \Phi_{w_2^{-1}}^+\right|.$$

Hence $\ell(w_1 w_2) = \ell(w_1) + \ell(w_2)$ if and only if $\Phi_{w_1}^+ \cap \Phi_{w_2^{-1}}^+ = \emptyset$.
(ii) *If $\Phi_{w_1}^+ = \Phi_{w_2}^+$, then $w_1 = w_2$.*

Proof. For $\varepsilon_1, \varepsilon_2, \varepsilon_3 \in \{+, -\}$, let $\Phi_{\varepsilon_1 \varepsilon_2 \varepsilon_3}$ be the set of all $\gamma \in \Phi^{\varepsilon_3}$ such that $w_2 \cdot \gamma \in \Phi^{\varepsilon_2}$, $w_1 w_2 \cdot \gamma \in \Phi^{\varepsilon_1}$. Then we have

$$\Phi_{w_1 w_2}^+ = \Phi_{-++} \sqcup \Phi_{--+},\ \Phi_{w_1}^+ = w_2 \cdot (\Phi_{-++} \sqcup \Phi_{-+-}),\ \Phi_{w_2}^+ = \Phi_{+-+} \sqcup \Phi_{--+},$$

so $\ell(w_1) + \ell(w_2) - \ell(w_1 w_2) = |\Phi_{-+-}| + |\Phi_{+-+}|$. Further, we have

$$\Phi_{-+-} = w_2^{-1} \cdot (\Phi_{w_1}^+ \cap \Phi_{w_2^{-1}}^+),\ \Phi_{+-+} = -\Phi_{-+-},$$

so we have $\ell(w_1) + \ell(w_2) - \ell(w_1 w_2) = 2\left|\Phi_{w_1}^+ \cap \Phi_{w_2^{-1}}^+\right|$, as required. Hence (i) holds. For (ii), we have $\ell(w_1 {w_2}^{-1}) = 0$ by (i), so $w_1 {w_2}^{-1} = 1$. $\square$

As shown in [11], for any Coxeter system (W, S), there are a reflection representation space V and a root basis Π such that Π is a basis of V (as a vector space) and $\langle \alpha_s, \alpha_{s'} \rangle = -1$ whenever $m_{s,s'} = \infty$. In this case, the root system Φ is called *standard*. From now on, Φ is assumed to be standard unless otherwise specified.

For $v = \sum_{s \in S} c_s \alpha_s \in V$, write $[\alpha_s]\, v = c_s$ and let $\operatorname{supp} v$ be the set of all $s \in S$ such that $c_s \neq 0$.

For any $\gamma = w \cdot \alpha_s \in \Phi$, the *reflection* $s_\gamma \in W$ about γ is defined as $s_\gamma = wsw^{-1}$ and acts on V by $s_\gamma \cdot v = v - 2\langle v, \gamma \rangle \gamma$. Then:

Proposition 2.2 (cf. [11]). (i) *Let $w \in W$, $\gamma \in \Phi^+$. Then $\ell(ws_\gamma) > \ell(w)$ if $w \cdot \gamma > 0$, and $\ell(ws_\gamma) < \ell(w)$ otherwise.*
(ii) *For $\gamma, \gamma' \in \Phi$ and $w \in W$, $ws_\gamma w^{-1} = s_{\gamma'}$ if and only if $w \cdot \gamma = \pm\gamma'$.*

Further, there are two useful theorems about reflections: the former is Theorem 5.4 of [6], and the latter is a special case of Theorem 1.20 of [18].

Theorem 2.3. *If $w \in W$, $w^2 = 1$, then w can be written as a product of reflections about pairwise orthogonal positive roots (so these reflections commute with each other). Hence $w \cdot \gamma = -\gamma$ for some $\gamma \in \Phi$ whenever $w^2 = 1$, $w \neq 1$.*

Theorem 2.4. *Suppose that $|W| < \infty$, $w \in W$ and w fixes some roots $\gamma_1, \ldots, \gamma_k$ of W. Then w can be written as a product of reflections which also fix all γ_i.*

For $\Psi \subset \Phi$, let $W(\Psi)$ be the subgroup of W generated by all s_γ, $\gamma \in \Psi$ (such subgroup is called a *reflection subgroup*) and let $\overline{\Psi} = W(\Psi) \cdot \Psi$. Then:

Theorem 2.5 ([7]). *$W(\Psi)$ is a Coxeter group with (not necessarily standard) root system $\overline{\Psi}$ and length function $\tilde{\ell}$ such that $\tilde{\ell}(w) = \left|\Phi_w^+ \cap \overline{\Psi}\right|$.*

Further, let $W^\Psi = \{w \in W \mid \Phi_w^+ \cap \overline{\Psi} = \emptyset\}$. Then Theorem 4.1 of [12], stated only for Weyl groups, is improved for arbitrary Coxeter groups by similar proof:

Theorem 2.6. *Each $w \in W$ is written uniquely as $w = w^{\Psi} w_{\Psi}$, $w^{\Psi} \in W^{\Psi}$, $w_{\Psi} \in W(\Psi)$. Further, w^{Ψ} is the unique element of minimal length in $wW(\Psi)$.*

In particular, let $\Pi_I = \{\alpha_s \mid s \in I\}$ for each $I \subset S$. Then $W(\Pi_I)$, W^{Π_I} are denoted by W_I, W^I respectively. W_I is called a *parabolic subgroup*, and (W_I, I) forms a Coxeter system with root system $\Phi_I = W_I \cdot \Pi_I$ (this is standard whenever Φ is), root basis Π_I and length function $\ell\,|_{W_I}$. Note that W is the direct product of all W_I where I runs over all connected components of S; so (W, S) is called *irreducible* if S is connected. Now we give a simple proof of the following well-known facts:

Proposition 2.7. **(i)** $\Phi_I = \{\gamma \in \Phi \mid \operatorname{supp}\gamma \subset I\}$ *for all* $I \subset S$.
(ii) $\operatorname{supp}\gamma$ *is connected for all* $\gamma \in \Phi$.

Proof. Let $\gamma \in \Phi$ such that $\operatorname{supp}\gamma \subset I$. Put $w = (s_\gamma)^I$. Then $w \cdot \alpha_s > 0$ for all $s \in I$, while $w \cdot \alpha_s = \alpha_s - 2\langle \gamma, \alpha_s \rangle \gamma > 0$ for all $s \in S \setminus I$ since $s \notin \operatorname{supp}\gamma$. These yield $\Phi_w^+ = \emptyset$ and so $w = 1$, therefore $s_\gamma \in W_I$. Then $s_\gamma \cdot \gamma' = -\gamma'$ for some $\gamma' \in \Phi_I$ by Theorem 2.3. This implies $\gamma = \pm\gamma'$ and so $\gamma \in \Phi_I$. The converse is obvious, so (i) holds. Further, (i) implies $\gamma \in \Phi_{\operatorname{supp}\gamma}$, so (ii) follows. $\square$

Note that $\Phi_{w^I}^+ \cap \Phi_{w_I^{-1}}^+ = \emptyset$ for any $w \in W$, so $\ell(w) = \ell(w^I) + \ell(w_I)$ by Lemma 2.1 (i). Further, we have the following:

Proposition 2.8. $\Phi_{w_I}^+ = \Phi_w^+ \cap \Phi_I^+$ *for* $w \in W$, $I \subset S$.

Proof. $\Phi_{w_I}^+ \subset \Phi_I^+$ by Proposition 2.7 (i). Further, note that $w^I \cdot \gamma$ is positive, negative whenever $\gamma \in \Phi_I^+$, $\gamma \in \Phi_I^-$ respectively. Then for $\gamma \in \Phi_I^+$, $w^I w_I \cdot \gamma < 0$ if and only if $w_I \cdot \gamma < 0$. Hence our claim holds. $\square$

For $I \subset S$, we say that I is of *finite type* if $|W_I| < \infty$, or equivalently $|\Phi_I| < \infty$. For such I, let $w_0(I)$ denote the *longest element* of W_I; this satisfies $\Phi_{w_0(I)}^+ = \Phi_I^+$. Moreover, it is known (cf. [18]) that $w_0(I) \cdot \Pi_I = -\Pi_I$; so we can define a permutation $\sigma_I : I \to I$ by $w_0(I) \cdot \alpha_s = -\alpha_{\sigma_I(s)}$ for any $s \in I$. Since $w_0(I)$ is involutive and preserves the bilinear form, this σ_I is an involutive graph automorphism of Γ_I. Note that for $I \subset S$ of finite type, $w_0(I) = w_0(I_1) \cdots w_0(I_k)$ and $\sigma_{I_i} = \sigma_I|_{I_i}$ for all i, where $I_1, \dots, I_k$ are the connected components of I. Now a certain numbering on S and the action of σ_S for each finite irreducible Coxeter system (cf. [11] for the classification) are listed in Fig. 1; each $\sigma_S(s_i)$ is denoted there by $\overline{s_i}$, and $\overline{s_i}$ is omitted if $\overline{s_i} = s_i$.

Now we have the following proposition, which is a slight improvement of a result proved by Deodhar in the proof of [6], Proposition 4.2:

Proposition 2.9. *Let $I \subsetneq J \subset S$ and suppose that J is connected and not of finite type. Then $|\Phi_J \smallsetminus \Phi_I| = \infty$. In addition, if $|J| < \infty$, then there exist infinitely many $\gamma \in \Phi^+$ such that* $\operatorname{supp}\gamma = J$.

Proof. First, suppose $|J| < \infty$, so the power set of J is also finite. Since $|\Phi_J| = \infty$, there is some $J' \subset J$ such that $|\Psi_{J'}| = \infty$, where $\Psi_{J'} = \{\gamma \in \Phi \mid \operatorname{supp}\gamma = J'\}$. Take such J' as large as possible with respect to inclusion, and assume $J' \neq J$. Then there is some $s \in J \smallsetminus J'$ adjacent to an element of J' since J is connected. Now the action of s maps $\Psi_{J'}$ injectively into $\Psi_{J' \cup \{s\}}$, while $\left|\Psi_{J' \cup \{s\}}\right| < \infty$ by the choice of J'. But this is contradiction, so we have

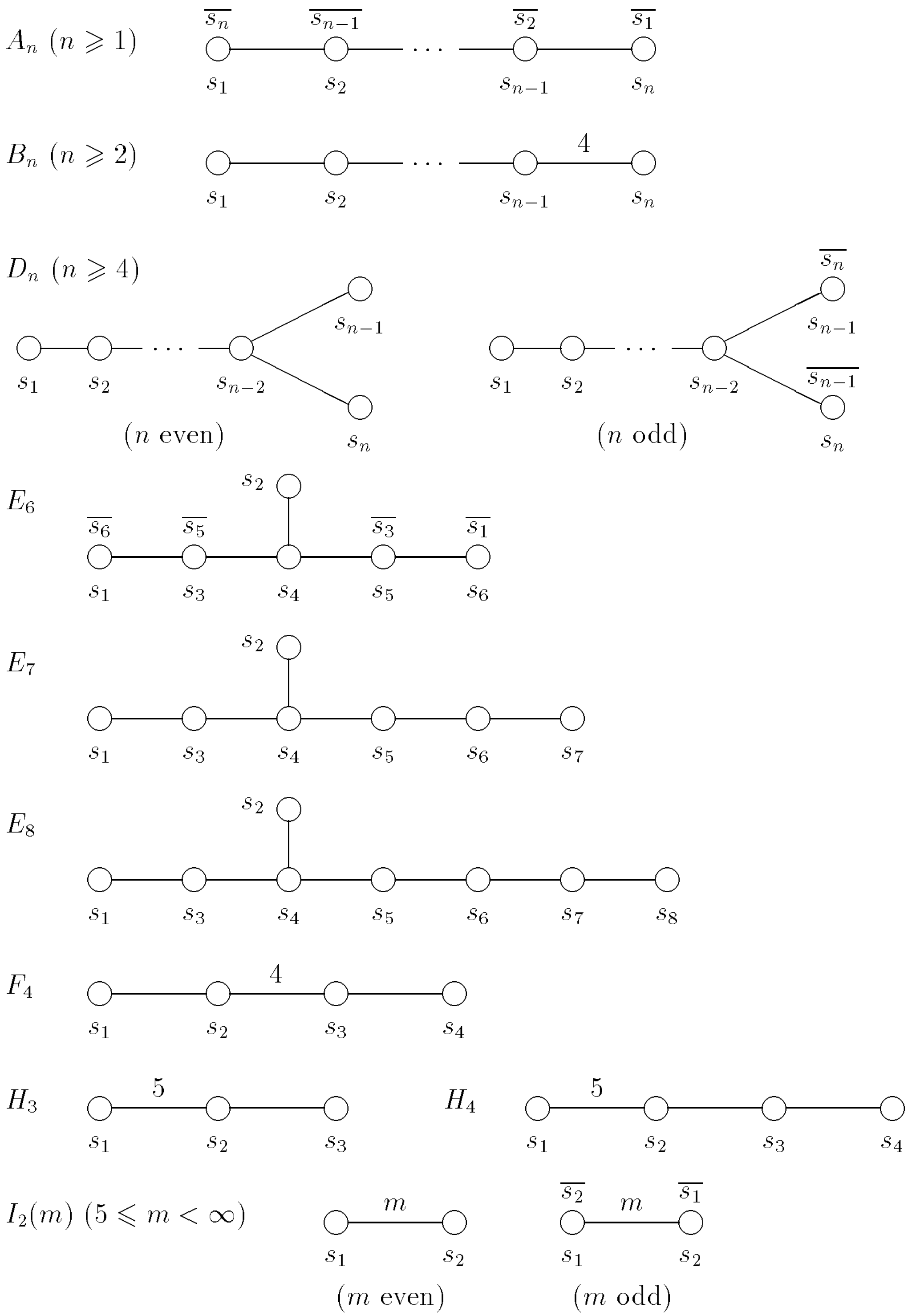


Fig. 1. Actions of longest elements

$J' = J$, as required.

