UTMS 2002–13 March 20, 2002

Uniqueness of an inverse source problem

by

Kim Sungwhan

UNIVERSITY OF TOKYO

GRADUATE SCHOOL OF MATHEMATICAL SCIENCES

KOMABA, TOKYO, JAPAN

UNIQUENESS OF AN INVERSE SOURCE PROBLEM

KIM SUNGWHAN

Abstract. We consider an inverse source problem of determining the shape and location of inhomogeneity in a Neumann boundary value problem for an elliptic equation

$$-\Delta u + \chi_D u = 0 \quad \text{in} \quad \Omega \qquad \text{and} \qquad \frac{\partial u}{\partial \nu} = g \quad \text{on} \quad \partial\Omega,$$

where $\overline{D} \subset \Omega$ and χ_D is the characteristic function of a subdomain D. For the determination of D, we measure Dirichlet data u on $\partial\Omega$. We prove the uniqueness in this inverse problem within some classes of subdomains of Ω .

Department of Mathematical Sciences
The University of Tokyo
3-8-1 Komaba, Meguro, Tokyo 153 Japan
e-mail : sungwhan@ms.u-tokyo.ac.jp

1. Introduction and Main Results

In this paper, we consider an inverse problem arising in the determination of transistor contact resistivity and contact window location of planar electronic devices. For the physical interpretation, see [4], [15], [16]. We are interested in the inverse problem of recovering an unknown contact subdomain D from a single boundary measurement of the voltage potential. More precisely, let $\Omega \subset \mathbb{R}^n$, $n = 2$ or 3, be a bounded domain with C^2 boundary and D be a subdomain compactly contained in Ω with Lipschitz boundary ∂D. Let g be a given applied current on $\partial\Omega$. Then the corresponding electric potential u satisfies the Neumann boundary value problem :

$$\begin{cases} -\Delta u \quad + \quad \chi_D u & = 0 \qquad \text{in} \quad \Omega \\ \qquad\qquad \frac{\partial u}{\partial \nu} & = g \qquad \text{on} \quad \partial\Omega, \end{cases} \tag{1.1}$$

where χ_D is the characteristic function of D and ν is the unit outward normal vector to $\partial\Omega$. Since, for given domain D and $g \in H^{-\frac{1}{2}}(\partial\Omega)$, there exists a unique solution $u \in H^1(\Omega)$ of (1.1), the Neumann-to-Dirichlet map $\Lambda_D : H^{-\frac{1}{2}}(\partial\Omega) \to H^{\frac{1}{2}}(\partial\Omega)$ can be defined by

$$\Lambda_D(g) := u|_{\partial\Omega}. \tag{1.2}$$

The inverse problem is to identify the unknown domain D by a single boundary measurement $(g, \Lambda_D(g))$ on $\partial\Omega$.

There are extensive studies for the determination of contact resistivity in both mathematical and engineering fields (e.g., [4], [14], [15], [16]). In particular, a uniqueness result within a one-parameter monotone family from a one-point boundary measurement of the potential is obtained in [4]. Moreover [14] provides a global uniqueness result within the class of two- or three- dimensional balls from a single boundary measurement. On the other hand, there are many inverse problems similar to our problem. For example, inverse problems of the determination of the potential q in the Schrödinger equation $-\Delta u + qu = 0$ in Ω have been studied in [1], [2], [6], [7]. The inverse conductivity problem, which is the determination of the coefficient γ in the equation $\nabla \cdot (\gamma \nabla u) = 0$ in Ω, is also a related significant problem (e.g., [3], [8], [11], [12], [13], [18]). As for monographs concerning inverse problem, see [9], [10].

Our purpose in this paper is to prove the uniqueness for the inverse problem where we are required to determine an unknown domain D by a single boundary measurement $(g, \Lambda_D(g))$. To our knowledge, the uniqueness result [14] in the ball case is the latest one, and the general uniqueness by a single boundary measurement is still open.

