UTMS 2002–9 February 15, 2002

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ABSTRACT. Hara [Ha3] and Smith [Sm2] independently proved that in a normal $\mathbb{Q}$-Gorenstein ring of characteristic $p \gg 0$, the test ideal coincides with the multiplier ideal associated to the trivial divisor. We extend this result for a pair (R, Δ) of a normal ring R and an effective $\mathbb{Q}$-Weil divisor Δ on $\operatorname{Spec} R$. As a corollary, we obtain the equivalence of strongly F-regular pairs and klt pairs.

1. INTRODUCTION

Recently it turned out that there exists a relation between multiplier ideals and tight closure. Precisely speaking, it was proved that some algebraic statements established by multiplier ideals could also be understood via tight closure, for example, Briançon-Skoda theorem (see [BS], [HH1], [La]), the problem concerning the growth of symbolic powers of ideals in regular local rings (see [ELS], [HH3]), etc. The purpose of this paper is to give an interpretation of multiplier ideals via tight closure.

The theory of tight closure was introduced by Hochster and Huneke [HH1], using the Frobenius map in characteristic $p > 0$. In this theory, test ideals play a central role. On the other hand, multiplier ideals, for which we have the strong vanishing theorem, are fundamental tools in birational geometry. Hara [Ha3] and Smith [Sm2] independently proved that in a normal $\mathbb{Q}$-Gorenstein ring of characteristic $p \gg 0$, the test ideal coincides with the multiplier ideal associated to the trivial divisor. Since the real worth of multiplier ideals is displayed in considering pairs, we attempt to extend this result for pairs. Here, by a *pair*, we mean a pair (R, Δ) of a normal ring R and an effective $\mathbb{Q}$-Weil divisor Δ on $\operatorname{Spec} R$.

Hara generalized the notion of tight closure to that for pairs, which is called Δ-tight closure. Using the Δ-tight closure operation, we introduce the geometric test ideal for pairs which is a generalization of the notion of test ideal. We denote by $\tau(R, \Delta)$ the geometric test ideal of (R, Δ) and by $\mathcal{J}(Y, \Delta)$ the multiplier ideal of (Y, Δ). Then, our main result is stated as follows:

Theorem 4.2. *Let $(R, \mathfrak{m})$ be a normal local ring essentially of finite type over a field of characteristic zero and Δ an effective $\mathbb{Q}$-Weil divisor on $Y = \operatorname{Spec} R$ such that $K_Y + \Delta$ is $\mathbb{Q}$-Cartier. Then, in characteristic $p \gg 0$,*

$$\tau(R, \Delta) = \mathcal{J}(Y, \Delta).$$

As a corollary of the main theorem, we get the equivalence of "F-singulari-ties of pairs" and "singularities of pairs."

The notions of F-regular and F-pure rings, which are closely related to the theory of tight closure, were defined by Hochster and Huneke [HH1] and Hochster and Roberts [HR] respectively. Recently it became clear that F-singularities (F-regular and F-pure rings) correspond to singularities arising in birational geometry (Kawamata log terminal and log canonical singularities). See [Ha2], [HW], [MS], [Sm1]. The notions of Kawamata log terminal (or klt for short) and log canonical (or for short lc) singularities are defined not only for normal rings but also for pairs, and it is these "singularities of pairs" that play a very important role in birational geometry. Therefore Hara and K.-i. Watanabe [HW] generalized the notions of F-singularities to those for pairs, and they conjectured the equivalence of "F-singularities of pairs" and "singularities of pairs." We prove their conjectures.

The geometric test ideal $\tau(R, \Delta)$ defines the locus of non-F-regular points of (R, Δ) in $\operatorname{Spec} R$. Likewise, the multiplier ideal $\mathcal{J}(\operatorname{Spec} R, \Delta)$ defines the locus of non-klt points of $(\operatorname{Spec} R, \Delta)$. Hence we obtain the following result as a direct consequence of the main result.

Corollary 4.3 ([HW, Conjecture 5.1.1]). *Let $(R, \mathfrak{m})$ be a normal local ring essentially of finite type over a field of characteristic zero and Δ an effective $\mathbb{Q}$-Weil divisor on $Y = \operatorname{Spec} R$ such that $K_Y + \Delta$ is $\mathbb{Q}$-Cartier. Then, (Y, Δ) is klt if and only if (R, Δ) is of open strongly F-regular type.*

Acknowledgements. The author wishes to thank Professor Toshiyuki Katsura, his research supervisor, for helpful advice and warm encouragement. He is also grateful to Professor Kei-ichi Watanabe for various comments and many suggestions, Professor Nobuo Hara for valuable information about the subadditivity theorem for "$\mathfrak{a}$-test ideals" and suggestions about the proof of Theorem 3.13, and Yasunari Nagai for many discussions about birational geometry.

2. Preliminaries

2.1. **F-singularities of pairs.** First we briefly review definitions and basic properties on "F-singularities of pairs," which were defined by Hara and K.-i. Watanabe. Refer to [HW] for details.

Throughout this paper, all rings are commutative Noetherian integral domains with identity. Let R be an integral domain of characteristic $p > 0$ and $F : R \to R$ the Frobenius map which sends x to x^p. Since R is reduced, we can identify $F : R \to R$ with the natural inclusion map $R \hookrightarrow R^{1/p}$. R is called *F-finite* if $R \hookrightarrow R^{1/p}$ is a finite map. For example, any algebra essentially of finite type over a perfect field is F-finite. We also remark that if R is F-finite, then R is excellent [Ku].

Notation. Let R be a normal domain with quotient field K. A $\mathbb{Q}$-Weil divisor D on $Y = \operatorname{Spec} R$ is a linear combination $D = \sum_{i=1}^{r} a_i D_i$ of irreducible reduced subschemes $D_i \subset Y$ of codimension 1 with rational coefficients a_i. The round-up and round-down of D is defined by $\lceil D \rceil = \sum_{i=1}^r \lceil a_i \rceil D_i$ and $\lfloor D \rfloor = \sum_{i=1}^r \lfloor a_i \rfloor D_i$. We also denote

$$R(D) = \{x \in K \mid \operatorname{div}_R(x) + D \geq 0\}.$$

Definition 2.1 ([HW, Definition 2.1])**.** Let R be an F-finite normal domain of characteristic $p > 0$ and Δ an effective $\mathbb{Q}$-Weil divisor on $\operatorname{Spec} R$.

(1) (R, Δ) is said to be *F-pure* if the inclusion map $R \hookrightarrow R((q-1)\Delta)^{1/q}$ splits as an R-module homomorphism for every $q = p^e$.
(2) (R, Δ) is said to be *strongly F-regular* if for every nonzero element $c \in R$, there exists $q = p^e$ such that $c^{1/q}R \hookrightarrow R((q-1)\Delta)^{1/q}$ splits as an R-module homomorphism.

Remark 2.2. (i) R is F-pure (resp. strongly F-regular) if and only if $(R, 0)$ is F-pure (resp. strongly F-regular). Refer to [HH1], [HH2] or [HR] for strongly F-regular and F-pure rings. Here we only note that the following implications hold for rings.

$$\begin{array}{ccccc} \text{regular} & \Rightarrow & \text{strongly F-regular} & \Rightarrow & \text{normal and Cohen-Macaulay} \\ & & \Downarrow & & \\ & & \text{F-pure} & & \end{array}$$

(ii) We can replace $R((q-1)\Delta)^{1/q}$ by $R(\lceil q\Delta \rceil)^{1/q}$ in the above definition of strong F-regularity.

Basic Properties ([HW, Proposition 2.2]). Let (R, Δ) be as above.

(i) Strong F-regularity implies F-purity.
(ii) (R, Δ) is strongly F-regular if and only if for every nonzero element $c \in R$, there exists q' such that $c^{1/q}R \to R(q\Delta)^{1/q}$ splits as an R-module homomorphism for all $q = p^e \geq q'$.
(iii) If (R, Δ) is strongly F-regular, then $\lfloor \Delta \rfloor = 0$.
(iv) If (R, Δ) is F-pure, then $\lceil \Delta \rceil$ is reduced.
(v) If (R, Δ) is F-pure (resp. strongly F-regular), so is (R, Δ') for every effective $\mathbb{Q}$-Weil divisor $\Delta' \leq \Delta$.

Example 2.3. Let $R = k[[x_1, \dots, x_n]]$ be an n-dimensional complete regular local ring over a field k of characteristic $p > 0$ and $\Delta = \operatorname{div}_R(x_1^{d_1} + \cdots + x_n^{d_n})$. Assume that p is sufficiently large and let $t_0 = \min\{1, \sum_{i=1}^n \frac{1}{d_i}\}$. Then, $(R, t\Delta)$ is strongly F-regular if and only if $t < t_0$. If $(R, t\Delta)$ is F-pure, then $t \leq t_0$. When $\sum_{i=1}^n \frac{1}{d_i} > 1$, then $(R, t_0\Delta)$ is always F-pure. On the other hand when $\sum_{i=1}^n \frac{1}{d_i} \leq 1$, then $(R, t_0\Delta)$ is F-pure if and only if $p \equiv 1 \mod d_i$ for every $i = 1, \dots, n$.

