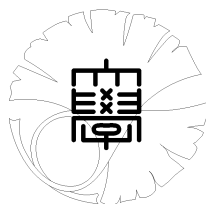# Abelian étale coverings of curves over
local fields and its application to modular curves

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ABELIAN ÉTALE COVERINGS OF CURVES OVER LOCAL FIELDS AND ITS APPLICATION TO MODULAR CURVES

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ABSTRACT. We give a geometric bound of a part of the abelian étale fundamental group of curves over local fields, in particular in the case of bad reduction. We apply the result to the modular curve $X_0(p)/\mathbb{Q}_p$ to show that the unramified abelian covering between $X_1(p) \to X_0(p)$ (Shimura covering) uses up all the possible ramification over the special fiber.

CONTENTS

1. INTRODUCTION

The problem of classifying the abelian étale coverings of a proper smooth curve is the central problem of the geometric class field theory, and there is an arithmetical interest when the base field is a number field or a local field. There is a well-formulated class field theory for these curves (e.g. Kato-Saito[6], Saito[14]) but there are very few examples where the abelian étale coverings are explicitly classified. The most remarkable known example is the case of modular curve $X_0(p)$ over $\mathbb{Q}$ (Mazur[8], Introduction, Theorem (2)), which makes use of the deep theory of integral Hecke algebras. We give the corresponding result for $X_0(p)$ over $\mathbb{Q}_p$, by proving general results for the curves over local fields.

For a proper variety X over a field K, let $\pi_1^{ab}(X)$ be the abelian étale fundamental group of X. There is a natural surjection $\pi_1^{ab}(X) \to G_K^{ab}$ where $G_K^{ab} =$

$\mathrm{Gal}(K_{ab}/K)$ is the Galois group of the maximal abelian extension of K, and we denote the kernel by $\pi_1^{ab}(X)^{geo}$:

$$0 \longrightarrow \pi_1^{ab}(X)^{geo} \longrightarrow \pi_1^{ab}(X) \longrightarrow G_K^{ab} \longrightarrow 0$$

For a proper smooth geometrically irreducible variety X over a *local field* K, i.e. a complete discrete valuation field K with finite residue field F with $\mathrm{char}F = p$, we showed in our previous paper [15] that $\pi_1^{ab}(X)^{geo}$ is an extension of $\widehat{\mathbb{Z}}^r$ by a finite torsion group $\pi_1^{ab}(X)_{tor}^{geo}$. Here the rank r is the F-rank (the dimension of the maximal F-split subtorus) of the special fiber of the Néron model of the Albanese variety of X.

In this paper, we confine ourselves in the case of *curves*, and investigate the most mysterious part of $\pi_1^{ab}(X)_{tor}^{geo}$, namely $\pi_1^{ab}(X)_{ram}^{geo}$ which classifies the abelian étale coverings of the generic fiber which are "completely ramified over the special fiber", defined as follows. Let X_F be the special fiber of the proper flat regular model $\mathscr{X}$ over the integer ring O_K (Abyhankar[1]). Then we have a natural map $\pi_1^{ab}(X)^{geo} \longrightarrow \pi_1^{ab}(X_F)^{geo}$ which is surjective if X has a K-rational point, and denote the kernel by $\pi_1^{ab}(X)_{ram}^{geo}$:

$$0 \longrightarrow \pi_1^{ab}(X)_{ram}^{geo} \longrightarrow \pi_1^{ab}(X)^{geo} \longrightarrow \pi_1^{ab}(X_F)^{geo} \longrightarrow 0$$

Then our main result is :

Theorem 1.1. *Assume that X admits a K-rational point.*

(i) (Theorem 3.2) *The dual of prime-to-p part of the finite abelian group $\pi_1^{ab}(X)_{ram}^{geo}$ injects to the component group Φ of the Néron model of the jacobian variety of X.*

(ii) (Theorem 4.1) *Assume moreover that the absolute ramification index e of K is less than $p-1$, $\mathrm{char}K = 0$, and X has semistable reduction over O_K. Then the p-primary part of $\pi_1^{ab}(X)_{ram}^{geo}$ vanishes.*

Note that if X_F is smooth, i.e. X has good reduction, $\pi_1^{ab}(X)^{geo} \to \pi_1^{ab}(X_F)^{geo}$ is isomorphic on the prime-to-p part by the proper smooth base change theorem on H_{et}^1. As $\pi_1^{ab}(X_F)^{geo}$ can be calculated in principle from the special fiber X_F, we have control on the whole of $\pi_1^{ab}(X)$ in the case $e < p-1$, $\mathrm{char}K = 0$ and X has semistable reduction, in terms of the special fiber of the Néron model of the jacobian of X.

In the latter part of this paper, we give an application of these results to the modular curve $X_0(p)$, where $X_0(p)$ is the usual modular curve classifying the elliptic curves with $\Gamma_0(p)$-structure (p is a prime). For the curve $X_0(p)/\mathbb{Q}_p$, above theorem enables us to compute the group $\pi_1^{ab}(X_0(p)/\mathbb{Q}_p)^{geo}$ completely :

Theorem 1.2. $\pi_1^{ab}(X_0(p)/\mathbb{Q}_p)^{geo}$ *has the following structure :*

$$0 \longrightarrow \Phi(J_0(p)) \longrightarrow \pi_1^{ab}(X_0(p)/\mathbb{Q}_p)^{geo} \longrightarrow \widehat{\mathbb{Z}}^r \longrightarrow 0$$

where $\Phi(J_0(p)) \cong \pi_1^{ab}(X_0(p)/\mathbb{Q}_p)_{ram}^{geo} = \pi_1^{ab}(X_0(p)/\mathbb{Q}_p)_{tor}^{geo}$ is a cyclic group of order equal to the numerator of $\frac{p-1}{12}$, and $r = \frac{g+h-1}{2}$ where g is the genus of $X_0(p)$ and h is the number of the supersingular points defined over $\mathbb{F}_p$.