We will prove the global uniqueness within two classes $\mathcal{S}$ and $\mathcal{T}$ of subdomains of Ω which are defined as follows : Let A and B be any domains in $\mathbb{R}^n$, $n = 2$ or 3. For a unit vector a and $t \in \mathbb{R}$, let us set

$$\begin{aligned} L_t^a[A,B] &:= \{x \in A \cup B \mid x \cdot a = t\}, \\ U_{t^-}^a[A,B] &:= \{x \in A \cup B \mid x \cdot a < t\}, \\ U_{t^+}^a[A,B] &:= \{x \in A \cup B \mid x \cdot a > t\}, \end{aligned}$$

where the notation " $\cdot$ " denotes the inner product in $\mathbb{R}^n$.

Let $\mathcal{S}$ be a class of simply connected subdomains of Ω with C^2 boundary so that for any $A, B \in \mathcal{S}$, $A \neq B$ implies that either one is contained in the other, or there exist a unit vector a and a real number t such that

$$A \setminus \overline{B} \subset U_{t^-}^a[A,B] \quad \text{and} \quad B \setminus \overline{A} \subset U_{t^+}^a[A,B].$$

Example 1 : As $\mathcal{S}$, we can take the class of all balls contained in Ω.

Example 2 : The class of convex subdomains of Ω of which any two distinct elements A, B satisfy

either $i) A \subset B$ or $B \subset A$
or $ii) A \setminus \overline{B}$ and $B \setminus \overline{A}$ are non-empty and simply connected.

Next let $\mathcal{T}$ be a class of simply connected subdomains of Ω with C^2 boundary so that for any $A, B \in \mathcal{T}$, $A \neq B$ implies that either one is contained in the other, or there exist unit vectors a, b and t_1, t_2, s_1, $s_2 \in \mathbb{R}$ with $t_1 < t_2$ and $s_1 < s_2$ such that

$$A \setminus \overline{B} \subset U_{t_1-}^a \cup U_{t_2+}^a \qquad \text{and} \qquad B \setminus \overline{A} \subset U_{t_1+}^a \cap U_{t_2-}^a, \tag{1.3}$$

and

$$A \setminus \overline{B} \subset U_{s_1+}^b \cap U_{s_2-}^b \qquad \text{and} \qquad B \setminus \overline{A} \subset U_{s_1-}^b \cup U_{s_2+}^b. \tag{1.4}$$

The class of ellipses with common center in $\mathbb{R}^n$ contained in Ω is one example of $\mathcal{T}$.

Throughout this paper, we assume that $g \in C^{0,\alpha}(\partial\Omega)$ for some $0 < \alpha < 1$, $g \geq 0$ and $g \not\equiv 0$ on $\partial\Omega$. Then the maximum principle and Hopf's lemma (e.g., [5]) imply that

$$u(x) \geq 0 \qquad \text{for all} \quad x \in \overline{\Omega}. \tag{1.5}$$

Also it is well known (e.g., [17]) that

$$u \in C^{1,\alpha}(\Omega) \qquad \text{for some} \quad 0 < \alpha < 1 \tag{1.6}$$

and

$$u \quad \text{is analytic in} \quad \Omega \setminus \partial D. \tag{1.7}$$

Now we are ready to state our main results.

Theorem 1.1. *Suppose that D_1, D_2 belong to the family $\mathcal{S}$ and $\Omega \setminus (\overline{D_1 \cup D_2})$ is connected. If $\Lambda_{D_1}(g) = \Lambda_{D_2}(g)$, then $D_1 = D_2$.*

Theorem 1.2. *Suppose that D_1, D_2 belong to the family $\mathcal{T}$ and $\Omega \setminus (\overline{D_1 \cup D_2})$ is connected. If $\Lambda_{D_1}(g) = \Lambda_{D_2}(g)$, then $D_1 = D_2$.*

Before proving these theorems in Section 3, we will make some preliminary discussion for the one-dimensional case, which is motivating for the proofs.