The notions of F-regularity and F-purity are also defined for rings of characteristic zero as follows.

Definition 2.4. Let R be a finitely generated algebra over a field k of characteristic zero and Δ an effective $\mathbb{Q}$-Weil divisor on $\operatorname{Spec} R$. The pair (R, Δ) is said to be of *open F-pure type* (resp. *open strongly F-regular type*) if there exist a finitely generated $\mathbb{Z}$-subalgebra A of k, a finitely generated A-algebra R_A and an effective $\mathbb{Q}$-Weil divisor Δ_A on $\operatorname{Spec} R_A$, with a flat structure map $A \to R_A$ such that

(1) $(A \to R_A) \otimes_A k \cong (k \to R)$ and $\Delta_A \otimes_A k \cong \Delta$.
(2) $(R_\kappa, \Delta_\kappa)$ is F-pure (resp. strongly F-regular) for every closed point s in a dense open subset of $\operatorname{Spec} A$, where $\kappa = \kappa(s)$ denotes the residue field of $s \in \operatorname{Spec} A$, $R_\kappa = R_A \otimes_A \kappa(s)$ and $\Delta_\kappa = \Delta_A \otimes_A \kappa(s)$.

(R, Δ) is said to be of *dense F-pure type* if in the above condition (2) "dence open" is replaced by "dense."

Example 2.5. Let $R = k[[x_1, \dots, x_n]]$ be an n-dimensional complete regular local ring over a field k of characteristic zero and $\Delta = \operatorname{div}_R(x_1^{d_1} + \cdots + x_n^{d_n})$. Then, $(R, t\Delta)$ is of open strongly F-regular type (resp. of dense F-pure type) if and only if $\min\{1, \sum_{i=1}^n \frac{1}{d_i}\} > t$ (resp. $\min\{1, \sum_{i=1}^n \frac{1}{d_i}\} \geq t$).

2.2. **Birational Geometry.** We recall the definition and fundamental properties of singularities which appear in the Mori theory, and of multiplier ideal sheaves. Refer to [KM] and [La] for details.

Let Y be a normal variety over an algebraically closed field of characteristic zero and Δ a $\mathbb{Q}$-Weil divisor on Y such that $K_Y + \Delta$ is $\mathbb{Q}$-Cartier, that is, $r(K_Y + \Delta)$ is a Cartier divisor for some positive integer r, where K_Y is the canonical divisor of Y. Let $f : X \to Y$ be a resolution of singularities such that $\cup_{i=1}^s E_i + f_*^{-1}\Delta$ has simple normal crossing support, where $\operatorname{Exc}(f) = \cup_{i=1}^s E_i$ is the exceptional divisor of f and $f_*^{-1}\Delta$ is the strict transform of Δ in X. We denote by K_X the canonical divisor of X. Then, for some integers $b_1, \dots, b_s$,

$$r(K_X + f_*^{-1}\Delta) \underset{\text{lin.}}{\sim} f^*(r(K_Y + \Delta)) + \sum_{i=1}^s b_i E_i.$$

Hence we have

$$K_X + f_*^{-1}\Delta \underset{\mathbb{Q}\text{-lin.}}{\sim} f^*(K_Y + \Delta) + \sum_{i=1}^s a_i E_i,$$

where $a_i = \frac{b_i}{r}$ $(i = 1, \dots, s)$.

Definition 2.6. Under the same notation as above:

(1) We say that the pair (Y, Δ) is *Kawamata log terminal* (or *klt* for short) if $a_i > -1$ for every $i = 1, \dots, s$ and $\lfloor \Delta \rfloor \leq 0$.
(2) We say that the pair (Y, Δ) is *log canonical* (or *lc* for short) if $a_i \geq -1$ for every $i = 1, \dots, s$ and the coefficient of Δ in each irreducible component is less than or equal to 1.
(3) The *multiplier ideal sheaf* $\mathcal{J}(Y, \Delta)$ associated to Δ is defined to be

$$\mathcal{J}(Y, \Delta) = f_*\mathcal{O}_X(\lceil K_X - f^*(K_Y + \Delta) \rceil).$$

Remark 2.7. (i) The above definitions do not depend on the choice of a desingularization $f : X \to Y$.

(ii) When Δ is effective, $\mathcal{J}(Y,\Delta)$ is indeed an ideal sheaf. However in case Δ is not effective, it is generally not a submodule of $\mathcal{O}_Y$ but a fractional ideal sheaf.

Basic Properties. In the situation of the above definition:

(i) For every $\mathbb{Q}$-Weil divisor $\Delta' \le \Delta$ on Y, $\mathcal{J}(Y,\Delta') \supseteq \mathcal{J}(Y,\Delta)$.
(ii) For every Cartier divisor Δ' on Y, $\mathcal{J}(Y,\Delta+\Delta') = \mathcal{J}(Y,\Delta) \otimes_Y \mathcal{O}_Y(-\Delta')$.
(iii) The pair (Y,Δ) is klt if and only if $\mathcal{J}(Y,\Delta) \supseteq \mathcal{O}_Y$. In particular when Δ is effective, (Y,Δ) is klt if and only if $\mathcal{J}(Y,\Delta) = \mathcal{O}_Y$.

Proposition 2.8. *Let $f : X \to Y$ be a finite covering of normal varieties which is étale in codimension* 1 *and Δ an effective $\mathbb{Q}$-Weil divisor on Y. Then*

$$\mathcal{J}(Y,\Delta) = \mathcal{J}(X, f^*\Delta) \cap \mathcal{O}_Y.$$

Proposition 2.9 ([DEL])**.** *(1) (Restriction Theorem) Let Y be a normal Cohen-Macaulay quasi-projective variety, Δ an effective divisor on Y such that $K_Y + \Delta$ is $\mathbb{Q}$-Cartier, and H a normal irreducible Cartier divisor which is not in the support of Δ. Then*

$$\mathcal{J}(H, \Delta|_H) \subseteq \mathcal{J}(Y,\Delta) \cdot \mathcal{O}_H.$$

(2) (Subadditivity Theorem) Let Y be a smooth quasi-projective variety, and Δ_1 and Δ_2 be any two effective $\mathbb{Q}$-divisors on Y. Then

$$\mathcal{J}(Y,\Delta_1+\Delta_2) \subseteq \mathcal{J}(Y,\Delta_1) \cdot \mathcal{J}(Y,\Delta_2).$$

Example 2.10. (1) When Y is non-singular and Δ is any $\mathbb{Q}$-Weil divisor on Y with simple normal crossing support, then $\mathcal{J}(Y,\Delta) = \mathcal{O}_Y(-\lfloor \Delta \rfloor)$.
(2) Let $Y = \mathbb{C}^n$ with coordinates $x_1,\dots,x_n$ and $\Delta = \operatorname{div}_Y(x_1^{d_1} + \cdots + x_n^{d_n})$. Then, $(Y, t\Delta)$ is klt (resp. lc) if and only if $\min\{1, \sum_{i=1}^n \frac{1}{d_i}\} > t$ (resp. $\min\{1, \sum_{i=1}^n \frac{1}{d_i}\} \ge t$).
(3) Let $Y = \mathbb{C}^d$ with coordinates $x_1,\dots,x_d$ and $\Delta = \operatorname{div}_Y(x_1^{d+1} + \cdots + x_d^{d+1})$. Then, $\mathcal{J}(Y, \frac{d}{d+1}\Delta) = (x_1,\dots,x_d)$.

3. Δ-tight closure

In this section, we introduce the notion of Δ-tight closure defined by Hara and see that the Δ-tight closure operation satisfies properties similar to those of the "usual" tight closure operation (see [Hu] for the theory of "usual" tight closure). Moreover, using the Δ-tight closure operation, we define the geometric test ideal which is a generalization of the "usual" test ideal.

Notation. Let R be a normal domain of characteristic $p > 0$ and Δ an effective $\mathbb{Q}$-Weil divisor on $\operatorname{Spec} R$.

- We always use the letter q (resp. q', q'') for a power p^e (resp. $p^{e'}$, $p^{e''}$) of p.
- For any ideal I in R, we denote by $I^{[q]}$ the ideal of R generated by the q th powers of elements of I.

- For any divisorial ideal J of R (i.e., $J = R(D)$ for some unique integral Weil divisor D), we denote by $J^{(m)}$ the reflexive hull of J^m. If $J = R(D) = R(\lfloor D \rfloor)$, then $J^{(m)} = R(m\lfloor D \rfloor)$.
- The notation ${}^eR((q-1)\Delta)$ denotes $R((q-1)\Delta)$ itself, but viewed as an R-module via the e-times Frobenius map $F^e : R \to R((q-1)\Delta)$.
- When $(R, \mathfrak{m})$ is local, we denote by E_R the injective hull of the residue field $R/\mathfrak{m}$.