Notations. Throughout this paper, a *local field K* means a complete discrete valuation field with finite residue field F, with $\text{char}F = p$. O_K is the integer ring of K. For any field K, $\overline{K}$ is a separable closure of K, and $G_K = \text{Gal}(\overline{K}/K)$ is the absolute Galois group of K. For a variety X over any field K, $\overline{X} = X \times_K \text{Spec}(\overline{K})$.

For an abelian group X, its Pontrjagin dual is denoted by $X^\vee = \text{Hom}(X, \mathbb{Q}/\mathbb{Z})$. $\widehat{\mathbb{Z}} \cong \prod_p \mathbb{Z}_p$ is the profinite completion of $\mathbb{Z}$.

For a scheme X over O_K, we denote the generic fiber and the special fiber of X respectively by X_K, X_F. For any group scheme X, $X[m]$ denotes the kernel of the multiplication-by-m map, which is also a group scheme. We use the notation $\mu_N = \mathbb{G}_m[N]$ for the group scheme of N-th roots of unity over an arbitrary base, without specifying the base scheme.

2. Preliminaries on abelian étale fundamental groups

Here we review some generalities on abelian étale fundamental groups, and fix the notations.

2.1. **Geometric abelian étale fundamental groups.** For any noetherian connected scheme X, the étale fundamental group $\pi_1(X)$ is a profinite group classifying finite étale coverings of X ([SGA1]), and we denote the maximal abelian quotient by $\pi_1^{ab}(X)$. When X is a proper variety over a field K, there is an exact sequence (*ibid.*, Exposé IX, Th. 6.1) :

$$(2.1) \qquad 1 \longrightarrow \pi_1(\overline{X}) \longrightarrow \pi_1(X) \longrightarrow G_K \longrightarrow 1$$

where $\overline{K}$ is a separable closure of K, $\overline{X} = X \times_K \text{Spec}(\overline{K})$, and $G_K = \text{Gal}(\overline{K}/K)$ is the absolute Galois group of K. Therefore we have surjection $\pi_1^{ab}(X) \to G_K^{ab}$ where $G_K^{ab} = \text{Gal}(K_{ab}/K)$ is the Galois group of the maximal abelian extension of K, and denote the kernel by $\pi_1^{ab}(X)^{geo}$:

$$(2.2) \qquad 0 \longrightarrow \pi_1^{ab}(X)^{geo} \longrightarrow \pi_1^{ab}(X) \longrightarrow G_K^{ab} \longrightarrow 0$$

Remark 2.1. When X has a K-rational point x, (2.1) and consequently also (2.2) has a splitting, and $\pi_1^{ab}(\overline{X})_{G_K} \cong \pi_1^{ab}(X)^{geo}$ where the former group is the coinvariant with respect to the G_K-action by inner automorphisms. In this case,

$\pi_1^{ab}(X)^{geo}$ has a geometric interpretation as the group classifying the abelian finite étale coverings of X in which x splits completely.

Now assume that K is a *local field* with the finite residue field F with $\operatorname{char} F = p$, and let $\mathscr{X}$ be a proper flat model of X over the integer ring O_K, i.e. a proper flat scheme $\mathscr{X} \to \operatorname{Spec} O_K$ with $\mathscr{X} \times_{O_K} \operatorname{Spec} K \cong X$, and let $X_F = \mathscr{X} \times_{O_K} \operatorname{Spec} F$ be the special fiber. Then we have a canonical surjection :

$$\pi_1^{ab}(X) \longrightarrow \pi_1^{ab}(\mathscr{X}) \cong \pi_1^{ab}(X_F)$$

Here the latter isomorphism follows by [SGA1] Exposé X, Th. 2.1 (a part of proper base change theorem for H_{et}^1). We have the corresponding map on the geometric part by the commutative diagram :

$$(2.3) \qquad \begin{array}{ccccccccc} 0 & \longrightarrow & \pi_1^{ab}(X)^{geo} & \longrightarrow & \pi_1^{ab}(X) & \longrightarrow & G_K^{ab} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \pi_1^{ab}(X_F)^{geo} & \longrightarrow & \pi_1^{ab}(X_F) & \longrightarrow & G_F & \longrightarrow & 0 \end{array}$$

Definition 2.2. Denote the kernel of left and middle vertical maps of (2.3) by $\pi_1^{ab}(X)_{ram}$, $\pi_1^{ab}(X)_{ram}^{geo}$ respectively. (These groups classify the abelian étale coverings of X which become completely ramified over a part of X_F when extended to the coverings over X_F.)

By definition, we have an exact sequence :

$$(2.4) \qquad 0 \longrightarrow \pi_1^{ab}(X)_{ram}^{geo} \longrightarrow \pi_1^{ab}(X)^{geo} \longrightarrow \pi_1^{ab}(X_F)^{geo}$$

As $\pi_1^{ab}(\overline{X})$ is the Pontrjagin dual of the étale cohomology group $H_{et}^1(\overline{X}, \mathbb{Q}/\mathbb{Z})$, we can pass to the dual and translate the above definition into the language of H_{et}^1. By the Hochschild-Serre spectral sequece, we have a short exact sequence :

$$0 \longrightarrow H^1(G_K, \mathbb{Q}/\mathbb{Z}) \longrightarrow H_{et}^1(X, \mathbb{Q}/\mathbb{Z}) \longrightarrow H_{et}^1(\overline{X}, \mathbb{Q}/\mathbb{Z})^{G_K} \longrightarrow 0$$

where $(-)^{G_K}$ is the Galois invariant. (Here we used $H^2(G_K, \mathbb{Q}/\mathbb{Z}) = 0$ for local fields K, and the same holds when we replace K by the finite field F.) We can view this exact sequence as the Pontrjagin dual of (2.2), and therefore we have a canonical isomorphism :

$$(2.5) \qquad (\pi_1^{ab}(X)^{geo})^\vee \cong H_{et}^1(\overline{X}, \mathbb{Q}/\mathbb{Z})^{G_K}$$