2. One-Dimensional Case

We consider the one-dimensional case. Let $\Omega = (0, L)$ be a bounded open interval in $\mathbb{R}$ and let $D_1 = (a_1, a_2)$, $D_2 = (b_1, b_2)$ be open subintervals compactly contained in Ω with $a_1 \leq b_1$. Let u_j, $j = 1, 2$, be the solution of the second order ordinary differential equation :

$$\begin{cases} -u_j'' + \chi_{D_j} u_j = 0 & \text{in} \quad \Omega \\ u_j'(0) = d_0 \quad \text{and} & u_j'(L) = d_L. \end{cases} \tag{2.1}$$

To guarantee the non-negative of solutions u_j, we assume that $d_0 < 0$ and $d_L > 0$. Then we can show the uniqueness in the one-dimensional case : If $u_1(0) = u_2(0)$ and $u_1(L) = u_2(L)$, then $D_1 = D_2$. We want to prove this uniqueness by an argument which is extended to the two- or three-dimensional case in Section 3. Suppose that $D_1 \neq D_2$. We should consider two cases :

Case 1. either $D_1 \subset D_2$ or $D_2 \subset D_1$

Case 2. there exists $x \in \Omega$ such that

$$\emptyset \neq D_1 \setminus \overline{D_2} \subset (0, x) \qquad \text{and} \qquad \emptyset \neq D_2 \setminus \overline{D_1} \subset (x, L).$$

In Case 1, the result in [4] already yields the uniqueness. Hence we exclusively discuss Case 2. Let us define $y := u_1 - u_2$ in Ω. Since $y'' = 0$ in $(0, a_1)$ and $y(0) = y'(0) = 0$, the function y must be identically zero in $(0, a_1)$. Therefore y satisfies the nonhomogeneous equation :

$$y'' = u_1 \chi_{D_1 \setminus D_2} + y \chi_{D_1 \cap D_2} \qquad \text{in} \quad (a_1, x), \tag{2.2}$$

$$y(a_1) = y'(a_1) = 0. \tag{2.3}$$

By setting $z := \min\{a_2, x\}$, the equation (2.2) can be converted into

$$y'' = y + u_2 \chi_{D_1 \setminus D_2} \qquad \text{in} \quad (a_1, z), \tag{2.4}$$

$$y'' = 0 \qquad \text{in} \quad (z, x). \tag{2.5}$$

The solution y of the nonhomogeneous equation (2.4) with boundary data (2.3) has the form

$$y(t) = \int_{a_1}^{t} \sinh(t - s) u_2(s) \chi_{D_1 \setminus D_2}(s) ds \qquad \text{in} \quad (a_1, z). \tag{2.6}$$

By differentiating (2.6), we have

$$y'(t) = \int_{a_1}^{t} \cosh(t-s) u_2(s) \chi_{D_1 \setminus D_2}(s) ds \qquad \text{in} \quad (a_1, z). \tag{2.7}$$

Since $u_2 \not\equiv 0$ is non-negative and $D_1 \setminus D_2 \neq \emptyset$, it follows from (2.6) and (2.7) that

$$y(z) > 0 \qquad \text{and} \qquad y'(z) > 0. \tag{2.8}$$

Furthermore by (2.5) and (2.8), we can find that

$$y(x) > 0. \tag{2.9}$$

On the other hand, y also satisfies the following nonhomogeneous equation

$$\begin{cases} y'' = -u_2 \chi_{D_2 \setminus D_1} + y \chi_{D_1 \cap D_2} & \text{in} \quad (x, b_2), \\ y(b_2) = y'(b_2) = 0. \end{cases} \tag{2.10}$$

Solving the equation (2.10) in the same way, we can obtain

$$y(x) < 0. \tag{2.11}$$

Hence a contradiction occurs and we can conclude that $D_1 = D_2$. $\Box$

This one-dimensional case justifies the necessity of the introduction of classes such as $\mathcal{S}$ and $\mathcal{T}$. The set $L_t^a[A, B]$ plays the corresponding role in the two- or three- dimensional case as the real number x in Case 2.

