

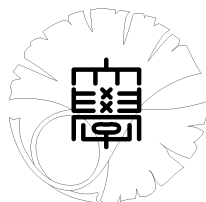
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**A mean value theorem for orders of
degree zero divisor class groups of
quadratic extensions over a function field**

by

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A MEAN VALUE THEOREM FOR ORDERS OF DEGREE ZERO DIVISOR CLASS GROUPS OF QUADRATIC EXTENSIONS OVER A FUNCTION FIELD

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ABSTRACT. Let k be a function field of one variable over a finite field with the characteristic not equal to two. In this paper, we consider the prehomogeneous representation of the space of binary quadratic forms over k . We have two main results. The first result is on the principal part of the global zeta function associated with the prehomogeneous vector space. The second result is on a mean value theorem for degree zero divisor class groups of quadratic extensions over k , which is a consequence of the first one.

0. INTRODUCTION

Let k be a function field of one variable over a finite field and h_k be the order of degree zero divisor class group. We assume that $\text{char}(k) \neq 2$. For a quadratic extension L of k , we denote by $\mathcal{N}(\mathfrak{D}_L)$ the norm of its relative discriminant. In this paper, we will give a mean value of h_L with respect to $\mathcal{N}(\mathfrak{D}_L)$.

Our main result is Theorem 6.7. We briefly state our results here. Let q and $\zeta_k(s)$ be the order of the constant field and the Dedekind zeta function of k . We denote by \mathfrak{M} the complete set of places of k , and by q_v the absolute norm of $v \in \mathfrak{M}$. The following theorem is a special case of Theorem 6.7.

Theorem 0.1.

$$\lim_{n \rightarrow \infty} \frac{1}{q^{3n}} \sum_{\substack{[L:k]=2 \\ \mathcal{N}(\mathfrak{D}_L)=q^{2n}}} h_L = 2h_k \frac{h_k}{q-1} \zeta_k(2) \prod_{v \in \mathfrak{M}} (1 - q_v^{-2} - q_v^{-3} + q_v^{-4}).$$

Theorems of this kind are called density theorems. Today, many density theorems are known. Among them, theorems about the asymptotic behavior of the mean value of the number of equivalence classes of quadratic forms are very classical and studied by many mathematicians including Gauss, Siegel, and Shintani. The density theorem for the case of binary quadratic forms is known as Gauss' conjecture. This was firstly proved by Lipschitz for imaginary case, and by Siegel for real case. Siegel [11] also proved the density theorem for integral equivalence classes of quadratic forms in general.

M.Sato and T.Shintani formulated this kind of density problems using the notion of prehomogeneous vector spaces. In [13], Shintani considered a zeta function associated with the space of quadratic forms. There, he reproved the Gauss' conjecture, and improved the error estimate. Shintani [12] also considered the space of binary cubic forms, and gave the density of the class number of integral binary cubic forms.

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Datskovsky and Wright [19], [1], [3], [4] treated the zeta function associated with the space of binary cubic forms over an arbitrary global field using the adelic language. Then they obtained the Davenport-Heilbronn [5], [6] density theorem of cubic fields from the viewpoint of prehomogeneous vector spaces.

Also for the space of binary quadratic forms, Shintani's global theory was extended to an arbitrary number field by Yukie [15]. Then Datskovsky [1] carried out the local theory, and gave another proof of the Goldfeld-Hoffstein [7] mean value theorem of "class number times regulator" of quadratic extensions over any fixed number field. However, the problem for binary quadratic forms over a function field was left unanswered. In this paper, we will study this case.

For the general process from a prehomogeneous vector space to its density theorem, see [16] or [17].

This paper is organized as follows. In Section 1, we collect notation we use throughout this paper. In Section 2, we review structures of the space of binary quadratic forms. The rational orbit decomposition and the structure of the stabilizers are discussed. In Sections 3 and 4, we treat the global theory. In Section 3, we define the global zeta function $Z(\Phi, s)$. This is a function of a Schwartz-Bruhat function Φ , and a complex variable s . The main purpose of this section is to show that it converges if $\text{Re}(s)$ is sufficiently large. In Section 4, we study analytic properties of the global zeta function. We show its rationality and determine the pole structure. Our main result is Theorem 4.20. The residues at the poles are described by means of some distributions. We later need its rightmost pole, which is rather simple; it is a constant multiple of $\hat{\Phi}(0)$ where $\hat{\Phi}$ is the Fourier transform of Φ .

In Section 5, we define various invariant measures, and consider local zeta functions. Then in Section 6, we define certain Dirichlet series and study their analytic properties by putting together the results we have obtained before. And by using the filtering process, we obtain a density theorem which is a generalized version of Theorem 0.1.

Our discussion is quite similar to [1], which is the case of number fields. However, as A.Yukie pointed out at [16], p.12, there is an incomplete argument in his paper. His choice of the measure on the stabilizer at [1], p.218 is wrong because it does not satisfy the functorial property, which he implicitly used in [1], p.230. We will correct the argument in Section 5 following [9]. It is easy to see that our choice can also be applied to his paper, and the final results of [1] need no modification.

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1. NOTATION

Here, we will prepare basic notation. For a finite set X we denote by $\#X$ its cardinality. If f, g are functions on a set Z , and $|f(z)| \leq Cg(z)$ for some constant C independent of $z \in Z$, we denote $f(z) \ll g(z)$. The standard symbols $\mathbb{Q}, \mathbb{R}, \mathbb{C}$, and \mathbb{Z} will denote the set of rational, real, complex numbers and the rational integers, respectively. If R is any ring then R^\times is the group of units of R , and if X is a variety defined over R then X_R denotes its R -points.

Suppose that G is a locally compact group and Γ is a discrete subgroup of G contained in the maximal unimodular subgroup of G . For any left invariant measure dg on G , we

choose a left invariant measure dg (we use the same notation, but the meaning will be clear from the context) on $X = G/\Gamma$ so that

$$\int_G f(g)dg = \int_X \sum_{\gamma \in \Gamma} f(g\gamma)dg.$$

Throughout this paper, we agree that k denotes any fixed function field of one variable over a finite field of constants \mathbb{F}_q , $q \neq 2^n$. Denote by \mathfrak{M} the complete set of places of k . For $v \in \mathfrak{M}$, k_v denotes the completion of k at v and $|\cdot|_v$ the normalized absolute value on k_v . We denote by \mathcal{O}_v, π_v and q_v , the ring of integers of k_v , a fixed uniformizer in \mathcal{O}_v , and the cardinality of $\mathcal{O}_v/\pi_v\mathcal{O}_v$, respectively.

Returning to k , let \mathfrak{g}, h_k be the genus of k , the order of degree zero divisor class group of k , respectively. It will be convenient to set

$$\mathfrak{C}_k = \frac{h_k}{q-1}.$$

Let $\zeta_k(s)$ denote the Dedekind zeta function $\prod_{v \in \mathfrak{M}} (1 - q_v^{-1})^{-s}$ of k . It is known that $\zeta_k(s)$ is a rational function of q^s and

$$\lim_{s \rightarrow 1} (1 - q^{1-s})\zeta_k(s) = q^{1-\mathfrak{g}}\mathfrak{C}_k.$$

Later, we will consider quadratic extensions of k . For an extension L of k , the symbols q_L, h_L, \mathfrak{C}_L and \mathfrak{M}_L are defined in similar way. We will denote by $\mathcal{N}(\mathfrak{D}_L)$ the ideal norm of the relative discriminant of L over k .

Let $\mathbb{A}, \mathbb{A}^\times$ be the ring of adeles and the group of ideles of k , respectively. The group \mathbb{A}^\times is endowed with the idele norm, denoted by $|\cdot|_{\mathbb{A}}$, where $|x|_{\mathbb{A}} = \prod_{v \in \mathfrak{M}} |x_v|_v, x = (x_v)_v \in \mathbb{A}^\times$. The field k , identified with a subset of \mathbb{A} via the diagonal embedding, forms a lattice in \mathbb{A} . Let $\mathbb{A}^\times \supset \mathbb{A}^n$ be the set of elements with idele norm q^n . If V is a vector space over k , let $\mathcal{S}(V_{\mathbb{A}}), \mathcal{S}(V_{k_v})$ be the space of Schwartz-Bruhat functions.

We choose a Haar measure dx on \mathbb{A} , $d^\times t$ on \mathbb{A}^\times , dx_v on \mathcal{O}_v , $d^\times t_v$ on \mathcal{O}_v^\times , so that

$$\int_{\mathbb{A}/k} dx = 1, \quad \int_{\mathbb{A}^\times/k^\times} d^\times t = 1, \quad \int_{\mathcal{O}_v} dx_v = 1, \quad \int_{\mathcal{O}_v^\times} d^\times t_v = 1,$$

respectively. We later have to compare the global measure and the product of local measures. It is well known that

$$(1.1) \quad dx = q^{1-\mathfrak{g}} \prod_{v \in \mathfrak{M}} dx_v \quad \text{and} \quad dt = \mathfrak{C}_k^{-1} \prod_{v \in \mathfrak{M}} d^\times t_v.$$

Finally, for the sake of convenience, we denote the element $G = GL(1) \times GL(2)$ in the following manner:

$$n(u) = \left(1, \begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix} \right), \quad d(t, t_1) = \left(t, \begin{pmatrix} t_1 & 0 \\ 0 & t_1 \end{pmatrix} \right), \quad a(\tau) = \left(1, \begin{pmatrix} 1 & 0 \\ 0 & \tau \end{pmatrix} \right).$$

We note that $a(\tau)n(u) = n(\tau u)a(\tau)$.

2. THE SPACE OF BINARY QUADRATIC FORMS

Let V be the 3-dimensional affine space. We define V with the space of binary quadratic forms via the correspondence:

$$x = (x_0, x_1, x_2) \in V \longleftrightarrow F_x(z_1, z_2) = x_0 z_1^2 + x_1 z_1 z_2 + x_2 z_2^2.$$

The group $G = GL(1) \times GL(2)$ acts on V from the left, the $GL(1)$ -part acts on V by the usual scalar multiplication, and the $GL(2)$ -part acts on V by the linear change of variables. Explicitly, the action of G on V is given by

$$F_{g \cdot x}(z_1, z_2) = tF_x(az_1 + cz_2, bz_1 + dz_2), \quad g = \left(t, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) \in G, x \in V.$$

For any field K the space V_K is a 3-dimensional vector space over K . The action of G on V then gives a representation $\varrho : G \rightarrow GL(V)$ defined over K . The kernel of ϱ is a one-dimensional torus T_ϱ in the center of G . Let

$$H = \text{Im}(\varrho).$$

H is a closed reductive subgroup of $GL(V)$.

For $x \in V$, let $P(x)$ denote the discriminant of F_x :

$$P(x) = x_1^2 - 4x_0x_2.$$

For $g = (t, \begin{pmatrix} a & b \\ c & d \end{pmatrix}) \in G$, set $\chi(g) = t(ad - bc)$. Since $\chi(T_\varrho) = 1$, we can regard χ as a character on H . It is easy to see that

$$P(h \cdot x) = (\chi(h))^2 P(x) \quad \text{and} \quad \det(h) = \chi(h)^3.$$

We call a form x non-singular if $P(x) \neq 0$ and singular otherwise. Let V' be the set of all non-singular forms in V . It is easy to see that two forms in V'_K are G_K -equivalent if and only if their splitting fields over K are the same. Thus non-singular G_K -orbits in V_K are in one-to-one correspondence with the extensions of K of degree less or equal to two. For $x \in V_K$, $K(x)$ denotes the splitting field of x over K . And for $x \in V'_K$, let $G_x \subset G$ denote the stabilizer subgroup of x , and G_x° the identity component.

Proposition 2.1. (i) $|G_x/G_x^\circ| = 2$.

$$(ii) \quad G_x^\circ = \begin{cases} GL(1) \times GL(1) & K(x) = K, \\ G_x^\circ = R_{K(x)/K}(GL(1)) & [K(x) : K] = 2. \end{cases}$$

Proof. Here, we write the summary which will be needed in Section 5. For detail, see [1].

First for the case $K(x) = K$. Let $F_x(z_1, z_2) = (pz_1 + qz_2)(rz_1 + sz_2)$, then for $g = (t, \begin{pmatrix} a & b \\ c & d \end{pmatrix}) \in G_x^\circ$, $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ acts on $pz_1 + qz_2, rz_1 + sz_2$ as scalar multiplications, hence it has two eigenvectors and they are $\begin{pmatrix} p \\ q \end{pmatrix}, \begin{pmatrix} r \\ s \end{pmatrix}$. Set these eigenvalues α, β , then $t = (\alpha\beta)^{-1}$ and

$$G_x^\circ \rightarrow GL(1) \times GL(1) \quad g \mapsto (\alpha, \beta)$$

gives an isomorphism of groups. Note that $g \mapsto (\beta, \alpha)$ also gives an isomorphism.

Next for the case $[K(x) : K] = 2$. Let

$$F_x(z_1, z_2) = x_0z_1^2 + x_1z_1z_2 + x_2z_2^2 = x_0(z_1 + \theta z_2)(z_1 + \theta' z_2),$$

where θ' is the Galois conjugate of θ over K . Then for $g = (t, \begin{pmatrix} a & b \\ c & d \end{pmatrix}) \in G_x^\circ$, $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ acts on $z_1 + \theta z_2$ as scalar multiplication by $a + b\theta$ and $t = N_{K(x)/K}(a + b\theta)^{-1}$. Now, let

$$\varphi : G_x^\circ \rightarrow GL(1) \quad g \mapsto a + b\theta.$$

Then φ is a morphism of linear algebraic groups, defined over $K(x)$, and as an abstract group, $G_x^\circ(K) \cong GL(1)_{K(x)}$. Also note that $g \mapsto a + b\theta'$ gives another isomorphism. \square

3. THE GLOBAL ZETA FUNCTION: DEFINITION AND CONVERGENCE

§ 3.1. Definition of the global zeta function. From now on, let k be a function field of one variable over a finite field of constants \mathbb{F}_q such that $\text{char}(k) \neq 2$. Then H_k becomes a discrete subgroup of $H_{\mathbb{A}}$ and V_k a H_k -invariant lattice of $V_{\mathbb{A}}$.

Definition 3.1. Let $V_k'' = \{x \in V_k' \mid [k(x) : k] = 2\}$. For $\Phi \in \mathcal{S}(V_{\mathbb{A}})$ and $s \in \mathbb{C}$, we define the global zeta function $Z(\Phi, s)$ by

$$Z(\Phi, s) = \int_{H_{\mathbb{A}}/H_k} |\chi(h)|_{\mathbb{A}}^s \sum_{x \in V_k''} \Phi(h \cdot x) dh$$

where dh is a Haar measure on $H_{\mathbb{A}}$ that will be normalized in §3.2.

§ 3.2. A Haar measure on $H_{\mathbb{A}}$. Firstly, we describe a Haar measure on $G_{\mathbb{A}}$. Let $G_{\mathbb{A}} = \mathcal{K}B_{\mathbb{A}}$ be the Iwasawa decomposition, where \mathcal{K} is the standard maximal compact subgroup of $G_{\mathbb{A}}$ and $B_{\mathbb{A}}$ a Borel subgroup of $G_{\mathbb{A}}$. More precisely, $\mathcal{K} = \prod_{v \in \mathfrak{M}} \mathcal{K}_v$, $\mathcal{K}_v = G_{\mathcal{O}_v}$, and $B = \{(t, \begin{pmatrix} t_1 & 0 \\ a & t_2 \end{pmatrix})\}$.