On the other hand, suppose $|J| = \infty$ and let $s \in J \setminus I$. Take a maximal tree on Γ_J. For $t \in J$, let $s = s_0, s_1, \ldots, s_k = t$ be the unique reduced path in this maximal tree from s to t, and let $\gamma_t = s_k s_{k-1} \cdots s_1 \cdot \alpha_s$. Then we have $\operatorname{supp} \gamma_t = \{s, s_1, \ldots, s_k\}$ and so all γ_t are distinct and satisfy $\gamma_t \in \Phi_J \setminus \Phi_I$. Hence we have $|\Phi_J \setminus \Phi_I| = \infty$. $\square$

In the rest of this section, we recall the terminology on groupoids (cf. [4], [9]). A *groupoid* is a small category such that every morphism is invertible; in other words, a family $G = \{G_{x,y}\}_{x,y \in V(G)}$ of sets with index set $V(G) \times V(G)$ for which a partial multiplication is defined and satisfies associativity, existence of identities and inverses. We often identify such G with the (disjoint) union of all $G_{x,y}$, $x, y \in V(G)$. The partial multiplication $w_1 w_2 \in G_{x,z}$ is defined for $w_1 \in G_{x,y}$, $w_2 \in G_{y,z}$. The identity exists uniquely in each $G_{x,x}$. For $w \in G_{x,y}$, its inverse w^{-1} is unique and belongs to $G_{y,x}$.

A *subgroupoid* H of G is a subcategory of G which forms a groupoid. Such H is called *full* if $H_{i,j} = G_{i,j}$ for all $i, j \in V(H)$, called *wide* if $V(H) = V(G)$, and called *normal* if H is wide and $gxg^{-1} \in H_{j,j}$ for all $x \in H_{i,i}$, $g \in G_{j,i}$. Further, any full subgroupoid $H \subset G$ is called the *restriction* of G to $V(H)$. Then for a groupoid G, the restriction of G to V', where V' is one of the maximal subsets of $V(G)$ such that $G_{x,y} \neq \emptyset$ for all $x, y \in V'$, is called a *connected component* of G. G is called *connected* if G consists of only one connected component.

The *intersection* of subgroupoids H_λ, $\lambda \in \Lambda$ of G is defined naturally, and also forms a subgroupoid of G. This becomes normal in G whenever all H_λ are. Further, for any subset $X \subset G$, the *subgroupoid of G generated by X* is the intersection of all subgroupoids of G containing X (or equivalently, the smallest subgroupoid of G containing X). The *normal subgroupoid of G generated by X* is defined similarly.

A *groupoid homomorphism* $G \to G'$ is a covariant functor of groupoids G, G' considered as categories, while a *groupoid anti-homomorphism* is similar but it is contravariant instead of covariant. Note that for a groupoid homomorphism $f : G \to G'$, its image $f(G)$ may *not* be a subgroupoid of G', but it forms a subgroupoid whenever f maps $V(G)$ injectively to $V(G')$. On the other hand, its kernel $\ker f$, the inverse image of identities of G', always forms a normal subgroupoid of G.

For a groupoid G and its normal subgroupoid N, the *quotient groupoid* G/N is defined as follows. Let $V(G/N)$ be the set of all equivalence classes $[x]$ in $V(G)$, where x is equivalent to y if and only if $N_{x,y} \neq \emptyset$. Further, let $[w]$ be the equivalence class of $w \in G$ in G, where w is equivalent to w' if and only if $w = uw'v$ for some $u, v \in N$. Then define

$$(G/N)_{[x],[y]} = \{[w] \mid w \in G_{x',y'} \text{ for some } x' \in [x]\,, y' \in [y]\}.$$

Now for $[w_1] \in (G/N)_{[x],[y]}$ and $[w_2] \in (G/N)_{[y],[z]}$, define $[w_1]\,[w_2] = [w_1{}'w_2{}']$ where $w_1{}' \in [w_1]$, $w_2{}' \in [w_2]$ and $w_1{}'w_2{}' \in G$ is defined. Then this multiplication makes G/N a groupoid.

The groupoid version of "The First Isomorphism Theorem" is as follows:

Theorem 2.10 (cf. [4], [9]). *Let $f : G \to G'$ be a groupoid homomorphism such that f is injective on $V(G)$. Then f induces an isomorphism $\overline{f} : G/\ker f \to f(G)$.*

One of the important examples of groupoids is the *fundamental groupoid* of a graph, which is used in the following sections. Let $\mathcal{G}$ be any undirected graph with vertex set $V(\mathcal{G})$. We define an equivalence relation $\sim$ on directed paths of $\mathcal{G}$; $\sim$ is generated by the relation

$$e_1 \cdots e_{k-1} e_k e_k^{-1} e_{k+1} \cdots e_n \sim e_1 \cdots e_{k-1} e_{k+1} \cdots e_n,$$

where each e_i is a directed edge (that is, a directed path of length one) and e_k^{-1} is the opposite edge of e_k. This relation $\sim$ is called the *homotopy equivalence*. Then the concatenation of paths induces a partial multiplication of homotopy classes, and $\overline{\mathcal{P}}(\mathcal{G}) = \{\overline{\mathcal{P}}(\mathcal{G})_{x,y}\}_{x,y \in V(\mathcal{G})}$ forms a groupoid with this multiplication, where $\overline{\mathcal{P}}(\mathcal{G})_{x,y}$ is the set of all homotopy classes of paths from x to y. $\overline{\mathcal{P}}(\mathcal{G})$ is called the fundamental groupoid of $\mathcal{G}$.

3 Decomposition into three factors

In this section, we show that the centralizer of W_I admits a decomposition $W_{[x]^{\mathrm{iso}}} \times (W(\Phi_{[x]}^{\perp}) \rtimes G_{x,x})$ as described in Introduction. This is done in two steps; Corollary 3.2 and Theorem 3.5. Further, we examine the first two factors $W_{[x]^{\mathrm{iso}}}$ and $W(\Phi_{[x]}^{\perp})$. The remaining factor $G_{x,x}$ is described in Section 5.

We prepare some notations. In order to deal with the case $|I| = \infty$ as well, we fix a family Ω of index sets such that for each $I \subset S$, there is a unique $\Lambda \in \Omega$ having the same cardinality with I. Then let $S^{(\Lambda)}$ be the set of all injective maps ("Λ-tuples" with no repetitions) $x : \Lambda \to S$ for each $\Lambda \in \Omega$, and let $S^{(*)}$ be the union of all $S^{(\Lambda)}$. For $x \in S^{(\Lambda)}$, write $x_\lambda = x(\lambda)$ for any $\lambda \in \Lambda$, and let $\Lambda(x) = \Lambda$, $\ell(x) = |\Lambda|$ and $[x] = \{x_\lambda \mid \lambda \in \Lambda(x)\}$. Note that $\Lambda(x) = \Lambda(y)$ whenever $\ell(x) = \ell(y)$.

If $\ell(x) < \infty$, then we take $\Lambda(x) = \{1, 2, \dots, \ell(x)\}$. So we write $x = (x_1, x_2, \dots, x_{\ell(x)})$, and $[x] = \{x_1, x_2, \dots, x_{\ell(x)}\}$ in this case.

For $I \subset S$, let I^{iso} be the set of all $s \in I$ which commutes with every $s' \in I$, or equivalently, the set of all isolated vertices of Γ_I. Let $x, y \in S^{(*)}$. Then define

$$C_{x,y} = \begin{cases} \{w \in W \mid \alpha_{x_\lambda} = \pm w \cdot \alpha_{y_\lambda} \text{ for all } \lambda \in \Lambda(x)\} & \text{if } \ell(x) = \ell(y), \\ \emptyset & \text{otherwise.} \end{cases}$$

Note that the condition $\alpha_{x_\lambda} = \pm w \cdot \alpha_{y_\lambda}$ is equivalent to $x_\lambda = w y_\lambda w^{-1}$, so the centralizer of each W_I occurs as $C_{x,x}$ by taking $x \in S^{(*)}$ such that $[x] = I$. Further, define $C'_{x,y} = C_{x,y} \cap W^{[y]^{\mathrm{iso}}}$ and $C''_{x,y} = C_{x,y} \cap W^{[y]}$; that is,

$$C'_{x,y} = \{w \in C_{x,y} \mid \alpha_{x_\lambda} = w \cdot \alpha_{y_\lambda} \text{ if } y_\lambda \in [y]^{\mathrm{iso}}\},$$
$$C''_{x,y} = \{w \in W \mid \alpha_{x_\lambda} = w \cdot \alpha_{y_\lambda} \text{ for all } \lambda \in \Lambda(x)\}$$

whenever $\ell(x) = \ell(y)$.

Proposition 3.1. *$C = \{C_{x,y}\}_{x,y}$, $C' = \{C'_{x,y}\}_{x,y}$ and $C'' = \{C''_{x,y}\}_{x,y}$ are groupoids on $S^{(*)}$.*

Proof. By definition, the claim is obvious for C and C''.

Let $w_1 \in C'_{x,y}$, $w_2 \in C'_{y,z}$ and $z_\lambda \in [z]^{\mathrm{iso}}$. Since w_2 preserves the bilinear form, we have $y_\lambda \in [y]^{\mathrm{iso}}$ and so $\alpha_{x_\lambda} = w_1 w_2 \cdot \alpha_{z_\lambda}$. This implies $w_1 w_2 \in C'_{x,z}$. Similarly, we have ${w_1}^{-1} \in C'_{y,x}$; hence C' is also a groupoid. $\square$

Corollary 3.2. *$C_{x,x} = W_{[x]^{\mathrm{iso}}} \times C'_{x,x}$ for every $x \in S^{(*)}$.*

Proof. Note that $C'_{x,x}$ forms a group by this proposition; then our claim is deduced by $W_{[x]^{\mathrm{iso}}} \subset Z(C_{x,x})$ and $C'_{x,x} = C_{x,x} \cap W^{[x]^{\mathrm{iso}}}$. $\square$

Note that $W_{[x]^{\mathrm{iso}}}$ is an elementary abelian 2-group generated by $[x]^{\mathrm{iso}}$.

For $I \subset S$, let $\Phi_I^\perp$ denote the set of all roots $\gamma \in \Phi$ which are orthogonal to every $\alpha_s \in \Pi_I$. Then the following lemma follows immediately from definition of $C_{x,y}$:

Lemma 3.3. $\Phi^{\perp}_{[x]} = w \cdot \Phi^{\perp}_{[y]}$ *for any* $x, y \in S^{(*)}$, $w \in C_{x,y}$.

Note that for any $x \in S^{(*)}$, the reflection subgroup $W(\Phi^{\perp}_{[x]})$ (cf. Section 2) is a subgroup of $C''_{x,x}$, so $W(\Phi^{\perp}_{[x]}) \cdot \Phi^{\perp}_{[x]} = \Phi^{\perp}_{[x]}$ by this lemma. Now define $G_{x,y} = C'_{x,y} \cap W^{\Phi^{\perp}_{[y]}}$ for $x, y \in S^{(*)}$; that is,

$$G_{x,y} = \{w \in C'_{x,y} \mid \Phi^{+}_{w} \cap \Phi^{\perp}_{[y]} = \emptyset\}.$$

Lemma 3.4. $G_{x,y} = \{w \in C'_{x,y} \mid (\Phi^{\perp}_{[x]})^{+} = w \cdot (\Phi^{\perp}_{[y]})^{+}\}$ *for any* $x, y \in S^{(*)}$. *Hence* $G = \{G_{x,y}\}_{x,y}$ *is a wide subgroupoid of* $C'_{x,x}$.

Proof. Let $w \in G_{x,y}$. Then $\Phi^{\perp}_{[x]} = w \cdot \Phi^{\perp}_{[y]}$ by Lemma 3.3, so $w \cdot (\Phi^{\perp}_{[y]})^{+} \subset (\Phi^{\perp}_{[x]})^{+}$ by definition. This implies $w \cdot (\Phi^{\perp}_{[y]})^{-} \subset (\Phi^{\perp}_{[x]})^{-}$. Hence we have $(\Phi^{\perp}_{[x]})^{\pm} = w \cdot (\Phi^{\perp}_{[y]})^{\pm}$ respectively. The converse is clear. $\square$

Theorem 3.5. $C'_{x,x} = W(\Phi^{\perp}_{[x]}) \rtimes G_{x,x}$ *for every* $x \in S^{(*)}$.