3. Proof of Theorems 1.1 and 1.2

Proof of Theorem 1.1. Let u_j, $j = 1, 2$, be the solution of (1.1) corresponding to the domain D_j. By (1.5), u_1 and u_2 are non-negative functions on $\overline{\Omega}$. Let us define $y := u_1 - u_2$ in Ω. Then the function y satisfies

$$\Delta y = 0 \quad \text{in} \quad \Omega \setminus (\overline{D_1 \cup D_2}), \tag{3.1}$$

$$\Delta y = u_1 \geq 0 \quad \text{in} \quad D_1 \setminus \overline{D_2}, \tag{3.2}$$

$$\Delta y = -u_2 \leq 0 \quad \text{in} \quad D_2 \setminus \overline{D_1}, \tag{3.3}$$

$$\Delta y = y \quad \text{in} \quad D_1 \cap D_2, \tag{3.4}$$

$$\text{and} \qquad y = \frac{\partial y}{\partial \nu} = 0 \quad \text{on} \quad \partial\Omega. \tag{3.5}$$

$$\tag{3.6}$$

Since y is harmonic in $\Omega \setminus (\overline{D_1 \cup D_2})$ and $y = \frac{\partial y}{\partial \nu} = 0$ on $\partial\Omega$, the unique continuation implies that

$$y = 0 \quad \text{in} \quad \Omega \setminus (\overline{D_1 \cup D_2}) \quad \text{and} \quad y = \nabla y = 0 \quad \text{on} \quad \partial(D_1 \cup D_2).$$

We will show that the domains D_1 and D_2 coincide by contradiction. Suppose that $D_1 \neq D_2$. By [4], we see that the monotone case can not occur. Since D_1 and D_2 belong to the family $\mathcal{S}$, we may assume that there exist a unit vector a_0 and a real number t_0 such that

$$D_1 \setminus \overline{D_2} \subset U^{a_0}_{t_0{}^-}[D_1, D_2] \quad \text{and} \quad D_2 \setminus \overline{D_1} \subset U^{a_0}_{t_0{}^+}[D_1, D_2]. \tag{3.7}$$

For the simplicity of notations, $L^{a_0}_t[D_1, D_2]$, $U^{a_0}_{t^-}[D_1, D_2]$, and $U^{a_0}_{t^+}[D_1, D_2]$ are denoted by $L^{a_0}_t$, $U^{a_0}_{t^-}$, and $U^{a_0}_{t^+}$, respectively. For any $t \in \mathbb{R}$, let us define two functions

$$\phi(t) := \int_{L^{a_0}_t} y d\sigma \quad \text{and} \quad v_t(x) = t - a_0 \cdot x, \tag{3.8}$$

where the surface $L^{a_0}_t$ is oriented so that the orientation is the same as in (3.9). Note that the function v_t is harmonic in $\mathbb{R}^n$ and positive in $U^{a_0}_{t^-}$. It follows from

Green's second identity that

$$\int_{U_{t^-}^{a_0}} (\Delta y) v_t dx = \int_{\partial U_{t^-}^{a_0}} (\frac{\partial y}{\partial \nu} v_t - y \frac{\partial v_t}{\partial \nu}) d\sigma, \tag{3.9}$$

where ν is the outward unit vector normal to $\partial U_{t^-}^{a_0}$. Since $y = \nabla y = 0$ on $\partial(D_1 \cup D_2)$ and $v_t = 0$, $\frac{\partial v_t}{\partial \nu} = -1$ on $L_t^{a_0}$, we have

$$\int_{\partial U_{t^-}^{a_0}} (\frac{\partial y}{\partial \nu} v_t - y \frac{\partial v_t}{\partial \nu}) d\sigma = \phi(t). \tag{3.10}$$