Every element of $B_{\mathbb{A}}$ can be written uniquely as $b = d(t, t_1)a(\tau)n(u)$, where $u \in \mathbb{A}$, $t, t_1, \tau \in \mathbb{A}^{\times}$, and it is easy to check that $db = |\tau|_{\mathbb{A}} d^{\times} t d^{\times} t_1 d^{\times} \tau du$ is a right invariant measure on $B_{\mathbb{A}}$. We normalize the Haar measure $d\kappa$ on \mathcal{K} by $\int_{\mathcal{K}} d\kappa = 1$. Then $dg = d\kappa db$ gives a normalization of the Haar measure on $G_{\mathbb{A}}$.

Let $\mathbb{A}(\emptyset) = \prod_{v \in \mathfrak{M}} \mathcal{O}_v$, $\mathbb{A}^{\times}(\emptyset) = \prod_{v \in \mathfrak{M}} \mathcal{O}_v^{\times}$. Note that t, t_1, u, τ of the Iwasawa decomposition $g = \kappa d(t, t_1)a(\tau)n(u)$ are not unique, but t, t_1, τ are unique up to multiplication by elements of $\mathbb{A}^{\times}(\emptyset)$ (hence $|t(g)|_{\mathbb{A}}, |t_1(g)|_{\mathbb{A}}$, and $|\tau(g)|_{\mathbb{A}}$ are well-defined), and u is uniquely determined modulo $\tau^{-1}\mathbb{A}(\emptyset)$.

Recall that $H \cong G/T_{\rho}$ where $T_{\rho} = \{d(t_1^2, t_1^{-1}) \in G\}$. Define dh on $H_{\mathbb{A}}$ by setting $dg = d^{\times} t_1 dh$. More explicitly, if we write

$$h = \rho(\kappa d(t, 1)a(\tau)n(u)),$$

then

$$dh = |\tau|_{\mathbb{A}} d\kappa d^{\times} t d^{\times} \tau du.$$

Let $\mathcal{B} = \rho(B)$ and $\mathcal{T} = \rho(T)$ where T is the maximal torus in B . In Section 4, we will compute integrals over $H_{\mathbb{A}}/\mathcal{B}_k$ and $H_{\mathbb{A}}/\mathcal{T}_k$ with respect to the measure $H_{\mathbb{A}}$. Such integrals are explicitly given as follows:

$$(3.1) \quad \int_{H_{\mathbb{A}}/\mathcal{B}_k} \Psi(h) dh = \int_{\mathcal{K}} d\kappa \int_{\mathbb{A}^{\times}/k^{\times}} d^{\times} t \int_{\mathbb{A}^{\times}/k^{\times}} d^{\times} \tau \int_{\mathbb{A}/k} du \Psi(\rho(\kappa d(t, 1)a(\tau)n(u))) |_{\tau}|_{\mathbb{A}},$$

$$(3.2) \quad \int_{H_{\mathbb{A}}/\mathcal{T}_k} \Psi(h) dh = \int_{\mathcal{K}} d\kappa \int_{\mathbb{A}^{\times}/k^{\times}} d^{\times} t \int_{\mathbb{A}^{\times}/k^{\times}} d^{\times} \tau \int_{\mathbb{A}} du \Psi(\rho(\kappa d(t, 1)a(\tau)n(u))) |_{\tau}|_{\mathbb{A}},$$

where $\Psi \in L^1(H_{\mathbb{A}}/\mathcal{B}_k), L^1(H_{\mathbb{A}}/\mathcal{T}_k)$, respectively.

Let $\widehat{\mathcal{T}}_k$ be the subgroup of H_k generated by \mathcal{T}_k and $\iota = \rho(1, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix})$. We will also consider an integral over $H_{\mathbb{A}}/\widehat{\mathcal{T}}_k$. Note that \mathcal{T}_k is a subgroup of $\widehat{\mathcal{T}}_k$ of index 2. In the same way as [15], p362, we can prove the next lemma.

Lemma 3.2. For $u = (u_v)_{v \in \mathfrak{M}} \in \mathbb{A}$, define

$$\alpha(u) = \prod_{v \in \mathfrak{M}} \max(1, |u_v|_v).$$

Then for $g = \kappa d(t, 1)n(u)a(\tau) \in G_{\mathbb{A}}$, $|\tau(g\iota)|_{\mathbb{A}} = \alpha(u)^2 |\tau(g)|_{\mathbb{A}}^{-1}$.

By this lemma, we can write down the integral over $H_{\mathbb{A}}/\widehat{\mathcal{T}}_k$ as follows:

$$(3.3) \quad \int_{H_{\mathbb{A}}/\widehat{\mathcal{T}}_k} \Psi(h) dh = \int_{\mathcal{K}} d\kappa \int_{\mathbb{A}^{\times}/k^{\times}} d^{\times}t \left(\int_{\substack{\mathbb{A}^{\times}/k^{\times} \\ |\tau|_{\mathbb{A}} < \alpha(u)}} d^{\times}\tau + \frac{1}{2} \int_{\substack{\mathbb{A}^{\times}/k^{\times} \\ |\tau|_{\mathbb{A}} = \alpha(u)}} d^{\times}\tau \right) \int_{\mathbb{A}} du \Psi(\varrho(\kappa d(t, 1)n(u)a(\tau))),$$

where $\Psi \in L^1(H_{\mathbb{A}}/\widehat{\mathcal{T}}_k)$.

§ 3.3. The convergence of the global zeta function. Firstly, we describe a fundamental domain of $H_{\mathbb{A}}/H_k$. Set $\Theta_2 = \{g \in GL(2)_{\mathbb{A}} \mid |\tau(g)|_{\mathbb{A}} < q^{2g}\}$ and $\Theta = \mathbb{A}^{\times} \times \Theta_2 \subset G_{\mathbb{A}}$. Then by a reduction theorem of Harder [8], $\Theta G_k = G_{\mathbb{A}}$. The set Θ is invariant under multiplication by elements of B_k on the right. Hence a fundamental domain of B_k in Θ contains a fundamental domain of G_k in $G_{\mathbb{A}}$. Then

$$|Z(\Phi, s)| \ll \int_{\varrho(\Theta)/\mathcal{B}_k} |\chi(h)|_{\mathbb{A}}^{\text{Re}(s)} \sum_{x \in V_k''} |\Phi(h \cdot x)| dh.$$

Let us describe the fundamental domain of $\varrho(\Theta)/\mathcal{B}_k$. [2] Lemma 2.2 immediately leads to the following.

Lemma 3.3. *Every element of $\varrho(\Theta)$ is right \mathcal{B}_k -equivalent to an element of $\varrho(\mathcal{S})$, where*

$$\mathcal{S} = \bigcup_u \bigcup_{t, \tau} \mathcal{K} n(u) d(t, 1) a(\tau) \subset G_{\mathbb{A}},$$

t, τ run over a set of representative of $\mathbb{A}^{\times}/k^{\times}$, $|\tau|_{\mathbb{A}} \leq q^{2g}$, and u runs over a finite set in \mathbb{A} .

Next, we will estimate $\sum_{x \in V_k''} \Phi(h \cdot x)$.

Lemma 3.4. *There exists an integer $N > 0$ such that $\sum_{x \in V_k''} |\Phi(d(t, 1)a(\tau) \cdot x)| = 0$ whenever $|t|_{\mathbb{A}} > q^N$.*

Proof. Let $x = (x_0, x_1, x_2) \in V_k''$. Suppose

$$\varrho(d(t, 1)a(\tau)) \cdot x = (tx_0, t\tau x_1, t\tau^2 x_2) \in \text{supp}(\Phi).$$

Since $\text{supp}(\Phi)$ is compact, the first coordinate is bounded, that is, there exists an integer $N > 0$ such that $|tx_0|_{\mathbb{A}} \leq q^N$. On the other hand, $x_0 \neq 0$ follows from $x \in V_k''$, and hence $|x_0|_{\mathbb{A}} = 1$. This completes the proof. \square

We recall the following well known fact.

Lemma 3.5. (i) *Let Ψ be a Schwartz-Bruhat function on \mathbb{A} . Then,*

$$\sum_{x \in k} |\Psi(tx)| \ll \max(1, |t|_{\mathbb{A}}^{-1}), \quad \sum_{x \in k^{\times}} |\Psi(tx)| \ll |t|_{\mathbb{A}}^{-1}.$$

(ii) *Let Ψ be a Schwartz-Bruhat function on $\text{Aff}_{\mathbb{A}}^n$. Then there exist Schwartz-Bruhat functions $\Psi_1, \dots, \Psi_n \geq 0$ such that*

$$|\Psi(x_1, \dots, x_n)| \leq \Psi_1(x_1) \cdots \Psi_n(x_n)$$

for x_1, \dots, x_n .

We are now ready to prove

Theorem 3.6. $Z(\Phi, s)$ converges absolutely and locally uniformly in the region of $\operatorname{Re}(s) > 3$. In particular, $Z(\Phi, s)$ is a holomorphic function of s in the region.

Proof. Let $\sigma = \operatorname{Re}(s)$. We have

$$\begin{aligned} Z(\Phi, s) &\ll \int_{\mathbb{A}^\times/k^\times} d^\times t \int_{\substack{\mathbb{A}^\times/k^\times \\ |\tau|_{\mathbb{A}} \leq q^2}} d^\times \tau \left| \tau \right|_{\mathbb{A}} |t\tau|_{\mathbb{A}}^\sigma \sum_{x \in V_k''} |\Phi(tx_0, t\tau x_1, t\tau^2 x_2)| && \text{by Lemma 3.3} \\ &= \int_{\substack{\mathbb{A}^\times/k^\times \\ |t|_{\mathbb{A}} \leq q^N}} d^\times t \int_{\substack{\mathbb{A}^\times/k^\times \\ |\tau|_{\mathbb{A}} \leq q^2}} d^\times \tau \left| \tau \right|_{\mathbb{A}} |t\tau|_{\mathbb{A}}^\sigma \sum_{x \in V_k''} |\Phi(tx_0, t\tau x_1, t\tau^2 x_2)| && \text{by Lemma 3.4.} \end{aligned}$$

For $x = (x_0, x_1, x_2) \in V_k''$, both $x_0 \neq 0$ and $x_2 \neq 0$. Hence,

$$\begin{aligned} &\ll \int_{\substack{\mathbb{A}^\times/k^\times \\ |t|_{\mathbb{A}} \leq q^N}} d^\times t \int_{\substack{\mathbb{A}^\times/k^\times \\ |\tau|_{\mathbb{A}} \leq q^2}} d^\times \tau \left| \tau \right|_{\mathbb{A}} |t\tau|_{\mathbb{A}}^\sigma |t|_{\mathbb{A}}^{-1} |t\tau^2|_{\mathbb{A}}^{-1} \max(1, |t\tau|_{\mathbb{A}}^{-1}) && \text{by Lemma 3.5} \\ &= \int_{\substack{\mathbb{A}^\times/k^\times \\ |t|_{\mathbb{A}} \leq q^{N+2}}} d^\times t \int_{\substack{\mathbb{A}^\times/k^\times \\ |\tau|_{\mathbb{A}} \leq q^2}} d^\times \tau \left| \tau \right|_{\mathbb{A}} |t|_{\mathbb{A}}^\sigma |t/\tau|_{\mathbb{A}}^{-1} |t\tau|_{\mathbb{A}}^{-1} \max(1, |t|_{\mathbb{A}}^{-1}) \\ &\ll \int_{\substack{\mathbb{A}^\times/k^\times \\ |t|_{\mathbb{A}} \leq q^{N+2}}} |t|_{\mathbb{A}}^{\sigma-3} d^\times t \cdot \int_{\substack{\mathbb{A}^\times/k^\times \\ |\tau|_{\mathbb{A}} \leq q^2}} | \tau |_{\mathbb{A}} d^\times \tau. \end{aligned}$$

Hence, we obtain the desired result. \square

§ 3.4. The definition of $Z_+(\Phi, s)$ and $Z_-(\Phi, s)$. For $t \in \mathbb{A}^\times$, set

$$\lambda_+(t) = \begin{cases} 0, & |t|_{\mathbb{A}} < 1, \\ \frac{1}{2}, & |t|_{\mathbb{A}} = 1, \\ 1, & |t|_{\mathbb{A}} > 1, \end{cases} \quad \lambda_-(t) = \begin{cases} 1, & |t|_{\mathbb{A}} < 1, \\ \frac{1}{2}, & |t|_{\mathbb{A}} = 1, \\ 0, & |t|_{\mathbb{A}} > 1, \end{cases}$$

and for $h \in H_{\mathbb{A}}$, the same symbol λ_+, λ_- denotes

$$\lambda_+(h) = \lambda_+(\chi(h)) \quad \lambda_-(h) = \lambda_-(\chi(h)).$$

Using these symbols, define

$$\begin{aligned} Z_+(\Phi, s) &= \int_{H_{\mathbb{A}}/H_k} |\chi(h)|_{\mathbb{A}}^s \lambda_+(h) \sum_{x \in V_k''} \Phi(h \cdot x) dh, \\ Z_-(\Phi, s) &= \int_{H_{\mathbb{A}}/H_k} |\chi(h)|_{\mathbb{A}}^s \lambda_-(h) \sum_{x \in V_k''} \Phi(h \cdot x) dh. \end{aligned}$$

Then we have

$$Z(\Phi, s) = Z_+(\Phi, s) + Z_-(\Phi, s).$$

Moreover, the following proposition holds. We can obtain this just as in [1] Proposition 2.1, and we omit the proof.

Proposition 3.7. $Z_+(\Phi, s)$ is a polynomial in q^s .

4. THE GLOBAL ZETA FUNCTION: ANALYTIC CONTINUATION AND THE PRINCIPAL PART FORMULA

§ 4.1. The principal part $I(\Phi, s)$. Let $\langle \cdot \rangle : \mathbb{A} \rightarrow \mathbb{C}^\times$ be a fixed non-trivial additive character on \mathbb{A} trivial on k . Let $[\cdot, \cdot]$ be a nondegenerate symmetric bilinear form on $V_{\mathbb{A}}$ given by

$$[x, y] = x_0y_2 - \frac{1}{2}x_1y_1 + x_2y_0,$$

and we identify $V_{\mathbb{A}}^*$ with $V_{\mathbb{A}}$ via the pairing $(x, y) = \langle [x, y] \rangle$. Then, the lattice $V_k \subset V_{\mathbb{A}}$ becomes self-dual. For $g = (t, \begin{pmatrix} a & b \\ c & d \end{pmatrix}) \in G$, set $g' = (t^{-1}, \frac{1}{ad-bc} \begin{pmatrix} a & b \\ c & d \end{pmatrix})$. Then the above form satisfies

$$[x, y] = [\varrho(g) \cdot x, \varrho(g') \cdot y].$$

Set $dy = dy_0dy_1dy_2$, for $y = (y_0, y_1, y_2) \in V_{\mathbb{A}}$. For $\Phi \in \mathcal{S}(V_{\mathbb{A}})$, we define the Fourier transform $\hat{\Phi}$ of Φ by

$$\hat{\Phi}(x) = \int_{V_{\mathbb{A}}} \Phi(y)(x, y)dy.$$

Then $\hat{\Phi} \in \mathcal{S}(V_{\mathbb{A}})$, and $\hat{\hat{\Phi}}(x) = \Phi(-x)$. For $h = \varrho(g) \in H$, set $\Phi_h(x) = \Phi(h \cdot x)$ and $h' = \varrho(g')$. Then it is easy to see that the Fourier transform of $\Phi(h \cdot \cdot)$ is $|\chi(h)|_{\mathbb{A}}^{-3} \hat{\Phi}(h' \cdot \cdot)$.