Definition 3.1 (Hara)**.** Let $N \subseteq M$ be modules over an F-finite normal domain R of characteristic $p > 0$ and Δ an effective $\mathbb{Q}$-Weil divisor on $\operatorname{Spec} R$. We denote by $F^e : M = M \otimes_R R \to M \otimes_R {}^eR((q-1)\Delta)$ the e-times Frobenius map induced on M which sends $z \in M$ to $z^q := F^e(z) = z \otimes 1 \in M \otimes_R {}^eR((q-1)\Delta)$. Set $N_M^{[q]\Delta} := \operatorname{Im}(F^e(N) \to F^e(M))$. Then the Δ-*tight closure* $N_M^{*\Delta} \subseteq M$ of N in M is defined as follows: $z \in N_M^{*\Delta}$ if and only if there exists a nonzero element $c \in R$ such that $cz^q := z \otimes c \in N_M^{[q]\Delta}$ for all $q = p^e \gg 0$.

$$\begin{tikzcd} N = N \otimes_R R \arrow[r, hook] \arrow[d, "F^e"] & M = M \otimes_R R \arrow[d, "F^e"] \\ N \otimes_R {}^eR((q-1)\Delta) \arrow[r] & M \otimes_R {}^eR((q-1)\Delta) \end{tikzcd}$$

Moreover, the *finitistic* Δ-*tight closure* $N_M^{*\Delta fg} \subseteq M$ of N in M is defined to be $N_M^{*\Delta fg} := \bigcup_{M'} N_{M'}^{*\Delta}$, where M' runs through all finitely generated R-submodules of M which contain N.

Remark 3.2. (i) When $\Delta = 0$, Δ-tight closure coincides with "usual" tight closure.
(ii) In general, $N_M^{*\Delta} \subsetneq (N_M^{*\Delta})_M^{*\Delta}$. In this sense, the Δ-tight closure operation is not a "closure operation."
(iii) Let I be an ideal in R. Then, $I_R^{[q]\Delta} = I^{[q]}R((q-1)\Delta)$.
(iv) We can replace ${}^eR((q-1)\Delta)$ by ${}^eR(\lceil q\Delta \rceil)$ in the above definition.
(v) In general, $N_M^{*\Delta fg} \subseteq N_M^{*\Delta}$. If M itself is finitely generated, then $N_M^{*\Delta fg} = N_M^{*\Delta}$.

Basic Properties. In the situation of the above definition,
(i) $N \subseteq N_M^{*\Delta}$.
(ii) $N_M^{*\Delta}/N = 0_{M/N}^{*\Delta}$.
(iii) For any effective $\mathbb{Q}$-Weil divisor $\Delta' \le \Delta$ on $\operatorname{Spec} R$, $N_M^{*\Delta'} \subseteq N_M^{*\Delta}$.
(iv) For any effective Cartier divisor Δ' on $\operatorname{Spec} R$, $N_M^{*(\Delta+\Delta')} = N_M^{*\Delta} : R(-\Delta')$.
(v) If (R, Δ) is strongly F-regular, then $I^{*\Delta} = I$ for every ideal I in R.

Strong F-regularity can be characterized via Δ-tight closure.

Lemma 3.3. *Let $(R, \mathfrak{m})$ be an F-finite normal local ring of characteristic $p > 0$ and Δ an effective $\mathbb{Q}$-Weil divisor on* $\operatorname{Spec} R$*. Then (R, Δ) is strongly F-regular if and only if $0_E^{*\Delta} = 0$, where $E = E(R/\mathfrak{m})$ is the injective hull of the residue field $R/\mathfrak{m}$.*

Proof. The proof is essentially the same as that for the no boundary case [Ha1, Proposition 2.1].

Assume that (R, Δ) is strongly F-regular. If $z \in 0_E^{*\Delta}$, then there exists a nonzero element $c \in R$ such that $cF^e(z) = 0$ for all $q = p^e \gg 0$. Let

$$\phi_c^{(e)} : \mathrm{Hom}_R(R((q-1)\Delta)^{1/q}, R) \to \mathrm{Hom}_R(R, R) = R$$

be an R-module homomorphism induced by the R-linear map $R \xrightarrow{c^{1/q}} R((q-1)\Delta)^{1/q}$ for each $q = p^e$. Since (R, Δ) is strongly F-regular, $\phi_c^{(e)}$ is surjective for all $q = p^e \gg 0$. Since the R-module homomorphism $cF^e : E \to E \otimes_R {}^eR((q-1)\Delta)$ which sends z to cz^q is the Matlis dual of $\phi_c^{(e)}$, cF^e is injective for every $q = p^e \gg 0$. Hence $z = 0$. Conversely, suppose that $0_E^{*\Delta} = 0$, and fix any nonzero element $c \in R$. If z is a nonzero element of the socle $(0 : \mathfrak{m})_E$ of E, then there exists $q = p^e$ such that $cF^e(z) \neq 0$. Since $(0 : \mathfrak{m})_E$ is an one-dimensional $R/\mathfrak{m}$-vector space, we can take q which works for every $z \in (0 : \mathfrak{m})_E$. Then cF^e is injective on $(0 : \mathfrak{m})_E$. Since E is an essential extension of $(0 : \mathfrak{m})_E$, cF^e itself is injective. Taking the Matlis dual of cF^e, $\phi_c^{(e)}$ is surjective, namely (R, Δ) is strongly F-regular. $\square$

We introduce Δ-test elements which are very useful to show the propositions about Δ-tight closure.

Definition 3.4. Let R be an F-finite normal domain of characteristic $p > 0$ and Δ an effective $\mathbb{Q}$-Weil divisor on $\mathrm{Spec}\, R$. A nonzero element $c \in R$ is called Δ-*test element* if for each ideal I in R, $x \in I^{*\Delta}$ if and only if $cx^q \in I^{[q]}R(\lceil (q-1)\Delta \rceil)$ for all $q = p^e$.

By the following lemma, Δ-test elements always exist. The following lemma generalizes [HH2, Theorem 3.3].

Lemma 3.5. *Let R be an F-finite normal domain of characteristic $p > 0$ and Δ an effective $\mathbb{Q}$-Weil divisor on* $\mathrm{Spec}\, R$. *Let $c \in R(-\Delta_{\mathrm{red}})$ be any nonzero element such that the localization R_c with respect to c is strongly F-regular, where Δ_{red} is the reduced divisor whose support is equal to that of Δ.*

(1) *(R, Δ) is strongly F-regular if and only if there exists $q = p^e$ such that $c^{1/q}R \hookrightarrow R(\lceil (q-1)\Delta \rceil)^{1/q}$ splits as an R-module homomorphism.*
(2) *c^n is a Δ-test element for some positive integer n.*

Proof. Let $d \in R$ be any nonzero element.

Claim 1. For some q' and q'' there exists an R-module homomorphism

$$R(\lceil (q'-1)\Delta \rceil)^{1/q'} \to R, \quad d^{1/q'} \mapsto c^{q''}.$$

In particular when $d = 1$, then we can take p as q'.

Proof of Claim 1. Since $(R_c, \Delta_c) = (R_c, 0)$ is strongly F-regular, for some q' there exists an R_c-module homomorphism

$$g : (R((q'-1)\Delta)^{1/q'})_c \to R_c, \quad d^{1/q'} \mapsto 1.$$

By the F-finiteness of R, $c^{q''-1}g(R((q'-1)\Delta)^{1/q'}) \subseteq R$ for sufficiently large q''. Since $c \cdot R(\lceil (q'-1)\Delta \rceil)^{1/q'} \subset R((q'-1)\Delta)^{1/q'}$, we obtain an R-linear map $R(\lceil (q'-1)\Delta \rceil)^{1/q'} \to R$ taking $d^{1/q'}$ to $c^{q''}$. □

First we will show (1). Suppose that there exists $q = p^e$ such that $c^{1/q}R \hookrightarrow R(\lceil (q-1)\Delta \rceil)^{1/q}$ splits as an R-module homomorphism. By Claim 1, we get an $R^{1/qq''}$-linear map

$$R(\lceil (qq'q''-1)\Delta \rceil)^{1/qq'q''} \to R(\lceil (qq''-1)\Delta \rceil)^{1/qq''}, \quad d^{1/qq'q''} \mapsto c^{q''/qq''} = c^{1/q},$$

tensoring with $R(\lceil (qq''-1)\Delta \rceil)$ and taking qq''-th roots. On the other hand, replacing q'' suitably, there exists an R-module homomorphism $R(\lceil (q''-1)\Delta \rceil)^{1/q''} \to R$ which sends 1 to 1. Tensoring this with $R(\lceil (q-1)\Delta \rceil)$ and taking q-th roots, we obtain an $R^{1/q}$-linear map $R(\lceil (qq''-1)\Delta \rceil)^{1/qq''} \to R(\lceil (q-1)\Delta \rceil)^{1/q}$ which sends 1 to 1. By composing these maps, we get the following R-module homomorphism

$$\begin{gathered} R(\lceil (qq'q''-1)\Delta \rceil)^{1/qq'q''} \to R(\lceil (qq''-1)\Delta \rceil)^{1/qq''} \to R(\lceil (q-1)\Delta \rceil)^{1/q} \to R, \\ d^{1/qq'q''} \longmapsto c^{1/q} \longmapsto c^{1/q} \longmapsto 1. \end{gathered}$$

This establishes (1).