Therefore, by (3.9) and (3.10), we obtain

$$\begin{aligned}
\phi(t) &= \int_{U_{t^-}^{a_0}} (\Delta y) v_t dx \\
&= \int_{U_{t^-}^{a_0} \setminus D_2} u_1 v_t dx + \int_{U_{t^-}^{a_0} \cap D_1 \cap D_2} y v_t dx - \int_{U_{t^-}^{a_0} \setminus D_1} u_2 v_t dx \\
&= \int_{U_{t^-}^{a_0} \setminus D_2} (u_1 - u_2) v_t dx + \int_{U_{t^-}^{a_0} \setminus D_2} u_2 v_t dx + \int_{U_{t^-}^{a_0} \cap D_1 \cap D_2} y v_t dx \\
&\quad + \int_{U_{t^-}^{a_0} \setminus D_1} (u_1 - u_2) v_t dx - \int_{U_{t^-}^{a_0} \setminus D_1} u_1 v_t dx \\
&= \int_{U_{t^-}^{a_0}} y v_t dx + \int_{U_{t^-}^{a_0} \setminus D_2} u_2 v_t dx - \int_{U_{t^-}^{a_0} \setminus D_1} u_1 v_t dx.
\end{aligned} \tag{3.11}$$

Moreover, the differentiation of the equation (3.11) with respect to the variable t yields

$$\phi'(t) = \int_{U_{t^-}^{a_0}} y dx + \int_{U_{t^-}^{a_0} \setminus D_2} u_2 dx - \int_{U_{t^-}^{a_0} \setminus D_1} u_1 dx. \tag{3.12}$$

Let $t_m := \sup\{t \in \mathbb{R} \mid U_{t^-}^{a_0} = \emptyset\}$ and $t_M := \inf\{t \in \mathbb{R} \mid U_{t^+}^{a_0} = \emptyset\}$. Since $\int_{U_{t^-}^{a_0}} y dx = \int_{t_m}^t (\int_{L_s^{a_0}} y d\sigma) ds = \int_{t_m}^t \phi(s) ds$, we can write $\phi'(t)$ as follows :

$$\phi'(t) = \int_{t_m}^t \phi(s) ds + \int_{U_{t^-}^{a_0} \setminus D_2} u_2 dx - \int_{U_{t^-}^{a_0} \setminus D_1} u_1 dx. \tag{3.13}$$

Differentating the equation (3.13) with respect to the variable t, we obtain

$$\phi''(t) = \phi(t) + \int_{L_t^{a_0} \setminus D_2} u_2 d\sigma - \int_{L_t^{a_0} \setminus D_1} u_1 d\sigma. \tag{3.14}$$

Since $U_{t^-}^{a_0} \setminus D_1 = \emptyset$ for any $t \in (t_m, t_0)$, by (3.13), the function ϕ satisfies the nonhomogeneous differential equation

$$\begin{cases} \phi'' = \phi + r & \text{in} \quad (t_m, t_0) \\ \phi(t_m) = \phi'(t_m) = 0, \end{cases} \tag{3.15}$$

where $r(t) = \int_{L_t^{a_0}\setminus D_2} u_2 d\sigma$. By solving the equation (3.15), the solution ϕ is given by

$$\phi(t) = \int_{t_m}^{t} \sinh(t-s)r(s)ds \qquad \text{in} \quad (t_m, t_0). \tag{3.16}$$

Since r is non-negative and not identically zero, it follows from (3.16) that

$$\phi(t_0) > 0.$$

On the other hand, we can argue similarly in $U_{t_0^+}^{a_0}$, and we obtain that $\phi(t_0) < 0$ by $\Delta y = -u_2 < 0$ in $D_2 \setminus \overline{D_1}$. Hence a contradiction occurs and it leads to the conclusion that $D_1 = D_2$. □

Proof of Theorem 1.2. We continue to use notations in the proof of Theorem 1.1. Suppose that $D_1 \neq D_2$. Due to [4], we can consider only the non-monotone case, i.e., $D_1 \setminus \overline{D_2} \neq \emptyset$ and $D_2 \setminus \overline{D_1} \neq \emptyset$. Let u_j, $j = 1, 2$, be the solution of (1.1) corresponding to the domain D_j. Without loss of generality, we may assume that

$$\int_{D_1\setminus D_2} u_2 dx - \int_{D_2\setminus D_1} u_1 dx \geq 0. \tag{3.17}$$