Set $S_k = V_k \setminus V_k''$, and $H_{\mathbb{A}}^0 = \{h \in H_{\mathbb{A}} \mid |\chi(h)|_{\mathbb{A}} = 1\}$. For $n \in \mathbb{Z}$ and $\Phi \in \mathcal{S}(V_{\mathbb{A}})$, we define $\Phi_n(x) = \Phi(\pi^n x)$. Let

$$(4.1) \quad I^0(\Phi) = \int_{H_{\mathbb{A}}^0/H_k} \left(\sum_{x \in S_k} \hat{\Phi}(h' \cdot x) - \sum_{x \in S_k} \Phi(h \cdot x) \right) dh,$$

$$(4.2) \quad I(\Phi, s) = \frac{1}{2}I^0(\Phi) + \sum_{n \geq 1} q^{-ns} I^0(\Phi_{-n}).$$

Then by applying the Poisson summation formula to $Z_-(\Phi, s)$, we obtain the following.

Proposition 4.1.

$$Z(\Phi, s) = Z_+(\Phi, s) + Z_+(3-s, \hat{\Phi}) + I(\Phi, s).$$

From now on, we will study the integral $I^0(\Phi)$ in §4.2–§4.8 and then compute $I(\Phi, s)$ in §4.9.

§ 4.2. The smoothed Eisenstein series. To compute $I^0(\Phi)$, it seems natural to divide the index set S_k of the summation into its H_k -orbits and perform integration separately. However, we cannot put this into practice because the corresponding integrals diverge. This is the main difficulty when one calculates the global zeta functions of the prehomogeneous vector spaces. To surmount this problem Shintani [12] introduced the smoothed Eisenstein series of $GL(2)$. Then he determined the principal part in the case of the space of binary cubic forms and the space of binary quadratic forms over \mathbb{Q} . Later A.Yukie [17] generalized the theory of Eisenstein series to the groups of products of $GL(n)$'s, and determined the principal part in some more cases. In this subsection, we essentially repeat their argument in our settings.

For $g \in G_{\mathbb{A}}$ and $z \in \mathbb{C}$, we define the Eisenstein series $E(z, g)$ by

$$E(z, g) = \sum_{\gamma \in G_k/B_k} |\tau(g\gamma)|_{\mathbb{A}}^{-\frac{z+1}{2}}.$$

This is left $\mathcal{K}Z(G)_{\mathbb{A}}$ -invariant and right $Z(G)_{\mathbb{A}}G_k$ -invariant. Since the kernel of ϱ is contained in the center of G , we can regard $E(z, g)$ as a function of $h \in H_{\mathbb{A}}$. That is to say, we can write

$$(4.3) \quad E(z, h) = \sum_{\gamma \in H_k/\mathcal{B}_k} |\tau(h\gamma)|_{\mathbb{A}}^{-\frac{z+1}{2}}.$$

This series converges for $\operatorname{Re}(z) > 1$ and is, in fact, a rational function of $q^{z/2}$.

Write $h = \varrho(\kappa d(t, 1)a(\tau)n(u))$. Then $E(z, h)$ depends only on τ, u , and does not on κ, t . Moreover, $E(z, h)$ is right H_k -invariant, hence its value remains unchanged if we replace u by $u + a$ for $a \in k$. Hence $E(z, h) = E_{\tau}(z, u)$ has the following Fourier expansion.

$$(4.4) \quad E_{\tau}(z, u) = C_0(z, \tau) + \sum_{a \in k^{\times}} C_a(z, \tau) \langle au \rangle,$$

where

$$C_a(z, \tau) = \int_{\mathbb{A}/k} E_{\tau}(z, u) \langle -au \rangle du.$$

We define

$$\phi(z) = q^{1-\mathfrak{g}} \frac{\zeta_k(z)}{\zeta_k(z+1)}.$$

Note that $\phi(z)$ is a rational function of q^z and is holomorphic in the region $\operatorname{Re}(z) \geq 1 - \delta$ for some $\delta > 0$ except for a simple pole at $z \in \mathbb{C}$ satisfying $q^{1-z} = 1$. The following lemma about the Fourier coefficients $C_a(z, \tau)$ is well known and we omit the proof.

Lemma 4.2. (i) *The constant term $C_0(z, \tau)$ has the following explicit formula.*

$$C_0(z, \tau) = |\tau|_{\mathbb{A}}^{-\frac{z+1}{2}} + |\tau|_{\mathbb{A}}^{\frac{z-1}{2}} \phi(z).$$

(ii) *Let $[\tau] = \sum_{v \in \mathfrak{M}} (\operatorname{ord}_v(\tau))_v$ denote the divisor of τ , and \mathfrak{c} a canonical divisor, associated with the character $\langle \cdot \rangle$. Then, $C_a(z, \tau) = 0$ for all $a \notin L(\mathfrak{c} - [\tau])$. If $a \in L(\mathfrak{c} - [\tau]), a \neq 0$, then*

$$C_a(z, \tau) = |\tau|_{\mathbb{A}}^{\frac{z-1}{2}} \frac{P_a(z, \tau)}{\zeta_k(z+1)},$$

where $P_a(z, \tau)$ is a polynomial in q^{-z} . In particular, $C_a(z, \tau)$ is a holomorphic function of z in the half-plane $\operatorname{Re}(z) > 0$.

Note that the number of $a \in k^{\times}$ such that $a \in L(\mathfrak{c} - [\tau])$ is finite. From this, we immediately obtain the following.

Corollary 4.3. *The function $E(z, \tau)$ is holomorphic in the region $\operatorname{Re}(z) > 0$ with an exception of a simple pole at $z \in \mathbb{C}$ satisfying $q^{1-z} = 1$, and*

$$\lim_{z \rightarrow 1} (1 - q^{1-z})E(z, \tau) = \lim_{z \rightarrow 1} (1 - q^{1-z})\phi(z) = \frac{q^{2-2\mathfrak{g}} \mathfrak{c}_k}{\zeta_k(2)}.$$

We denote the value in the formula of Corollary 4.3 by ρ_0 . Let ψ be an entire function such that for any $c_1, c_2 \in \mathbb{R}$ and $N > 0$,

$$\sup_{c_1 < \operatorname{Re}(w) < c_2} (1 + |w|^N) |\psi(w)| < \infty.$$

Let $\operatorname{Re}(w) > 1$. Following [12], we define the smoothed Eisenstein series $\mathcal{E}(w, h)$ by

$$(4.5) \quad \mathcal{E}(w, h) = \frac{1}{2\pi i} \int_{\operatorname{Re}(z)=r} \frac{E(z, h)}{w-z} \psi(z) dz$$

for some $1 < r < \operatorname{Re}(w)$. Note that this integral does not depend on the choice of r . Similarly as $E(z, h)$, $\mathcal{E}(w, h)$ has the Fourier expansion

$$\mathcal{E}(w, h) = \sum_{a \in k} \mathcal{C}_a(w, \tau) \langle au \rangle,$$

where

$$\mathcal{C}_a(w, \tau) = \frac{1}{2\pi i} \int_{\operatorname{Re}(z)=r} \frac{C_a(z, \tau)}{w-z} \psi(z) dz.$$

This $\mathcal{E}(w, h)$ satisfies the following property.

Lemma 4.4. (i) *As a function of w , $\mathcal{E}(w, h)$ is holomorphic in the region $\operatorname{Re}(w) > 1$.*

(ii) *For any w such that $\operatorname{Re}(w) > 1$, $\mathcal{E}(w, h) \ll |\tau(h)|_{\mathbb{A}}^{(\operatorname{Re}(w)-1)/2}$.*

(iii) $\lim_{w \rightarrow 1+0} (1 - q^{1-w}) \mathcal{E}(w, h) = \rho_0 \psi(1)$.

(iv) *For any $M > 1$,*

$$\sup_{\substack{1 < \operatorname{Re}(w) < M \\ h \in H_{\mathbb{A}}}} |(1 - q^{1-w}) \mathcal{E}(w, h)| < \infty.$$

Since the proof is similar to that of Lemma 3.2 of [2], we omit it. (Note that the convergence of (iii) is not uniform.) As a result of this, we have the following.

Corollary 4.5. *For $f \in L^1(H_{\mathbb{A}}^0/H_k)$,*

$$\lim_{w \rightarrow 1+0} (1 - q^{1-w}) \int_{H_{\mathbb{A}}^0/H_k} f(h) \mathcal{E}(w, h) dh = \rho_0 \psi(1) \int_{H_{\mathbb{A}}^0/H_k} f(h) dh.$$

§ 4.3. Decomposition of $I^0(\Phi, w)$. Set

$$(4.6) \quad I^0(\Phi, w) = \int_{H_{\mathbb{A}}^0/H_k} \sum_{x \in S_k} \left(\hat{\Phi}(h' \cdot x) - \Phi(h \cdot x) \right) \mathcal{E}(w, h) dh.$$

By Corollary 4.5, we have

$$\lim_{w \rightarrow 1+0} (1 - q^{1-w}) I^0(\Phi, w) = \rho_0 \psi(1) I^0(\Phi).$$

We have the following lemma on the structure of S_k . This can be easily proved and we simply state the result here.

Lemma 4.6. *Let $S_k^i, i = 0, 1, 2$, be the subsets of V_k given by*

$$S_k^0 = \{0\}, \quad S_k^1 = \{x \in V_k \mid x \neq 0, P(x) = 0\}, \quad S_k^2 = \{x \in V_k \mid k(x) = k, P(x) \neq 0\}.$$

Then $S_k = S_k^0 \amalg S_k^1 \amalg S_k^2$ and moreover,

$$S_k^1 = H_k \times_{B_k} \{(0, 0, a) \mid a \in k^\times\}, \quad S_k^2 = H_k \times_{\hat{T}_k} \{(0, a, 0) \mid a \in k^\times\}.$$

Definition 4.7. *For $i = 0, 1, 2$, we define*

$$J_i(\Phi, w) = \int_{H_{\mathbb{A}}^0/H_k} \sum_{x \in S_k^i} \Phi(h \cdot x) \mathcal{E}(w, h) dh.$$

The next lemma shows that each $J_i(\Phi, w)$ converges and is holomorphic in the right half-plane $\operatorname{Re}(w) > 1$.

Lemma 4.8. *The integral*

$$\int_{H_{\mathbb{A}}^0/H_k} \sum_{x \in V_k} |\Phi(h \cdot x)| |\mathcal{E}(w, h)| dh$$

converges absolutely and locally uniformly in the region $\operatorname{Re}(w) > 1$.

Proof. Set $\Theta^0 = \{g \in \Theta \mid |\det(\varrho(g))|_{\mathbb{A}} = 1\}$. For $|t\tau|_{\mathbb{A}} = 1$ and $|\tau|_{\mathbb{A}} \leq q^{2g}$, by Lemma 3.5,

$$\begin{aligned} \sum_{x \in V_k} |\Phi(\varrho(d(t, 1)a(\tau))x)| &= \sum_{x \in V_k} |\Phi(tx_0, t\tau x_1, t\tau^2 x_2)| \\ &\ll \max\{1, |t|_{\mathbb{A}}^{-1}\} \max\{1, |t\tau|_{\mathbb{A}}^{-1}\} \max\{1, |t\tau^2|_{\mathbb{A}}^{-1}\} \ll |\tau|_{\mathbb{A}}^{-1}. \end{aligned}$$

Then, by Lemma 4.4(ii), the integral is bounded by a constant multiple of

$$\int_{\mathcal{K}} d\kappa \int_{\substack{\mathbb{A}^\times/k^\times \\ |t\tau|_{\mathbb{A}}=1, |\tau|_{\mathbb{A}} \leq q^2}} d^\times t \int_{\substack{\mathbb{A}^\times/k^\times \\ |t\tau|_{\mathbb{A}}=1, |\tau|_{\mathbb{A}} \leq q^2}} d^\times \tau \int_{\mathbb{A}/k} du |\tau|_{\mathbb{A}}^{\frac{\operatorname{Re}(w)-1}{2}} = \int_{\substack{\mathbb{A}^\times/k^\times \\ |\tau|_{\mathbb{A}} \leq q^2}} |\tau|_{\mathbb{A}}^{\frac{\operatorname{Re}(w)-1}{2}} d^\times \tau < \infty.$$

□

Note that if $h \in H_{\mathbb{A}}^0$, $h = h'$. Then, by Lemma 4.6, we have the following.

Proposition 4.9.

$$(4.7) \quad I^0(\Phi, w) = J_0(\hat{\Phi}, w) - J_0(\Phi, w) + J_1(\hat{\Phi}, w) - J_1(\Phi, w) + J_2(\hat{\Phi}, w) - J_2(\Phi, w).$$

From now on, we will compute J_0, J_1, J_2 in §4.4, 4.5, 4.7, respectively. For this purpose, we will introduce some notation. For meromorphic functions $f_1(w), f_2(w)$, we will use the notation $f_1 \sim f_2$ if $f_1 - f_2$ can be continued meromorphically to a right half plane $\operatorname{Re}(w) > 1 - \sigma$ for some $\sigma > 0$ and becomes holomorphic in the region.

§4.4. Computation of $J_0(\Phi, w)$.