Hence by taking the Pontrjagin dual of the exact sequence (2.4), we have :

Lemma 2.3. *Let X be a proper geometrically irreducible variety over a local field K, and let $\mathscr{X}$ be its proper flat model over O_K, and denote the special fiber by X_F. Then there is a canonical exact sequence of abelian groups :*

$$(2.6) \qquad H_{et}^1(\overline{X_F}, \mathbb{Q}/\mathbb{Z})^{G_F} \longrightarrow H_{et}^1(\overline{X}, \mathbb{Q}/\mathbb{Z})^{G_K} \longrightarrow (\pi_1^{ab}(X)_{ram}^{geo})^\vee \longrightarrow 0$$

Remark 2.4. Note that when X has a K-rational point, it gives a splitting of the exact sequence (2.2), and consequently all the vertical arrows of (2.3) are surjective, and exact sequences (2.4) and (2.6) turn out to be the short exact sequences.

2.2. Étale cohomology groups.

Here we also review some properties of the group $H^1_{et}(\overline{X}, \mathbb{Q}/\mathbb{Z})^{G_K}$ for a proper geometrically irreducible *curve* X over a field K.

First recall that, for "prime to p" part, we have :

Lemma 2.5 ([SGA4], Exposé IX, Corollaire 4.7). *Let* $\ell \neq \mathrm{char} K$ *be a prime. For any proper geometrically connected curve X over a field K, $H^1_{et}(\overline{X}, \mathbb{Q}_\ell/\mathbb{Z}_\ell) \cong \mathrm{Pic}^0(X)[\ell^\infty](-1)$ as G_K-modules.*

More generally, considering the long exact sequence induced from the short exact sequence :

$$0 \longrightarrow \tfrac{1}{N}\mathbb{Z}/\mathbb{Z} \longrightarrow \mathbb{Q}/\mathbb{Z} \overset{N}{\longrightarrow} \mathbb{Q}/\mathbb{Z} \longrightarrow 0$$

we have $H^1_{et}(\overline{X}, \tfrac{1}{N}\mathbb{Z}/\mathbb{Z}) = H^1_{et}(\overline{X}, \mathbb{Q}/\mathbb{Z})[N]$. As $\mathbb{Q}/\mathbb{Z} = \varinjlim_N \tfrac{1}{N}\mathbb{Z}/\mathbb{Z}$ and the étale cohomology commutes with the directed inductive limits ([SGA4] Exposé VII, Théorème 5.7), we deduce that :

$$H^1_{et}(\overline{X}, \mathbb{Q}/\mathbb{Z}) = \varinjlim_N H^1_{et}(\overline{X}, \frac{1}{N}\mathbb{Z}/\mathbb{Z}) = \bigcup_N H^1_{et}(\overline{X}, \frac{1}{N}\mathbb{Z}/\mathbb{Z})$$

(In particular, $H^1_{et}(\overline{X}, \mathbb{Q}/\mathbb{Z})$ is an torsion module.) Therefore we have :

$$(2.7) \qquad H^1_{et}(\overline{X}, \mathbb{Q}/\mathbb{Z})^{G_K} = \bigcup_N H^1_{et}(\overline{X}, \mathbb{Q}/\mathbb{Z})[N]^{G_K} = \bigcup_N H^1_{et}(\overline{X}, \frac{1}{N}\mathbb{Z}/\mathbb{Z})^{G_K}$$

and can reduce the calculation of $H^1_{et}(\overline{X}, \mathbb{Q}/\mathbb{Z})^{G_K}$ to that of $H^1_{et}(\overline{X}, \mathbb{Z}/N\mathbb{Z})^{G_K}$, for which we can make use of the following generalization of Lemma 2.5 :

Lemma 2.6 (Milne[10], Theorem 3.9, Proposition 4.16). *For any proper geometrically irreducible curve over a field K and integer $N \geq 1$, we have a canonical isomorphism of G_K-modules :*

$$H^1_{et}(\overline{X}, \mathbb{Z}/N\mathbb{Z}) \cong \mathrm{Hom}_{\mathrm{gp}}(\mu_N, \mathrm{Pic}^0(X)[N])$$

where $\mathrm{Hom}_{\mathrm{gp}}$ *denotes the G_K-module consisting of homomorphisms as group schemes. In particular, we have :*

$$H^1_{et}(\overline{X}, \mathbb{Z}/N\mathbb{Z})^{G_K} \cong \mathrm{Hom}_{K-\mathrm{gp}}(\mu_N, \mathrm{Pic}^0(X)[N])$$

where $\mathrm{Hom}_{K-\mathrm{gp}}$ *denotes the abelian group consisting of homomorphisms defined over K.*

Remark 2.7. This interpretation shows that, $(\pi_1^{ab}(X)^{geo})^\vee \cong H_{et}^1(\overline{X}, \mathbb{Q}/\mathbb{Z})^{G_K}$ is the *maximal μ-type subgroup* of $\mathrm{Pic}^0(X)$ in the terminology of Mazur[8], I-3. This re-interpretes the Mazur's theory of the μ-type subgroup of the jacobian of the modular curves as the theory of $\pi_1^{ab}(X)^{geo}$ of the modular curves.