Proof. By Lemma 3.3, we have $w s_{\gamma} w^{-1} = s_{w \cdot \gamma} \in W(\Phi^{\perp}_{[x]})$ for all $w \in C'_{x,x}$, $\gamma \in \Phi^{\perp}_{[x]}$; thus $W(\Phi^{\perp}_{[x]})$ is normal in $C'_{x,x}$, while $G_{x,x}$ is a subgroup of $C'_{x,x}$ by Lemma 3.4. Further, Theorem 2.6 implies that each $w \in C'_{x,x}$ is written uniquely as $w = w'w''$, $w' \in G_{x,x}$, $w'' \in W(\Phi^{\perp}_{[x]})$. Hence our claim holds. $\square$

In the rest of this section, we examine the factor $W(\Phi^{\perp}_{[x]})$ and the action of $G_{x,x}$ on $W(\Phi^{\perp}_{[x]})$. By Theorem 2.5, $W(\Phi^{\perp}_{[x]})$ is a Coxeter group with root system $\Phi^{\perp}_{[x]}$ (note that $W(\Phi^{\perp}_{[x]}) \cdot \Phi^{\perp}_{[x]} = \Phi^{\perp}_{[x]}$). Moreover, its Coxeter generator is determined by the result of [7]. Namely, let $\widetilde{\Pi}_x$ denote the set of all $\gamma \in (\Phi^{\perp}_{[x]})^{+}$

which cannot be written as a positive linear combination of other elements of $(\Phi^{\perp}_{[x]})^{+}$, and let $\widetilde{S}_x = \{s_\gamma \mid \gamma \in \widetilde{\Pi}_x\}$. Then:

Proposition 3.6. *$(W(\Phi^{\perp}_{[x]}), \widetilde{S}_x)$ is a Coxeter system with (not necessarily standard) root system $\Phi^{\perp}_{[x]}$ and length function $\widetilde{\ell}_x$ such that $\widetilde{\ell}_x(w) = \left|\Phi^{+}_{w} \cap \Phi^{\perp}_{[x]}\right|$.*

According to [14], the set $\widetilde{\Pi}_x$ can be determined concretely (note that $\widetilde{\Pi}_x$ can become infinite even for the case $|S| < \infty$; cf. [14] for detail), even if the root system of original (W, S) is not well understood. Moreover, we have the following:

Proposition 3.7. *Each $w \in G_{x,x}$ acts on $W(\Phi^{\perp}_{[x]})$ as an automorphism of the Coxeter graph $\widetilde{\Gamma}_x$ of $(W(\Phi^{\perp}_{[x]}), \widetilde{S}_x)$. Moreover, this yields a group homomorphism $G_{x,x} \to \mathrm{Aut}\widetilde{\Gamma}_x$. In particular, the semidirect product $W(\Phi^{\perp}_{[x]}) \rtimes G_{x,x}$ becomes direct whenever $\mathrm{Aut}\widetilde{\Gamma}_x = 1$.*

Proof. First, we show $w \cdot \gamma \in \widetilde{\Pi}_x$ for all $w \in G_{x,x}$, $\gamma \in \widetilde{\Pi}_x$. We have $w \cdot \gamma \in (\Phi^{\perp}_{[x]})^{+}$ since $\gamma \in (\Phi^{\perp}_{[x]})^{+}$. Assume $w \cdot \gamma \notin \widetilde{\Pi}_x$. Then $w \cdot \gamma$ can be written as a positive linear combination of $\gamma' \in (\Phi^{\perp}_{[x]})^{+}$, $\gamma' \neq w \cdot \gamma$. This implies that γ is also written as a positive linear combination of $w^{-1} \cdot \gamma'$, and we have $w^{-1} \cdot \gamma' \in (\Phi^{\perp}_{[x]})^{+}$, $w^{-1} \cdot \gamma' \neq \gamma$. Thus $\gamma \notin \widetilde{\Pi}_x$, but this is contradiction. Hence $w \cdot \gamma \in \widetilde{\Pi}_x$.

Let $w \in G_{x,x}$. Then for any $\gamma \in \widetilde{\Pi}_x$, we have $w s_\gamma w^{-1} = s_{w \cdot \gamma}$ and $w \cdot \gamma \in \widetilde{\Pi}_x$ as above. This implies that w induces a permutation $\sigma_w : s_\gamma \mapsto w s_\gamma w^{-1}$ on $\widetilde{S}_x$. Further, $\sigma_w \sigma_{w'} = \sigma_{ww'}$ and $\sigma_1 = \mathrm{id}_{\widetilde{S}_x}$ by definition, while $\sigma_w(s_\gamma)\sigma_w(s_{\gamma'}) = w s_\gamma s_{\gamma'} w^{-1}$ and $s_\gamma s_{\gamma'}$ have the same order. Hence $w \mapsto \sigma_w$ is a group homomorphism from $G_{x,x}$ to $\mathrm{Aut}\widetilde{\Gamma}_x$. $\square$

4 Transition graph

In this section, we define an undirected graph $\mathcal{G}$ on $S^{(*)}$, which we call the *transition graph*. This graph is constructed from the information about actions of the longest elements of finite parabolic subgroups only. In later sections, $\mathcal{G}$ and its subgraph $\mathcal{H}$ are used for describing the structure of the centralizers.

In what follows, it is important that we work with *ordered tuples* $x \in S^{(*)}$. Similar arguments appeared in [3] or [16], but they dealt with subsets of S only, in which the order was not relevant.

For $I, J \subset S$, let $I_{\sim J}$ denote the union of all connected components of $I \cup J$ containing some $s \in J$. We write $I_{\sim s}$ as a shorthand for $I_{\sim \{s\}}$. Further, if $x \in S^{(*)}$, then we write $x_{\sim J}$ instead of $[x]_{\sim J}$. Now define

$$\mathcal{B} = \{(x, s) \mid x \in S^{(*)}, s \in S \smallsetminus [x], x_{\sim s} \text{ is of finite type}\}.$$

For $(x, s) \in \mathcal{B}$, put

$$w_x^s = w_0(x_{\sim s}) w_0(x_{\sim s} \smallsetminus \{s\}).$$

Then, since $w_0(I) \cdot \Pi_I = -\Pi_I$ for any $I \subset S$ of finite type, there is a unique $y \in S^{(*)}$ such that $\ell(y) = \ell(x)$ and $\alpha_{y_\lambda} = w_x^s \cdot \alpha_{x_\lambda}$ for all $\lambda \in \Lambda(x)$. Now define $\varphi(x, s) = (\varphi_v(x, s), \varphi_l(x, s))$, where

$$\varphi_v(x, s) = y, \ \ \varphi_l(x, s) = \sigma_{x_{\sim s}}(s)$$

(cf. Section 2 for definition of σ). Then:

Remark 4.1. $[x] \sqcup \{s\} = [\varphi_v(x, s)] \sqcup \{\varphi_l(x, s)\}$ for any $(x, s) \in \mathcal{B}$.

Lemma 4.2. *If $I \subset J \subset S$ and J is of finite type, then $(w_0(J) w_0(I))^{-1} = w_0(J) w_0(\sigma_J(I))$ and $\Phi^+_{w_0(J) w_0(I)} = \Phi^+_J \smallsetminus \Phi^+_I$.*

Proof. We show $\Phi^+_{w_0(J)w_0(I)} = \Phi^+_{w_0(\sigma_J(I))w_0(J)}$; then we have $w_0(J)w_0(I) = w_0(\sigma_J(I))w_0(J)$ and so $(w_0(J)w_0(I))^{-1} = w_0(J)w_0(\sigma_J(I))$ since the longest elements are involutive.

Obviously, we have $\Phi^+_{w_0(J)w_0(I)} \subset \Phi^+_J$, $\Phi^+_{w_0(\sigma_J(I))w_0(J)} \subset \Phi^+_J$. Let $\gamma \in \Phi^+_J$. Then $w_0(J)w_0(I)\cdot\gamma < 0$ if and only if $w_0(I)\cdot\gamma > 0$ (since $w_0(I)\cdot\gamma \in \Phi_J$), or equivalently $\gamma \notin \Phi^+_I$. Thus we have $\Phi^+_{w_0(J)w_0(I)} = \Phi^+_J \setminus \Phi^+_I$. Similarly, $w_0(\sigma_J(I))w_0(J)\cdot\gamma < 0$ if and only if $w_0(J)\cdot\gamma \notin \Phi^-_{\sigma_J(I)}$ (since $w_0(J)\cdot\gamma < 0$), or equivalently $\gamma \notin \Phi^+_I$; thus $\Phi^+_{w_0(\sigma_J(I))w_0(J)} = \Phi^+_J \setminus \Phi^+_I$. Hence we have $\Phi^+_{w_0(J)w_0(I)} = \Phi^+_{w_0(\sigma_J(I))w_0(J)}$, as required. □

Corollary 4.3. *φ is an involution on $\mathcal{B}$. Further, $w^{\varphi_l(x,s)}_{\varphi_v(x,s)} = (w^s_x)^{-1}$ for any $(x,s)\in\mathcal{B}$.*

Proof. For $(x,s)\in\mathcal{B}$, we have

$$x_{\sim s} = \sigma_{x_{\sim s}}(x_{\sim s}\setminus\{s\}) \sqcup \{\varphi_l(x,s)\},$$
$$[\varphi_v(x,s)] = \sigma_{x_{\sim s}}(x_{\sim s}\setminus\{s\}) \sqcup ([x]\setminus x_{\sim s})$$

by definition of φ. Then $\varphi_v(x,s)_{\sim\varphi_l(x,s)} = x_{\sim s}$ and so $\varphi(x,s)\in\mathcal{B}$. Thus φ maps $\mathcal{B}$ to itself. Further, $\varphi_l(\varphi_v(x,s),\varphi_l(x,s)) = s$ since $\sigma_{x_{\sim s}}$ is involutive. Finally, the previous lemma yields

$$\begin{aligned}(w^s_x)^{-1} &= w_0(x_{\sim s})w_0(\sigma_{x_{\sim s}}(x_{\sim s}\setminus\{s\}))\\ &= w_0(\varphi_v(x,s)_{\sim\varphi_l(x,s)})w_0(\varphi_v(x,s)_{\sim\varphi_l(x,s)}\setminus\{\varphi_l(x,s)\}) = w^{\varphi_l(x,s)}_{\varphi_v(x,s)},\end{aligned}$$

which implies $\varphi_v(\varphi_v(x,s),\varphi_l(x,s)) = x$ by definition. □

Let

$$\mathcal{B}^\varphi = \{(x,s)\in\mathcal{B} \mid \varphi(x,s) = (x,s)\}.$$

Then by Remark 4.1, $(x,s) \in \mathcal{B}^{\varphi}$ if and only if $(x,s) \in \mathcal{B}$ and $\varphi_v(x,s) = x$. Now let $\widetilde{\mathcal{H}}$ be the directed graph on $S^{(*)}$ having an edge e_x^s from x to $\varphi_v(x,s)$ with label s for each $(x,s) \in \mathcal{B} \setminus \mathcal{B}^{\varphi}$. Then the above corollary implies that for each edge e_x^s, the edge $e_{\varphi_v(x,s)}^{\varphi_l(x,s)}$, denoted by $(e_x^s)^{-1}$, exists and goes from $\varphi_v(x,s)$ to x. Note that $((e_x^s)^{-1})^{-1} = e_x^s$. Then let $\mathcal{H}$ be the undirected graph on $S^{(*)}$ obtained from $\widetilde{\mathcal{H}}$ by identifying each edge e_x^s with its inverse.

When we draw the picture of $\mathcal{H}$, an edge, obtained from e_x^s and its inverse, is represented as an edge with labels s close to the vertex x and $\varphi_l(x,s)$ close to $\varphi_v(x,s)$; moreover, for the case $s = \varphi_l(x,s)$, the repeated s's may be replaced by a single s. See Fig. 2 below for example.

For $x \in S^{(*)}$, define

$$\begin{aligned} \mathrm{CO}(x) &= \{A \subset \Lambda(x) \mid x_A \text{ is a union of connected components of } [x]\}, \\ \mathrm{CO}^{>1}_{<\infty}(x) &= \{A \in \mathrm{CO}(x) \mid x_A \text{ is of finite type, } {x_A}^{\mathrm{iso}} = \emptyset\}, \end{aligned}$$

where

$$x_A = \{x_\lambda \mid \lambda \in A\}.$$

These families form elementary abelian 2-groups with symmetric difference of sets as multiplication. This multiplication of A and A' is written as AA'. Further, for $A \in \mathrm{CO}^{>1}_{<\infty}(x)$, let x^A be the unique element of $S^{(*)}$ satisfying $\ell(x^A) = \ell(x)$, $(x^A)_\lambda = \sigma_{x_A}(x_\lambda)$ for all $\lambda \in A$ and $(x^A)_\lambda = x_\lambda$ for all $\lambda \in \Lambda(x) \setminus A$. Then we have $(x^A)_\lambda = w_0(x_A) x_\lambda w_0(x_A)$ for all $\lambda \in \Lambda(x)$ since x_A is a union of connected components of $[x]$.

Lemma 4.4. *Let $x \in S^{(*)}$ and $A, A' \in \mathrm{CO}^{>1}_{<\infty}(x)$. Then:*

(i) $\mathrm{CO}(x^A) = \mathrm{CO}(x)$ *and* $\mathrm{CO}^{>1}_{<\infty}(x^A) = \mathrm{CO}^{>1}_{<\infty}(x)$.

(ii) $(x^A)^{A'} = x^{AA'}$.

Proof. By the above remark, we have $m_{(x^A)_\lambda,(x^A)_\mu} = m_{x_\lambda,x_\mu}$ for all $\lambda, \mu \in \Lambda(x)$; hence (i) holds. Further, (ii) follows immediately from the fact that $AA' \in \mathrm{CO}^{>1}_{<\infty}(x)$ and $\sigma_{x_A}(x_\lambda) = \sigma_{x_{A'}}(x_\lambda)$ for any $\lambda \in A \cap A'$. $\square$

For any graph $\mathcal{G}'$ with vertex set $V(\mathcal{G}')$ and $x \in V(\mathcal{G}')$, let $\mathcal{G}'_{\sim x}$ denote the connected component of $\mathcal{G}'$ containing x. Then for $x \in S^{(*)}$, let

$$\mathcal{A}_x = \{A \in \mathrm{CO}^{>1}_{<\infty}(x) \mid x^A \in V(\mathcal{H}_{\sim x})\}.$$

Now let $\widetilde{\mathcal{G}}$ be the graph obtained from $\widetilde{\mathcal{H}}$ by adding the edge e_x^A, for each $x \in S^{(*)}$ and $A \in \mathcal{A}_x$, from x to x^A with label A. Since $\mathcal{A}_{x^A} = \mathcal{A}_x$ for any $A \in \mathcal{A}_x$ (as checked in Lemma 5.15 of the next section), the edge $e_{x^A}^A$, denoted by $(e_x^A)^{-1}$, exists and goes from x^A to x for any edge e_x^A. Note that $((e_x^A)^{-1})^{-1} = e_x^A$. Then let $\mathcal{G}$ be the undirected graph obtained from $\widetilde{\mathcal{G}}$ by identifying each edge with its inverse, so $\mathcal{G}$ contains $\mathcal{H}$ as a subgraph. Note that $V(\mathcal{G}_{\sim x}) = V(\mathcal{H}_{\sim x})$. Further, as showed in the next section, the structure of the factor $G_{x,x}$ is indeed deduced from only the connected component $\mathcal{G}_{\sim x}$; so we need to compute only the component $\mathcal{G}_{\sim x}$, not the whole of $\mathcal{G}$.

Example 4.5. Let (W, S) be a finite Coxeter system of type B_5 with numbering on S in Fig. 1, and let $x = (s_1, s_3, s_4)$. Then $\mathcal{G}_{\sim x}$ is as in Fig. 2, where $x' = (s_1, s_4, s_3)$, $y = (s_4, s_1, s_2)$ and $y' = (s_4, s_2, s_1)$. (For simplicity, every loop $e_z^{\emptyset}$ is omitted in this figure.)