Proposition 4.10.

$$J_0(\Phi, w) \sim \Phi(0) \frac{\psi(w)}{1 - q^{\frac{1-w}{2}}}.$$

Proof. By the formula of §3.2, (3.1), we have

$$\begin{aligned} J_0(\Phi, w) &= \Phi(0) \int_{H_{\mathbb{A}}^0/H_k} \left(\frac{1}{2\pi i} \int_{\operatorname{Re}(z)=r} \frac{\sum_{\gamma \in H_k/B_k} |\tau(h\gamma)|_{\mathbb{A}}^{-\frac{z+1}{2}}}{w-z} \psi(z) dz \right) dh \\ &= \Phi(0) \int_{H_{\mathbb{A}}^0/B_k} \left(\frac{1}{2\pi i} \int_{\operatorname{Re}(z)=r} \frac{|\tau(h)|_{\mathbb{A}}^{-\frac{z+1}{2}}}{w-z} \psi(z) dz \right) dh \\ &= \Phi(0) \int_{\substack{\mathbb{A}^\times/k^\times \\ |t\tau|_{\mathbb{A}}=1}} d^\times t \int_{\substack{\mathbb{A}^\times/k^\times \\ |t\tau|_{\mathbb{A}}=1}} d^\times \tau \int_{\mathbb{A}/k} du \left(\frac{1}{2\pi i} \int_{\operatorname{Re}(z)=r} \frac{|\tau|_{\mathbb{A}}^{-\frac{z+1}{2}}}{w-z} \psi(z) dz \right) |\tau|_{\mathbb{A}}. \end{aligned}$$

By replacing τ by τ/t ,

$$\begin{aligned} &= \Phi(0) \int_{\mathbb{A}^\times/k^\times} d^\times t \left(\frac{1}{2\pi i} \int_{\operatorname{Re}(z)=r} \frac{|t|_{\mathbb{A}}^{\frac{z-1}{2}}}{w-z} \psi(z) dz \right) \\ &= \Phi(0) \left(\int_{\mathbb{A}^+/k^\times} d^\times t + \int_{\mathbb{A}^-/k^\times} d^\times t \right) \left(\frac{1}{2\pi i} \int_{\operatorname{Re}(z)=r} \frac{|t|_{\mathbb{A}}^{\frac{z-1}{2}}}{w-z} \psi(z) dz \right), \end{aligned}$$

where, $\mathbb{A}^+ = \{t \in \mathbb{A}^\times \mid |t|_{\mathbb{A}} > 1\}$, $\mathbb{A}^- = \{t \in \mathbb{A}^\times \mid |t|_{\mathbb{A}} \leq 1\}$. Let $r_1 < 1$. The former integral is equal to

$$\begin{aligned} \int_{\mathbb{A}^+/k^\times} d^\times t \frac{1}{2\pi i} \int_{\operatorname{Re}(z)=r_1} \frac{|t|_{\mathbb{A}}^{\frac{z-1}{2}}}{w-z} \psi(z) dz &= \sum_{m=1}^{\infty} \frac{1}{2\pi i} \int_{\operatorname{Re}(z)=r_1} \frac{q^{\frac{z-1}{2}m}}{w-z} \psi(z) dz \\ &= \frac{1}{2\pi i} \int_{\operatorname{Re}(z)=r_1} \frac{(q^{\frac{1-z}{2}} - 1)^{-1}}{w-z} \psi(z) dz \end{aligned}$$

and is holomorphic for $\operatorname{Re}(w) > r_1$. Hence,

$$\begin{aligned} J_0(\Phi, w) &\sim \Phi(0) \int_{\mathbb{A}^-/k^\times} d^\times t \frac{1}{2\pi i} \int_{\operatorname{Re}(z)=r} \frac{|t|_{\mathbb{A}}^{\frac{z-1}{2}}}{w-z} \psi(z) dz \\ &= \Phi(0) \frac{1}{2\pi i} \int_{\operatorname{Re}(z)=r} \frac{(1 - q^{\frac{1-z}{2}})^{-1}}{w-z} \psi(z) dz \\ &= \Phi(0) \frac{\psi(w)}{1 - q^{\frac{1-w}{2}}} + \Phi(0) \frac{1}{2\pi i} \int_{\operatorname{Re}(z)=r_2 > \operatorname{Re}(w)} \frac{(1 - q^{\frac{1-z}{2}})^{-1}}{w-z} \psi(z) dz \\ &\sim \Phi(0) \frac{\psi(w)}{1 - q^{\frac{1-w}{2}}}. \end{aligned}$$

We have thus proved the proposition. □

§ 4.5. Computation of $J_1(\Phi, w)$. For $\Phi \in \mathcal{S}(V_{\mathbb{A}})$, we define $M\Phi \in \mathcal{S}(V_{\mathbb{A}})$ by

$$(4.8) \quad M\Phi(x) = \int_{\mathcal{K}} \Phi(\varrho(\kappa) \cdot x) d\kappa.$$

Since $Z(\Phi, s) = Z(M\Phi, s)$, we assume that $\Phi = M\Phi$ for the rest of this section.

For $\Psi \in \mathcal{S}(\mathbb{A})$ and $s \in \mathbb{C}$, Tate's zeta function $\Sigma(\Psi, s)$ is defined by

$$(4.9) \quad \Sigma(\Psi, s) = \int_{\mathbb{A}^\times} |t|_{\mathbb{A}}^s \Psi(t) d^\times t.$$

It is well known (see [14],[18]) that $\Sigma(\Psi, s)$ can be written as follows:

$$(4.10) \quad \Sigma(\Psi, s) = P(\Psi, s) + \frac{\hat{\Psi}(0)}{1 - q^{1-s}} - \frac{\Psi(0)}{1 - q^{-s}},$$

where $P(\Psi, s)$ is a polynomial in q^s, q^{-s} .

For $\Phi \in \mathcal{S}(V_{\mathbb{A}})$, define $R_1\Phi \in \mathcal{S}(\mathbb{A})$ as $R_1\Phi(t) = \Phi(0, 0, t)$.

Proposition 4.11.

$$J_1(\Phi, w) \sim \phi(w) \psi(w) \Sigma \left(R_1\Phi, \frac{w+1}{2} \right).$$

Proof. By Lemma 4.6 and the formula §3.2,(3.2), we have

$$\begin{aligned}
J_1(\Phi, w) &= \int_{H_{\mathbb{A}}^0/H_k} \sum_{\gamma \in H_k/\mathcal{B}_k} \sum_{a \in k^\times} \Phi(h\gamma \cdot (0, 0, a)) \mathcal{E}(w, h) dh \\
&= \int_{H_{\mathbb{A}}^0/\mathcal{B}_k} \sum_{a \in k^\times} \Phi(h \cdot (0, 0, a)) \mathcal{E}(w, h) dh \\
&= \int_{\mathcal{K}} d\kappa \int_{\mathbb{A}^\times/k^\times} d^\times t \int_{\mathbb{A}^\times/k^\times} d^\times \tau \int_{\mathbb{A}/k} du \sum_{a \in k^\times} \Phi(\varrho(\kappa) \cdot (0, 0, at\tau^2)) \mathcal{E}(w, a(\tau)n(u)) |\tau|_{\mathbb{A}} \\
&\quad |t\tau|_{\mathbb{A}}=1 \\
&= \int_{\mathbb{A}^\times} d^\times t \int_{\mathbb{A}^\times/k^\times} d^\times \tau R_1\Phi(t\tau^2) \mathcal{C}_0(w, \tau) |\tau|_{\mathbb{A}}. \\
&\quad |t\tau|_{\mathbb{A}}=1
\end{aligned}$$

Here, in the last expression, we replaced $\mathcal{E}(w, h)$ by $\mathcal{C}_0(w, \tau)$ by the orthogonality of characters. By replacing τ by τ/t and afterwards t by τ^2/t , we have

$$\begin{aligned}
J_1(\Phi, w) &= \int_{\mathbb{A}^\times} d^\times t \int_{\substack{\mathbb{A}^\times/k^\times \\ |\tau|_{\mathbb{A}}=1}} d^\times \tau R_1\Phi(t) \mathcal{C}_0\left(w, \frac{t}{\tau}\right) \left|\frac{t}{\tau}\right|_{\mathbb{A}} \\
&= \int_{\mathbb{A}^\times} d^\times t R_1\Phi(t) \frac{1}{2\pi i} \int_{\operatorname{Re}(z)=r} \frac{|t|_{\mathbb{A}}^{-\frac{z-1}{2}} + |t|_{\mathbb{A}}^{\frac{z+1}{2}} \phi(z)}{w-z} \psi(z) dz.
\end{aligned}$$

We now break the above integral into two parts. Let $r_1 < -1$. The first part is equal to

$$\frac{1}{2\pi i} \int_{\operatorname{Re}(z)=r_1} \Sigma\left(R_1\Phi, -\frac{z-1}{2}\right) \frac{\psi(z)}{w-z} dz.$$

By the theory of Tate's zeta function, $\Sigma\left(R_1\Phi, -\frac{z-1}{2}\right)$ is a holomorphic function in the region $\operatorname{Re}(z) < -1$. Therefore this part is an entire function of w . Hence,

$$\begin{aligned}
J_1(\Phi, w) &\sim \frac{1}{2\pi i} \int_{\operatorname{Re}(z)=r} \Sigma\left(R_1\Phi, \frac{z+1}{2}\right) \frac{\phi(z)}{w-z} \psi(z) dz \\
&\sim \phi(w) \psi(w) \Sigma\left(R_1\Phi, \frac{w+1}{2}\right).
\end{aligned}$$

□

§ 4.6. Unstable distributions. Here, we introduce some distributions and consider its analytic properties, which will be needed later. The results in this subsection are essentially due to [15] Section 2.

Definition 4.12. For $s, w \in \mathbb{C}$ and $\Psi \in \mathcal{S}(\text{Aff}_{\mathbb{A}}^2)$, we define

$$\begin{aligned} T(\Psi, s, w) &= \int_{\mathbb{A}^\times} \int_{\mathbb{A}} |t|_{\mathbb{A}}^s \alpha(u)^{-w} \Psi(t, tu) dud^\times t, \\ T^+(\Psi, s, w) &= \int_{\mathbb{A}^\times} \int_{\mathbb{A}} \lambda_+(t) |t|_{\mathbb{A}}^s \alpha(u)^{-w} \Psi(t, tu) dud^\times t, \\ T^-(\Psi, s, w) &= \int_{\mathbb{A}^\times} \int_{\mathbb{A}} \lambda_-(t) |t|_{\mathbb{A}}^s \alpha(u)^{-w} \Psi(t, tu) dud^\times t, \\ T^0(\Psi, w) &= \int_{\mathbb{A}^0} \int_{\mathbb{A}} \alpha(u)^{-w} \Psi(t, tu) dud^\times t, \end{aligned}$$

where $\alpha(u)$ is introduced in §3.2.

Lemma 4.13. (i) $T^+(\Psi, s, w)$ converges absolutely and locally uniformly for all $s, w \in \mathbb{C}$, and $T^-(\Psi, s, w)$ converges absolutely and locally uniformly for all $\text{Re}(s) + \text{Re}(w) > 2, \text{Re}(s) > 2$. In particular, $T^0(\Psi, w)$ converges absolutely and locally uniformly for all $w \in \mathbb{C}$.

(ii) As a function of s , $T^+(\Psi, s, w)$ is a polynomial in q^s .

Proof. Let $\sigma = \text{Re}(s), \sigma_1 = \text{Re}(w)$. Let $f, g \geq 0$ be Schwartz-Bruhat functions on \mathbb{A} such that $|\Psi(x_1, x_2)| \leq f(x_1)g(x_2)$ for $x_1, x_2 \in \mathbb{A}$. By changing u to ut^{-1} , we have

$$T^-(\Psi, s, w) \ll \int_{\mathbb{A}^\times/k^\times} d^\times t \int_{\mathbb{A}} du \lambda_-(t) |t|_{\mathbb{A}}^{\sigma-1} \sum_{x \in k^\times} f(tx)g(u) \alpha(t^{-1}x^{-1}u)^{-\sigma_1}.$$

Also, the same argument as the proof of Lemma 3.4 shows that there exists an integer $N > 0$ such that

$$T^+(\Psi, s, w) \ll \int_{\substack{\mathbb{A}^\times/k^\times \\ |t|_{\mathbb{A}} \leq q^N}} d^\times t \int_{\mathbb{A}} du \lambda_+(t) |t|_{\mathbb{A}}^{\sigma-1} \sum_{x \in k^\times} f(tx)g(u) \alpha(t^{-1}x^{-1}u)^{-\sigma_1}.$$

Let $C_1 = \text{supp } f, C_2 = \text{supp } g$. We will give an estimate $\alpha(t^{-1}x^{-1}u)^{-\sigma_1}$ for $t \in \mathbb{A}^\times, x \in k^\times, u \in \mathbb{A}$ such that $tx \in C_1, u \in C_2$. Since $\alpha(\cdot) \leq 1$, we have $\alpha(t^{-1}x^{-1}u)^{-\sigma_1} \leq 1$ if $\sigma_1 \geq 0$. Let $\sigma_1 \leq 0$. By the definition of $\alpha(\cdot)$, we have

$$\begin{aligned} \alpha(t^{-1}x^{-1}u) &= \prod_{v \in \mathfrak{M}} \sup(1, |t^{-1}x^{-1}u|_v) \\ &= \prod_{v \in \mathfrak{M}} |t^{-1}x^{-1}|_v \sup(|tx|_v, |u|_v) = |t|_{\mathbb{A}}^{-1} \prod_{v \in \mathfrak{M}} \sup(|tx|_v, |u|_v). \end{aligned}$$

Since C_1, C_2 are compact, $\prod_{v \in \mathfrak{M}} \sup(|tx|_v, |u|_v)$ is bounded by a constant. Hence,

$$\alpha(t^{-1}x^{-1}u)^{-\sigma_1} \ll |t|_{\mathbb{A}}^{\sigma_1}$$

Therefore (i) follows from Lemma 3.5.

Let $\Psi_n(\cdot) = \Psi(\pi^n \cdot)$. Then we have

$$T^+(\Psi, s, w) = \frac{1}{2} T^0(\Psi, w) + \sum_{n=1}^N q^{ns} T^0(\Psi_n, w).$$

This establishes (ii). □

From now on, we will give the explicit formula of $T(\Psi, s, w)$. Since $\alpha(u)$ is a product of local factors $\alpha_v(u)$, we can also define local distributions T_v for $\Psi_v \in \mathcal{S}(k_v^2)$:

$$T_v(\Psi_v, s, w) = \int_{k_v^\times} \int_{k_v} |t_v|_v^s \alpha_v(u_v)^{-w} \Psi_v(t_v, t_v u_v) du_v d^\times t_v.$$

Then by (1.1),

$$T(\Psi, s, w) = q^{1-g} \mathfrak{C}_k^{-1} \prod_{v \in \mathfrak{M}} T_v(\Psi_v, s, w)$$

for $\Psi = \prod_{v \in \mathfrak{M}} \Psi_v$. We recall Proposition 2.8 and 2.9 of [15].

Lemma 4.14. (i) $T_v(\Psi_v, s, w)$ converges absolutely and locally uniformly in the region $\operatorname{Re}(s) + \operatorname{Re}(w) > 1$, $\operatorname{Re}(s) > 0$, and is holomorphic in the region.
(ii) If Ψ_v is the characteristic function of \mathcal{O}_v^2 ,

$$T_v(\Psi_v, s, w) = \frac{1 - q_v^{-(s+w)}}{(1 - q_v^{-s})(1 - q_v^{-(s+w-1)})}.$$

(iii) For any Ψ_v , $(1 - q_v^{-s})(1 - q_v^{-(s+w-1)})T_v(\Psi_v, s, w)$ becomes a polynomial in $q_v^{\pm s}$, $q_v^{\pm w}$.

This lemma implies

Lemma 4.15. $T(\Psi, s, w)$ is holomorphic in the region $\operatorname{Re}(s) + \operatorname{Re}(w) > 2$, $\operatorname{Re}(s) > 1$, and moreover, is a rational function of q^{-s} , q^{-w} . More precisely, suppose that $\Psi = \otimes \Psi_v$ and that S a finite set of places such that Ψ_v are the characteristic functions of \mathcal{O}_v^2 for $v \notin S$, then

$$T(\Psi, s, w) = T_S(\Psi, s, w) \frac{\zeta_{k,S}(s) \zeta_{k,S}(s+w-1)}{\zeta_{k,S}(s+w)},$$

where $T_S(\Psi, s, w) = q^{1-g} \mathfrak{C}_k^{-1} \prod_{v \in S} T_v(\Psi_v, s, w)$ and

$$\zeta_{k,S}(s) = \prod_{v \notin S} (1 - q_v^{-s})^{-1}$$

is the truncated Dedekind zeta function.