3. The prime-to-p part of $\pi_1^{ab}(X)_{ram}^{geo}$

From this section, we assume that X is *a proper smooth geometrically irreducible curve over a local field K.*

In this case, the proper flat regular model $\mathscr{X}$ always exists by Abyhankar[1], and denote its special fiber by X_F. Let $J = \mathrm{Pic}^0(X)$ be the jacobian variety of X, and denote the Néron model and its special fiber respectively by $\mathscr{J}$ and J_F. The quotient of J_F by its connected component of the identity J_F^0 is a finite étale group $\Phi(J)$ over F, which we call the *group of components* of $\mathscr{J}$ (or of J, by an abuse of language).

Now we assume that X *admits a K-rational point* (see Remark 2.4), and our starting point of the investigation is the following :

Lemma 3.1 (Raynaud[11], [SGA7] Exposé IX, (12.1.12))**.** *Assume that X admits a K-rational point. Then we have $\mathrm{Pic}^0(X_F) \cong J_F^0$ as smooth group schemes over F.*

In this section we treat the prime-to-p part of $\pi_1^{ab}(X)_{ram}^{geo}$, and the main result is stated as follows :

Theorem 3.2. *Assume that X admits a K-rational point. Then there is an injection of finite abelian groups $(\pi_1^{ab}(X)_{ram}^{geo})^\vee_{\mathrm{not}\, p} \longrightarrow \Phi(J)$, where $(-)_{\mathrm{not}\, p}$ denotes the prime-to-p part $(p = \mathrm{char}F)$.*

In this section, we fix some notations on Galois modules, and give the proof of the above theorem. As the theorem can be proved by establishing the injectivity on ℓ-primary parts for each prime number $\ell \neq p = \mathrm{char}F$, we fix an $\ell \neq p$ in the remainder of this section.

3.1. **Galois modules.** For any smooth group scheme X of finite type over K, define pro-ℓ (resp. ind-ℓ) group scheme $T_\ell(X)$ (resp. $X[\ell^\infty]$) by :

$$T_\ell(X) = \varprojlim_n X[\ell^n], \quad X[\ell^\infty] = \varinjlim_n X[\ell^n]$$

where $X[m]$ denotes the kernel of the multiplication-by-m map which is an étale group scheme over K, and inductive (resp. projective) limit is taken with respect to the canonical inclusions (resp. multiplication-by-ℓ maps). We use the same notations for group schemes over O_K or F. We often identify $T_\ell(X), X[\ell^\infty]$ with the associated Galois modules, which are $\mathbb{Z}_\ell$-modules and satisfies :

$$X[\ell^\infty] \cong T_\ell(X) \otimes (\mathbb{Q}_\ell/\mathbb{Z}_\ell), \quad T_\ell(X) = \mathrm{Hom}(\mathbb{Q}_\ell/\mathbb{Z}_\ell, X[\ell^\infty])$$

Now we fix the notations concerning Galois modules. Let $G = G_K$ or G_F be the absolute Galois group, and let M be an arbitrary $\mathbb{Z}_\ell$-module with a continuous action of G. As usual, we write $\mathbb{Z}_\ell(1) = T_\ell(\mathbb{G}_m)$, and define the *Tate twist* $M(r)$ of M for $\forall r \in \mathbb{Z}$ by :

$$M(r) = \begin{cases} M \otimes \mathbb{Z}_\ell(r) & (r > 0) \\ M & (r = 0) \\ \text{Hom}(\mathbb{Z}_\ell(-r), M) & (r < 0) \end{cases}$$

as Galois modules. $M \mapsto M(r)$ gives exact functors of Galois modules for $\forall r \in \mathbb{Z}$, and we have canonical isomorphisms $M(r)(s) \cong M(r+s)$ for $\forall r, s \in \mathbb{Z}$. The Galois action on $\mathbb{Z}_\ell(1)$ is via the *cyclotomic character* $\chi : G \to \mathbb{Z}_\ell^\times$. The Galois invariant and coinvariant is denoted respectively by M^G, M_G.

Definition 3.3. The χ-*part* M^χ of M is defined by $M^\chi = (M(-1)^G)(1)$. It is canonically identified with the maximal subgroup of M on which G acts via the cyclotomic character χ.

Note that the functors $M \mapsto M^G$, $M \mapsto M^\chi$ are both left exact.

3.2. **Tate modules of the jacobians of curves.** Now we return to the curve X over a local field K. Applying Lemma 2.5 and Lemma 3.1, we can restate the Lemma 2.3 in the present case, in view of Remark 2.4, as follows :

Lemma 3.4. *For a proper smooth geometrically irreducible curve X over a local field K which admits a K-rational point, there is a canonical short exact sequence of abelian groups :*

$$0 \longrightarrow J_F^0[\ell^\infty](-1)^{G_F} \longrightarrow J[\ell^\infty](-1)^{G_K} \longrightarrow (\pi_1^{ab}(X)_{ram}^{geo})_\ell^\vee \longrightarrow 0$$

where $(\pi_1^{ab}(X)_{ram}^{geo})_\ell^\vee$ denotes the ℓ-primary part of $(\pi_1^{ab}(X)_{ram}^{geo})^\vee$.

Using the canonical isomorphism $M(-1)^G \cong M^\chi(-1)$ of G_F-modules, we can express this lemma by the short exact sequence of G_F-modules :

(3.1) $$0 \longrightarrow J_F^0[\ell^\infty]^\chi \longrightarrow J[\ell^\infty]^\chi \longrightarrow (\pi_1^{ab}(X)_{ram}^{geo})_\ell^\vee(1) \longrightarrow 0$$

Now we relate this group with the component group $\Phi(J)$. For this we need the description of $\Phi(J)$ by the Galois module associated to J, stated as follows :

Lemma 3.5 ([SGA7] Exposé IX, Proposition 11.2)**.** *For an abelian variety A over a local field K, denote the Néron model and its special fiber respectively by $\mathscr{A}, A_F$. Let A_F^0 be the connected component of the identity, and denote the component group by $\Phi(A) = A_F/A_F^0$. For any prime $\ell \neq \text{char}F$, there is a canonical isomorphism of G_F-modules :*

$$\Phi(A)_\ell \cong \text{Coker}\left(T_\ell(A)^I \otimes (\mathbb{Q}_\ell/\mathbb{Z}_\ell) \longrightarrow (T_\ell(A) \otimes (\mathbb{Q}_\ell/\mathbb{Z}_\ell))^I\right)$$

where $\Phi(A)_\ell$ is the ℓ-primary part of $\Phi(A)$, and $(-)^I$ denotes the invariant by the action of the inertia group $I \subset G_K$.