Now we will prove (2). It follows from Claim 1 that there exists an R-module homomorphism

$$h : R(\lceil (p-1)\Delta \rceil)^{1/p} \to R, \quad 1 \mapsto c,$$

replacing $c^{q''}$ by c. Then it is enough to show that c^3 is a Δ-test element.

Claim 2. For every $q = p^e$, there exists an R-module homomorphism

$$g_e : R(\lceil (q-1)\Delta \rceil)^{1/q} \to R, \quad 1 \mapsto c^2.$$

Proof of Claim 2. When $q = p$, then we may set $g_1 = c \cdot h$. Suppose that the assertion holds for $q = p^e$. Then, by the same argument as in the proof of (1), we obtain an $R^{1/p}$-linear map $R(\lceil (pq-1)\Delta \rceil)^{1/pq} \to R(\lceil (p-1)\Delta \rceil)^{1/p}$ which sends 1 to $c^{2/p}$. We may compose this with an $R^{1/p}$-module homomorphism $R(\lceil (p-1)\Delta \rceil)^{1/p} \to R(\lceil (p-1)\Delta \rceil)^{1/p}$ which sends 1 to $c^{(p-2)/p}$, and then with h.

$$\begin{gathered} R(\lceil (pq-1)\Delta \rceil)^{1/pq} \to R(\lceil (p-1)\Delta \rceil)^{1/p} \to R(\lceil (p-1)\Delta \rceil)^{1/p} \to R, \\ 1 \longmapsto c^{2/p} \longmapsto c \longmapsto c^2. \end{gathered}$$

This is a required homomorphism for $pq = p^{e+1}$. □

As we see in the proof of (1), there exists an $R^{1/qq''}$-module homomorphism

$$f : R(\lceil (qq'q''-1)\Delta \rceil)^{1/qq'q''} \to R(\lceil (qq''-1)\Delta \rceil)^{1/qq''}, \quad d^{1/qq'q''} \mapsto c^{1/q},$$

and thanks to Claim 2, we obtain an $R^{1/q}$-module homomorphism

$$g' : R(\lceil (qq''-1)\Delta \rceil)^{1/qq''} \to R(\lceil (q-1)\Delta \rceil)^{1/q}, \quad 1 \mapsto c^{2/q}.$$

Then the $R^{1/q}$-linear map

$$g' \circ f : R(\lceil (qq'q''-1)\Delta \rceil)^{1/qq'q''} \to R(\lceil (q-1)\Delta \rceil)^{1/q}$$

satisfies $g' \circ f(d^{1/qq'q''}) = c^{3/q}$. Thus, $dx^{qq'q''} \in I^{[qq'q'']}R(\lceil (qq'q''-1)\Delta \rceil)$ implies $c^3x^q \in I^{[q]}R(\lceil (q-1)\Delta \rceil)$. It follows that c^3 is a Δ-test element. $\square$

Under some condition, the finitistic Δ-tight closure coincides with the Δ-tight closure. In case $\Delta = 0$, the following proposition is proved in [AM], [Ha3], [Mc], [Sm2].

Proposition 3.6. *Let $(R, \mathfrak{m})$ be an F-finite normal local ring of characteristic $p > 0$ and Δ an effective $\mathbb{Q}$-Weil divisor on* $\operatorname{Spec} R$. *Let $J = R(D) \subseteq R$ be a divisorial ideal such that $D + \Delta$ is $\mathbb{Q}$-Cartier. Then $0^{*\Delta}_{H^d_{\mathfrak{m}}(J)} = 0^{*\Delta fg}_{H^d_{\mathfrak{m}}(J)}$.*

Proof. We use the same strategy as that of [Ha3]. For a sequence of elements $\mathbf{x} = x_1, \ldots, x_d$ of R and a positive integer t, we write $\mathbf{x}^t = x_1^t, \ldots, x_d^t$. For an R-module M, we denote

$$\mathcal{K}(\mathbf{x}, t, M) := \operatorname{Ker}\left(\frac{M}{(\mathbf{x})M} \xrightarrow{(x_1 \cdots x_d)^{t-1}} \frac{M}{(\mathbf{x}^t)M} \right).$$

We also denote

$$\mathcal{K}(\mathbf{x}, \infty, M) := \bigcup_{t \in \mathbb{N}} \mathcal{K}(\mathbf{x}, t, M).$$

Claim. Let $(R, \mathfrak{m})$ be a d-dimensional normal local ring of characteristic $p > 0$ and $J \subseteq R$ a divisorial ideal. Let $\mathbf{x} = x_1, \ldots, x_d$ be a system of parameters of R. Suppose that there exist a nonzero element $c \in R$ and integer $t_0 \geq 2$ such that

$$c\mathcal{K}(\mathbf{x}^{qs}, \infty, J^{[q]}R((q-1)\Delta)) \subseteq \mathcal{K}(\mathbf{x}^{qs}, t_0, J^{[q]}R((q-1)\Delta))$$

for all $s \geq 1$ and $q = p^e \gg 0$. Then $0^{*\Delta}_{H^d_{\mathfrak{m}}(J)} = 0^{*\Delta fg}_{H^d_{\mathfrak{m}}(J)}$.

Proof of Claim. It is similar to the proof of [Ha3, Theorem A.2]. $\square$

Since $r(D+\Delta)$ is a Cartier divisor for some positive integer r, let $R(r(D+\Delta)) = yR$. Fix any $a \in R(-r\Delta)$, and let $x_1 := ay \in J$. Then, by [Wi, Lemma 4.3], there exist an element $x_2 \in R$ which is not in any minimal prime divisor of x_1 and $c \in J$ such that $x_2^n J^{(n)} \subseteq c^n R$ for all $n > 0$. The sequence x_1, x_2 can be extended to a system of parameters $\mathbf{x} = x_1, \ldots, x_d$ for R. Now given any power $q = p^e$, write $q - 1 = kr + i$ for integers k and i with $0 \leq i \leq r-1$. Then we have

$$\begin{aligned} c^r x_2^q R(krD + (q-1)\Delta) &\subseteq c^r x_2^{kr} (J^{(kr)} \otimes_R R((q-1)\Delta))^{**} \\ &\subseteq c^{kr+r} R((q-1)\Delta) \\ &\subseteq J^{[q]} R((q-1)\Delta), \end{aligned}$$

where $(\ )^{**}$ denotes the reflexive hull. Since $x_1^q \in J^{[q]}$, this implies that

$$c^r (x_1 \cdots x_{i-1} x_{i+1} \cdots x_n)^{qs} R(krD + (q-1)\Delta) \subseteq J^{[q]} R((q-1)\Delta)$$

for every $s \geq 1$ and $i = 1, \dots, n$. Therefore, letting $c_1 = c^r$, we have

$$c_1 \cdot \operatorname{Im}\left(H^{d-1}\left(\mathbf{x}^{qst}; \frac{R(krD+(q-1)\Delta)}{J^{[q]}R((q-1)\Delta)}\right) \right.$$
$$\left. \to H^{d-1}\left(\mathbf{x}^{qst+qs}; \frac{R(krD+(q-1)\Delta)}{J^{[q]}R((q-1)\Delta)}\right)\right) = 0$$

for all $s, t \geq 1$. On the other hand, let $c' \in R$ be a test element (By Lemma 3.5, such c' always exists). If $z \in R$ is an element such that $z \bmod (\mathbf{x}^{qs})R(i\Delta) \in \mathcal{K}(\mathbf{x}^{qs}, \infty, R(i\Delta))$, then

$$az \in (x_1^{qst}, \dots, x_n^{qst}) : (x_1 \cdots x_n)^{qs(t-1)}$$

for some $t \geq 1$, so $az \in (x_1^{qs}, \dots, x_n^{qs})^*$ by colon-capturing [HH1]. Hence, letting $c_2 = a \cdot c'$,

$$c_2\mathcal{K}(\mathbf{x}^{qs}, \infty, R(krD+(q-1)\Delta)) \cong c_2\mathcal{K}(\mathbf{x}^{qs}, \infty, R(i\Delta)) = 0.$$

Thus, applying [Ha3, Lemma A.3] to the exact sequence

$$0 \to J^{[q]}R((q-1)\Delta) \to R(krD+(q-1)\Delta) \to \frac{R(krD+(q-1)\Delta)}{J^{[q]}R((q-1)\Delta)} \to 0,$$

we see that

$$c_1c_2\mathcal{K}(\mathbf{x}^{qs}, \infty, J^{[q]}R((q-1)\Delta)) \subseteq \mathcal{K}(\mathbf{x}^{qs}, 2, J^{[q]}R((q-1)\Delta))$$

for all $s, t \geq 1$. Thanks to the claim, we obtain the assertion. $\square$

Now we define the geometric test ideal.