Let us define the distribution $\tilde{T}(\Psi, s)$ by

$$\tilde{T}(\Psi, s) = \frac{1}{\log q} \frac{d}{dw} T(\Psi, s, w)|_{w=0}.$$

Also we define $\tilde{T}^+(\Psi, s)$, $\tilde{T}^-(\Psi, s)$ in similar way. These are rational functions of q^s . For later purposes, we will state the pole structure of $\tilde{T}(\Psi, s)$. Since

$$\tilde{T}(\Psi, s) = \zeta_{k,S}(s-1) \frac{d}{dw} T_S(\Psi, s, w)|_{w=0} + T_S(\Psi, s, 0) \left(\zeta'_{k,S}(s-1) - \zeta_{k,S}(s-1) \frac{\zeta'_{k,S}(s)}{\zeta_{k,S}(s)} \right)$$

and hence,

Lemma 4.16. $\tilde{T}(\Psi, s)$ is a rational function of q^s , and holomorphic in the region $\operatorname{Re}(s) > 2$. It has at most double pole at $s = 2$, and $(1 - q^{2-s})^2 \tilde{T}(\Psi, s)$ is holomorphic in the region $\operatorname{Re}(s) > 1$.

§ 4.7. **Computation of $J_2(\Phi, w)$.** For $\Phi \in \mathcal{S}(V_{\mathbb{A}})$, we define $R_2\Phi \in \mathcal{S}(\text{Aff}_{\mathbb{A}}^2)$ by $R_2\Phi(t, u) = \Phi(0, t, u)$.

Proposition 4.17.

$$J_2(\Phi, w) \sim \phi(w)\psi(w) \left(\frac{1}{1 - q^{\frac{1-w}{2}}} - \frac{1}{2} \right) T^0 \left(R_2\Phi, \frac{1-w}{2} \right).$$

Proof. By the formula of §3.2,(3.3),

$$\begin{aligned} J_2(\Phi, w) &= \int_{H_{\mathbb{A}}^0/H_k} \sum_{\gamma \in H_k/\widehat{T}_k} \sum_{a \in k^\times} \Phi(h\gamma \cdot (0, a, 0)) \mathcal{E}(w, h) dh \\ &= \int_{H_{\mathbb{A}}^0/\widehat{T}_k} \sum_{a \in k^\times} \Phi(h \cdot (0, a, 0)) \mathcal{E}(w, h) dh \\ &= \int_{\mathcal{K}} d\kappa \int_{\mathbb{A}} du \int d^\times \tau \int_{\substack{\mathbb{A}^\times/k^\times \\ |\tau|_{\mathbb{A}}=1}} d^\times t \sum_{a \in k^\times} \Phi(\varrho(\kappa) \cdot (0, at\tau, aut\tau)) \mathcal{E}(w, n(u)a(\tau)) \\ &= \int_{\mathbb{A}} du \int d^\times \tau \int_{\mathbb{A}^0} d^\times t R_2\Phi(t, tu) \mathcal{E}(w, a(\tau)n(\frac{u}{\tau})), \end{aligned}$$

where the last transformation can be obtained by changing t to t/τ , and including the sum $\sum_{a \in k^\times}$ into the integration of t . Here, we used the notation

$$\int d^\times \tau = \int_{\substack{\mathbb{A}^\times/k^\times \\ |\tau|_{\mathbb{A}} < \alpha(u)}} d^\times \tau + \frac{1}{2} \int_{|\tau|_{\mathbb{A}} = \alpha(u)} d^\times \tau$$

for simplicity. Now, we show the following claim.

Claim . *Set*

$$\mathcal{E}'(w, h) = \sum_{a \in k^\times} \mathcal{C}_a(w, \tau) \langle au \rangle = \mathcal{E}(w, h) - \mathcal{C}_0(w, \tau).$$

Then

$$P_1 = \int_{\mathbb{A}} du \int d^\times \tau \int_{\mathbb{A}^0} d^\times t R_2\Phi(t, tu) \mathcal{E}'(w, a(\tau)n(\frac{u}{\tau}))$$

is a holomorphic function of w in the half-plane $\text{Re}(w) > 0$.

Proof of Claim. By the Fourier expansion of the Eisenstein series,

$$P_1 = \sum_{a \in k^\times} \int_{\mathbb{A}} du \int d^\times \tau \int_{\mathbb{A}^0} d^\times t R_2\Phi(t, tu) \mathcal{C}_a(w, \tau) \langle \frac{ua}{\tau} \rangle.$$

We can see that there are finitely many a 's such that $\mathcal{C}_a(w, \tau) \neq 0$ for some τ with $|\tau|_{\mathbb{A}} < \alpha(u)$ for the following reason.

- (i) For each τ , the number of a 's such that $\mathcal{C}_a(w, \tau) \neq 0$ is finite. Moreover, for the same $k^\times \mathbb{A}^\times(\emptyset)$ -coset in \mathbb{A}^\times the Fourier coefficients $\mathcal{C}_a(w, \tau)$ are equal.
- (ii) By Lemma 4.2, $\mathcal{C}_a(w, \tau) = 0$ for all $a \in k^\times$ when $|\tau|_{\mathbb{A}} < q^{2-2g}$.
- (iii) The set $\{\tau \in \mathbb{A}^\times/k^\times \mathbb{A}^\times(\emptyset) \mid q^{2-2g} \leq |\tau|_{\mathbb{A}} \leq \alpha(u)\}$ is a finite set.

Therefore, it is enough to prove that each integral in the sum $\sum_{a \in k^\times}$ is holomorphic in the region $\text{Re}(w) > 0$. On the other hand, (ii) implies that the region of integration with respect to τ is compact. Since $\mathcal{C}_a(w, \tau)$ is holomorphic in the region $\text{Re}(w) > 0$, we obtain the claim. \square

Now by the claim above, we have

$$J_2(\Phi, w) = \int_{\mathbb{A}} du \int_{\mathbb{A}^0} d^\times t \int d^\times \tau R_2 \Phi(t, tu) \frac{1}{2\pi i} \int_{\operatorname{Re}(z)=r} \frac{|\tau|_{\mathbb{A}}^{-\frac{z+1}{2}} + |\tau|_{\mathbb{A}}^{\frac{z-1}{2}} \phi(z)}{w-z} \psi(z) dz.$$

We break the above integral into two parts. Note that

$$\int d^\times \tau |\tau|_{\mathbb{A}}^z = \left(\frac{1}{1-q^{-z}} - \frac{1}{2} \right) \alpha(u)^z$$

for $\operatorname{Re}(z) > 0$. Let $r_1 < -1$. Then the first one is equal to

$$\begin{aligned} & \int_{\mathbb{A}} du \int_{\mathbb{A}^0} d^\times t R_2 \Phi(t, tu) \int d^\times \tau \frac{1}{2\pi i} \int_{\operatorname{Re}(z)=r_1} |\tau|_{\mathbb{A}}^{-\frac{z+1}{2}} \frac{\psi(z)}{w-z} dz \\ &= \frac{1}{2\pi i} \int_{\operatorname{Re}(z)=r_1} \left(\frac{1}{1-q^{\frac{z+1}{2}}} - \frac{1}{2} \right) T^0 \left(R_2 \Phi, \frac{z+1}{2} \right) \frac{\psi(z)}{w-z} dz \end{aligned}$$

and hence, is holomorphic in the region $\operatorname{Re}(w) > r_1$. Therefore,

$$\begin{aligned} J_2(\Phi, w) &\sim \int_{\mathbb{A}} du \int_{\mathbb{A}^0} d^\times t \int d^\times \tau R_2 \Phi(t, tu) \frac{1}{2\pi i} \int_{\operatorname{Re}(z)=r} \frac{|\tau|_{\mathbb{A}}^{\frac{z-1}{2}} \phi(z)}{w-z} \psi(z) dz \\ &= \phi(w) \psi(w) \int_{\mathbb{A}} du \int_{\mathbb{A}^0} d^\times t R_2 \Phi(t, tu) \int d^\times \tau |\tau|_{\mathbb{A}}^{\frac{w-1}{2}} \\ &\quad + \int_{\mathbb{A}} du \int_{\mathbb{A}^0} d^\times t \int d^\times \tau R_2 \Phi(t, tu) \frac{1}{2\pi i} \int_{\operatorname{Re}(z)=r_2 > \operatorname{Re}(w)} \frac{|\tau|_{\mathbb{A}}^{\frac{z-1}{2}} \phi(z)}{w-z} \psi(z) dz \\ &= \phi(w) \psi(w) \left(\frac{1}{1-q^{\frac{1-w}{2}}} - \frac{1}{2} \right) T^0 \left(R_2 \Phi, \frac{1-w}{2} \right) \\ &\quad + \int_{\mathbb{A}} du \int_{\mathbb{A}^0} d^\times t \int d^\times \tau R_2 \Phi(t, tu) \frac{1}{2\pi i} \int_{\operatorname{Re}(z)=r_2 > \operatorname{Re}(w)} \frac{|\tau|_{\mathbb{A}}^{\frac{z-1}{2}} \phi(z)}{w-z} \psi(z) dz. \end{aligned}$$

Now, Similarly to P_2 , we can show that the second part of the last expression is an entire function of w . Hence we have the proposition. \square

§ 4.8. Explicit evaluation of $I^0(\Phi)$. Now, we turn to $I^0(\Phi)$. Recall that

$$(4.11) \quad \lim_{w \rightarrow 1+0} (1-q^{1-w}) I^0(\Phi, w) = \rho_0 \psi(1) I^0(\Phi)$$

by Corollary 4.5. We will compute $I^0(\Phi)$ by using Proposition 4.9 and 4.10, 4.11, 4.17. Recall that $\Sigma(\Psi_1, w)$ has a simple pole at $w = 1$, and $T^0(\Psi_2, w)$ is holomorphic at $w = 0$. We will write their Laurent expansion in q^{1-w}, q^{-w} , respectively, at the values $q^{1-w} = 1, q^{-w} = 1$, by

$$\Sigma(\Psi_1, w) = \sum_{i=-1}^{\infty} \Sigma_{(i)}(\Psi_1, 1) (1-q^{1-w})^i, \quad T^0(\Psi_2, w) = \sum_{i=0}^{\infty} T_{(i)}^0(\Psi_2, 0) (1-q^{-w})^i.$$

Obviously, $T_{(0)}^0(\Psi_2, 0) = T^0(\Phi_2, 0)$, $T_{(1)}^0(\Psi_2, 0) = \frac{1}{\log q} \frac{d}{dw} T^0(\Psi_2, w)|_{w=0}$.

Since the limit (4.11) exists, the double poles of the right-hand side of (4.7) at $w = 1$ cancel out on the whole. Hence we get the following lemma. Note that some straightforward calculation also shows the following equality. (See [1].)

Lemma 4.18.

$$T_{(0)}^0(R_2\hat{\Phi}, 0) - T_{(0)}^0(R_2\Phi, 0) = \Sigma_{(-1)}(R_1\Phi, 1) - \Sigma_{(-1)}(R_1\hat{\Phi}, 1).$$

Hence, we have

$$\begin{aligned} I(\Phi, w) &\sim \frac{\psi(w)}{1 - q^{\frac{1-w}{2}}}(\hat{\Phi}(0) - \Phi(0)) \\ &\quad + \phi(w)\psi(w) \left\{ \frac{\Sigma_{(-1)}(R_1\hat{\Phi}, 1)}{2} + \Sigma_{(0)}(R_1\hat{\Phi}, 1) - \frac{\Sigma_{(-1)}(R_1\Phi, 1)}{2} - \Sigma_{(0)}(R_1\Phi, 1) \right\} \\ &\quad - \phi(w)\psi(w) \left\{ T_{(1)}^0(R_2\hat{\Phi}, 0) - T_{(1)}^0(R_2\Phi, 0) \right\}. \end{aligned}$$

Then, together with (4.11), we obtain the following.

Proposition 4.19.

$$\begin{aligned} I^0(\Phi) &= \frac{2}{\rho_0}(\hat{\Phi}(0) - \Phi(0)) \\ &\quad + \left\{ \frac{\Sigma_{(-1)}(R_1\hat{\Phi}, 1)}{2} + \Sigma_{(0)}(R_1\hat{\Phi}, 1) - \frac{\Sigma_{(-1)}(R_1\Phi, 1)}{2} - \Sigma_{(0)}(R_1\Phi, 1) \right\} \\ &\quad - T_{(1)}^0(R_2\hat{\Phi}, 0) + T_{(1)}^0(R_2\Phi, 0). \end{aligned}$$

§ 4.9. The principal part formula.

Theorem 4.20. *Suppose that $\Phi = M\Phi$. Then $Z(\Phi, s)$ is a rational function of q^s . More precisely,*

$$Z(\Phi, s) = Z_+(\Phi, s) + Z_+(\hat{\Phi}, 3-s) + I(\Phi, s),$$

where $Z_+(\Phi, s)$ and $Z_+(\hat{\Phi}, 3-s)$ are polynomials in q^s and q^{-s} , respectively, and $I(\Phi, s)$ is given by

$$\begin{aligned} (4.12) \quad &\frac{2}{\rho_0} \left\{ \left(\frac{1}{2} + \frac{q^{3-s}}{1 - q^{3-s}} \right) \hat{\Phi}(0) - \left(\frac{1}{2} + \frac{q^{-s}}{1 - q^{-s}} \right) \Phi(0) \right\} \\ &+ \left\{ \left(\frac{\Sigma_{(-1)}(R_1\hat{\Phi}, 1)}{2} + \Sigma_{(0)}(R_1\hat{\Phi}, 1) \right) \left(\frac{1}{2} + \frac{q^{2-s}}{1 - q^{2-s}} \right) - \Sigma_{(-1)}(R_1\hat{\Phi}, 1) \frac{q^{2-s}}{(1 - q^{2-s})^2} \right\} \\ &- \left\{ \left(\frac{\Sigma_{(-1)}(R_1\Phi, 1)}{2} + \Sigma_{(0)}(R_1\Phi, 1) \right) \left(\frac{1}{2} + \frac{q^{1-s}}{1 - q^{1-s}} \right) + \Sigma_{(-1)}(R_1\Phi, 1) \frac{q^{1-s}}{(1 - q^{1-s})^2} \right\} \\ &- (\tilde{T}^+(R_2\hat{\Phi}, 3-s) - \tilde{T}^-(R_2\Phi, s)). \end{aligned}$$

Proof. We will compute $I(\Phi, s)$ by (4.2) and Proposition 4.19. Note that $\widehat{\Phi}_n(t) = q^{-3n}\hat{\Phi}_{-n}(t)$. Then we have

$$(4.13) \quad \Phi_n(0) = \Phi(0), \quad \widehat{\Phi}_n(0) = q^{-3n}\hat{\Phi}(0).$$

Also since

$$\Sigma(\Phi_n, w) = q^{-nw}\Sigma(\Phi, w), \quad \Sigma(R_1\widehat{\Phi}_n, w) = q^{-n(3-w)}\Sigma(R_1\hat{\Phi}, w),$$

we have

$$\begin{aligned}
 (4.14) \quad & \Sigma_{(-1)}(R_1\Phi_n, 1) = q^{-n}\Sigma_{(-1)}(R_1\Phi, 1), \\
 & \Sigma_{(0)}(R_1\Phi_n, 1) = q^{-n}\Sigma_{(0)}(R_1\Phi, 1) - nq^{-n}\Sigma_{(-1)}(R_1\Phi, 1), \\
 & \Sigma_{(-1)}(R_1\widehat{\Phi}_n, 1) = q^{-2n}\Sigma_{(-1)}(R_1\hat{\Phi}, 1), \\
 & \Sigma_{(0)}(R_1\widehat{\Phi}_n, 1) = q^{-2n}\Sigma_{(0)}(R_1\hat{\Phi}, 1) + nq^{-2n}\Sigma_{(-1)}(R_1\hat{\Phi}, 1).
 \end{aligned}$$

Hence, By (4.13) and (4.14), we obtain the first three terms of (4.12). To get the last term of (4.12), note that simple calculation shows

$$\begin{aligned}
 (4.15) \quad & \frac{1}{2}T^0(R_2\Phi, w) + \sum_{n \geq 1} q^{-ns}T^0(R_2\Phi_{-n}, w) = T^-(R_2\Phi, s, w), \\
 & \frac{1}{2}T^0(R_2\hat{\Phi}, w) + \sum_{n \geq 1} q^{-ns}T^0(R_2\widehat{\Phi}_{-n}, w) = T^+(R_2\hat{\Phi}, 3-s, w).
 \end{aligned}$$

On the other hand, by Lemma 4.13, 4.15, $\tilde{T}^+(R_2\hat{\Phi}, 3-s)$ and $\tilde{T}^-(R_2\Phi, s)$ are rational functions of q^s , and hence $Z(\Phi, s)$ is a rational function of q^s . This finishes the proof. \square

We define

$$(4.16) \quad \mathcal{R}_1 = \frac{1}{\log q} \frac{2}{\rho_0} = \frac{1}{\log q} \frac{2\zeta_k(2)}{q^{2-2g}\mathfrak{C}_k}.$$

Together with Theorem 4.20 and Lemma 4.16, we obtain the following.