By the canonical isomorphism of G_F-modules $T_\ell(A)^I \cong T_\ell(A_F^0)$ ([SGA7] Exposé IX, Proposition 2.2.5, (2.2.3.3)), we can restate the above lemma by the short exact sequence of G_F-modules :

$$(3.2) \qquad 0 \longrightarrow A_F^0[\ell^\infty] \longrightarrow A[\ell^\infty]^I \longrightarrow \Phi(A)_\ell \longrightarrow 0$$

Proof of Theorem 3.2. Applying the left exact functor $M \mapsto M^\chi$ to the short exact sequence (3.2) for $A = J$, we have the exact sequence :

$$(3.3) \qquad 0 \longrightarrow J_F^0[\ell^\infty]^\chi \longrightarrow J[\ell^\infty]^\chi \longrightarrow \Phi(J)_\ell^\chi$$

Comparing this with the short exact sequence (3.1), we have a canonical injection $(\pi_1^{ab}(X)_{ram}^{geo})_\ell^\vee \longrightarrow \Phi(J)_\ell^\chi(-1)$. As $\Phi(J)_\ell^\chi(-1) \cong \Phi(J)_\ell(-1)^G$ injects to $\Phi(J)_\ell$ as finite abelian groups and $\ell \neq p$ was arbitrary, we have proven the theorem. $\qquad \square$

4. THE p-PRIMARY PART FOR LOW ABSOLUTE RAMIFICATION CASE

As in the previous section, we consider a proper smooth geometrically irreducible curve X over a local field K which admits a K-rational point. Let $e \in \mathbb{N}$ denote the *absolute ramification index* of K, i.e. the normalized valuation of $p = \mathrm{char} F$ in K. In this section, we prove the following :

Theorem 4.1. *Assume $e < p - 1$ and $\mathrm{char} K = 0$. If X has semistable reduction over O_K, then $(\pi_1^{ab}(X)_{ram}^{geo})_p = 0$, where $(-)_p$ denotes the p-primary part ($p = \mathrm{char} F$).*

Remark 4.2. This theorem is a generalization of the Prop. 7 of Kato-Saito[6], where it is proved in the good reduction case. Here we employ a completely different method of proof.

Our task here is to analyze the $H_{et}^1(\overline{X}, \mathbb{Z}/p^n\mathbb{Z})^{G_K} \cong \mathrm{Hom}_{K-\mathrm{gp}}(\mu_{p^n}, \mathrm{Pic}^0(X)[p^n])$ (Lemma 2.6). For this purpose, we recall the structure of the p^n-torsion points $J[p^n]$ of the jacobian $J = \mathrm{Pic}^0(X)$, following [SGA7], Exposé IX. In the rest of this section, we assume that X *has semistable reduction over O_K*. In this case, J has semistable reduction, i.e. the connected component of the identity J_F^0 of the special fiber J_F of the Néron model $\mathscr{J}$ is a semiabelian scheme over F. Moreover, if we denote by $\mathscr{J}^0$ the connected component of the identity of $\mathscr{J}$, we know that $\mathscr{J}^0[p^n]$ is a quasi-finite group scheme over O_K ([SGA7] Exposé IX, Lemme 2.2.1)

Now recall that any quasi-finite group scheme $\mathscr{G}$ over O_K has the canonical and functorial decomposition $\mathscr{G} = \mathscr{G}^f \coprod \mathscr{G}'$ where $\mathscr{G}^f$ (*the fixed part* of $\mathscr{G}$) is finite flat over O_K and $\mathscr{G}'$ has empty special fiber ([EGA] II,(6.2.6), [SGA7] Exposé IX, (2.2.3.1)).

Coming back to our case, $\mathscr{J}^0[p^n]$ has the decomposition :

$$\mathscr{J}^0[p^n] = \mathscr{J}^0[p^n]^f \coprod \mathscr{J}^0[p^n]'$$

where $\mathscr{J}^0[p^n]^f$ is a finite flat group scheme. We denote the generic fiber of $\mathscr{J}^0[p^n]^f$ by $J^0[p^n]^f$, which is a subgroup of $J[p^n]$, the generic fiber of $\mathscr{J}^0[p^n]$ (recall $(\mathscr{J}^0)_K = J$). Note that $\mathscr{J}^0[p^n]$, $\mathscr{J}^0[p^n]^f$ have a common special fiber $J_F^0[p^n]$, which is canonically isomorphic to $\mathrm{Pic}^0(X_F)[p^n]$ by Lemma 3.1 (recall the running hypothesis that X has a K-rational point).

Moreover, we have additional information by the semi-stability hypothesis ([SGA7] Exposé IX, (5.5.8)) :

$$(4.1) \qquad J[p^n]/J^0[p^n]^f \cong M_K \otimes (\mathbb{Z}/p^n\mathbb{Z}) = (M \otimes (\mathbb{Z}/p^n\mathbb{Z}))_K$$

where M denotes the character group of the toric part of J_F, which is an unramified Galois module, free of finite rank over $\mathbb{Z}$. We consider M as an étale group scheme over O_K, therefore justifying the notation M_K. (Note that in general we have to take the corresponding object for the dual abelian variety, but we have the autoduality of the jacobian in the present case.)