For example, $(x, s_5) \in \mathcal{B}$ and $\varphi(x, s_5) = (x', s_5)$; in fact, $x_{\sim s_5} = \{s_3, s_4, s_5\}$ is of finite type, and we have

$$\sigma_{\{s_3,s_4,s_5\}}\sigma_{\{s_3,s_4\}}(s_3) = \sigma_{\{s_3,s_4,s_5\}}(s_4) = s_4,$$

$$\sigma_{\{s_3,s_4,s_5\}}\sigma_{\{s_3,s_4\}}(s_4) = \sigma_{\{s_3,s_4,s_5\}}(s_3) = s_3$$

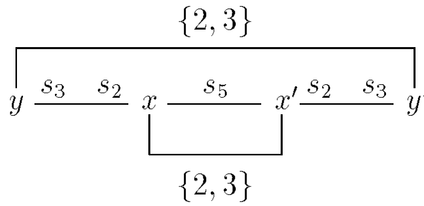


Fig. 2. Transition graph for Example 4.5

(cf. Fig. 1). On the other hand, we have $\mathrm{CO}^{\geq 1}_{<\infty}(y) = \{\emptyset, \{2,3\}\}$, and $y_{\{2,3\}} = \{s_1, s_2\}$, $\sigma_{\{s_1,s_2\}}(s_1) = s_2$, $\sigma_{\{s_1,s_2\}}(s_2) = s_1$. Then $y^{\{2,3\}} = y'$. Other edges are obtained similarly (note that $(y, s_5), (y', s_5) \in \mathcal{B}^{\varphi}$).

5 The factor $G_{x,x}$

In this section, the remaining factor $G_{x,x}$ of the decomposition is examined. We give a presentation of this group by using the graph $\mathcal{G}$ defined in Section 4.

The main results of this section, which we prove later, are as follows. First, we introduce a subgroupoid H of G; define

$$H = G \cap C'',$$

so H is a wide subgroupoid of G and C''. Then:

Proposition 5.1. *H is a normal subgroupoid of G.*

Further, let G' be the wide subgroupoid of G such that $G'_{x,y} = G_{x,y}$ if $H_{x,y} \neq \emptyset$, $G'_{x,y} = \emptyset$ otherwise. Then $H \subset G'$ and $G'_{x,x} = G_{x,x}$ for all $x \in S^{(*)}$, so we treat G' instead of G in this section.

For any undirected graph $\mathcal{G}'$, let $\mathcal{P}(\mathcal{G}')$ ($\mathcal{P}(\mathcal{G}')_{x,y}$) denote the set of all directed paths of $\mathcal{G}'$ (from x to y, respectively), and let $\overline{\mathcal{P}}(\mathcal{G}')$ denote the fundamental groupoid of $\mathcal{G}'$ (cf. Section 2). The homotopy class of $p \in \mathcal{P}(\mathcal{G}')$ is denoted by $[p]$. Secondly, we define groupoid anti-homomorphisms $g : \overline{\mathcal{P}}(\mathcal{G}) \to G'$ and $h : \overline{\mathcal{P}}(\mathcal{H}) \to H$ which is the restriction of g, and show that these are both surjective. Recall that $w_x^s = w_0(x_{\sim s})w_0(x_{\sim s} \setminus \{s\})$ for $(x, s) \in \mathcal{B}$ and let

$$w_x^A = w_0(x_A)$$

for each $A \in \mathrm{CO}_{<\infty}^{\geq 1}(x)$. Then:

Theorem 5.2. **(i)** $w_x^s \in H_{\varphi_v(x,s),x}$ *for any* $(x, s) \in \mathcal{B} \setminus \mathcal{B}^{\varphi}$.
(ii) $w_x^A \in G_{x^A,x}$ *for any* $A \in \mathrm{CO}_{<\infty}^{\geq 1}(x)$.
(iii) *There exists a unique groupoid anti-homomorphism h from $\overline{\mathcal{P}}(\mathcal{H})$ to H which sends each $[e_x^s]$ to w_x^s. Moreover, this map is identity on $S^{(*)}$ and surjective.*
(iv) *There exists a unique groupoid anti-homomorphism g from $\overline{\mathcal{P}}(\mathcal{G})$ to G' which is an extension of h and sends each $\left[e_x^A\right]$ to w_x^A. Moreover, this map is identity on $S^{(*)}$ and surjective.*

By this theorem, G', H are anti-isomorphic to the quotients $\overline{\mathcal{P}}(\mathcal{G})/\ker g$, $\overline{\mathcal{P}}(\mathcal{H})/\ker h$ respectively. In particular, let g_x, h_x be the restriction of g, h to $\overline{\mathcal{P}}(\mathcal{G}_{\sim x})$, $\overline{\mathcal{P}}(\mathcal{H}_{\sim x})$ respectively. Then $G_{x,x}$, $H_{x,x}$ are anti-isomorphic to $(\overline{\mathcal{P}}(\mathcal{G}_{\sim x})/\ker g_x)_{x,x}$, $(\overline{\mathcal{P}}(\mathcal{H}_{\sim x})/\ker h_x)_{x,x}$ respectively. So thirdly, we describe the structure of $\ker g_x$, $\ker h_x$. For $I \subset S$, let $\mathcal{H}^{(I)}$ be the 'restriction' of $\mathcal{H}$ to I; that is, the subgraph of $\mathcal{H}$ consisting of all $y \in S^{(*)}$ such that $[y] \subset I$ and all e_y^s such that $[y] \cup \{s\} \subset I$. Note that $(e_y^s)^{-1} \in \mathcal{H}^{(I)}$ if and only if $e_y^s \in \mathcal{H}^{(I)}$. Then:

Theorem 5.3. **(i)** $\ker h_x$ *is generated as a normal subgroupoid by all $[p]$, $p \in R_1(x)$, where $R_1(x)$ is the set of all nontrivial simple closed reduced paths*

$p = e_{y_1}^{s_1} \cdots e_{y_n}^{s_n} \in \mathcal{P}(\mathcal{H}_{\sim x})$ *contained in some* $\mathcal{H}^{(I)}$ *such that* $|I \setminus [y_1]| = 2$ *and* $I_{\sim\{s_1,s_2\}}$ *is of finite type.*

(ii) $\ker g_x$ *is generated as a normal subgroupoid by all* $[p]$, $p \in R_1(x) \cup R_2(x) \cup R_3(x)$, *where* $R_2(x)$ *consists of all paths* $e_y^A e_{y^A}^{A'} (e_y^{AA'})^{-1}$, $y \in V(\mathcal{G}_{\sim x})$, $A, A' \in \mathcal{A}_x$, *and* $R_3(x)$ *consists of all paths* $e_y^s e_{\varphi_v(y,s)}^A (e_{y^A}^s)^{-1} (e_y^A)^{-1}$, $y \in V(\mathcal{G}_{\sim x})$, $s \in S$, $A \in \mathcal{A}_x$.

Now certain presentations of $(\overline{\mathcal{P}}(\mathcal{G}_{\sim x})/\ker g_x)_{x,x}$ and $(\overline{\mathcal{P}}(\mathcal{H}_{\sim x})/\ker h_x)_{x,x}$, therefore of $G_{x,x}$ and $H_{x,x}$, are obtained from these results. Let T_x be a maximal tree in $H_{\sim x}$ (this is also a maximal tree in $\mathcal{G}_{\sim x}$). For each $y \in V(\mathcal{G}_{\sim x})$, let p_y be the unique reduced path in T_x from x to y. Moreover, let $E(\mathcal{G}_{\sim x})$, $E(\mathcal{H}_{\sim x})$ denote the set of all directed edges of $\mathcal{G}_{\sim x}$, $\mathcal{H}_{\sim x}$ respectively; so every element of $R_i(x)$, $i = 1, 2, 3$ can be regarded as a word on $E(\mathcal{G}_{\sim x})$. Then Theorem 5.17 of [5] yields the following:

Theorem 5.4. **(i)** $(\overline{\mathcal{P}}(\mathcal{H}_{\sim x})/\ker h_x)_{x,x}$ *has the following presentation:*

$$\left\langle E(\mathcal{H}_{\sim x}) \mid \{ee^{-1} \mid e \in E(\mathcal{H}_{\sim x})\} \cup \{e \mid e \in T_x\} \cup R_1(x) \right\rangle.$$

(ii) $(\overline{\mathcal{P}}(\mathcal{G}_{\sim x})/\ker g_x)_{x,x}$ *has the following presentation:*

$$\left\langle E(\mathcal{G}_{\sim x}) \mid \{ee^{-1} \mid e \in E(\mathcal{G}_{\sim x})\} \cup \{e \mid e \in T_x\} \cup R_1(x) \cup R_2(x) \cup R_3(x) \right\rangle.$$

Moreover, in each presentation, a generator $e \in E(\mathcal{G}_{\sim x})$ *corresponds to the coset containing* $[p_y e {p_z}^{-1}]$ *where* e *is an edge from* y *to* z.

Finally, in order to simplify the above presentation of $G_{x,x}$, a certain smaller generating set of $G_{x,x}$ and their multiplication are described in Corollary 5.19.

From now on, we start proving the above results.

Lemma 5.5. *If* $G_{x,y} \neq \emptyset$, *then* $\mathrm{CO}(x) = \mathrm{CO}(y)$ *and* $\mathrm{CO}_{<\infty}^{\geq 1}(x) = \mathrm{CO}_{<\infty}^{\geq 1}(y)$.

Proof. Take any $w \in G_{x,y}$. For $\lambda, \mu \in \Lambda(x)$, we have

$$\left\langle \alpha_{x_\lambda}, \alpha_{x_\mu} \right\rangle = \left\langle \pm w \cdot \alpha_{y_\lambda}, \pm w \cdot \alpha_{y_\mu} \right\rangle = \pm \left\langle \alpha_{y_\lambda}, \alpha_{y_\mu} \right\rangle,$$

and so $\left\langle \alpha_{x_\lambda}, \alpha_{x_\mu} \right\rangle = \left\langle \alpha_{y_\lambda}, \alpha_{y_\mu} \right\rangle$ since they have the same signature. Thus we have $m_{x_\lambda, x_\mu} = m_{y_\lambda, y_\mu}$ and so our claim holds. $\square$

Remark 5.6. By this proof, each $w \in C_{x,y}$ induces a graph isomorphism $y_\lambda \mapsto x_\lambda$ from $\Gamma_{[y]}$ to $\Gamma_{[x]}$.

Owing to this lemma, we define a groupoid $\mathrm{CO}_{<\infty}^{>1}$ on the set of equivalence classes in $S^{(*)}$ by

$$(\mathrm{CO}_{<\infty}^{>1})_{\overline{x},\overline{y}} = \mathrm{CO}_{<\infty}^{>1}(x) \text{ if } \overline{x} = \overline{y},\ (\mathrm{CO}_{<\infty}^{>1})_{\overline{x},\overline{y}} = \emptyset \text{ otherwise},$$

where x is defined to be equivalent to y if and only if $G_{x,y} \neq \emptyset$.

Proposition 5.7. *The map* $G \to \mathrm{CO}_{<\infty}^{>1}$, $G_{x,y} \ni w \mapsto A_w \in (\mathrm{CO}_{<\infty}^{>1})_{\overline{x}}$ *is a groupoid homomorphism, where*

$$A_w = \{\lambda \in \Lambda(x) \mid \alpha_{x_\lambda} = -w \cdot \alpha_{y_\lambda}\}.$$

Proof. Let $w \in G_{x,y}$. If y_λ is adjacent to y_μ (or equivalently $\left\langle \alpha_{y_\lambda}, \alpha_{y_\mu} \right\rangle < 0$), then $\lambda \in A_w$ if and only if $\mu \in A_w$ since $\left\langle w \cdot \alpha_{y_\lambda}, w \cdot \alpha_{y_\mu} \right\rangle = \left\langle \alpha_{y_\lambda}, \alpha_{y_\mu} \right\rangle$ and $\left\langle \alpha_{x_\lambda}, \alpha_{x_\mu} \right\rangle \leqslant 0$. This implies $A_w \in \mathrm{CO}(y)$.

If $y_\lambda \in {y_{A_w}}^{\mathrm{iso}}$, then we have $y_\lambda \in [y]^{\mathrm{iso}}$ and $\alpha_{x_\lambda} = -w \cdot \alpha_{y_\lambda}$, but this is impossible since $w \in C'_{x,y}$. So ${y_{A_w}}^{\mathrm{iso}} = \emptyset$. Further, we have $\Phi^+_{y_{A_w}} \subset \Phi^+_w$, so y_{A_w} is of finite type since $\left|\Phi^+_{y_{A_w}}\right| \leqslant |\Phi^+_w| = \ell(w) < \infty$. Thus we have $A_w \in \mathrm{CO}_{<\infty}^{>1}(y)$ and so this map is well-defined.

The rest of our claim, namely $A_{ww'} = A_w A_{w'}$ for $w \in G_{x,y}$, $w' \in G_{y,z}$ and $A_{w^{-1}} = A_w$ $(= {A_w}^{-1})$, immediately follows from definition. $\square$

Note that $H = G \cap C''$ is the kernel of this homomorphism, so Proposition 5.1 follows.

Lemma 5.8. *Let $x \in S^{(*)}$, $I \subset S$ and suppose $w \in W_{[x]\cup I}$, $w \cdot \Pi_{[x]} \subset \Pi$ and $\Pi_I \subset \Phi_w^+$. Then $[x] \cap I = \emptyset$, $x_{\sim I}$ is of finite type and $w = w_0(x_{\sim I})w_0(x_{\sim I} \smallsetminus I)$.*

Proof. $[x] \cap I = \emptyset$ and $w \in C''_{y,x}$ for some $y \in S^{(*)}$ by the hypothesis. Now we show $\Phi_w^+ = \Phi_{x_{\sim I}}^+ \smallsetminus \Phi_{x_{\sim I} \smallsetminus I}^+$.