Corollary 4.21. *$Z(\Phi, s)$ is a rational function of q^s , and $(1 - q^{3-s})Z(\Phi, s)$ is holomorphic in the region $\text{Re}(s) > 2$. Moreover,*

$$\text{Res}_{s=3} Z(\Phi, s) = \mathcal{R}_1 \hat{\Phi}(0).$$

We have the following functional equation for the global zeta function $Z(\Phi, s)$.

Corollary 4.22. *Let $Z_{ad}(\Phi, s) = Z(\Phi, s) - \tilde{T}(R_2\Phi, s)$, then $Z_{ad}(\Phi, s)$ satisfies the functional equation*

$$Z_{ad}(\Phi, s) = Z_{ad}(\hat{\Phi}, 3-s).$$

5. LOCAL THEORY

§ 5.1. The canonical measure on the stabilizer. For an algebraic group G , Let G° denote its identity component. In this subsection, we normalize invariant measures on $H_{k_v}, H_{k_v}/H_{x_{k_v}}^\circ, H_{x_{k_v}}^\circ$ for $x \in V'_{k_v}$, and $H_{\mathbb{A}}/H_{x_{\mathbb{A}}}^\circ, H_{x_{\mathbb{A}}}^\circ$ for $x \in V''_k$, following the method of [9]. We also define a constant $b_{x,v}$, and compute the volume of $H_{x_{\mathbb{A}}}^\circ/H_{x_k}^\circ$ with respect to this measure (Proposition 5.3).

We define the invariant measure dh_v on H_{k_v} similarly as in §3.2. Let $\mathcal{K}_v = G_{\mathcal{O}_v}$ be the standard maximal compact subgroup of G_{k_v} . For $h_v \in H_{k_v}$, let $h_v = \varrho(\kappa_v d(t_v, 1)a(\tau_v)n(u_v))$ be its Iwasawa decomposition. Define an invariant measure dh_v on H_{k_v} by $dh_v = |\tau|_v d\kappa_v d^\times t_v d^\times \tau_v du_v$. This normalization is equivalent to $\int_{\varrho(\mathcal{K}_v)} dh_v = 1$.

If we write $d_{\text{pr}}h = \prod_{v \in \mathfrak{M}} dh_v$, then by (1.1) we can see

$$(5.1) \quad dh = q^{1-g}\mathfrak{C}_k^{-2}d_{\text{pr}}h,$$

where dh is defined in §3.2.

Next, we choose a left invariant measure $dh'_{x,v}$ on $H_{k_v}/H_{xk_v}^\circ$ for $x \in V'_{k_v}$. Let dy_v be the Haar measure on V_{k_v} such that the volume of $V_{\mathcal{O}_v}$ is one. Note that $\frac{dy_v}{|P(y_v)|_v^{3/2}}$ is a left H_{k_v} -invariant measure on V'_{k_v} and $H_{k_v}/H_{xk_v}^\circ$ is a double cover of $H_{k_v}x \subset V'_{k_v}$. Therefore there exists a unique left H_{k_v} -invariant measure $dh'_{x,v}$ on $H_{k_v}/H_{xk_v}^\circ$ such that for $\Psi \in L^1(H_{k_v}x, \frac{dy_v}{|P(y_v)|_v^{3/2}})$,

$$(5.2) \quad \int_{H_{k_v}x} \Psi(y_v) \frac{dy_v}{|P(y_v)|_v^{3/2}} = \int_{H_{k_v}/H_{xk_v}^\circ} \Psi(h'_{x,v} \cdot x) dh'_{x,v}.$$

If $h_{xy} \in H_{k_v}$ satisfies $y = h_{xy}x$ and $i_{h_{xy}}$ is the inner automorphism $h \mapsto h_{xy}^{-1}hh_{xy}$ of H_{k_v} , then $i_{h_{xy}}(H_{y k_v}^\circ) = H_{x k_v}^\circ$ and hence $i_{h_{xy}}$ induces the map $i_{h_{xy}} : H_{k_v}/H_{y k_v}^\circ \rightarrow H_{k_v}/H_{x k_v}^\circ$. Since the integral on the right hand side of (5.2) depends only on the orbit of x , it follows that $i_{h_{xy}}^*(dh'_{x,v}) = dh'_{y,v}$.

We divide V'_{k_v} into three subsets for conveniences. Let

$$V'_{k_v} = V_{k_v}^{\text{sp}} \amalg V_{k_v}^{\text{ur}} \amalg V_{k_v}^{\text{rm}},$$

where each subset consists of orbits corresponding to k_v , quadratic unramified extension of k_v , and quadratic ramified extension of k_v , respectively. Only $V_{k_v}^{\text{rm}}$ has two orbits and $V_{k_v}^{\text{sp}}, V_{k_v}^{\text{ur}}$ has a single orbit.

If we define

$$K_{x,k_v} = \begin{cases} k_v^\times \times k_v^\times & x \in V_{k_v}^{\text{sp}}, \\ k_v(x)^\times & x \in V_{k_v}^{\text{ur}}, V_{k_v}^{\text{rm}}, \end{cases}$$

then by Proposition 2.1, $G_{xk_v}^\circ \cong K_{x,k_v}$. Let $\mathcal{O}_{k_v(x)}$ denote the ring of integers of $k_v(x)$. We will normalize the measure $dk_{x,v}$ on K_{x,k_v} so that

$$\int_{\mathcal{O}_v^\times \times \mathcal{O}_v^\times} dk_{x,v} = 1, \quad \int_{\mathcal{O}_{k_v(x)}^\times} dk_{x,v} = 1$$

for each case. This induces a Haar measure $dg''_{x,v}$ on $G_{xk_v}^\circ$ via the isomorphism $G_{xk_v}^\circ \cong K_{x,k_v}$. Though there are two isomorphisms $\psi_{x,v}^{(1)}, \psi_{x,v}^{(2)}$ for each case (see the proof of Proposition 2.1), since

$$\begin{aligned} \psi_{x,v}^{(1)-1}(\mathcal{O}_v^\times \times \mathcal{O}_v^\times) &= \psi_{x,v}^{(2)-1}(\mathcal{O}_v^\times \times \mathcal{O}_v^\times) & x \in V_{k_v}^{\text{sp}}, \\ \psi_{x,v}^{(1)-1}(\mathcal{O}_{k_v(x)}^\times) &= \psi_{x,v}^{(2)-1}(\mathcal{O}_{k_v(x)}^\times) & x \in V_{k_v}^{\text{ur}}, V_{k_v}^{\text{rm}}, \end{aligned}$$

we can define $dg''_{x,v}$ without ambiguity. Then, define $dh''_{x,v}$ by setting $dh''_{x,v} d^\times t_{1v} = dg''_{x,v}$ via the isomorphism $H_{xk_v}^\circ \cong G_{xk_v}^\circ/T_{k_v}$.

The next proposition shows that $dh''_{x,v}$ satisfies the functorial property. In this sense, our choice $dh''_{x,v}$ is canonical.

Proposition 5.1. *Suppose that $x, y \in V'_{k_v}$ and that $y = \varrho(g_{xy})x$ for some $g_{xy} \in G_{k_v}$. Let $i_{g_{xy}} : G_{y k_v}^\circ \rightarrow G_{x k_v}^\circ$ be the isomorphism given by $i_{g_{xy}}(g) = g_{xy}^{-1}gg_{xy}$. Then,*

$$(5.3) \quad dg''_{y,v} = i_{g_{xy}}^*(dg''_{x,v}) \quad \text{and} \quad dh''_{y,v} = i_{\varrho(g_{xy})}^*(dh''_{x,v}).$$

Proof. We only consider the case $[k_v(x) : k_v] = 2$. We can prove the case $k_v(x) = k_v$ similarly. One can easily show that for any g_{xy} , there exist $\psi_{x,v}, \psi_{y,v}$ such that the

following diagram is commutative.

$$\begin{array}{ccc} G_{yk_v}^\circ & \xrightarrow{\psi_{y,v}} & k_v(y)^\times \\ g_{xy} \downarrow & & \parallel \\ G_{xk_v}^\circ & \xrightarrow{\psi_{x,v}} & k_v(x)^\times \end{array}$$

This establishes the first claim and the second claim follows from the observation that $i_{g_{xy}}|_{T_{k_v}}$ is the identity map. \square

Define a constant $b_{x,v} > 0$ such that $dh_v = b_{x,v} dh'_{x,v} dh''_{x,v}$. Then, the following proposition shows that $b_{x,v}$ depends only on the orbit of x .

Proposition 5.2. *If $x, y \in V'_{k_v}$ are in the same H_{k_v} -orbit, then $b_{x,v} = b_{y,v}$.*

Proof. Since the group H_{k_v} is unimodular, $i_{h_{x,y}}^* dh_v = dh_v$. Hence,

$$\begin{aligned} dh_v &= b_{y,v} dh'_{y,v} dh''_{y,v} \\ &= b_{y,v} i_{h_{x,y}}^* dh'_{x,v} i_{h_{x,y}}^* dh''_{x,v} = b_{y,v} b_{x,v}^{-1} i_{h_{x,y}}^* dh_v \\ &= b_{y,v} b_{x,v}^{-1} dh_v. \end{aligned}$$

Therefore $b_{x,v} = b_{y,v}$. \square

For $x \in V''_k$, Let

$$dh'_x = \prod_{v \in \mathfrak{M}} b_{x,v} dh'_{x,v}, \quad dh''_x = \prod_{v \in \mathfrak{M}} dh''_{x,v}$$

be measures on $H_{\mathbb{A}}/H_{x\mathbb{A}}^\circ, H_{x\mathbb{A}}^\circ$, respectively. Then,

$$dh'_x dh''_x = d_{\text{pr}} h.$$

We will conclude this subsection by computing the volume of $H_{x\mathbb{A}}^\circ/H_{xk}^\circ$ with respect to the measure dh''_x defined above.

Proposition 5.3. *For $x \in V''_k$,*

$$\int_{(H_x^\circ)_{\mathbb{A}}/(H_x^\circ)_k} dh''_x = 2 \frac{\mathfrak{C}_{k(x)}}{\mathfrak{C}_k}.$$

Proof. Recall that $(H_x^\circ)_{\mathbb{A}}/(H_x^\circ)_k \cong (G_x^\circ)_{\mathbb{A}}/T_{\mathbb{A}}(G_x^\circ)_k$. One can easily see that the inclusion

$$(G_x^\circ)_{\mathbb{A}}^0/T_{\mathbb{A}}^0(G_x^\circ)_k \hookrightarrow (G_x^\circ)_{\mathbb{A}}/T_{\mathbb{A}}(G_x^\circ)_k$$

has index two, and the sequence

$$1 \longrightarrow T_{\mathbb{A}}^0/T_k \longrightarrow (G_x^\circ)_{\mathbb{A}}^0/(G_x^\circ)_k \longrightarrow (G_x^\circ)_{\mathbb{A}}^0/T_{\mathbb{A}}^0(G_x^\circ)_k \longrightarrow 1$$

is exact. Since

$$\int_{T_{\mathbb{A}}^0/T_k} d^\times t = \mathfrak{C}_k \quad \text{and} \quad \int_{(G_x^\circ)_{\mathbb{A}}^0/(G_x^\circ)_k} dg_x'' = \mathfrak{C}_{k(x)},$$

we can get

$$\int_{(H_x^\circ)_{\mathbb{A}}/(H_x^\circ)_k} dh''_x = 2 \int_{(G_x^\circ)_{\mathbb{A}}^0/T_{\mathbb{A}}^0(G_x^\circ)_k} dh''_x = 2 \frac{\mathfrak{C}_{k(x)}}{\mathfrak{C}_k}.$$

\square

§ 5.2. **Local zeta function.** Here, we will define the local zeta function, and compute the local zeta function and the constant $b_{x,v}$ for some orbital representatives.

Definition 5.4. Let $x \in V'_{k_v}$, $\Phi_v \in \mathcal{S}(V_{k_v})$, and $s \in \mathbb{C}$. We define

$$Z_{x,v}(\Phi_v, s) = b_{x,v} \int_{H_{k_v}/H_{x k_v}^\circ} |\chi(h'_{x,v})|_v^s \Phi_v(h'_{x,v}x) dh'_{x,v},$$

$$\Omega_{x,v}(\Phi_v, s) = \int_{H_{k_v}x} |P(y)|_v^{s/2} \Phi_v(y) \frac{dy}{|P(y)|_v^{3/2}}.$$

The function $\Omega_{x,v}(\Phi_v, s)$ is called the local zeta function. By the definition of $dh'_{x,v}$,

$$Z_{x,v}(\Phi_v, s) = b_{x,v} \int_{H_{k_v}/H_{x k_v}^\circ} \frac{|P(h'_{x,v}x)|_v^{s/2}}{|P(x)|_v^{s/2}} \Phi_v(h'_{x,v}x) dh'_{x,v}$$

$$= b_{x,v} |P(x)|_v^{-s/2} \Omega_{x,v}(\Phi_v, s).$$

Since $\Omega_{x,v}(\Phi_v, s)$ depends only on the orbit of x , for $x, y \in V'_{k_v}$ in the same orbit,

$$(5.4) \quad Z_{x,v}(\Phi_v, s) = \frac{|P(y)|_v^{s/2}}{|P(x)|_v^{s/2}} Z_{y,v}(\Phi_v, s).$$

Now, we will express $Z_{x,v}$ and $b_{x,v}$ explicitly for some representative x .