Now we apply the result of Raynaud[12] to obtain the following lemma :

Lemma 4.3. *Under the hypothesis $e < p - 1$ and $\mathrm{char}\, K = 0$, the following natural homomorphism is an isomorphism :*

$$H_{et}^1(\overline{X_F}, \mathbb{Z}/p^n\mathbb{Z})^{G_F} \xrightarrow{\cong} H_{et}^1(\overline{X}, \mathbb{Z}/p^n\mathbb{Z})^{G_K}$$

Proof. By Lemma 2.6, we have :

$$H_{et}^1(\overline{X_F}, \mathbb{Z}/p^n\mathbb{Z})^{G_F} \cong \mathrm{Hom}_{F-\mathrm{gp}}(\mu_{p^n}, J_F^0[p^n])$$
$$H_{et}^1(\overline{X}, \mathbb{Z}/p^n\mathbb{Z})^{G_K} \cong \mathrm{Hom}_{K-\mathrm{gp}}(\mu_{p^n}, J[p^n])$$

By (4.1), we have an exact sequence :

$$0 \longrightarrow \mathrm{Hom}_{K-\mathrm{gp}}(\mu_{p^n}, J^0[p^n]^f) \longrightarrow \mathrm{Hom}_{K-\mathrm{gp}}(\mu_{p^n}, J[p^n])$$
$$\longrightarrow \mathrm{Hom}_{K-\mathrm{gp}}(\mu_{p^n}, (M \otimes (\mathbb{Z}/p^n\mathbb{Z}))_K)$$

But by Corollaire 3.3.6 of Raynaud[12], we have :

$$\mathrm{Hom}_{K-\mathrm{gp}}(\mu_{p^n}, (M \otimes (\mathbb{Z}/p^n\mathbb{Z}))_K) \cong \mathrm{Hom}_{O_K-\mathrm{gp}}(\mu_{p^n}, M \otimes (\mathbb{Z}/p^n\mathbb{Z})) = 0$$

which vanishes simply because μ_{p^n} is connected and $M \otimes (\mathbb{Z}/p^n\mathbb{Z})$ is étale. Therefore we have, using [12], Corollaire 3.3.6 again,

$$\mathrm{Hom}_{K-\mathrm{gp}}(\mu_{p^n}, J[p^n]) \cong \mathrm{Hom}_{K-\mathrm{gp}}(\mu_{p^n}, J^0[p^n]^f) \cong \mathrm{Hom}_{O_K-\mathrm{gp}}(\mu_{p^n}, \mathscr{J}^0[p^n]^f)$$

Therefore it suffices to see that $\mathrm{Hom}_{O_K-\mathrm{gp}}(\mu_{p^n}, \mathscr{J}^0[p^n]^f) \cong \mathrm{Hom}_{F-\mathrm{gp}}(\mu_{p^n}, J_F^0[p^n])$, which is equivalent by Cartier duality ([SGA3] Exposé VIIA, (3.3.1)) to :

$$\mathrm{Hom}_{O_K-\mathrm{gp}}(\mathscr{J}^0[p^n]^{f*}, \mathbb{Z}/p^n\mathbb{Z}) \cong \mathrm{Hom}_{F-\mathrm{gp}}(J_F^0[p^n]^*, \mathbb{Z}/p^n\mathbb{Z})$$

where * denotes the Cartier dual $\mathscr{G}^* = \underline{\mathrm{Hom}}(\mathscr{G}, \mathbb{G}_m)$. But as $\mathbb{Z}/p^n\mathbb{Z}$ is étale, this is clear by the Hensel's lemma (e.g. [EGA] IV, (18.5.12)). $\square$

Proof of Theorem 4.1. By Lemma 2.3, it is enough to show that the canonical homomorphism $H^1_{et}(\overline{X_F}, \mathbb{Q}_p/\mathbb{Z}_p)^{G_F} \longrightarrow H^1_{et}(\overline{X}, \mathbb{Q}_p/\mathbb{Z}_p)^{G_K}$ is an isomorphism. But by exactly the same argument as in (2.7) we have :

$$H^1_{et}(\overline{X_F}, \mathbb{Q}_p/\mathbb{Z}_p)^{G_F} = \bigcup_n H^1_{et}(\overline{X_F}, \mathbb{Q}_p/\mathbb{Z}_p)[p^n]^{G_F} = \bigcup_n H^1_{et}(\overline{X_F}, \frac{1}{p^n}\mathbb{Z}/\mathbb{Z})^{G_F}$$

$$H^1_{et}(\overline{X}, \mathbb{Q}_p/\mathbb{Z}_p)^{G_K} = \bigcup_n H^1_{et}(\overline{X}, \mathbb{Q}_p/\mathbb{Z}_p)[p^n]^{G_K} = \bigcup_n H^1_{et}(\overline{X}, \frac{1}{p^n}\mathbb{Z}/\mathbb{Z})^{G_K}$$

Therefore the desired isomorphism is deduced from the corresponding result in the each p-power torsion level, namely the Lemma 4.3. $\square$

Combining with Th. 3.2, we have :

Corollary 4.4. *Assume $e < p - 1$ and $\mathrm{char}K = 0$. If X admits a K-rational point and has semistable reduction over O_K, then there is an injection of finite abelian groups $(\pi_1^{ab}(X)^{geo}_{ram})^\vee \longrightarrow \Phi(J)$.*

Remark 4.5. We have the canonical perfect dualiy :

$$\Phi \otimes \Phi \longrightarrow \mathbb{Q}/\mathbb{Z}$$

deduced by the autoduality of the jacobian and [SGA7] Exposé IX, (11.4.1) or Conjecture 1.3 (proven in this case). This gives the canonical identification $\Phi \cong \Phi^\vee$, which enables us to state Cor. 4.4 as the *surjection* $\Phi(J) \longrightarrow \pi_1^{ab}(X)^{geo}_{ram}$.