Since $w \in W_{[x]\cup I}$, we have $\Phi_w^+ \subset \Phi_{[x]\cup I}^+$. If $\gamma \in \Phi_{[x]\cup I}^+$ and $w \cdot \gamma < 0$, then $\operatorname{supp}\gamma \cap I \neq \emptyset$ since $w \in C''_{y,x}$, and so $\operatorname{supp}\gamma \subset x_{\sim I}$ since $\operatorname{supp}\gamma$ is connected. Thus $\gamma \in \Phi_{x_{\sim I}}^+ \smallsetminus \Phi_{x_{\sim I} \smallsetminus I}^+$. Conversely, suppose $\gamma \in \Phi_{x_{\sim I}}^+ \smallsetminus \Phi_{x_{\sim I} \smallsetminus I}^+$. Take any $s \in \operatorname{supp}\gamma \cap I$. Then we have $[\alpha_t]\, w \cdot \alpha_s \neq 0$ for some $t \in S \smallsetminus [y]$; otherwise, we have $\alpha_s = w^{-1} w \cdot \alpha_s \in w^{-1} \cdot \Phi_{[y]} \subset \Phi_{[x]}$ since $w \in C''_{y,x}$, but this is contradiction. Then, since $[\alpha_t]\, w \cdot \alpha_{x_\lambda} = 0$ for any $\lambda \in \Lambda(x)$ and $[\alpha_t]\, w \cdot \alpha_{s'} \leqslant 0$ for any $s' \in I$, we have $[\alpha_t]\, w \cdot \gamma < 0$ and so $w \cdot \gamma < 0$. Thus $\Phi_w^+ = \Phi_{x_{\sim I}}^+ \smallsetminus \Phi_{x_{\sim I} \smallsetminus I}^+$.

Since $|\Phi_w^+| = \ell(w) < \infty$ and $x_{\sim I} \smallsetminus I \subsetneq x_{\sim I}$, Proposition 2.9 implies that $x_{\sim I}$ is of finite type. Further, Lemma 4.2 implies that $\Phi^+_{w_0(x_{\sim I})w_0(x_{\sim I} \smallsetminus I)} = \Phi_w^+$, so we have $w = w_0(x_{\sim I})w_0(x_{\sim I} \smallsetminus I)$. $\square$

Proposition 5.9. *Let $(x, s) \in \mathcal{B}$. Then $(x, s) \in \mathcal{B}^\varphi$ if and only if $\Phi^+_{w_x^s} \cap \Phi^\perp_{[x]} \neq \emptyset$.*

Proof. Suppose $\varphi(x, s) = (x, s)$. Then we have $w^{\varphi_l(x,s)}_{\varphi_v(x,s)} = w_x^s$ and so $(w_x^s)^2 = 1$. So by Theorem 2.3, $w_x^s \cdot \gamma = -\gamma$ for some $\gamma \in \Phi^+_{x_{\sim s}}$. Now we have

$$\langle \gamma, \alpha_{x_\lambda} \rangle = \langle w_x^s \cdot \gamma, w_x^s \cdot \alpha_{x_\lambda} \rangle = \left\langle -\gamma, \alpha_{\varphi_v(x,s)_\lambda} \right\rangle = -\langle \gamma, \alpha_{x_\lambda} \rangle$$

for all $\lambda \in \Lambda(x)$ since $\varphi_v(x, s) = x$, so $\gamma \in \Phi^+_{w_x^s} \cap \Phi^\perp_{[x]}$.

Conversely, suppose $\gamma \in \Phi^+_{w_x^s} \cap \Phi^\perp_{[x]}$, so $\gamma \in \Phi^+_{x_{\sim s}}$ and $s_\gamma \in C''_{x,x}$. Further, since $\gamma \in \Phi_{[x]}^{\perp}$, we have

$$1 = \langle \gamma, \gamma \rangle = \sum_{t \in \operatorname{supp} \gamma} ([\alpha_t]\,\gamma)\,\langle \gamma, \alpha_t \rangle = ([\alpha_s]\,\gamma)\,\langle \gamma, \alpha_s \rangle$$

and so $\langle \gamma, \alpha_s \rangle > 0$. This implies $s_\gamma \cdot \alpha_s < 0$; in fact, this is obvious if $\gamma = \alpha_s$. On the other hand, if $\gamma \neq \alpha_s$, then $[\alpha_t]\,\gamma > 0$ for some $t \in S \setminus \{s\}$, so we have

$$[\alpha_t]\,(s_\gamma \cdot \alpha_s) = [\alpha_t]\,(\alpha_s - 2\,\langle \gamma, \alpha_s \rangle\,\gamma) = -2\,\langle \gamma, \alpha_s \rangle\,[\alpha_t]\,\gamma < 0.$$

Thus $s_\gamma \cdot \alpha_s < 0$. Hence we have $s_\gamma = w_x^s$ by Lemma 5.8, and so $\varphi_v(x, s) = x$ since $s_\gamma \in C''_{x,x}$. $\square$

Then Theorem 5.2 (i) now follows immediately; for such (x, s), we have $w_x^s \in C''_{\varphi_v(x,s),x}$ by definition of φ_v, while $\Phi^+_{w_x^s} \cap \Phi^{\perp}_{[x]} = \emptyset$ by Proposition 5.9. Hence $w_x^s \in H_{\varphi_v(x,s),x}$. Moreover, since $(w_x^s)^{-1} = w_{\varphi_v(x,s)}^{\varphi_l(x,s)}$, the groupoid anti-homomorphism h, as in Theorem 5.2 (iii), exists uniquely and is identity on $S^{(*)}$ (by the fact that $\overline{\mathcal{P}}(\mathcal{H})$ is a free groupoid on $\mathcal{H}$; cf. [4], [9]). For each $p \in \mathcal{P}(\mathcal{H})$, we write $h(p)$ as a shorthand for $h([p])$. Now we show that h is surjective, which completes Theorem 5.2 (iii). For any $p = e_{x_1}^{s_1} e_{x_2}^{s_2} \cdots e_{x_n}^{s_n} \in \mathcal{P}(\mathcal{H})$, we say that p is *nondegenerate* if $\ell(h(p)) = \sum_{i=1}^n \ell(w_{x_i}^{s_i})$, *degenerate* otherwise. Note that $\ell(h(p)) \leqslant \sum_{i=1}^n \ell(w_{x_i}^{s_i})$ for any $p \in \mathcal{P}(\mathcal{H})$.

Lemma 5.10. *Suppose $w \in C''_{x,y}$, $I \subset S$ and $\Pi_I \subset \Phi^+_w$. Then $[y] \cap I = \emptyset$ and $J = y_{\sim I}$ is of finite type. Further, $\ell(w) = \ell(w^J) + \ell(w_J)$, $\Phi^+_{w_J} = \Phi^+_w \cap \Phi^+_J$, $w_J = w_0(J) w_0(J \setminus I)$ and $w_J \in C''_{z,y}$ for some $z \in S^{(*)}$.*

Proof. $[y] \cap I = \emptyset$ since $\Pi_I \subset \Phi^+_w$ and $w \in C''_{x,y}$. Further, $\ell(w) = \ell(w^J) + \ell(w_J)$ and $\Phi^+_{w_J} = \Phi^+_w \cap \Phi^+_J$ by Proposition 2.8. So $\Pi_I \subset \Phi^+_{w_J}$.

Let $\lambda \in \Lambda(y)$, $y_\lambda \in J$. Then we have

$$\alpha_{x_\lambda} = w \cdot \alpha_{y_\lambda} = w^J w_J \cdot \alpha_{y_\lambda} = \sum_{t \in J} ([\alpha_t]\, w_J \cdot \alpha_{y_\lambda}) w^J \cdot \alpha_t.$$

Since $w^J \cdot \alpha_t > 0$ for all $t \in J$ and $\alpha_{y_\lambda} \notin \Phi^+_w \supset \Phi^+_{w_J}$, the right side of the above equality is a nonnegative linear combination of positive roots, while the left side is a simple root. This means that $[\alpha_t]\, w_J \cdot \alpha_{y_\lambda} > 0$ for exactly one $t \in J$; that is, $w_J \cdot \alpha_{y_\lambda}$ is a simple root. Thus we have $w_J \in C''_{z,y}$ for some $z \in S^{(*)}$. Now Lemma 5.8 implies that J is of finite type and $w_J = w_0(J) w_0(J \smallsetminus I)$, as required. $\square$

The surjectivity of h is deduced from the following:

Corollary 5.11. *For any $w \in H_{x,y}$, there exists a nondegenerate path $p \in \mathcal{P}(\mathcal{H})_{y,x}$ such that $h(p) = w$. Further, if $s \in S$ and $w \cdot \alpha_s < 0$, then we can take such p containing e^s_y as its first edge; in particular, $(y,s) \in \mathcal{B} \smallsetminus \mathcal{B}^\varphi$.*

Proof. The case $\ell(w) = 0$ is obvious, so suppose $\ell(w) > 0$. Let $s \in S$, $w \cdot \alpha_s < 0$. Since $w \in C''_{x,y}$, Lemma 5.10 implies that $(y,s) \in \mathcal{B}$, $\ell(w(w^s_y)^{-1}) = \ell(w) - \ell(w^s_y) < \ell(w)$ and $\Phi^+_{w^s_y} = \Phi^+_w \cap \Phi^+_{y \sim s}$. Then we have $\Phi^+_{w^s_y} \cap \Phi^\perp_{[y]} \subset \Phi^+_w \cap \Phi^\perp_{[y]} = \emptyset$ since $w \in H_{x,y}$. So we have $(y,s) \notin \mathcal{B}^\varphi$ by Proposition 5.9, and $w^s_y \in H_{\varphi_v(y,s),y}$, therefore $w(w^s_y)^{-1} \in H_{x,\varphi_v(y,s)}$. Now by induction on $\ell(w)$, there is a nondegenerate $p' \in \mathcal{P}(\mathcal{H})_{\varphi_v(y,s),x}$ such that $h(p') = w(w^s_y)^{-1}$. Then $p = e^s_y p'$ is a required path (note that h is an *anti*-homomorphism). $\square$

Remark 5.12. Although Proposition 5.5 of [6] is similar to Corollary 5.11, they have the following differences. In this corollary we deal with transition between *ordered tuples*, while [6] deals with transition between subsets of S

only. Moreover, this corollary gives a decomposition of w in H, while [6] only gives a decomposition in C'', a larger groupoid than H.

Remark 5.13. $\mathcal{A}_x = \{A \in \mathrm{CO}^{\geq 1}_{<\infty}(x) \mid H_{x^A,x} \neq \emptyset\}$ since h is surjective.

On the other hand, Theorem 5.2 (ii) is deduced from the following:

Lemma 5.14. *$w_x^A \in G_{x^A,x}$ and $A_{w_x^A} = A$ for any $A \in \mathrm{CO}^{\geq 1}_{<\infty}(x)$.*

Proof. Since ${x_A}^{\mathrm{iso}} = \emptyset$, we have $w_0(x_A) \in C'_{x^A,x}$ by definition of x^A, while $\Phi^+_{w_0(x_A)} \cap \Phi^\perp_{[x]} = \emptyset$ since $w_0(x_A) \in W_{[x]}$. Thus $w_0(x_A) \in G_{x^A,x}$. $A_{w_0(x_A)} = A$ is obvious. $\square$

Now we show Theorem 5.2 (iv).

Lemma 5.15. **(i)** *If $G_{x,y} \neq \emptyset$, then $\mathcal{A}_x = \mathcal{A}_y$. (So $\mathcal{A}_{x^A} = \mathcal{A}_x$ for any $A \in \mathrm{CO}^{\geq 1}_{<\infty}(x)$ by Lemma 5.14.)*
(ii) *$A_w \in \mathcal{A}_x$ for any $w \in G'_{x,y}$.*

Proof. **(i)** Take any $w \in G_{x,y}$, and let $A \in \mathcal{A}_y$. For $\lambda \in \Lambda(y)$, let ε_λ be $+$ or $-$ such that $\alpha_{x_\lambda} = \varepsilon_\lambda w \cdot \alpha_{y_\lambda}$. If $\lambda \notin A$, then we have

$$\alpha_{(x^A)_\lambda} = \alpha_{x_\lambda} = \varepsilon_\lambda w \cdot \alpha_{y_\lambda} = \varepsilon_\lambda w \cdot \alpha_{(y^A)_\lambda}.$$

Suppose $\lambda \in A$, and let $\sigma_{y_A}(y_\lambda) = y_\mu$. Then we have $\sigma_{x_A}(x_\lambda) = x_\mu$ by Remark 5.6. Further, $\varepsilon_\lambda = \varepsilon_\mu$ since y_λ and y_μ belong the same connected component of $[y]$. So we have

$$\alpha_{(x^A)_\lambda} = \alpha_{x_\mu} = \varepsilon_\lambda w \cdot \alpha_{y_\mu} = \varepsilon_\lambda w \cdot \alpha_{(y^A)_\lambda}.$$

Moreover, $\Phi_w^+ \cap \Phi_{[y^A]}^\perp = \Phi_w^+ \cap \Phi_{[y]}^\perp = \emptyset$ since $w \in G_{x,y}$. Thus $w \in G_{x^A,y^A}$. We distinguish this w by writing w'. Then the above argument shows $A_{w'} = A_w$.

Since $A \in \mathcal{A}_y$, there is a path $p \in \mathcal{P}(\mathcal{H})_{y,y^A}$. Now we have $w'h(p)w^{-1} \in G_{x^A,y^A} H_{y^A,y} G_{y,x} \subset G_{x^A,x}$ and $A_{w'h(p)w^{-1}} = A_w \emptyset (A_w)^{-1} = \emptyset$, so $w'h(p)w^{-1} \in H_{x^A,x}$. This implies $A \in \mathcal{A}_x$, so we have $\mathcal{A}_y \subset \mathcal{A}_x$. The converse is similar.