Definition 5.5. We call $w_v \in V'_{k_v}$ a standard orbital representative if

$$F_{w_v}(z_1, z_2) = \begin{cases} z_1 z_2 & w_v \in V_{k_v}^{\text{sp}}, \\ (z_1 + \theta z_2)(z_1 + \theta' z_2) & w_v \in V_{k_v}^{\text{ur}}, V_{k_v}^{\text{rm}}, \text{ where } \mathcal{O}_{k_v(w_v)} = \mathcal{O}_v[\theta]. \end{cases}$$

For each orbit in V'_{k_v} , we take one of the standard orbital representatives and denote the fixed set of representatives by SR_v .

Note that for a standard orbital representative w_v , $P(w_v)$ is the discriminant of $k_v(x)$ over k_v . Hence,

$$(5.5) \quad |P(w_v)|_v = \begin{cases} 1 & w_v \in V_{k_v}^{\text{sp}}, V_{k_v}^{\text{ur}}, \\ q_v^{-1} & w_v \in V_{k_v}^{\text{rm}}. \end{cases}$$

Let $\Phi_{v,0}$ be the characteristic function of $V_{\mathcal{O}_v}$. Firstly, we will give the explicit formula of $Z_{w_v,v}(\Phi_{v,0}, s)$. Although our choice of the measure on the stabilizers is different from that of [1] in general, they coincide for standard orbital representatives. Hence we can use his result directly. Note that our local zeta functions are $b_{x,v}$ times that of Datskovsky's.

Proposition 5.6 ([1]Proposition 4.1). For a standard orbital representative w_v ,

$$Z_{w_v,v}(\Phi_{v,0}, s) = \begin{cases} \frac{1}{1 - q^{1-s}} & w_v \in V_{k_v}^{\text{sp}}, \\ \frac{1 + q^{-s}}{(1 - q^{-s})(1 - q^{1-s})} & w_v \in V_{k_v}^{\text{ur}}, \\ \frac{1}{(1 - q^{-s})(1 - q^{1-s})} & w_v \in V_{k_v}^{\text{rm}}. \end{cases}$$

Secondly, we will give the values of $b_{w_v,v}$. To use Datskovsky's result, we need some discussion.

Lemma 5.7. *For a standard orbital representative w_v ,*

$$\int_{\varrho(\mathcal{K}_v) \cap H_{w_v k_v}^\circ} dh''_{w_v, v} = 1.$$

Proof. Here, we only consider the case $[k_v(w_v) : k_v] = 2$. The case $k_v(w_v) = k_v$ can be proved similarly. Let $N = N_{k_v(w_v)/k_v} : k_v(w_v)^\times \rightarrow k_v^\times$ be the norm map.

Let $F_{w_v}(z_1, z_2) = (z_1 + \theta z_2)(z_1 + \theta' z_2)$. Recall that for $g = (t, \begin{pmatrix} a & b \\ c & d \end{pmatrix}) \in G_{k_v}$, the condition $g \in G_{w_v k_v}^\circ$ is equivalent to

$$tN(a + b\theta) = 1, \quad c + d\theta = \theta(a + b\theta)$$

and that the isomorphism $\psi_{w_v, v} : G_{w_v k_v}^\circ \rightarrow k_v(w_v)^\times$ is defined by $g \mapsto a + b\theta$ for this g . This map also gives the isomorphism $T_{k_v} \cong k_v^\times$. One can easily see that $\psi_{w_v, v}(T_{k_v} \cap \mathcal{K}_v) = \mathcal{O}_v^\times$. We claim that

$$\psi_{w_v, v}(G_{w_v k_v}^\circ \cap \mathcal{K}_v) = \mathcal{O}_{k_v(w_v)}^\times.$$

Let $g = (t, \begin{pmatrix} a & b \\ c & d \end{pmatrix}) \in G_{w_v k_v}^\circ \cap \mathcal{K}_v$. Then $N(a + b\theta) = t^{-1} \in \mathcal{O}_v^\times$ and hence we have $a + b\theta \in \mathcal{O}_{k_v(w_v)}^\times$. On the other hand, any element of $\mathcal{O}_{k_v(w_v)}^\times$ can be written as $\gamma = a + b\theta$ with $a, b \in \mathcal{O}_v$. For this γ , take $t, c, d \in k_v$ such that

$$tN(a + b\theta) = 1, \quad c + d\theta = \theta(a + b\theta),$$

and let $g = (t, \begin{pmatrix} a & b \\ c & d \end{pmatrix}) \in G_{k_v}$. Then, clearly $t \in \mathcal{O}_v^\times, c, d \in \mathcal{O}_v, g \in G_{w_v k_v}^\circ$, and $\gamma = \psi_{w_v, v}(g)$. Also, $P(w_v) = P(g \cdot w_v) = \chi(g)^3 P(w_v)$ shows $\chi(g) = t(ad - bc) \in \mathcal{O}_v^\times$. Hence, we have $g \in \mathcal{K}_v$. This establishes the claim and now by the definition of $dh''_{w_v, v}$, we have

$$\int_{\varrho(\mathcal{K}_v) \cap G_{w_v k_v}^\circ} dh''_{w_v, v} = \frac{\text{vol}(\mathcal{O}_{k_v(w_v)}^\times)}{\text{vol}(\mathcal{O}_v^\times)} = \frac{1}{1} = 1.$$

□

Proposition 5.8. *For a standard orbital representative w_v ,*

$$b_{w_v, v} = \frac{|P(w_v)|_v^{3/2}}{\int_{\varrho(\mathcal{K}_v)w_v} dx_v}.$$

Proof. Recall that dh_v is the measure on H_{k_v} such that $\text{vol}(\varrho(\mathcal{K}_v)) = 1$. Hence,

$$\begin{aligned} 1 &= \int_{\varrho(\mathcal{K}_v)} dh_v = b_{w_v, v} \int_{\varrho(\mathcal{K}_v)H_{w_v}^\circ/H_{w_v}^\circ} dh'_{w_v, v} \int_{\varrho(G_{w_v}^\circ \cap \mathcal{K}_v)} dh''_{w_v, v} \\ &= b_{w_v, v} \int_{\varrho(\mathcal{K}_v)H_{w_v}^\circ/H_{w_v}^\circ} dh'_{w_v, v} \\ &= b_{w_v, v} \int_{\varrho(\mathcal{K}_v)w_v} \frac{dx_v}{|P(x_v)|_v^{3/2}}. \end{aligned}$$

On the other hand, for $x_v \in \varrho(\mathcal{K}_v)w_v$, $|P(x_v)|_v = |P(w_v)|_v$. This finishes the proof. □

Under the above preparation, we can describe $b_{w_v, v}$. For our purpose, $|P(w_v)|_v^{3/2}/b_{w_v, v}$ (which is equal to $\int_{\varrho(\mathcal{K}_v)w_v} dx_v$ by Proposition 5.8) is more important than $b_{w_v, v}$ itself, and the following proposition gives the desired description.

Proposition 5.9. *For a standard orbital representative w_v ,*

$$\frac{|P(w_v)|_v^{3/2}}{b_{w_v,v}} = \begin{cases} \frac{1}{2}(1 - q_v^{-2}) & w_v \in V_{k_v}^{\text{sp}}, \\ \frac{1}{2}(1 - q_v^{-1})^2 & w_v \in V_{k_v}^{\text{ur}}, \\ \frac{1}{2}q_v^{-1}(1 - q_v^{-1})(1 - q_v^{-2}) & w_v \in V_{k_v}^{\text{rm}}. \end{cases}$$

This proposition is essentially proved in [1] Proposition 4.2, but he did not give a detailed account of the argument for determining the volume of each of $\varrho(\mathcal{K}_v)w_v$ from the sum for two orbits in $V_{k_v}^{\text{rm}}$. We can directly determine each volume by following the argument in [10] Section 4. For the convenience of the reader, we briefly sketch their argument by indicating the main steps of the proof. Note that the action of $\mathcal{K}_v = G_{\mathcal{O}_v}$ on $V_{\mathcal{O}_v}$ induces the action of $G_{\mathcal{O}_v/\mathfrak{p}_v^2}$ on $V_{\mathcal{O}_v/\mathfrak{p}_v^2}$. Let x be one of the standard orbital representatives in $V_{k_v}^{\text{rm}}$ and denote by $\bar{x} \in V_{\mathcal{O}_v/\mathfrak{p}_v^2}$ its reduction modulo \mathfrak{p}_v^2 .

1. Set $\mathcal{D} = \{y \in V_{\mathcal{O}_v} \mid y \equiv x(\mathfrak{p}_v^2)\}$.
2. If $y \in \mathcal{D}$, $k_v(y) = k_v(x)$. Moreover, $\mathcal{D} \subset \mathcal{K}_v x$.
3. $\text{vol}(\mathcal{K}_v x) = \text{vol}(\mathcal{D}) \#(G_{\mathcal{O}_v/\mathfrak{p}_v^2}/G_{\mathcal{O}_v/\mathfrak{p}_v^2, \bar{x}})$.
4. $\text{vol}(\mathcal{D}) = q_v^{-6}$, $\#G_{\mathcal{O}_v/\mathfrak{p}_v^2} = q_v^6(q_v - 1)^2(q_v^2 - 1)$,
5. $\#G_{\mathcal{O}_v/\mathfrak{p}_v^2, \bar{x}} = 2(q_v - 1)q_v^4$.

From this proposition, we can obtain $b_{w_v,v}$ easily using (5.5).

6. THE MEAN VALUE THEOREM

In this section, we will deduce our mean value theorem by putting together the results we have obtained before. In §6.1, we will see that the global zeta function is approximately the Dirichlet generating series for the sequence $\text{vol}(H_{x_{\mathbb{A}}}^{\circ}/H_{x_k}^{\circ})$. If it were exactly this generating series, the theory of partial fraction would allow us to extract the mean value of the coefficients from the analytic behavior of this series. However, our global zeta function contains an additional factor in each term. In §6.2 we will surmount this difficulty by using the technique called the filtering process, which was formulated by Datskovsky-Wright [3].

§6.1. The adelic synthesis. We will introduce some notation. For the rest of this paper, we suppose $\Phi \in \mathcal{S}(V_{\mathbb{A}})$ is of the form $\Phi = \otimes \Phi_v$, where $\Phi_v \in \mathcal{S}(V_{k_v})$. For $x \in V_k''$, define $Z_x(\Phi, s) = \prod_{v \in \mathfrak{M}} Z_{x,v}(\Phi_v, s)$. For each $v \in \mathfrak{M}$, take $w_{v,x} \in \text{SR}_v$ which lies in the orbit of x . We write $\Xi_{x,v}(\Phi_v, s) = Z_{w_{v,x},v}(\Phi_v, s)$ and $\Xi_x(\Phi, s) = \prod_{v \in \mathfrak{M}} \Xi_{x,v}(\Phi_v, s)$.

Then, as is well known, our global zeta function has the following expansion. (See [9] Section 6, for example. Note that $|H_x/H_x^{\circ}| = 2$ for all $x \in V_k''$.)

$$\begin{aligned} Z(\Phi, s) &= \frac{q^{1-g}}{2\mathfrak{e}_k^2} \sum_{x \in H_k \setminus V_k''} \int_{(H_x^{\circ})_{\mathbb{A}}/(H_x^{\circ})_k} dh_x'' \int_{H_{\mathbb{A}}/(H_x^{\circ})_{\mathbb{A}}} |\chi(h_x')|_{\mathbb{A}}^s \Phi(h_x' \cdot x) dh_x' \\ &= q^{1-g} \mathfrak{e}_k^{-3} \sum_{x \in H_k \setminus V_k''} \mathfrak{e}_{k(x)} Z_x(\Phi, s). \end{aligned}$$

We will consider $Z_x(\Phi, s)$. By (5.4), we have

$$Z_x(\Phi, s) = \prod_{v \in \mathfrak{M}} Z_{x,v}(\Phi_v, s) = \prod_{v \in \mathfrak{M}} \frac{|P(w_{v,x})|_v^{s/2}}{|P(x)|_v^{s/2}} \Xi_{x,v}(\Phi_v, s).$$

Observe that since $P(x) \in k^\times$, $\prod_{v \in \mathfrak{M}} |P(x)|_v = |P(x)|_{\mathbb{A}} = 1$. Also since $P(w_v)$ is the local discriminant of $k_v(x)$ over k_v , $\prod_{v \in \mathfrak{M}} |P(w_{v,x})|_v = \mathcal{N}(\mathfrak{D}_{k(x)})^{-1}$. Hence, we obtain the following:

$$(6.1) \quad Z(\Phi, s) = q^{1-g} \mathfrak{e}_k^{-3} \sum_{x \in H_k \setminus V_k''} \frac{\mathfrak{e}_{k(x)}}{\mathcal{N}(\mathfrak{D}_{k(x)})^{s/2}} \Xi_x(\Phi, s).$$

Let T be a finite set of places of k , and we denote by SR_T the Cartesian product $\prod_{v \in T} \text{SR}_v$. We consider T -tuples $w_T = (w_v)_{v \in T}$ of an element of SR_T . We say that $x \in V_k''$ is equivalent to w_T if x lies in the H_{k_v} -orbit of w_v for each $v \in T$ and denoted by $x \sim w_T$. Let $\Phi|_T = \prod_{v \in T} \Phi_v$, and

$$(6.2) \quad Z_{w_T, T}(\Phi|_T, s) = \prod_{v \in T} Z_{w_v, v}(\Phi_v, s).$$

Then,

$$(6.3) \quad Z(\Phi, s) = q^{1-g} \mathfrak{e}_k^{-3} \sum_{w_T \in \text{SR}_T} Z_{w_T, T}(\Phi|_T, s) \left(\sum_{x \sim w_T} \frac{\mathfrak{e}_{k(x)}}{\mathcal{N}(\mathfrak{D}_{k(x)})^{s/2}} \prod_{v \notin T} \Xi_{x, v}(\Phi_v, s) \right).$$

For $v \notin T$, we take Φ_v as the characteristic function $\Phi_{v,0}$ of $V_{\mathcal{O}_v}$. Then, for $v \notin T$, $\Xi_{x, v}(\Phi_{v,0}, s)$ is given in Proposition 5.6 and hence we have

$$(6.4) \quad \prod_{v \notin T} \Xi_{x, v}(\Phi_{v,0}, s) = \frac{\zeta_{k, T}(s-1) \zeta_{k, T}^2(s)}{\zeta_{k(x), T}(s)},$$

where $\zeta_{k, T}(s)$ and $\zeta_{k(x), T}(s)$ are the truncated Dedekind zeta function

$$\zeta_{k, T}(s) = \prod_{v \notin T} (1 - q_v^{-s})^{-1}, \quad \zeta_{k(x), T}(s) = \prod_{\substack{\mu \in \mathfrak{M}_{k(x)} \\ \mu|v, v \notin T}} (1 - q_\mu^{-s})^{-1}.$$

We will denote the function (6.4) by $\eta_{k(x), T}(s)$. Note that $\eta_{k(x), T}(s)$ is a Dirichlet series. Set the Dirichlet series $\xi_{w_T}(s)$ by

$$(6.5) \quad \xi_{w_T}(s) = \sum_{\substack{[k(x):k]=2 \\ x \sim w_T}} \frac{\mathfrak{e}_{k(x)}}{\mathcal{N}(\mathfrak{D}_{k(x)})^{s/2}} \eta_{k(x), T}(s).$$

Then (6.3) becomes

$$(6.6) \quad Z(\Phi, s) = q^{1-g} \mathfrak{e}_k^{-3} \sum_{w_T \in \text{SR}_T} Z_{w_T}(\Phi|_T, s) \xi_{w_T}(s).$$

In order to determine the analytic properties of $\xi_{w_T}(s)$, we require the following lemma. This is quite similar to [9] Lemma 6.17, and we omit the proof.