5. THE CASE OF THE MODULAR CURVE $X_0(p)$

In this section, we apply the results of preceding section to the modular curve $X_0(p)/\mathbb{Q}_p$ for a prime p, which is a proper smooth geometrically irreducible curve over $\mathbb{Q}_p$. For the definition and basic properties of the modular curve $X_0(p)$, we refer to Mazur [8]. The result of this section is summarized as follows :

Theorem 5.1. $\pi_1^{ab}(X_0(p)/\mathbb{Q}_p)^{geo}$ *has the following structure :*

$$0 \longrightarrow \Phi(J_0(p)) \longrightarrow \pi_1^{ab}(X_0(p)/\mathbb{Q}_p)^{geo} \longrightarrow \widehat{\mathbb{Z}}^r \longrightarrow 0$$

where $\Phi(J_0(p)) \cong \pi_1^{ab}(X_0(p)/\mathbb{Q}_p)^{geo}_{ram} = \pi_1^{ab}(X_0(p)/\mathbb{Q}_p)^{geo}_{tor}$ is a cyclic group of order equal to the numerator of $\frac{p-1}{12}$, and $r = \frac{g+h-1}{2}$ where g is the genus of $X_0(p)$ and h is the number of the supersingular points defined over $\mathbb{F}_p$.

Remark 5.2. This result should be interpreted as the local analogue of the theorem of Mazur which asserts that $\pi_1^{ab}(X_0(p)/\mathbb{Q})^{geo} \cong \Phi(J_0(p))$ ([8], Introduction, Theorem (2)). The fact that $\pi_1^{ab}(X_0(p)/\mathbb{Q})^{geo}$ is equal to $\pi_1^{ab}(X_0(p)/\mathbb{Q}_p)^{geo}_{ram} = (\pi_1^{ab}(X_0(p)/\mathbb{Q}_p)^{geo}_{tor}$ shows that the maximal abelian étale covering of $X_0(p)/\mathbb{Q}$ "uses up" all the ramification allowed at the special fiber at p. The origin of this phenomenon remains to be clarified.

5.1. **The part** $\pi_1^{ab}(X_0(p)/\mathbb{Q}_p)_{ram}^{geo}$. First, we apply the results of the preceding sections to determine the part $\pi_1^{ab}(X_0(p)/\mathbb{Q}_p)_{ram}^{geo}$. As $e = 1$ for $\mathbb{Q}_p$ and $X_0(p)$ admits a $\mathbb{Q}_p$-rational point (e.g. the ∞-cusp), and moreover $X_0(p)$ has semistable reduction over $\mathbb{Z}_p$ (Deligne-Rapoport[5], V-6, or Mazur[8], II-1), we know by Cor. 4.4 that $(\pi_1^{ab}(X_0(p)/\mathbb{Q}_p)_{ram}^{geo})^\vee$ injects to $\Phi(J_0(p))$, where $J_0(p)$ is the jacobian of $X_0(p)$. By Mazur-Rapoport[9], we know that :

Proposition 5.3 ([9], Theorem (A.1), b)). $\Phi(J_0(p))$ *is a cyclic group of order equal to the numerator of* $\frac{p-1}{12}$.

Because the Shimura covering ([8], Cor. (2.3)) gives a cyclic étale covering of $X_0(p)/\mathbb{Q}_p$ of order equal to the numerator of $\frac{p-1}{12}$, which is by definition completely ramified over one of the component of the special fiber, we have :

Proposition 5.4. $\pi_1^{ab}(X_0(p)/\mathbb{Q}_p)_{ram}^{geo}$ *is isomorphic to* $\Phi(J_0(p))$.

5.2. **The part** $\pi_1^{ab}(X_0(p)/\mathbb{F}_p)^{geo}$. We denote by $X_0(p)/\mathbb{Z}_p$ the minimal regular model of $X_0(p)/\mathbb{Q}_p$ following [8], and denote the special fiber by $X_0(p)/\mathbb{F}_p$. Here we determine the part $\pi_1^{ab}(X_0(p)/\mathbb{F}_p)^{geo}$, thereby completing the proof of Th. 5.1. The result is :

Proposition 5.5. $\pi_1^{ab}(X_0(p)/\mathbb{F}_p)^{geo}$ *is a free module of over* $\widehat{\mathbb{Z}}$ *with the rank equal to* $\frac{g+h-1}{2}$, *where g is the genus of $X_0(p)$ and h is the number of the supersingular points defined over $\mathbb{F}_p$.*

Proof. Put $G = \mathrm{Gal}(\overline{\mathbb{F}_p}/\mathbb{F}_p)$. As each of the component of the special fiber $X_0(p)/\mathbb{F}_p$ of $X_0(p)$ is (geometrically) isomorphic to $\mathbb{P}^1$, we know that $\mathrm{Pic}^0(X_0(p)/\mathbb{F}_p)$ is a torus with the character group canonically isomorphic (as the G-module) to the first homology group $H_1(\Gamma, \mathbb{Z})$ of the graph Γ of $X_0(p)/\overline{\mathbb{F}_p}$ ([SGA7] Exposé IX, 12.3). This graph is described in Mazur-Rapoport[9], §3. In particular, if we denote the genus of $X_0(p)$ by g, the total number of supersingular points are $g + 1$, and $H_1(\Gamma, \mathbb{Z})$ is a free module over $\mathbb{Z}$ of rank g.