Under (3.17), we will use only the condition (1.3). More precisely, since D_1 and D_2 belong to $\mathcal{T}$, there exist a unit vector a_0 and two real numbers t_1, t_2 with $t_1 < t_2$ such that

$$D_1 \setminus \overline{D}_2 \subset U_{t_1^-}^{a_0} \cup U_{t_2^+}^{a_0} \tag{3.18}$$

and

$$D_2 \setminus \overline{D}_1 \subset U_{t_1^+}^{a_0} \cap U_{t_2^-}^{a_0}. \tag{3.19}$$

Let $t_m := \sup\{t \in \mathbb{R} \mid U_{t^-}^{a_0} = \emptyset\}$ and $t_M := \inf\{t \in \mathbb{R} \mid U_{t^+}^{a_0} = \emptyset\}$. Note that $t_m < t_1 < t_2 < t_M$. As in the proof of Theorem 1.1, let $y := u_1 - u_2$ on Ω. Setting $\phi(t) := \int_{L_t^{a_0}} y d\sigma$ and $v_t(x) = t - a_0 \cdot x$, $t \in \mathbb{R}$, we can obtain two equations similar to (3.11) and (3.13)

$$\phi(t) = \int_{U^{a_0}_{t^-}} y v_t dx + \int_{U^{a_0}_{t^-}\setminus D_2} u_2 v_t dx - \int_{U^{a_0}_{t^-}\setminus D_1} u_1 v_t dx \tag{3.20}$$

and

$$\phi'(t) = \int_{t_m}^{t} \phi(s) ds + \int_{U^{a_0}_{t^-}\setminus D_2} u_2 dx - \int_{U^{a_0}_{t^-}\setminus D_1} u_1 dx. \tag{3.21}$$

We can find from the proof of Theorem 1.1 that ϕ is a strictly increasing and positive function on $(t_m, t_1]$, and a strictly decreasing and positive one on $[t_2, t_M)$. If ϕ is nonnegative on (t_1, t_2), then, by noting that $U^{a_0}_{t_M^-} \setminus D_2 = D_1 \setminus D_2$ and $U^{a_0}_{t_M^-} \setminus D_1 = D_2 \setminus D_1$, the assumption (3.17) and the equation (3.21) imply that

$$\phi'(t_M) > 0, \tag{3.22}$$

which is impossible. Therefore there exists a real number $t_* \in (t_1, t_2)$ such that $\phi(t_*) < 0$ and $\phi'(t_*) = 0$. Since $U^{a_0}_{t^-} \setminus D_2 = U^{a_0}_{t_1^-} \setminus D_2$ for any $t \in [t_1, t_2]$, it follows that $\int_{U^{a_0}_{t^-}\setminus D_2} u_2 dx$ is constant on $[t_1, t_2]$. Therefore, we have that for any $t \in (t_*, t_2)$

$$\begin{aligned} &\phi'(t) \\ = \; & \textstyle\int_{t_m}^{t_*} \phi(s)ds + \int_{t_*}^{t} \phi(s)ds + \int_{U^{a_0}_{t_*^-}\setminus D_2} u_2 dx - \int_{U^{a_0}_{t_*^-}\setminus D_1} u_1 dx - \int_{(U^{a_0}_{t^-}\setminus U^{a_0}_{t_*^-})\setminus D_1} u_1 dx \\ = \; & \textstyle\phi'(t_*) + \int_{t_*}^{t} \phi(s)ds - \int_{(U^{a_0}_{t^-}\setminus U^{a_0}_{t_*^-})\setminus D_1} u_1 dx \\ = \; & \textstyle\int_{t_*}^{t} \phi(s)ds - \int_{(U^{a_0}_{t^-}\setminus U^{a_0}_{t_*^-})\setminus D_1} u_1 dx. \end{aligned} \tag{3.23}$$

By using (3.23), we can prove that

$$\phi(t_2) < 0. \tag{3.24}$$

This is a contradiction to $\phi(t_2) > 0$ and we can conclude that $D_1 = D_2$. In fact, assume contrarily that $\phi(t_2) \geq 0$. Setting $\eta := \inf\{t \in (t_*, t_2] \mid \phi(t) \geq 0\}$, we have $t_* < \eta \leq t_2$, $\phi(t) < 0$ for any $t \in (t_*, \eta)$ and $\phi(\eta) = 0$. By the equation (3.23), we have that ϕ' is a decreasing function on (t_*, η). Since $\phi'(t_*) = 0$, we obtain

$$\phi'(t) < 0 \qquad \text{for any} \quad t \in (t_*, \eta), \tag{3.25}$$

which implies that $\phi(\eta) < \phi(t_*) < 0$, which is a contradiction to $\phi(\eta) = 0$ and proves our claim (3.24). □