(ii) $w_x^{A_w} \in G_{x^{A_w},x}$ by Lemma 5.14, while some $w' \in H_{x,y}$ exists by definition of G'. Then we have $w_x^{A_w} w w'^{-1} \in G_{x^{A_w},x}$ and $A_{w_x^{A_w} w w'^{-1}} = A_w A_w \emptyset = \emptyset$, so $w_x^{A_w} w w'^{-1} \in H_{x^{A_w},x}$. This implies $A_w \in \mathcal{A}_x$. $\square$

Proof of Theorem 5.2 (iv). Let $A \in \mathcal{A}_x$. Then by Lemmas 5.14 and 5.15, we have $w_x^A \in G'_{x^A,x}$ (since $H_{x^A,x} \neq \emptyset$) and $(w_x^A)^{-1} = w_{x^A}^A$ (since $(x^A)_A = x_A$). So, similarly to the case of h, such g exists uniquely and is identity on $S^{(*)}$. We write $g(p)$, $p \in \mathcal{P}(\mathcal{G})$ as a shorthand for $g([p])$.

Now we show that g is surjective. Let $w \in G'_{x,y}$. Then $w_x^{A_w} \in G_{x^{A_w},x}$ and $A_{w_x^{A_w} w} = A_w A_w = \emptyset$. So we have $w_x^{A_w} w \in H_{x^{A_w},y}$ and then $w_x^{A_w} w = h(p)$ for some $p \in \mathcal{P}(\mathcal{H})_{y,x^{A_w}}$ since h is surjective. Further, the edge $e_x^{A_w}$ exists by Lemma 5.15 (ii). Thus we have $p' = p(e_x^{A_w})^{-1} \in \mathcal{P}(\mathcal{G})_{y,x}$ and $g(p') = (w_x^{A_w})^{-1} h(p) = w$. Hence g is surjective. $\square$

From now on, we prove Theorem 5.3. Note that $\mathcal{A}_x = \mathcal{A}_y$ whenever $y \in V(\mathcal{G}_{\sim x}) = V(\mathcal{H}_{\sim x})$, by Lemma 5.15 (i).

Lemma 5.16. *Let $A \in \mathcal{A}_x$ and $e_y^s \in \mathcal{H}_{\sim x}$. Then $e_{y^A}^s \in \mathcal{H}_{\sim x}$ and $\varphi_v(y^A, s) = \varphi_v(y,s)^A$. Further, $w_y^s = w_{y^A}^s$ in W and $w_{\varphi_v(y,s)}^A w_y^s (w_y^A)^{-1} = w_{y^A}^s$ in G.*

Proof. Put $\varphi(y,s) = (z,s')$. Then we have $w_y^s \in C''_{z^A,y^A}$ similarly to the proof of Lemma 5.15 (i), while $w_y^s \cdot \alpha_s < 0$. So Lemma 5.8 implies that $(y^A, s) \in \mathcal{B}$,

$w_y^s = w_{y^A}^s$ and $\varphi_v(y^A, s) = z^A$. Now $y \neq z$ implies $y^A \neq z^A$, so $(y^A, s) \notin \mathcal{B}^\varphi$. Further, since $y \in V(\mathcal{H}_{\sim x})$ and $A \in \mathcal{A}_x$, we have $y^A \in V(\mathcal{H}_{\sim x})$ and so $e_{y^A}^s \in \mathcal{H}_{\sim x}$.

Now we have $w_z^A w_y^s (w_y^A)^{-1} \in G_{z^A,z} H_{z,y} G_{y,y^A} \subset G_{z^A,y^A}$. Further, since $w_y^A \in W_{[y]}$, we have $(w_y^A)^{-1} \cdot \alpha_s \in \Phi^+_{y_{\sim s}} \setminus \Phi^+_{y_{\sim s} \setminus \{s\}}$ and so $w_y^s (w_y^A)^{-1} \cdot \alpha_s \in \Phi^-_{z_{\sim s'}} \setminus \Phi^-_{z_{\sim s'} \setminus \{s'\}}$, therefore $w_z^A w_y^s (w_y^A)^{-1} \cdot \alpha_s < 0$. Then, since $w_z^A w_y^s (w_y^A)^{-1} \in W_{[y^A] \cup \{s\}}$, Lemma 5.8 implies that $w_z^A w_y^s (w_y^A)^{-1} = w_{y^A}^s$, as required. $\square$

Corollary 5.17. **(i)** *For any edge e_y^s of $\mathcal{H}_{\sim x}$ and $A \in \mathcal{A}_x$, $e_{y^A}^s$ is also an edge of $\mathcal{H}_{\sim x}$ from y^A to $\varphi_v(y,s)^A$ and $e_{\varphi_v(y,s)^A}^{\varphi_l(y,s)} = (e_{y^A}^s)^{-1}$. Hence we can define $\rho_A \in \mathrm{Aut}\mathcal{H}_{\sim x}$ by $\rho_A(y) = y^A$ and $\rho_A(e_y^s) = e_{y^A}^s$, and it induces a groupoid automorphism $\widetilde{\rho}_A$ on $\overline{\mathcal{P}}(\mathcal{H}_{\sim x})$.*

(ii) $w_z^A h(p) (w_y^A)^{-1} = h(\widetilde{\rho}_A(p))$ *for any $p \in \mathcal{P}(\mathcal{H}_{\sim x})_{y,z}$ and $A \in \mathcal{A}_y$.*

(iii) *$\mathcal{A}_x$ is a subgroup of $\mathrm{CO}^{\geq 1}_{<\infty}(x)$.*

(iv) *Both $\mathcal{A}_x \ni A \mapsto \rho_A \in \mathrm{Aut}\mathcal{H}_{\sim x}$, $\mathcal{A}_x \ni A \mapsto \widetilde{\rho}_A \in \mathrm{Aut}\overline{\mathcal{P}}(\mathcal{H}_{\sim x})$ are group homomorphisms.*

Proof. For any edge e_y^s of $\mathcal{H}_{\sim x}$ and $A \in \mathcal{A}_x$, we have $\varphi_v(y^A, s) = \varphi_v(y,s)^A$ by Lemma 5.16. So $\left[\varphi_v(y^A, s)\right] = [\varphi_v(y,s)]$, while $\left[y^A\right] \cup \{s\} = [y] \cup \{s\}$. Then we have $\varphi_l(y^A, s) = \varphi_l(y,s)$ by Remark 4.1; thus (i) holds. (ii) is deduced by repeated use of Lemma 5.16.

Now we show (iii). Let $A, A' \in \mathcal{A}_x$. Then some $p \in \mathcal{P}(\mathcal{H})_{x,x^A}$ exists, so $\rho_{A'}(p) \in \mathcal{P}(\mathcal{H})_{x^{A'},x^{AA'}}$. Thus we have $\mathcal{P}(\mathcal{H})_{x^{A'},x^{AA'}} \neq \emptyset$, therefore $x^{AA'} \in V(\mathcal{H}_{\sim x^{A'}}) = V(\mathcal{H}_{\sim x})$ since $A' \in \mathcal{A}_x$. This implies that $AA' \in \mathcal{A}_x$; so (iii) holds. Finally, (iv) immediately follows from definition. $\square$

Proof of Theorem 5.3 (i). Let N_x be the normal subgroupoid of $\overline{\mathcal{P}}(\mathcal{H}_{\sim x})$

generated by all $[p]$, $p \in R_1(x)$. Then we show $\ker h_x = N_x$.

Claim 1. $N_x \subset \ker h_x$.

Proof of Claim 1. Let $p \in R_1(x)$. Then p is of length at least two since $\mathcal{H}$ has no loops. Now $s_2 \neq \varphi_l(y_1, s_1)$ since p is reduced, and so $s_2 \notin [y_2] \cup \{\varphi_l(y_1, s_1)\} = [y_1] \cup \{s_1\}$, therefore $I = [y_1] \cup \{s_1, s_2\}$. This implies that $I \smallsetminus [y_i] \subset J$ for all i by induction, where $J = I_{\sim\{s_1,s_2\}}$. Thus we have $h(p) \in W_J$, while $h(p) \in H_{y_1,y_1}$ and J is of finite type. So Theorem 2.4 implies that $h(p)$ can be written as a product of reflections in W_J which fix $\Pi_{[y_1]\cap J}$ pointwise. Now since no element of $[y_1] \smallsetminus J$ is adjacent to an element of J, these reflections in fact fix $\Pi_{[y_1]}$ pointwise. This yields $h(p) \in W(\Phi^{\perp}_{[y_1]}) \cap H_{y_1,y_1}$, while $W(\Phi^{\perp}_{[y_1]}) \cap H_{y_1,y_1} = 1$ by Theorem 3.5. Hence $h(p) = 1$. (End of proof of Claim 1)

We show $\ker h_x \subset N_x$. Let $p = e^{s_1}_{y_1} \cdots e^{s_n}_{y_n} \in \mathcal{P}(\mathcal{H}_{\sim x})$ such that $h(p) = 1$ (so p is a closed path). We show $[p] \in N_x$ by induction on $|p| = \sum_{i=1}^{n} \ell(w^{s_i}_{y_i})$. This is obvious when $|p| = 0$, so suppose $|p| > 0$. Then p is degenerate since $h(p) = 1$, while $e^{s_1}_{y_1}$ is nondegenerate. So there is some index $1 \leqslant k \leqslant n-1$ such that $p^{(1,k)}$ is nondegenerate and $p^{(1,k+1)}$ is degenerate, where $p^{(i,j)} = e^{s_i}_{y_i} e^{s_{i+1}}_{y_{i+1}} \cdots e^{s_j}_{y_j}$ for any indices i, j. Put $w = h(p^{(1,k)})^{-1}$.

If $s_{k+1} = \varphi_l(y_k, s_k)$, then we have $[p] = \left[p^{(1,k-1)} p^{(k+2,n)}\right] \in N_x$ by induction. So we may assume $s_{k+1} \notin [y_{k+1}] \cup \{\varphi_l(y_k, s_k)\} = [y_k] \cup \{s_k\}$.

Claim 2. $w \cdot \alpha_{\varphi_l(y_k,s_k)} < 0$ and $w \cdot \alpha_{s_{k+1}} < 0$.

Proof of Claim 2. Since $p^{(1,k-1)}$ is nondegenerate, Lemma 2.1 (i) implies that $\Phi^+_{h(p^{(1,k-1)})^{-1}} \cap \Phi^+_{w^{s_k}_{y_k}} = \emptyset$, while $(w^{s_k}_{y_k})^{-1} \cdot \alpha_{\varphi_l(y_k,s_k)} < 0$. Thus we have

$$-w \cdot \alpha_{\varphi_l(y_k,s_k)} = h(p^{(1,k-1)})^{-1} \cdot (-(w^{s_k}_{y_k})^{-1} \cdot \alpha_{\varphi_l(y_k,s_k)}) > 0.$$

On the other hand, $\ell(h(p^{(1,k+1)})) < \ell(w^{-1}) + \ell(w^{s_{k+1}}_{y_{k+1}})$ by the choice of k, and so $\Phi^+_w \cap \Phi^+_{w^{s_{k+1}}_{y_{k+1}}} \neq \emptyset$ by Lemma 2.1 (i). This implies that $w \cdot \alpha_{s_{k+1}} < 0$ since

$w \in C''_{y_1,y_{k+1}}$ and $\Phi^+_{w^{s_{k+1}}_{y_{k+1}}} \subset \Phi^+_{(y_{k+1})_{\sim s_{k+1}}}$. (End of proof of Claim 2)

Put $I = [y_{k+1}] \cup \{\varphi_l(y_k, s_k), s_{k+1}\}$ and $J = I_{\sim\{\varphi_l(y_k,s_k),s_{k+1}\}}$. Then Lemma 5.10 implies that J is of finite type, $w_J \cdot \alpha_{\varphi_l(y_k,s_k)} < 0$, $w_J \cdot \alpha_{s_{k+1}} < 0$, $w_J \in H_{z,y_{k+1}}$ and $w^J \in H_{y_1,z}$ for some $z \in S^{(*)}$. Further, by Corollary 5.11, there are nondegenerate paths $q_1 \in \mathcal{P}(\mathcal{H})_{z,y_1}$, $(e^{s_k}_{y_k})^{-1}q_2 \in \mathcal{P}(\mathcal{H})_{y_{k+1},z}$ such that $h(q_1) = w^J$, $h((e^{s_k}_{y_k})^{-1}q_2) = w_J$ and $(e^{s_k}_{y_k})^{-1}q_2$ is contained in $\mathcal{H}^{(I)}$ (since $w_J \in W_I$). Note that $\left|(e^{s_k}_{y_k})^{-1}q_2q_1\right| = \ell(w^J) + \ell(w_J) = \ell(w)$. Similarly, there is a nondegenerate path $e^{s_{k+1}}_{y_{k+1}}q_3 \in \mathcal{P}(\mathcal{H})_{y_{k+1},z}$ contained in $\mathcal{H}^{(I)}$ and satisfying $h(e^{s_{k+1}}_{y_{k+1}}q_3) = w_J$. Note that $\left|e^{s_{k+1}}_{y_{k+1}}q_3q_1\right| = \ell(w)$.

Claim 3. $\left[e^{s_{k+1}}_{y_{k+1}}q_3{q_2}^{-1}e^{s_k}_{y_k}\right] \in N_x$.

Proof of Claim 3. By the above argument, $p' = e^{s_{k+1}}_{y_{k+1}}q_3{q_2}^{-1}e^{s_k}_{y_k} \in \mathcal{P}(\mathcal{H}_{\sim x})$ is a closed path contained in $\mathcal{H}^{(I)}$, $|I \smallsetminus [y_{k+1}]| = 2$ and $I_{\sim I \smallsetminus [y_{k+1}]} = J$ is of finite type. Now we can write $[p'] = [p_1c_1{p_1}^{-1} \cdots p_mc_m{p_m}^{-1}]$, where $p_i \in \mathcal{P}(\mathcal{H}^{(I)})$ and $c_i \in \mathcal{P}(\mathcal{H}^{(I)})$ is a nontrivial simple closed reduced path. Thus $c_i \in R_1(x)$. Hence we have $[p'] \in N_x$. (End of proof of Claim 3)

Claim 4. $\left[p^{(1,k-1)}q_2q_1\right] \in N_x$ and $\left[{q_1}^{-1}{q_3}^{-1}p^{(k+2,n)}\right] \in N_x$.