Definition 3.7. Let $(R, \mathfrak{m})$ be an F-finite normal local ring of characteristic $p > 0$ and Δ an effective $\mathbb{Q}$-Weil divisor on $Y = \operatorname{Spec} R$. Then the *geometric test ideal* $\tau(R, \Delta)$ associated to Δ is defined to be $\tau(R, \Delta) := \operatorname{Ann}_R(0^{*\Delta}_{E_R}) \subseteq R$.

Remark 3.8. When $\Delta = 0$ and R is $\mathbb{Q}$-Gorenstein, the geometric test ideal coincides with the "usual" test ideal which is generated by (Δ-)test elements (see [AM], [Ha3], [Mc], [Sm2]). However even if $K_Y + \Delta$ is $\mathbb{Q}$-Cartier, the geometric test ideal may not be generated by Δ-test elements.

Basic Properties. In the situation of the above definition,

(i) For any effective $\mathbb{Q}$-Weil divisor $\Delta' \leq \Delta$ on $\operatorname{Spec} R$, $\tau(R, \Delta') \supseteq \tau(R, \Delta)$.
(ii) For any effective Cartier divisor Δ' on $\operatorname{Spec} R$, $\tau(R, \Delta + \Delta') = \tau(R, \Delta) \otimes_R R(-\Delta')$.
(iii) (R, Δ) is strongly F-regular if and only if $\tau(R, \Delta) = R$.

Proposition 3.9. *Let $(R, \mathfrak{m})$ be an F-finite normal local ring of characteristic $p > 0$ and Δ an effective $\mathbb{Q}$-Weil divisor on $Y = \operatorname{Spec} R$ such that $K_Y + \Delta$ is $\mathbb{Q}$-Cartier. Then*

$$\tau(R, \Delta) = \bigcap_I (I : I^{*\Delta}) = \bigcap_M \operatorname{Ann}_R(0^{*\Delta}_M),$$

where the intersection in the middle term is taken over all ideal I in R and the intersection in the last term is taken over all finitely generated R-module M.

Proof. By the same arguement as in the proof of [HH1, Proposition 8.23], we see that $\cap(I : I^{*\Delta}) = \cap \mathrm{Ann}_R(0_M^{*\Delta}) = \mathrm{Ann}_R(0_E^{*\Delta fg})$. Thus we may show that $\tau(R, \Delta) = \mathrm{Ann}_R(0_E^{*\Delta fg})$, but it follows from Proposition 3.6. □

Corollary 3.10. *Let $(R, \mathfrak{m})$ be an F-finite normal local ring of characteristic $p > 0$ and Δ an effective $\mathbb{Q}$-Weil divisor on $\operatorname{Spec} R$ such that $K_Y + \Delta$ is $\mathbb{Q}$-Cartier. Then (R, Δ) is strongly F-regular if and only if $I^{*\Delta} = I$ for every ideal I in R.*

The following proposition corresponds to Proposition 2.8.

Proposition 3.11. *Let $(R, \mathfrak{m}) \hookrightarrow (S, \mathfrak{n})$ be a finite local homomorphism of F-finite normal local rings of characteristic $p > 0$ which is étale in codimension 1. Let Δ_R be an effective $\mathbb{Q}$-Weil divisor on $\operatorname{Spec} R$ and Δ_S the pullbuck of Δ_R by the induced morphism $\pi : \operatorname{Spec} S \to \operatorname{Spec} R$. Assume that $\deg \pi$ is not divisible by p. Then*

$$\tau(R, \Delta_R) = \tau(S, \Delta_S) \cap R.$$

Proof. Note that, by [KM, Proposition 5.7], R is a direct summand of S as R-module. Therefore we consider E_R as a direct summard of $E_S = S \otimes_R E_R$. If $\xi \in 0_{E_R}^{*\Delta_R}$, then there exists a nonzero element $c \in R$ such that $c\xi^q = 0$ in $E_R \otimes_R {}^eR((q-1)\Delta_R)$ for all $q = p^e \gg 0$, so $c\xi^q = 0$ in $E_S \otimes_R {}^eR((q-1)\Delta_R) = E_S \otimes_S {}^eS((q-1)\Delta_S)$. This implies that $\tau(S, \Delta_S) \cap R \subseteq \tau(R, \Delta)$.

Conversely, let c be a nonzero element of $\tau(R, \Delta)$, and fix any nonzero element $d \in R$. Let $F_R^e : E_R \to E_R \otimes_R {}^eR((q-1)\Delta_R)$ (resp. $F_S^e : E_S \to E_S \otimes_S {}^eS((q-1)\Delta_S)$) be the e-times Frobenius map induced on E_R (resp. E_S). Then $c \cdot \bigcap_{e \geq e'} \operatorname{Ker} dF_R^e$ for every $q' = p^{e'}$. Since E_R is Artinian, there exists $q'' = p^{e''}$ such that $\bigcap_{e \geq e'} \operatorname{Ker} dF_R^e = \bigcap_{e'' \geq e \geq e'} \operatorname{Ker} dF_R^e$. Now we consider the following exact sequence.

$$0 \to \bigcap_{e'' \geq e \geq e'} \operatorname{Ker} dF_R^e \to E_R \xrightarrow{dF_R^e} \bigoplus_{e'' \geq e \geq e'} E_R \otimes_R {}^eR((q-1)\Delta_R).$$

Since $R \hookrightarrow S$ is étale in codimension 1, by tensoring this sequence with S over R, we get the following exact sequence (cf. [HW, Theorem 4.8]).

$$\Big(\bigcap_{e'' \geq e \geq e'} \operatorname{Ker} dF_R^e\Big) \otimes_R S \to E_S \xrightarrow{dF_S^e} \bigoplus_{e'' \geq e \geq e'} E_S \otimes_S {}^eS((q-1)\Delta_S).$$

Hence $c \cdot \bigcap_{e'' \geq e \geq e'} \operatorname{Ker} dF_S^e = 0$ for every $d \in R$. Since $R \hookrightarrow S$ is a finite extension of normal domains, so $dS \cap R \neq 0$ for all nonzero elements $d \in S$. Therefore $c \cdot \bigcap_{e'' \geq e \geq e'} \operatorname{Ker} dF_S^e = 0$ for every $d \in S$, namely $c \in \tau(S, \Delta_S)$. □

Hara [HY] imformed the author of the restriction theorem and subadditivity theorem for "$\mathfrak{a}$-test ideals." The following results which correspond to Proposition 2.9 are obtained entirely by the same argument as in [HY].

Proposition 3.12 (cf. [HY]). *(1) (Restriction Theorem) Let $(R,\mathfrak{m})$ be a complete $\mathbb{Q}$-Gorenstein Cohen-Macaulay normal local ring of characteristic $p>0$ and Δ an effective $\mathbb{Q}$-Cartier divisor on* $\operatorname{Spec} R$*, that is, $r\Delta = \operatorname{div}_R(y)$ for some positive integer r and nonzero element $y \in R$. Let $x \in R$ be a nonzero divisor such that R/xR is normal and $y \notin xR$. Then, letting $S := R/xR$,*

$$\tau(S, \Delta|_{\operatorname{Spec} S}) \subseteq \tau(R,\Delta) \cdot S.$$

(2) (Subadditivity Theorem) Let $(R,\mathfrak{m})$ be a complete regular local ring of characteristic $p>0$, and Δ_1 and Δ_2 be any two effective $\mathbb{Q}$-divisors on $\operatorname{Spec} R$*. Then*

$$\tau(R, \Delta_1 + \Delta_2) \subseteq \tau(R,\Delta_1)\cdot \tau(R,\Delta_2).$$

Proof. (1)We identify E_S with $(0:x)_{E_R}$.

Claim.

$$0^{*\Delta}_{E_R} \cap E_S \subseteq 0^{*\Delta|_{\operatorname{Spec} S}}_{E_S}.$$

Proof of Claim. First we will look at the Frobenius actions on E_R and E_S. Since R is Cohen-Macaulay, we have the following commutative diagram with exact rows for every $q=p^e$ (See the proof of [HW, Theorem 4.9]).

$$\begin{tikzcd}
0 \arrow[r] & E_S \arrow[r] \arrow[d, "F_S^e"] & E_R \arrow[d, "x^{q-1}F_R^e"] \\
0 \arrow[r] & E_S \otimes_S {}^eS \arrow[r] & E_R \otimes_R {}^eR
\end{tikzcd}$$

If $\xi \in 0^{*\Delta}_{E_R} \cap E_S$, then for some nonzero element $c \in R$, $cF^e_R(\xi) = 0$ in $E_R \otimes_R {}^eR((q-1)\Delta)$ for all $q = p^e \gg 0$. We write $q-1 = kr+i$ for integers k and i with $0 \le i \le r-1$. Then there exist a nonzero element $c' \in R$ such that $c' \notin xR$ and $c'x^{q-1}y^kF^e_R(\xi) = 0$ in $E_R \otimes_R {}^eR$. Hence, by the above diagram, $c'y^kF^e_S(\xi) = 0$ in $E_S \otimes_S {}^eS$, so that $c'F^e_S(\xi) = 0$ in $E_S \otimes_S {}^eS((q-1)\Delta|_{\operatorname{Spec} S})$ for all $q=p^e \gg 0$. Since $c' \notin xR$, it implies that $\xi \in 0^{*\Delta|_{\operatorname{Spec} S}}_{E_S}$. ☐

Since R is complete, by [Ha3, Lemma 3.3], we have $0^{*\Delta}_{E_R} = (0:\tau(R,\Delta))_{E_R}$. Thus

$$\begin{aligned}(0:x)_{0^{*\Delta}_{E_R}} &= (0:\tau(R,\Delta)+xR)_{E_R} = (0:\frac{\tau(R,\Delta)+xR}{xR})_{E_S}\\ &= (0:\tau(R,\Delta)\cdot S)_{E_S}.\end{aligned}$$

In light of the claim, $\tau(R,\Delta)\cdot S = \operatorname{Ann}_S(0:x)_{0^{*\Delta}_{E_R}} \supseteq \tau(S,\Delta|_{\operatorname{Spec} S})$.