Now by (2.5), (2.7), and Lemma 2.6, we have :

$$
\begin{aligned}
\pi_1^{ab}(X_0(p)/\mathbb{F}_p)^{geo} &\cong \Big(\bigcup_N \mathrm{Hom}_{\mathbb{F}_p-\mathrm{gp}}(\mu_N, \mathrm{Pic}^0(X_0(p)/\mathbb{F}_p)[N])\Big)^\vee \\
&\cong \Big(\bigcup_N \mathrm{Hom}_G(H_1(\Gamma, \mathbb{Z}/N\mathbb{Z}), \mathbb{Z}/N\mathbb{Z})\Big)^\vee \\
&\cong \varprojlim_N \big(\mathrm{Hom}(H_1(\Gamma, \mathbb{Z}/N\mathbb{Z})_G, \mathbb{Q}/\mathbb{Z})^\vee\big) \\
&\cong \varprojlim_N H_1(\Gamma, \mathbb{Z}/N\mathbb{Z})_G \\
&\cong \varprojlim_N \big(H_1(\Gamma, \mathbb{Z})_G \otimes \mathbb{Z}/N\mathbb{Z}\big) \cong H_1(\Gamma, \mathbb{Z})_G \otimes \widehat{\mathbb{Z}}
\end{aligned}
$$

where $H_1(\Gamma, \mathbb{Z})_G$ is the G-coinvariant of $H_1(\Gamma, \mathbb{Z})$. The last isomorphism follows from the fact that $H_1(\Gamma, \mathbb{Z})_G$ is clearly a finitely generated $\mathbb{Z}$-module. Now it only remains to determine the group $H_1(\Gamma, \mathbb{Z})_G$ explicitly.

As we know that the vertices of Γ is fixed by G as the irreducible components are defined over $\mathbb{F}_p$, and the edges of Γ which correspond to the supersingular points of $X_0(p)$ are fixed or interchanged by pairs by the Frobenius automorphism of G, according to whether the field of definition of the corresponding supersingular point is $\mathbb{F}_p$ or $\mathbb{F}_{p^2}$ (these possibilities can be visibly read off from the Table 6 of [2]).

Therefore the Frobenius acts on the basis of the homology group formed by the pair of the edges corresponding to the pair of supersingular points defined over $\mathbb{F}_{p^2}$ by multipication by -1, which means that when we take the G-coinvariant, exactly these bases vanish. Therefore if we denote the number of the supersingular points defined over $\mathbb{F}_p$ and the pairs of supersingular points defined over $\mathbb{F}_{p^2}$ respectively by h and j (therefore $g + 1 = h + 2j$), the G-coinvariant of $H_1(\Gamma, \mathbb{Z})$ is a free module of rank $g - j = \frac{g+h-1}{2}$. (This rank is equal to the rank defined in the general setting in Saito[14], II-Def. 2.5.) $\square$

References

[1] S. Abyhankar, *Resolution of singularities for arithmetical surfaces*, in : *Arithmetical Algebraic Geometry* (Harper and Row, 1963), pp.111-152.

[2] B. J. Birch, W. Kuyk, ed. *Modular Functions of One Variable IV*, Lecture Notes in Math. **476**, Springer, 1975.

[3] S. Bloch, *Algebraic K-theory and classfield theory for arithmetic surfaces*, Annals of Math. **114** (1981), 229-265. (Erratum : Annals of Math. **126** (1987), 449.)

[4] P. Deligne, *Théorie de Hodge III*, Publ. Math. I.H.E.S. **44** (1974), 5-77.

[5] P. Deligne, M. Rapoport, *Schémas de modules des courbes elliptiques*, in : *Modular Functions of One Variable II* (Springer LNM 349, 1973), pp.143-316.

[6] K. Kato, S. Saito, *Unramified class field theory of arithmetical surfaces*, Annals of Math. **118** (1983), 241-275.

[7] N. Katz and S. Lang, *Finiteness theorems in geometric classfield theory*, L'Enseignement Math. **27** (1981), 285-319.

[8] B. Mazur, *Modular curves and the Eisenstein ideal*, Publ. Math. I.H.E.S. **47** (1977), 33-186.

[9] B. Mazur, M. Rapoport, *Behavior of the Néron model of the jacobian of $X_0(N)$ at bad primes*, Appendix to [8].

[10] J. S. Milne, *Étale Cohomology*, Princeton Univ. Press, 1980.

[11] M. Raynaud, *Spécialisation du foncteur de Picard*, Publ. Math. I.H.E.S. **38** (1971), 27-76.

[12] M. Raynaud, *Schéma en groupes de type $(p, \dots, p)$*, Bull. Soc. math. France **102** (1974), 241-280.

[13] J.-P. Serre, *Groups algébriques et corps de classes*, Hermann, 1959.

[14] S. Saito, *Class Field Theory for Curves over Local Fields*, J. of Number Theory **21** (1985), 44-80.

[15] T. Yoshida, *Finiteness theorem in class field theory of varieties over local fields*, preprint.

[EGA] A. Grothendieck, J. Dieudonné, *Éléments de Géométrie Algébrique*, Publ. Math. I.H.E.S. **4,8,11,17,20,24,28,32** (1960/61/63/64/65/66/67).

[SGA1] A. Grothendieck, et al., *Revêtements Étales et Groupe Fondamental (SGA 1)*, Lecture Notes in Math. **224**, Springer, 1971.

[SGA3] A. Grothendieck, et al., *Schémas en Groupes (SGA 3)*, Lecture Notes in Math. **151**,**152**,**153**, Springer, 1970.

[SGA4] A. Grothendieck, et al., *Théorie des Topos et Cohomologie Etale des Schémas (SGA 4)*, Lecture Notes in Math. **269**,**270**,**305**, Springer, 1972/73.

[SGA7] A. Grothendieck, et al., *Groupes de Monodromie en Géometrie Algébrique (SGA 7)*, Lecture Notes in Math. **288**,**340**, Springer, 1972/73.

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