Proof of Claim 4. By the choice of q_1, q_2, we have

$$h(p^{(1,k-1)}q_2q_1) = w^J(w_Jw^{s_k}_{y_k})h(p^{(1,k-1)}) = wh(p^{(1,k)}) = 1,$$

while $h(p^{(k+1,n)}) = h(p)h(p^{(1,k)})^{-1} = w$ implies that

$$\left|p^{(1,k-1)}q_2q_1\right| = \left|p^{(1,k-1)}\right| + (\ell(w) - \left|e^{s_k}_{y_k}\right|) \leqslant \left|p^{(1,k-1)}\right| + \left|p^{(k+1,n)}\right| - \left|e^{s_k}_{y_k}\right| < |p|.$$

Thus $\left[p^{(1,k-1)}q_2q_1\right] \in N_x$ by induction. Similarly, we have

$$h({q_1}^{-1}{q_3}^{-1}p^{(k+2,n)}) = (w(w^{s_{k+1}}_{y_{k+1}})^{-1})(w^{s_{k+1}}_{y_{k+1}}(w_J)^{-1})(w^J)^{-1} = 1,$$
$$\left|{q_1}^{-1}{q_3}^{-1}p^{(k+2,n)}\right| = (\ell(w) - \left|e^{s_{k+1}}_{y_{k+1}}\right|) + \left|p^{(k+2,n)}\right| < |p|,$$

so $\left[{q_1}^{-1}{q_3}^{-1}p^{(k+2,n)}\right] \in N_x$ by induction. (End of proof of Claim 4)

Since

$$[p] = \left[p^{(1,k-1)} q_2 q_1\right]$$
$$\cdot \left[{q_1}^{-1}{q_2}^{-1} e^{s_k}_{y_k}\right] \left[e^{s_{k+1}}_{y_{k+1}} q_3 {q_2}^{-1} e^{s_k}_{y_k}\right] \left[{q_1}^{-1}{q_2}^{-1} e^{s_k}_{y_k}\right]^{-1} \cdot \left[{q_1}^{-1}{q_3}^{-1} p^{(k+2,n)}\right],$$

we have $[p] \in N_x$ by Claims 3 and 4. Hence $\ker h_x \subset N_x$. □

Proof of Theorem 5.3 (ii). Let N_x be the normal subgroupoid of $\overline{\mathcal{P}}(\mathcal{G}_{\sim x})$ generated by all $[p]$, $p \in R_1(x) \cup R_2(x) \cup R_3(x)$. Then we show $\ker g_x = N_x$.

First, we show $N_x \subset \ker g_x$. Theorem 5.3 (i) shows that $[p] \in \ker h_x \subset \ker g_x$ if $p \in R_1(x)$. For $p = e^A_y e^{A'}_{y^A} (e^{AA'}_y)^{-1} \in R_2(x)$, we have $w_0(y_{AA'}) = w_0(y_{A'}) w_0(y_A)$, $w_0((y^A)_{A'}) = w_0(y_{A'})$ since A, A' are unions of connected components of $[y]$. So we have $g(p) = 1$. Further, by Lemma 5.16, we have $g(p) = 1$ for any $p \in R_3(x)$. Thus $N_x \subset \ker g_x$.

Conversely, we show $\ker g_x \subset N_x$. Take any $p \in \mathcal{P}(\mathcal{G}_{\sim x})$ such that $g(p) = 1$, and we show $[p] \in N_x$. Note that $N_x \subset \ker g_x$ as proved above. So by repeated use of elements of $R_3(x)$, we may assume that p is of the form $p' e^{A_1}_y e^{A_2}_{y^{A_1}} \cdots e^{A_k}_{y^{A_1 \cdots A_{k-1}}}$, $p' \in \mathcal{P}(\mathcal{H}_{\sim x})$ without changing whether $[p] \in N_x$ or not.

Since $g(p) = 1 \in H$, we have $\emptyset = A_{g(p)} = A_1 A_2 \cdots A_k$. Then we have

$$\begin{aligned}
&\left[e^{A_1}_y e^{A_2}_{y^{A_1}} \cdots e^{A_k}_{y^{A_1 \cdots A_{k-1}}}\right] \\
&\quad = \left[e^{A_1}_y e^{A_2}_{y^{A_1}} \cdots e^{A_{k-1}}_{y^{A_1 \cdots A_{k-2}}} e^{A_1 \cdots A_{k-1}}_{y^{A_1 \cdots A_{k-1}}}\right] \\
&\quad = \left[e^{A_1}_y \cdots e^{A_{k-2}}_{y^{A_1 \cdots A_{k-3}}} e^{A_1 \cdots A_{k-2}}_{y^{A_1 \cdots A_{k-2}}}\right] \left[e^{A_1 \cdots A_{k-2}}_y e^{A_{k-1}}_{y^{A_1 \cdots A_{k-2}}} (e^{A_1 \cdots A_{k-1}}_y)^{-1}\right]
\end{aligned}$$

and so $\left[e^{A_1}_y e^{A_2}_{y^{A_1}} \cdots e^{A_k}_{y^{A_1 \cdots A_{k-1}}}\right] \in N_x$ by induction on k. Since $N_x \subset \ker g_x$, this implies $h(p') = 1$; so $[p'] \in \ker h_x \subset N_x$ by Theorem 5.3 (i). Hence we have $[p] \in N_x$, as required. □

Finally, we examine the structure of $G_{x,x}$ more precisely. Recall that $H_{x,x}$

is normal in $G_{x,x}$.

Lemma 5.18. *The map* $G_{x,x}/H_{x,x} \ni wH_{x,x} \mapsto A_w \in \mathcal{A}_x$ *is an isomorphism of groups.*

Proof. The well-definedness and injectivity of this map are deduced from the definition of H and Lemma 5.15 (ii). For $A \in \mathcal{A}_x$, we have $h(p)^{-1}w_x^A \in G_{x,x}$ and $A_{h(p)^{-1}w_x^A} = A$ for some $p \in \mathcal{P}(\mathcal{H})_{x,x^A}$ since $x^A \in V(\mathcal{H}_{\sim x})$; thus this map is surjective. $\square$

Corollary 5.19. *Suppose that the group* $\mathcal{A}_x$ *is generated by certain elements* A_ν*. For each* ν*, let* $\widetilde{p}_\nu = p_{x^{A_\nu}}(e_x^{A_\nu})^{-1} \in \mathcal{P}(\mathcal{G})_{x,x}$.

(i) $G_{x,x}$ *is generated by all* $h(q)$, $q \in \mathcal{P}(\mathcal{H})_{x,x}$ *and all* $g(\widetilde{p}_\nu)$.

(ii) $g(\widetilde{p}_\nu)h(q)g(\widetilde{p}_\nu)^{-1} = h(\widetilde{\rho}_{A_\nu}({p_{x^{A_\nu}}}^{-1}qp_{x^{A_\nu}}))$.

(iii) $g(\widetilde{p}_\nu)^2 = h(p_{x^{A_\nu}}\widetilde{\rho}_{A_\nu}(p_{x^{A_\nu}}))$.

(iv) $g(\widetilde{p}_{\nu_1})^{-1}g(\widetilde{p}_{\nu_2})^{-1}g(\widetilde{p}_{\nu_1})g(\widetilde{p}_{\nu_2}) = h(p_{x^{A_{\nu_2}}}\widetilde{\rho}_{A_{\nu_2}}(p_{x^{A_{\nu_1}}})\widetilde{\rho}_{A_{\nu_1}}(p_{x^{A_{\nu_2}}})^{-1}{p_{x^{A_{\nu_1}}}}^{-1})$.

Proof. Let $w \in G_{x,x}$. Then $A_w = A_{\nu_1}\cdots A_{\nu_k}$ for some indices $\nu_1,\dots,\nu_k$ by the hypothesis. Since $A_{g(\widetilde{p}_\nu)} = A_\nu$, we have $wg(\widetilde{p}_{\nu_1})^{-1}\cdots(\widetilde{p}_{\nu_k})^{-1} \in H_{x,x}$, and so it is written as $h(q)$, $q \in \mathcal{P}(\mathcal{H})_{x,x}$ since h is surjective. Hence (i) holds. Further, (ii)-(iv) are deduced from Corollary 5.17 by direct computing. $\square$

6 Examples

Example 6.1. (W,S) is of type $\widetilde{B_7}$ and $x = (1,2,4,5,8) \in S^{(*)}$, as in Fig. 3 (in this section we write i as a shorthand for s_i). Then we compute the centralizer $C_{x,x}$ of $W_{\{1,2,4,5,8\}}$.

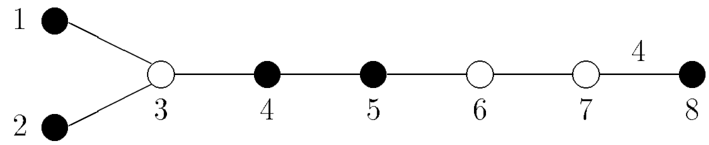
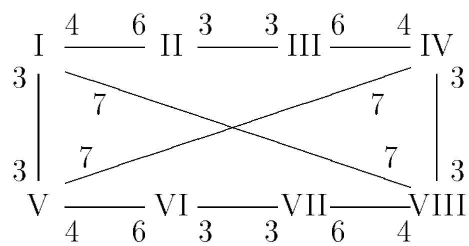


Fig. 3. Coxeter graph of type $\widetilde{B}_7$

1. Fig. 3 implies $[x]^{\mathrm{iso}} = \{1, 2, 8\}$. So by Corollary 3.2,

$$C_{x,x} = W_{\{1,2,8\}} \times C'_{x,x} \simeq (\mathbb{Z}/2\mathbb{Z})^3 \times C'_{x,x}.$$

2. The graph $\mathcal{H}_{\sim x}$ is as in Fig. 4. In this case, $\mathcal{H}_{\sim x}$ has no parallel edges,



$\mathrm{I} = (1, 2, 5, 6, 8)$
$\mathrm{II} = (1, 2, 4, 5, 8) = x$
$\mathrm{III} = (2, 1, 5, 4, 8)$
$\mathrm{IV} = (2, 1, 6, 5, 8)$
$\mathrm{V} = (2, 1, 5, 6, 8)$
$\mathrm{VI} = (2, 1, 4, 5, 8)$
$\mathrm{VII} = (1, 2, 5, 4, 8) = x^{A_0}$
$\mathrm{VIII} = (1, 2, 6, 5, 8)$

Fig. 4. Connected component of $\mathcal{H}$

so let $e(y, z)$ denote the unique directed edge of it from y to z. Now we give a presentation of $H_{x,x}$ by using Theorem 5.4. Let T_x be a maximal tree as in Fig. 5.

To determine $R_1(x)$, we have only to consider $\mathcal{H}^{(I)}_{\sim x}$ for $I = S \smallsetminus \{s\}$,

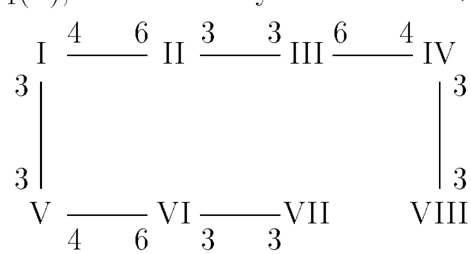


Fig. 5. Maximal tree in Fig. 4

$s \in S$ since $\ell(x) + 2 = 7 = |S| - 1$. For example, if $s = 4$, then we obtain $\mathcal{H}^{(I)}_{\sim x}$ from $\mathcal{H}_{\sim x}$ by deleting four vertices II, III, VI, VII and six edges

$e(\mathrm{I}, \mathrm{II}), e(\mathrm{II}, \mathrm{III}), e(\mathrm{III}, \mathrm{IV}), e(\mathrm{V}, \mathrm{VI}), e(\mathrm{VI}, \mathrm{VII}), e(\mathrm{VII}, \mathrm{VIII})$. By similar argument, $\mathcal{H}_{\sim x}^{(I)}$ is nonempty for $s = 3, 4, 6, 7$, as in Fig. 6, while this is empty for $s = 1, 2, 5, 8$.