(2) Fisrt we consider the following claim.

Claim. Let $R = k[[x_1,\dots,x_n]]$ (resp. $S = k[[y_1,\dots,y_m]]$) be an n-dimensional (resp. m-dimensional) complete regular local ring over a field k and Δ_R (resp. Δ_S) an effective $\mathbb{Q}$-Weil divisor on $\operatorname{Spec} R$ (resp. $\operatorname{Spec} S$). Let $T = R\hat{\otimes}_k S = k[[x_1,\dots,x_n,y_1,\dots,y_m]]$. Then

$$\tau(T, \Delta_R \otimes_k S + R \otimes_k \Delta_S) \subseteq (\tau(R,\Delta_R)\otimes_k \tau(S,\Delta_S))T.$$

Proof of Claim. It suffices to show that

$$0_{E_T}^{*(\Delta_R \otimes_k S + R \otimes_k \Delta_S)} \supseteq 0_{E_R}^{*\Delta_R} \otimes_k E_S + E_R \otimes_k 0_{E_S}^{*\Delta_S},$$

but it is clear since $E_T = E_R \otimes_k E_S$. $\square$

Let $\rho : T = R \hat{\otimes}_k R \to R$ be a diagonal map. Then $T \to T/\mathrm{Ker}\,\rho = R$ is a complete intersection, so it follows from the repeated application of the restriction theorem that $\tau(R, \Delta_1 + \Delta_2) \subseteq \tau(T, \Delta_1 \otimes_k R + R \otimes_k \Delta_2) \cdot R$. By the above claim, $\tau(T, \Delta_1 \otimes_k R + R \otimes_k \Delta_2) \cdot R \subseteq \tau(R, \Delta_1) \cdot \tau(R, \Delta_2)$. $\square$

Theorem 3.13. *Let $(R, \mathfrak{m})$ be an F-finite normal local ring of characteristic $p > 0$ and Δ an effective $\mathbb{Q}$-Weil divisor on $Y = \mathrm{Spec}\, R$ such that $K_Y + \Delta$ is $\mathbb{Q}$-Cartier. Let $f : X \to Y = \mathrm{Spec}\, R$ be a proper birational morphism with X normal. Then*

$$\tau(R, \Delta) \subseteq H^0(X, \mathcal{O}_X(\lceil K_X - f^*(K_Y + \Delta) \rceil)).$$

Proof. The essential idea of the proof is seen in [HW, Theorem 3.3] and [Wa]. Our proof consists of six steps.

(Step 1) Take any nonzero element $c \in \tau(R, \Delta)$, and fix a nonzero element $d \in R(-\lceil \Delta \rceil)$. As the proof of Proposition 3.11, for every $q' = p^{e'} > 0$, there exists $q'' = p^{e''}$ such that $c \cdot \bigcap_{e'' \geq e \geq e'} \mathrm{Ker}\, cdF^e = 0$, where $F^e : E_R \to E_R \otimes_R {}^e R((q-1)\Delta)$ is the e-times Frobenius map induced on E_R. Let

$$\varphi_e : \mathrm{Hom}_R(R((q-1)\Delta)^{1/q}, R) \to \mathrm{Hom}_R(R, R) = R$$

be an R-module homomorphism induced by the R-linear map $R \xrightarrow{(cd)^{1/q}} R((q-1)\Delta)^{1/q}$, and set $\varphi = \bigoplus_{e'' \geq e \geq e'} \varphi_e$. Since cdF^e is the Matlis dual of φ_e, $c \cdot \bigcap_{e'' \geq e \geq e'} \mathrm{Ker}\, cdF^e$ implies that $c \in \mathrm{Im}\, \varphi$. Hence, for every $e'' \geq e \geq e'$, there exists $c_e \in R$ and an R-module homomorphism ${\phi_e}' : R((q-1)\Delta)^{1/q} \to R$ sending $(cd)^{1/q}$ to c_e such that $\sum_{e'' \geq e \geq e'} c_e = c$.

(Step 2) We prove that $\lfloor \Delta - \mathrm{div}_R(c) \rfloor \leq 0$. Assume to the contrary that Δ has a component Δ_0 so that the coefficient of Δ in Δ_0 is at least $1 + v_{\Delta_0}(c)$, where v_{Δ_0} is the valuation of Δ_0. Since the coefficient of $q\Delta$ in Δ_0 is $q(1 + v_{\Delta_0}(c))$ or more, the R-linear map $R \xrightarrow{(cd)^{1/q}} R((q-1)\Delta)^{1/q}$ factors through $R \hookrightarrow R((1 + v_{\Delta_0}(c))\Delta_0)$. Hence, for every $e'' \geq e \geq e'$, ${\phi_e}'$ induces an R-module homomorphism $R((1 + v_{\Delta_0}(c))\Delta_0) \to R$ which sends 1 to c_e. Thus there exists an R-linear map $R((1 + v_{\Delta_0}(c))\Delta_0) \to R$ sending 1 to c. This is a contradiction.

(Step 3) Let $\phi_e = d^{1/q} {\phi_e}'$. ϕ_e (resp. ${\phi_e}'$) induces an R-linear map ψ_e (resp. ${\psi_e}'$) : $R((q-1)\Delta + \mathrm{div}_R(c))^{1/q} \to R(\mathrm{div}_R(c_e))$ which sends 1 (resp. $d^{1/q}$) to 1. We may assume without loss of generality that X is Gorenstein. Thanks to the adjunction formula, we may regard ψ_e (resp. ${\psi_e}'$) in

$$\begin{aligned} &\mathrm{Hom}_R(R((q-1)\Delta + \mathrm{div}_R(c))^{1/q}, R(\mathrm{div}_R(c_e))) \\ &\qquad \cong R(\lceil (1-q)(K_Y + \Delta) + q\mathrm{div}_R(c_e) - \mathrm{div}_R(c) \rceil)^{1/q} \end{aligned}$$

as a rational section of the sheaf $\mathcal{O}_X((1-q)K_X)$, and consider the corresponding divisor on X

$$D_e = D_{\psi_e} = (\psi_e)_0 - (\psi_e)_\infty \quad (\text{resp. } {D_e}' = D_{{\psi_e}'} = ({\psi_e}')_0 - ({\psi_e}')_\infty),$$

where $(\psi_e)_0$ and $(\psi_e)_\infty$ (resp. $({\psi_e}')_0$ and $({\psi_e}')_\infty$) are the divisors of zeros and poles of ψ_e (resp. ${\psi_e}'$) as a rational section of $\mathcal{O}_X((1-q)K_X)$. Clearly, $D_e = {D_e}' + \mathrm{div}_X(d)$. By the definition, D_e and ${D_e}'$ are linearly equivalent to $(1-q)K_X$, and $(\phi_e)_\infty$ and $({\phi_e}')_\infty$ are f-exceptional divisors. Hence f_*D_e (resp. $f_*{D_e}'$) is linearly equivalent to $(1-q)K_Y$ and f_*D_e(resp. $f_*{D_e}'$) $\geq \lfloor (q-1)\Delta \rfloor - q\mathrm{div}_R(c_e) + \mathrm{div}_R(c)$. We denote $X' = X \setminus \mathrm{Supp}\ (\psi_e)_\infty$. Then, ψ_e lies in

$$\begin{aligned}&\mathrm{Hom}_{\mathcal{O}_{X'}}(\mathcal{O}_{X'}(\mathrm{div}_{X'}(c))^{1/q}, \mathcal{O}_{X'}(\mathrm{div}_{X'}(c_e)))\\ &\qquad \cong H^0(X', \mathcal{O}_{X'}((1-q)K'_X + q\mathrm{div}_{X'}(c_e) - \mathrm{div}_{X'}(c))).\end{aligned}$$