Lemma 6.1. *Let $v \in \mathfrak{M}$, $x \in V_{k_v}'$ and $s_0 \in \mathbb{C}$. Then there exists $\Phi_v \in \mathcal{S}(V_{k_v})$ such that the support of Φ_v is contained in $H_{k_v}x$, $Z_{x, v}(\Phi_v, s)$ is a polynomial in q_v^s, q_v^{-s} and $Z_{x, v}(\Phi_v, s_0) \neq 0$.*

Let

$$\mathcal{R}_2 = \frac{2\zeta_k(2) \mathfrak{e}_k^2}{\log q}.$$

Also for $w_v \in \text{SR}_v$ and $w_T = (w_v)_{v \in T} \in \text{SR}_T$, we define

$$\varepsilon_v(w_v) = \frac{|P(w_v)|_v^{3/2}}{b_{w_v}}, \quad \varepsilon_T(w_T) = \prod_{v \in T} \varepsilon_v(w_v).$$

Now we can prove the following theorem.

Theorem 6.2. *For $w_T = (w_v)_{v \in T}$, the series $\xi_{w_T}(s)$ is a rational function of q^s , and is holomorphic in the region $\text{Re}(s) > 3$. Also $(1 - q^{3-s})\xi_{w_T}(s)$ is holomorphic in the region $\text{Re}(s) > 2$. Moreover,*

$$\text{Res}_{s=3} \xi_{w_T}(s) = \mathcal{R}_2 \varepsilon_T(w_T)$$

Proof. For each $v \in T$, we choose $\Phi_v \in \mathcal{S}(V_{k_v})$ such that $\text{supp}(\Phi_v) \subset H_{k_v} w_v$. Then, $Z_{w_v, v}(\Phi_v, s) = 0$ unless $x \sim w_v$. Hence (6.6) becomes

$$(6.7) \quad Z(\Phi, s) = q^{1-3g} \mathfrak{C}_k^{-3} Z_{w_T}(\Phi|_T, s) \xi_{w_T}(s).$$

Then the first two statements follows from Corollary 4.21 and Lemma 6.1. We will compute $\text{Res}_{s=3} \xi_{w_T}(s)$. By Corollary 4.21, $\text{Res}_{s=3} Z(\Phi, s) = \mathcal{R}_1 \hat{\Phi}(0)$. We consider

$$\hat{\Phi}(0) = \int_{V_{\mathbb{A}}} \Phi(x) dx = q^{3-3g} \prod_{v \in \mathfrak{M}} \int_{V_{k_v}} \Phi_v(x_v) dx_v.$$

For $v \notin T$, $\int_{V_{k_v}} \Phi_v(x_v) dx_v = 1$ since $\Phi_v = \Phi_{v,0}$ is the characteristic function of $V_{\mathcal{O}_v}$. For $v \in T$,

$$\int_{V_{k_v}} \Phi_v(x_v) dx_v = \int_{H_{k_v} w_v} \Phi_v(x_v) dx_v = \Omega_{w_v, v}(\Phi_v, 3) = \frac{|P(w_v)|_v^{3/2}}{b_{w_v}} Z_{w_v, v}(\Phi_v, 3).$$

Hence we have

$$\hat{\Phi}(0) = q^{3-3g} Z_{w_T}(\Phi|_T, 3) \varepsilon_T(w_T).$$

Together with (6.6), this yields the residue of $\xi_{w_T}(s)$. \square

§ 6.2. The filtering process. We fix a finite set T_0 of places of k and $w_{T_0} = (w_v)_{v \in T_0}$.

Definition 6.3. *For each finite subset $T \supset T_0$ of \mathfrak{M} , we define*

$$\xi_{w_{T_0}, T}(s) = \sum_{x \sim w_{T_0}} \frac{\mathfrak{C}_k(x)}{\mathcal{N}(\mathfrak{D}_{k(x)})^{s/2}} \eta_{k(x), T}(s).$$

For $v \in \mathfrak{M}$, let

$$E_v = \sum_{w_v \in \text{SR}_v} \varepsilon_v(w_v) = 1 - q_v^{-2} - q_v^{-3} + q_v^{-4},$$

and also for any subset T' of \mathfrak{M} , define

$$E_{T'} = \prod_{v \in T'} E_v.$$

Note that this product always converges to a positive number.

Proposition 6.4. *The Dirichlet series $\xi_{w_{T_0}, T}(s)$ becomes a rational function of q^s and holomorphic in the region $\text{Re}(s) > 3$. Also $(1 - q^{3-s})\xi_{w_{T_0}, T}(s)$ is holomorphic in the region $\text{Re}(s) > 2$. The residue of $\xi_{w_{T_0}, T}(s)$ at $s = 3$ is given by*

$$\mathcal{R}_2 \varepsilon_{T_0}(w_{T_0}) E_{T \setminus T_0}.$$

Proof. For $y_T = (y_v)_{v \in T} \in \text{SR}_T$, we denote $y_T|_{T_0} = (y_v)_{v \in T_0} \in \text{SR}_{T_0}$. Then

$$\xi_{w_{T_0}, T}(s) = \sum_{y_T|_{T_0} = w_{T_0}} \xi_{y_T}(s).$$

Now the proposition immediately follows from Theorem 6.2. \square

To deduce our mean value theorem, we have to show some properties of $\eta_{k(x), T}(s)$. For two Dirichlet series $\vartheta_i(s) = \sum_{n=0}^{\infty} r_{i,n}/q^{ns}$, $i = 1, 2$, we will indicate $\vartheta_1(s) \preceq \vartheta_2(s)$ or $\vartheta_2(s) \succeq \vartheta_1(s)$ if $r_{1,n} \leq r_{2,n}$ for all n . Especially, write $\vartheta(s) = \sum_{n=0}^{\infty} r_n/q^{ns} \succeq 0$ if $r_n \geq 0$ for all n . The following proposition is easy to prove.

Proposition 6.5. *The Dirichlet series $\eta_{k(x), T}(s)$ satisfies $\eta_{k(x), T}(s) \succeq 0$, and its first coefficient is 1. Also, for all $k(x)$,*

$$\eta_{k(x), T}(s) \preceq \eta_T(s) = \frac{\zeta_{k, T}(s-1)\zeta_{k, T}^2(s)}{\zeta_{k, T}(2s)}.$$

Moreover, $\eta_T(s)$ converges in the region $\text{Re}(s) > 2$ and

$$\lim_{T \uparrow \mathfrak{M}} (\eta_T(1) - 1) = 0.$$

Let us define $a_n \geq 0$ by

$$\sum_{n \geq 0} \frac{a_n}{q^{ns}} = \sum_{x \sim w_{T_0}} \frac{\mathfrak{C}_k(x)}{\mathcal{N}(\mathfrak{D}_{k(x)})^{s/2}}.$$

Now, we are ready to prove the following theorem.

Theorem 6.6.

$$\lim_{n \rightarrow \infty} \frac{a_n}{q^{3n}} = \log q \mathcal{R}_2 \varepsilon_{T_0}(w_{T_0}) E_{\mathfrak{M} \setminus T_0}.$$

Proof. Since $\eta_{k(x), T}(s) \succeq 0$, we have

$$\xi_{w_{T_0}, T}(s) \succeq \sum_{x \sim w_{T_0}} \frac{\mathfrak{C}_k(x)}{\mathcal{N}(\mathfrak{D}_{k(x)})^{s/2}} = \sum_{n \geq 0} \frac{a_n}{q^{ns}}.$$

Hence, if one write $\xi_{w_{T_0}, T}(s) = \sum r_{T,n}/q^{ns}$, then $r_{T,n} \geq a_n$. By the theory of partial fraction, $\lim_{n \rightarrow \infty} r_{T,n}/q^{3n} = \log q \mathcal{R}_2 \varepsilon_{T_0}(w_{T_0}) E_{T \setminus T_0}$. Hence,

$$\overline{\lim}_{n \rightarrow \infty} \frac{a_n}{q^{3n}} \leq \log q \mathcal{R}_2 \varepsilon_{T_0}(w_{T_0}) E_{T \setminus T_0}.$$

By letting T approach to \mathfrak{M} , we obtain

$$\overline{\lim}_{n \rightarrow \infty} \frac{a_n}{q^{3n}} \leq \log q \mathcal{R}_2 \varepsilon_{T_0}(w_{T_0}) E_{\mathfrak{M} \setminus T_0}.$$

This allows us to take $R' > 0$ such that $a_n \leq q^{3n} R'$ for all n . Let $\eta_T(s) = \sum_{n \geq 0} l_{T,n}/q^{ns}$. Then $l_{T,0} = 1$ and

$$\xi_{w_{T_0}, T}(s) \preceq \sum_{n \geq 0} \frac{a_n}{q^{ns}} \eta_{T_0}(s) = \sum_{n \geq 0} \frac{\sum_{n_1+n_2=n} a_{n_1} l_{T,n_2}}{q^{ns}}.$$

Since

$$\begin{aligned} \sum_{n_1+n_2=n} a_{n_1} l_{T,n_2} &= a_n + \sum_{n_2=1}^n l_{T,n_2} a_{n-n_2} \\ &\leq a_n + q^{3n} R' \sum_{n_2=1}^n \frac{l_{T,n_2}}{q^{3n_2}} \leq a_n + q^{3n} R' (\eta_T(1) - 1), \end{aligned}$$

we have

$$\underline{\lim}_{n \rightarrow \infty} \frac{a_n}{q^{3n}} \geq \log q \mathcal{R}_2 \varepsilon_{T_0}(w_{T_0}) E_{T \setminus T_0} - R' (\eta_T(1) - 1).$$

Again by letting T approach to \mathfrak{M} , we obtain

$$\underline{\lim}_{n \rightarrow \infty} \frac{a_n}{q^{3n}} \geq \log q \mathcal{R}_2 \varepsilon_{T_0}(w_{T_0}) E_{\mathfrak{M} \setminus T_0}.$$

Together with the estimate for the superior limit, we obtain the result. \square

§ 6.3. Main results. Let us rewrite Theorem 6.6 to a mean value theorem for the degree zero divisor class groups of quadratic extensions. Let \mathcal{B}_v be the index set of extensions of k_v of degree not greater than two. By assumption that $\text{char}(k) \neq 2$, the cardinality of this set is four for all v . We denote by k_{v,β_v} an extension corresponding to $\beta_v \in \mathcal{B}_v$. From now on, the letter L always denotes a quadratic extension of k . For $\beta_v \in \mathcal{B}_v$, we write $L \sim \beta_v$ when the extension of L/k at v is $k_{v,\beta_v}/k_v$. We fix a finite set T of places of k and $\beta_T = (\beta_v)_{v \in T} \in \prod_{v \in T} \mathcal{B}_v$. If $L \sim \beta_v$ for all $v \in T$ then we write $L \sim \beta_T$.

Define b_{β_v} and b_{β_T} as follows.

$$b_{\beta_v} = \begin{cases} \frac{1}{2}(1 - q_v^{-2}) & k_{v,\beta_v} = k_v, \\ \frac{1}{2}(1 - q_v^{-1})^2 & k_{v,\beta_v} \text{ is quadratic unramified over } k_v, \\ \frac{1}{2}q_v^{-1}(1 - q_v^{-1})(1 - q_v^{-2}) & k_{v,\beta_v} \text{ is quadratic ramified over } k_v, \end{cases}$$

$$b_{\beta_T} = \prod_{v \in T} b_{\beta_v}.$$

For all quadratic extensions L except $k \otimes_{\mathbb{F}_q} \mathbb{F}_{q^2}$, $q_L = q$. Then, we can rewrite Theorem 6.6 as follows.

Theorem 6.7.

$$\lim_{n \rightarrow \infty} \frac{1}{q^{3n}} \sum_{\substack{L \sim \beta_S \\ \mathcal{N}(\mathfrak{D}_L) = q^{2n}}} h_L = 2 \mathfrak{C}_k h_k \zeta_k(2) b_{\beta_S} \prod_{v \notin S} (1 - q_v^{-2} - q_v^{-3} + q_v^{-4}).$$

If we take $S = \emptyset$, we will obtain Theorem 0.1 in the introduction.

We conclude this paper with some modification of this formula. The next proposition about the density of quadratic extensions is well known. We can evaluate this formula by means of class field theory or an analysis of a slight variation of Tate's zeta function. We simply state the result here.

Proposition 6.8. *Set*

$$c_{\beta_S} = \prod_{v \in S} c_{\beta_v}, \quad c_{\beta_v} = \begin{cases} \frac{1}{2}(1 - q_v^{-1}) & k_{v,\beta_v} \text{ is unramified over } k_v, \\ \frac{1}{2}q_v^{-1}(1 - q_v^{-1}) & k_{v,\beta_v} \text{ is quadratic ramified over } k_v. \end{cases}$$

Then

$$\lim_{n \rightarrow \infty} \frac{1}{q^{2n}} \sum_{\substack{L \sim \beta_S \\ \mathcal{N}(\mathfrak{D}_L) = q^{2n}}} 1 = 2q^{1-g} \mathfrak{C}_k C_{\beta_S} \prod_{v \notin S} (1 - q_v^{-2}).$$

Therefore, together with Theorem 6.7, we can obtain the following formula:

Corollary 6.9.

$$\lim_{n \rightarrow \infty} \frac{1}{q^n} \frac{\sum_{L \sim \beta_S, \mathcal{N}(\mathfrak{D}_L) = q^{2n}} h_L}{\sum_{L \sim \beta_S, \mathcal{N}(\mathfrak{D}_L) = q^{2n}} 1} = q^{g-1} h_k \zeta_k(2) d_{\beta_S} \prod_{v \notin S} \frac{1 + q_v^{-1} - q_v^{-3}}{1 + q_v^{-1}},$$

where, d_{β_S} is given by

$$d_{\beta_S} = \prod_{v \in S} d_{\beta_v}, \quad d_{\beta_v} = \begin{cases} 1 + q_v^{-1} & k_{v, \beta_v} = k_v, \\ 1 - q_v^{-1} & k_{v, \beta_v} \text{ is quadratic unramified over } k_v, \\ 1 - q_v^{-2} & k_{v, \beta_v} \text{ is quadratic ramified over } k_v. \end{cases}$$

Let g_L be the genus of L . To avoid the notational confusion, here we denote the genus of k by g_k . If $\mathcal{N}(\mathfrak{D}_L) = q^{2n}$, $g_L - 1 = 2g_k - 2 + n$. Hence the preceding formula can also be expressed as follows:

Corollary 6.10.

$$\lim_{n \rightarrow \infty} \frac{1}{q^n} \frac{\sum_{L \sim \beta_S, g_L = n} h_L}{\sum_{L \sim \beta_S, g_L = n} 1} = \frac{h_k \zeta_k(2)}{q^{g_k}} d_{\beta_S} \prod_{v \notin S} \frac{1 + q_v^{-1} - q_v^{-3}}{1 + q_v^{-1}}.$$

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