Thus we have $R_1(x) = \{c_1, c_2\}$, where

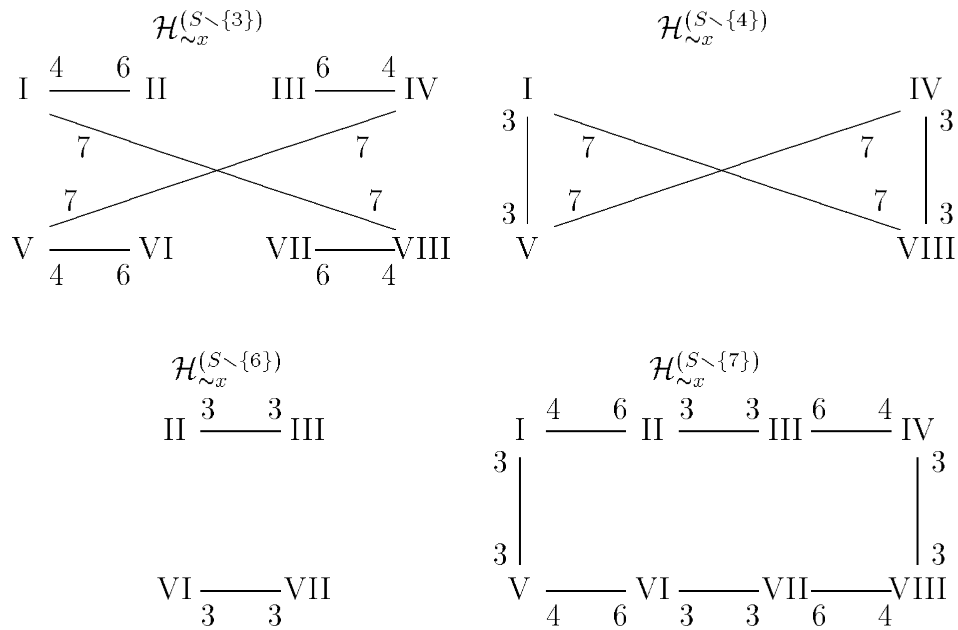


Fig. 6. Subgraphs $\mathcal{H}_{\sim x}^{(I)}$ of Fig. 4

$$c_1 = e(\mathrm{I}, \mathrm{VIII})e(\mathrm{VIII}, \mathrm{IV})e(\mathrm{IV}, \mathrm{V})e(\mathrm{V}, \mathrm{I}),$$

$$c_2 = e(\mathrm{I}, \mathrm{II})e(\mathrm{II}, \mathrm{III})e(\mathrm{III}, \mathrm{IV})e(\mathrm{IV}, \mathrm{VIII})e(\mathrm{VIII}, \mathrm{VII})e(\mathrm{VII}, \mathrm{VI})e(\mathrm{VI}, \mathrm{V})e(\mathrm{V}, \mathrm{I})$$

(note that in this case, every proper subset of S is of finite type). Now, by Theorem 5.4, $H_{x,x}$ is anti-isomorphic to

$$\begin{aligned}
&\Big\langle E(\mathcal{H}_{\sim x}) \mid \{ee^{-1} \mid e \in E(\mathcal{H}_{\sim x})\} \cup \{e \mid e \in T_x\} \cup \{c_1, c_2\}\Big\rangle \\
&\simeq \langle e(\mathrm{I}, \mathrm{VIII}), e(\mathrm{IV}, \mathrm{V}), e(\mathrm{VII}, \mathrm{VIII}) \\
&\qquad \mid e(\mathrm{I}, \mathrm{VIII})e(\mathrm{IV}, \mathrm{V}) = 1, e(\mathrm{VII}, \mathrm{VIII})^{-1} = 1\Big\rangle \\
&\simeq \langle e(\mathrm{IV}, \mathrm{V}) \mid\rangle \simeq \mathbb{Z}.
\end{aligned} \tag{6.1}$$

Further, $e(\mathrm{IV},\mathrm{V})$ in this presentation corresponds to $h(q) \in H_{x,x}$, where

$$q = p_{\mathrm{IV}} e(\mathrm{IV},\mathrm{V}) {p_{\mathrm{V}}}^{-1} = e(\mathrm{II},\mathrm{III}) e(\mathrm{III},\mathrm{IV}) e(\mathrm{IV},\mathrm{V}) e(\mathrm{V},\mathrm{I}) e(\mathrm{I},\mathrm{II}),$$

so $H_{x,x}$ is the free group of rank one generated by $h(q)$.

3. We describe the structure of $G_{x,x}$ by using Corollary 5.19. First, it follows from Fig. 3 that each $A \in \mathrm{CO}(x)$ is a union of some of $\{1\}$, $\{2\}$, $\{3,4\}$, $\{5\}$. Then we have $\mathrm{CO}^{>1}_{<\infty}(x) = \{\emptyset, \{3,4\}\}$. Put $A_0 = \{3,4\}$. Then we have $x^{A_0} = (1,2,5,4,8) = \mathrm{VII}$ and so $\mathcal{A}_x = \{\emptyset, A_0\}$. Let $\tilde{p}_{A_0} = p_{x^{A_0}} (e_x^{A_0})^{-1} \in \mathcal{P}(\mathcal{G})_{x,x}$.

By Corollary 5.19 (i), $G_{x,x}$ is generated by $h(q)$ and $g(\tilde{p}_{A_0})$ since $H_{x,x}$ is generated by $h(q)$. Further, by that corollary, we have

$$\begin{aligned} g(\tilde{p}_{A_0})^2 &= h(p_{x^{A_0}} \tilde{\rho}_{A_0}(p_{x^{A_0}})) \\ &= h(p_{x^{A_0}} e(\mathrm{VII},\mathrm{VIII}) e(\mathrm{VIII},\mathrm{IV}) e(\mathrm{IV},\mathrm{III}) e(\mathrm{III},\mathrm{II})) \end{aligned}$$

and this equals to 1; in fact, in the presentation (6.1), the path in the right side of the above equality is equal to $e(\mathrm{VII},\mathrm{VIII})$ and then vanishes. Similarly, we have

$$\begin{aligned} & g(\tilde{p}_{A_0}) h(q) g(\tilde{p}_{A_0})^{-1} \\ &= h(\tilde{\rho}_{A_0}({p_{x^{A_0}}}^{-1} q p_{x^{A_0}})) \\ &= h(\tilde{\rho}_{A_0}(e(\mathrm{VII},\mathrm{VI}) e(\mathrm{VI},\mathrm{V}) e(\mathrm{V},\mathrm{I}) e(\mathrm{I},\mathrm{II}) \\ &\qquad \cdot e(\mathrm{II},\mathrm{III}) e(\mathrm{III},\mathrm{IV}) e(\mathrm{IV},\mathrm{V}) e(\mathrm{V},\mathrm{VI}) e(\mathrm{VI},\mathrm{VII}))) \\ &= h(e(\mathrm{II},\mathrm{III}) e(\mathrm{III},\mathrm{IV}) e(\mathrm{IV},\mathrm{VIII}) e(\mathrm{VIII},\mathrm{VII}) \\ &\qquad \cdot e(\mathrm{VII},\mathrm{VI}) e(\mathrm{VI},\mathrm{V}) e(\mathrm{V},\mathrm{IV}) e(\mathrm{IV},\mathrm{III}) e(\mathrm{III},\mathrm{II})) \end{aligned}$$

and this equals to $h(q)^{-1}$ by (6.1). Thus we have

$$G_{x,x} = \left\langle h(q), g(\tilde{p}_{A_0}) \mid g(\tilde{p}_{A_0})^2 = 1, g(\tilde{p}_{A_0}) h(q) g(\tilde{p}_{A_0})^{-1} = h(q)^{-1} \right\rangle.$$

This implies that $\{1, g(\widetilde{p}_{A_0})\}$ forms a subgroup of $G_{x,x}$ isomorphic to $\mathcal{A}_x$, so we have

$$G_{x,x} \simeq H_{x,x} \rtimes \mathcal{A}_x \simeq \mathbb{Z} \rtimes (\mathbb{Z}/2\mathbb{Z}),$$

where $1 \in \mathbb{Z}/2\mathbb{Z}$ acts on $\mathbb{Z}$ as multiplication by -1. Moreover, put $a = g(\widetilde{p}_{A_0})$ and $b = ah(q)$. Then we have

$$G_{x,x} = \left\langle a, b \mid a^2 = 1, b^2 = 1 \right\rangle \simeq \widetilde{A}_1.$$

4. We determine the structure of the Coxeter system $(W(\Phi_{[x]}^{\perp}), \widetilde{S}_x)$. According to the result of [14], or by direct computing, we have

$$\widetilde{\Pi}_x = \{\widetilde{\gamma}, h(q) \cdot \widetilde{\gamma}\}, \text{ where } \widetilde{\gamma} = \sqrt{2}\,\alpha_7 + \alpha_8 = s_7 \cdot \alpha_8.$$

Put $\beta = \widetilde{\gamma}$, $\beta' = h(q) \cdot \widetilde{\gamma}$. Then $\beta' = \sqrt{2}\,\delta - \beta$, where

$$\delta = \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + 2\alpha_5 + 2\alpha_6 + 2\alpha_7 + \sqrt{2}\,\alpha_8$$

is the null root of $\widetilde{B}_7$.

This implies $\langle \beta, \beta' \rangle = -1$, so $(W(\Phi_{[x]}^{\perp}), \widetilde{S}_x)$ is of type $\widetilde{A}_1$. Further, we have $g(\widetilde{p}_{A_0}) \cdot \beta = \beta'$, $h(q) \cdot \beta = \beta'$, and so Proposition 3.7 implies $g(\widetilde{p}_{A_0}) \cdot \beta' = \beta$, $h(q) \cdot \beta' = \beta$. Then

$$a \cdot \beta = \beta',\ a \cdot \beta' = \beta,\ b \cdot \beta = \beta,\ b \cdot \beta' = \beta'.$$

So we have $C'_{x,x} \simeq \widetilde{A}_1 \rtimes \widetilde{A}_1$, where one of the generators of right $\widetilde{A}_1$ (that is, b) acts trivially on left $\widetilde{A}_1$ and the another (that is, a) acts as an involution of the Coxeter graph of left $\widetilde{A}_1$.

Summarizing, we have $C_{x,x} \simeq (\mathbb{Z}/2\mathbb{Z})^3 \times (\widetilde{A}_1 \rtimes \widetilde{A}_1)$.

In this example, $G_{x,x}$ is isomorphic to the semidirect product of $H_{x,x}$ by $\mathcal{A}_x$, and $H_{x,x}$ forms a free group. But these properties may fail in general.

Let (W, S) be as in Fig. 7 and let $x = (1, 2, 4, 5, 7, 8)$. Then, similarly to

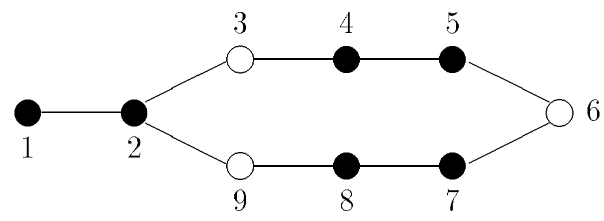


Fig. 7. Coxeter graph of another example

the previous example, it can be shown that

$$W_{[x]^{\mathrm{iso}}} = 1,\ W(\Phi_{[x]}^{\perp}) = 1,\ G_{x,x} \simeq \mathbb{Z}^2,\ H_{x,x} \simeq (2\mathbb{Z})^2 \subset \mathbb{Z}^2.$$

Thus $H_{x,x}$ is not a free group, and $G_{x,x}$ is not isomorphic to any semidirect product of $H_{x,x}$ by a group, since $G_{x,x}$ has no subgroup isomorphic to $G_{x,x}/H_{x,x} \simeq (\mathbb{Z}/2\mathbb{Z})^2$.

At last of this section, we give some remarks on the preceding results mentioned in Introduction.

First, let $x \in S^{(*)}$, $[x] = S$. Then by definition, $W_{[x]^{\mathrm{iso}}}$ is generated by all $w_0(J)$ where J runs over the connected components of S satisfying $|J| = 1$, while obviously $W(\Phi_{[x]}^{\perp}) = 1$. On the other hand, $\mathcal{H}_{\sim x}$ consists of only a single point x and so $H_{x,x} = 1$. This implies that $\mathcal{A}_x = \{A \subset \mathrm{CO}_{<\infty}^{>1}(x) \mid x^A = x\}$, so $\mathcal{A}_x$ is generated (as a group) by all $A \subset \Lambda(x)$ such that x_A is a connected component of $[x]$ of finite type, $|x_A| \geqslant 2$ and $\sigma_{x_A} = \mathrm{id}_{x_A}$. Hence by Corollary 5.19 (i), the well-known result on $Z_W(W_S)$ is in fact recovered.

Secondly, we also recover the result of Brink [2] by using our result; we check that $Z_W(s)$ $(s \in S)$ is the semidirect product of $W(\Phi_{\{s\}}^{\perp} \cup \{\alpha_s\})$ by a group isomorphic to the fundamental group of the odd Coxeter graph of (W, S). The *odd Coxeter graph* Γ^{odd} is the subgraph of Γ obtained by deleting all edges labeled an even number or ∞.

Now for $x = (s)$, we have

$$W_{[x]^{\mathrm{iso}}} \times W(\Phi_{[x]}^{\perp}) = \langle s \rangle \times W(\Phi_{\{s\}}^{\perp}) = W(\Phi_{\{s\}}^{\perp} \cup \{\alpha_s\}).$$

On the other hand, we have $G_{x,x} = H_{x,x}$ since $\mathcal{A}_x = \{\emptyset\}$. Further, $\mathcal{H}_{\sim x}$ is considered as a connected component of Γ^{odd} containing s; in fact, each $y \in V(\mathcal{H}_{\sim x})$ is identified with $t \in S$ such that $y = (t)$, and then y is adjacent to $z = (t')$ if and only if $(y, t') \in \mathcal{B} \smallsetminus \mathcal{B}^{\varphi}$, or equivalently $t \neq t'$ and $m_{t,t'}$ is odd (cf. Fig. 1). Now we have $R_1(x) = \emptyset$; in fact, if $p \in \mathcal{P}(\mathcal{H})_{y,y}$ satisfies the condition for $R_1(x)$, then p is a nontrivial cycle in Γ_I, but this is impossible since $I_{\sim y}$ is of finite type. This implies that $\ker h_x$ is trivial, so $H_{x,x}$ is anti-isomorphic to $\overline{\mathcal{P}}(\mathcal{H})_{x,x}$ which is a free group, as required.

Finally, we deal with the case where (W, S) is even (that is, every $m_{s,s'}$ is even or ∞) considered in the recent work by Bahls and Mihalik [1]. Our approach is different from that in [1], but gives the same generators of $Z_W(W_I)$ as [1] as follows.

In this case, $\mathcal{H}_{\sim x}$ consists of only a single point x for any $x \in S^{(*)}$, by Fig. 1. Now similarly to the case of $Z_W(W_S)$, $G_{x,x}$ is generated by all $w_0(J)$ where J runs over the connected components of $[x]$ of finite type satisfying $|J| \geqslant 2$ and $\sigma_J = \mathrm{id}_J$. Since (W, S) is even, a connected component J satisfies this condition if and only if $J = \{s, s'\}$ for some $s, s' \in S$ such that $m_{s,s'}$ is an even number greater than 2. For such J, we have $w_0(J) = (ss')^{m_{s,s'}/2} \in W_{[x]}$, and so $G_{x,x}$ is contained in the center of $C_{x,x}$. Note that $G_{x,x}$ is the direct products of copies of $\mathbb{Z}/2\mathbb{Z} \simeq A_1$ and so is an even Coxeter group. These generators $w_0(J)$ along with the generators of the remaining factor $W_{[x]^{\mathrm{iso}}} \times W(\Phi_{[x]}^{\perp})$ given by the result in [14] coincide with the generators given in [1].

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