(Step 4) We show that the coefficient of D_e in each irreducible component $D_{e,i}$ is $q-1$ or less. Assume to the contrary that there exists an irreducible component $D_{e,0}$ of D_e whose coefficient is at least q. Let $v_{D_{e,0}}$ be the valuation of $D_{e,0}$. Then, letting $\alpha = qv_{D_{e,0}}(c_e) - v_{D_{e,0}}(c) + q$, ψ_e lies in

$$\begin{aligned}&H^0(X', \mathcal{O}_{X'}(((1-q)K_{X'} + q\mathrm{div}_{X'}(c_e) - \mathrm{div}_{X'}(c)) - \alpha D_{e,0}))\\ &\qquad \cong \mathrm{Hom}_{\mathcal{O}_{X'}}(\mathcal{O}_{X'}(\alpha D_{e,0} + \mathrm{div}_{X'}(c))^{1/q}, \mathcal{O}_{X'}(\mathrm{div}_{X'}(c_e))).\end{aligned}$$

Therefore, we get the following commutative diagram.

$$\begin{tikzcd}
\mathcal{O}_{X'} \arrow[r, hook] \arrow[dr, hook] & \mathcal{O}_{X'}(q(v_{D_{e,0}}(c_e)+1)D_{e,0} + (\mathrm{div}_{X'}(c) - v_{D_{e,0}}(c)D_{e,0}))^{1/q} \arrow[d, "\psi_e"] \\
& \mathcal{O}_{X'}(\mathrm{div}_{X'}(c_e))
\end{tikzcd}$$

However the natural inclusion map

$$\mathcal{O}_{X'} \hookrightarrow \mathcal{O}_{X'}(q(v_{D_{e,0}}(c_e)+1)D_{e,0} + (\mathrm{div}_{X'}(c) - v_{D_{e,0}}(c)D_{e,0}))^{1/q}$$

factors through $\mathcal{O}_{X'}((v_{D_{e,0}}(c_e)+1)D_{e,0})$, and the above commutative diagram implies $(v_{D_{e,0}}(c_e)+1)D_{e,0} \leq \mathrm{div}_{X'}(c_e)$. This is absurd. Hence every coefficient of D_e in each irreducible component $D_{e,i}$ must be at most $q-1$.

(Step 5) We denote by $\cup_{j=1}^s E_j$ the exceptional divisor of f and by $f_*^{-1}\Delta'$ the strict transform of $\Delta' := \Delta - \mathrm{div}_R(c)$ in X. Then,

$$K_X + f_*^{-1}\Delta' \underset{\mathbb{Q}\text{-lin.}}{\sim} f^*(K_Y + \Delta') + \sum_{j=1}^s a_j E_j.$$

Let ${B_e}' = \frac{1}{q-1}{D_e}' - f_*^{-1}\Delta'$. Then ${B_e}'$ is $\mathbb{Q}$-linearly equivalent to $-(K_X + f_*^{-1}\Delta')$, so that $f_*{B_e}'$ is $\mathbb{Q}$-linearly equivalent to $-f_*(K_X + f_*^{-1}\Delta') = -(K_Y + \Delta')$. Hence $f_*{B_e}'$ is $\mathbb{Q}$-Cartier. Since ${B_e}' + \sum_{j=1}^s a_j E_j$ is $\mathbb{Q}$-linearly equivalent to $-f^*(K_Y + \Delta')$,

$(B_e{}' - f^*f_*B_e{}') + \sum_{j=1}^s a_jE_j$ is an f-exceptional divisor which is $\mathbb{Q}$-linearly trivial relative to f. Hence

$$(B_e{}' - f^*f_*B_e{}') + \sum_{j=1}^s a_jE_j = 0.$$

(Step 6) Now

$$\begin{aligned} f_*D_e{}' - (q-1)\Delta' &\geq (\lfloor (q-1)\Delta \rfloor - q\mathrm{div}_R(c_e) + \mathrm{div}_R(c)) - (q-1)\Delta' \\ &\geq -\Delta'' + q(\mathrm{div}_R(c) - \mathrm{div}_R(c_e)) \end{aligned}$$

for some effective $\mathbb{Q}$-Cartier divisor Δ'' on Y which is independent of q. This implies $f_*B_e{}' \geq -\frac{1}{q-1}\Delta'' + \frac{q}{q-1}(\mathrm{div}_R(c) - \mathrm{div}_R(c_e))$, whence

$$f^*f_*B_e \geq -\frac{1}{q-1}f^*\Delta'' + \frac{q}{q-1}(\mathrm{div}_X(c) - \mathrm{div}_X(c_e)).$$

On the other hand, we have seen in (Step 4) that the coefficient of D_e in E_j is at most $q-1$. Since $D_e = D_e{}' + \mathrm{div}_X(d)$ and we can assume that $\mathrm{div}_X(d) > f^*\Delta''$, the coefficient of $D_e{}' + f^*\Delta''$ in E_j is less than $q-1$. Hence the coefficient of $B_e{}' - f^*f_*B_e{}'$ is less than $1 - \frac{q}{q-1}(\mathrm{div}_X(c) - \mathrm{div}_X(c_e))$. For every $j = 1, \ldots, s$, there exists $e'' \geq e \geq e'$ such that $v_{E_j}(c) \geq v_{E_j}(c_e)$ where v_{E_j} is the valuation of E_j, and via (Step 5) it implies $a_j > -1$.

It follows from the above result and (Step 2) that

$$\mathrm{div}_X(c) + \lceil K_X - f^*(K_Y + \Delta) \rceil \geq 0,$$

that is, $c \in H^0(X, \mathcal{O}_X(\lceil K_X - f^*(K_Y + \Delta)\rceil))$. □

Example 3.14. (1) When R is a regular local ring and Δ is an effective $\mathbb{Q}$-Weil divisor with simple normal crossing support, then $\tau(R, \Delta) = R(-\lfloor \Delta \rfloor)$.

(2) Let $R = k[[x_1, \ldots, x_d]]$ be a d-dimensional complete regular local ring over a field k of characteristic $p > 0$ and $\Delta = \mathrm{div}_R(x_1^{d+1} + \cdots + x_d^{d+1})$. If the characteristic $p > d+1$, then $\tau(R, \frac{d}{d+1}\Delta) = (x_1, \ldots, x_d)$.

4. Main Theorem

To state the main result, we will explain the meaning of the phrase "in characteristic $p \gg 0$."

Let R be a normal domain which is finitely generated over a field k of characteristic zero and Δ an effective $\mathbb{Q}$-Weil divisor on $\mathrm{Spec}\, R$ such that $K_Y + \Delta$ is $\mathbb{Q}$-Cartier. Let $f : X \to \mathrm{Spec}\, R$ be a resolution of singularities such that $\mathrm{Exc}(f) + f^*(K_Y + \Delta)$ has simple normal crossing support. Choosing a suitable finitely generated $\mathbb{Z}$-subalgebra A of k, there exists a finitely generated flat A-algebra R_A, an effective $\mathbb{Q}$-Weil divisor Δ_A on $\mathrm{Spec}\, R_A$, a smooth A-scheme X_A and a birational A-morphism $f_A : X_A \to \mathrm{Spec}\, R_A$, such that $K_{R_A} + \Delta_A$ is $\mathbb{Q}$-Cartier, $\mathrm{Exc}(f_A) + f_A{}^*(K_{R_A} + \Delta_A)$ has simple normal crossing support, and by tensoring k over A one gets back R, Δ, X and $f : X \to \mathrm{Spec}\, R$. Given a closed point $s \in \mathrm{Spec}\, A$ with residue field $\kappa = \kappa(s)$, we denote the corresponding fibers over s by $f_\kappa : X_\kappa \to \mathrm{Spec}\, R_\kappa$, Δ_κ, etc. Then the pairs $(R_\kappa, \Delta_\kappa)$ over general closed points $s \in \operatorname{Spec} A$ inherit the properties possessed by the original one (R, Δ).

Now we fix a general closed point $s \in \operatorname{Spec} A$ with residue field $\kappa = \kappa(s)$ of sufficiently large characteristic $p \gg 0$. Then we refer to the fibers over $s \in \operatorname{Spec} A$ as "reduction modulo $p \gg 0$," and use the phrase "in characteristic $p \gg 0$" when we look at general closed fibers which are reduced from characteristic zero to characteristic $p \gg 0$ as above.

The following lemma is essential to prove "F-properties" in characteristic $p \gg 0$.

Lemma 4.1 ([Ha2])**.** *Let $(R, \mathfrak{m})$ be a normal local ring of dimension $d \geq 2$, essentially of finite type over a perfect field κ of characteristic $p > 0$. Let $f : X \to \operatorname{Spec} R$ be a resolution of singularities and D an f-ample $\mathbb{Q}$-Cartier divisor on X with simple normal crossing support. We denote the closed fiber of f by Z. If $(R, \mathfrak{m})$ is the localization at any prime ideal of a finitely generated κ-algebra which is a reduction modulo $p \gg 0$ as well as X, D and $f : X \to \operatorname{Spec} R$, then e-times Frobenius map*

$$F^e : H^d_Z(X, \mathcal{O}_X(-D)) \to H^d_Z(X, \mathcal{O}_X(-qD))$$

is injective for every $q = p^e$.