References

[1] J. Cheng and G. Nakamura, Stability for the inverse potential problem by finite measurements on the boundary, Inverse Problems, **17**(2001), 273-280

[2] J. R. Cannon and W. Rundell, An inverse problem for an elliptic partial differential equation, J.Math.Anal.Appl., **126**(1987), 329-340

[3] A. Friedman, Detection of mines by electric measurements, SIAM J.Appl. Math., **47**(1987), 201-212

[4] W. Fang and E. Cumberbatch, Inverse problems for metal oxide semiconductor field-effect transistor contact resistivity, SIAM J. Appl. Math., **52**(1992), 699-709

[5] D. Gilbarg and N.S. Trudinger, Elliptic Partial Differential Equations of Second Order, Springer-Verlag Berlin Heidelberg, 1977

[6] Hettlich F. and Rundell W., Ierative methods for the reconstruction of an inverse potential problem, Inverse Problems, **12**(1996), 251-266

[7] Hettlich F. and Rundell W., Recovery of the support of a source term in an elliptic differential equation, Inverse Problem, **13**(1997), 959-976

[8] V. Isakov, On uniqueness of recovery of a discontinuous conductivity coefficient, Commun. Pure Appl. Math., **41**(1988), 865-877

[9] V. Isakov, Inverse Source Problems, AMS, Providence, R.I., 1990

[10] V. Isakov, Inverse Problems for Partial Differential Equations, Springer-Verleg New York, 1998

[11] Kang H. and Seo J. K., Layer potential techique for the inverse conductiverty problem, Inverse Problems, **12**(1996), 267-278

[12] Kang H. and Seo J. K., Inverse conductiverty problem with one measurement : uniqueness for balls in $\mathbb{R}^3$, SIAM J. Appl. Math., **59**(1996), 1533-1539

[13] Kang H. and Seo J. K., Recent progress in the inverse conductivity problem with single measurement, Inverse Problems and Related Topics, (2000), 69-80

[14] Kang H., Kwon K., Yun K., Recovery of an inhomogeneity in an elliptic equation, Inverse Problems, **17**(2001), 25-44

[15] W. H. Loh, Modelling and measurement of contact resistances, Stanford Electronics Labs. Tech. Rep., No. G830-1, Stanford, CA, December 1987

[16] W. H. Loh, S. E. Swirhun, T. A. Schreyer, R.M. Swanson, and K. C. Saraswat, Modelling and measurement of contact resistances, IEEE Trans. Electron Devices, **34**(1987), 512-524

[17] Ladyzhenskaya O.A. and Ural'tseva N.N., Linear and Quasilinear Elliptic Equations, Academic Press, 1968

[18] Sylvester J. and Uhlmann G., A global uniqueness theorem for an inverse boundary value problem, Ann. Math., **125**(1987), 153-169

UTMS

The Graduate School of Mathematical Sciences was established in the University of Tokyo in April, 1992. Formerly there were two departments of mathematics in the University of Tokyo: one in the Faculty of Science and the other in the College of Arts and Sciences. All faculty members of these two departments have moved to the new graduate school, as well as several members of the Department of Pure and Applied Sciences in the College of Arts and Sciences. In January, 1993, the preprint series of the former two departments of mathematics were unified as the Preprint Series of the Graduate School of Mathematical Sciences, The University of Tokyo. For the information about the preprint series, please write to the preprint series office.

ADDRESS:
Graduate School of Mathematical Sciences, The University of Tokyo
3–8–1 Komaba Meguro-ku, Tokyo 153, JAPAN
TEL +81-3-5465-7001 FAX +81-3-5465-7012