Now we state our main result.

Theorem 4.2. *Let $(R, \mathfrak{m})$ be a normal local ring essentially of finite type over a field of characteristic zero and Δ an effective $\mathbb{Q}$-Weil divisor on $Y = \operatorname{Spec} R$ such that $K_Y + \Delta$ is $\mathbb{Q}$-Cartier. Then, in characteristic $p \gg 0$,*

$$\tau(R, \Delta) = \mathcal{J}(Y, \Delta).$$

Proof. Thanks to Theorem 3.13, it suffices to prove that $\tau(R, \Delta) \supseteq \mathcal{J}(Y, \Delta)$ in characteristic $p \gg 0$. Let $f : X \to Y = \operatorname{Spec} R$ be a resolution of singularities such that $\operatorname{Exc}(f) + f^{-1}_*\Delta$ has simple normal crossing support.

Take a nonzero element $c \in R(-\Delta_{\mathrm{red}})$ such that R_c is regular, where Δ_{red} is the reduced divisor whose support is equal to that of Δ. Let $\Delta' = \operatorname{div}_R(c)$. Then there is a rational number $0 \leq \epsilon \ll 1$ such that $\lfloor f^*(K_Y + \Delta) \rfloor = \lfloor f^*(K_Y + \Delta + \epsilon\Delta') \rfloor$. Take an f-ample $\mathbb{Q}$-Cartier divisor H on X which is supported on the exceptional locus of f such that $\lfloor f^*(K_Y + \Delta + \epsilon\Delta') - H \rfloor = \lfloor f^*(K_Y + \Delta) \rfloor$. Set $D = H - f^*(K_Y + \Delta + \epsilon\Delta')$ and we may assume that $\operatorname{Exc}(f) + f^{-1}_*(\Delta + \epsilon\Delta')$ has simple normal crossing support again, replacing f suitably. By Lemma 4.1, in characteristic $p \gg 0$, the e-times Frobenius map

$$F^e : H^d_Z(X, \mathcal{O}_X(f^*(K_Y + \Delta))) \to H^d_Z(X, \mathcal{O}_X(-qD))$$

is injective for every $q = p^e$, where Z is the closed fiber of f.

On the other hand, let

$$\delta : H^d_{\mathfrak{m}}(R(K_Y)) \to H^d_Z(X, \mathcal{O}_X(f^*(K_Y + \Delta)))$$

be the Matlis dual of the natural inclusion map

$$\mathcal{J}(Y, \Delta) = H^0(X, \mathcal{O}_X(\lceil K_X - f^*(K_Y + \Delta) \rceil)) \hookrightarrow R,$$

and

$$\delta_e : H^d_{\mathfrak{m}}(R(q(K_Y+\Delta+\epsilon\Delta'))) \to H^d_Z(X, \mathcal{O}_X(-qH+qf^*(K_Y+\Delta+\epsilon\Delta')))$$

the natural map induced by an edge map of the Leray spectral sequence

$$H^j_{\mathfrak{m}}(H^i(X, \mathcal{O}_X(qf^*(K_Y+\Delta+\epsilon\Delta')))) \Rightarrow H^{i+j}_Z(X, \mathcal{O}_X(qf^*(K_Y+\Delta+\epsilon\Delta'))).$$

Then by the Matlis duality,

$$\begin{aligned}
\ker(\delta) &= \mathrm{Hom}_R\left(\frac{R}{\mathcal{J}(Y,\Delta)}\,,\, E_R\right) = \mathrm{Ann}_{H^d_{\mathfrak{m}}(R(K_Y))}\mathcal{J}(Y,\Delta),\\
\ker(\delta_e) &= \mathrm{Hom}_R\left(\frac{R(\lceil K_Y - q(K_Y+\Delta+\epsilon\Delta')\rceil)}{H^0(X,\mathcal{O}_X(\lceil K_X+qD\rceil))}\,,\, E_R\right)\\
&= \mathrm{Ann}_{H^d_{\mathfrak{m}}(R(q(K_Y+\Delta+\epsilon\Delta')))}H^0(X,\mathcal{O}_X(\lceil K_X+qD\rceil)).
\end{aligned}$$

We obtain the following commutative diagram with exact rows for every $q=p^e$.

$$\begin{tikzcd}
0 \arrow[r] & \ker(\delta) \arrow[r] \arrow[d] & H^d_{\mathfrak{m}}(R(K_Y)) \arrow[r, "\delta"] \arrow[d, "F^e"] & H^d_Z(X,\mathcal{O}_X(-D)) \arrow[r] \arrow[d, "F^e"] & 0\\
0 \arrow[r] & \ker(\delta_e) \arrow[r] & H^d_{\mathfrak{m}}(R(q(K_Y+\Delta+\epsilon\Delta'))) \arrow[r, "\delta_e"] & H^d_Z(X,\mathcal{O}_X(-qD)) \arrow[r] & 0
\end{tikzcd}$$

Take any element $\xi \in H^d_{\mathfrak{m}}(R(K_Y)) \setminus \ker(\delta)$. By the above diagram, $\xi^q \notin \ker(\delta_e)$. By Lemma 3.5, c^n is a Δ-test element for some positive integer n, and for sufficiently large q,

$$\begin{aligned}
&H^0(X,\mathcal{O}_X(\lceil K_X+q(H-f^*(K_Y+\Delta+\epsilon\Delta'))\rceil))\\
&\qquad\subseteq c^{n+1}H^0(X,\mathcal{O}_X(\lceil K_X+q(H-f^*(K_Y+\Delta))\rceil)).
\end{aligned}$$

Hence

$$c^{n+1}\xi^q \notin \mathrm{Ann}_{H^d_{\mathfrak{m}}(R(q(K_Y+\Delta)))}H^0(X,\mathcal{O}_X(\lceil K_X+q(H-f^*(K_Y+\Delta))\rceil)).$$

If $\xi \in 0^{*\Delta}_{H^d_{\mathfrak{m}}(R(K_Y))}$, then by the proof of Lemma 3.5, $c^n\xi^q=0$ in $H^d_{\mathfrak{m}}(R(\lceil qK_Y+(q-1)\Delta\rceil))$. Therefore $c^{n+1}\xi^q=0$ in $H^d_{\mathfrak{m}}(R(q(K_Y+\Delta)))$, and this is a contradiction. It follows that

$$0^{*\Delta}_{H^d_{\mathfrak{m}}(R(K_Y))} \subseteq \ker(\delta) = \mathrm{Ann}_{H^d_{\mathfrak{m}}(R(K_Y))}\mathcal{J}(Y,\Delta),$$

and by Matlis duality (see [Ha3, Lemma 3.3]), $\tau(R,\Delta) \supseteq \mathcal{J}(Y,\Delta)$. $\square$

Remark 4.3. When $\Delta=0$, Theorem 4.2 coincides with the results of Hara [Ha3] and Smith [Sm2].

As a direct consequence of the main theorem, we get the equivalence of klt pairs and strongly F-regular pairs.

Corollary 4.4 ([HW, Conjecture 5.1.1]). *Let $(R,\mathfrak{m})$ be a normal local ring essentially of finite type over a field of characteristic zero and Δ an effective $\mathbb{Q}$-Weil divisor on $Y = \operatorname{Spec} R$ such that $K_Y+\Delta$ is $\mathbb{Q}$-Cartier. Then, (Y,Δ) is klt if and only if (R,Δ) is of open strongly F-regular type.*

Hara and K.-i. Watanabe [HW, Problem 5.1.2] conjectured that (Y,Δ) is lc if and only if (R,Δ) is of dense F-pure type. The following result about log canonical thresholds is a piece of evidence for their conjecture. See [Ko] for the basic properties of log canonical thresholds.

Corollary 4.5 ([HW, Conjecture 5.2.1])**.** *Let Y be a variety in characteristic zero with only klt singularity at a point $y \in Y$ and Δ an effective $\mathbb{Q}$-Cartier divisor on Y. We denote by $C_y(Y,\Delta)$ the log canonical threshold of Δ at $y \in Y$, that is,*

$$\begin{aligned} C_y(Y,\Delta) &= \sup\{t \in \mathbb{R} \mid (Y, t\Delta) \text{ is lc at } y \in Y\} \\ &= \sup\{t \in \mathbb{R} \mid (Y, t\Delta) \text{ is klt at } y \in Y\}. \end{aligned}$$

Then,

$$\begin{aligned} C_y(Y,\Delta) &= \sup\{t \in \mathbb{Q} \mid (\mathcal{O}_{Y,y}, t\Delta) \text{ is of dense F-pure type}\} \\ &= \sup\{t \in \mathbb{Q} \mid (\mathcal{O}_{Y,y}, t\Delta) \text{ is of open strongly F-regular type}\}. \end{aligned}$$

Proof. By [HW, Theorem 3.7], dense F-pure type implies lc. Hence the assertion is clear. □

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