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An arithmetic property of Shirosaki's hyperbolic projective hypersurface

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1 Introduction and main result

In 1974 S. Lang [La74] conjectured that an algebraic variety V defined over an arbitrarily fixed number field K carries only finitely many K-rational points (here referred as the *arithmetic finiteness property*, while it was termed to be "*Mordellic*" in [La86]) if the complex space $V_{\mathbf{C}}$ with some embedding $K \hookrightarrow \mathbf{C}$ is hyperbolic in the sense of Kobayashi [Ko70]. The conjecture was established for curves [Fa83] and for subvarieties of Abelian varieties [Fa91]. The analogue over function fields was proved by [No85] and [No92].

S. Kobayashi conjectured in 1970 that generic hypersurfaces of the complex projective space $\mathbf{P}^n(\mathbf{C})$ of high degree are hyperbolic ([Ko70]). In [MN96] the existence of hyperbolic projective hypersurfaces M was proved for all $n \geqq 2$ in a constructive way. Hence, it is interesting to study the arithmetic property of such M. In [No97] it was proved that those M satisfy the analogue of the arithmetic finiteness property over function fields, and have only finitely many S-unit points over number fields. Moreover, it was observed that "$abc\cdots$-Conjecture" ([No96a], [No96b], [Vo98]) would imply the arithmetic finiteness property of M. Cf. also Sarnak and Wang [SW95] for another arithmetic property of M. So far to the author's knowledge there had been no example, nor existence theorem of a projective hypersurface of dimension > 1 that carries the arithmetic finiteness property.

In 1998 Shirosaki [Sh98] found a simpler method to construct hyperbolic hypersurfaces X of degree d^n with some $d > 12$ by an idea using a unicity polynomial, which is different to that of [MN96]. The purpose of this paper is to prove the arithmetic finiteness property of Shirosaki's X.

To state the result, we take co-prime positive integers $d, e \in \mathbf{N}$ such that

$$d > 2e + 8, \tag{1.1}$$

and set

$$P(w_0, w_1) = w_0^d + w_1^d + w_0^e w_1^{d-e}. \tag{1.2}$$

Following to [Sh98], we define inductively

$$P_1(w_0, w_1) = P(w_0, w_1), \tag{1.3}$$
$$P_n(w_0, w_1, \ldots, w_n) = P_{n-1}(P(w_0, w_1), \ldots, P(w_{n-1}, w_n)), \quad n = 2, 3, \ldots .$$

Then, P_n is homogeneous and of degree d^n. Set

$$X = \{P_n(w_0, w_1, \ldots, w_n) = 0\} \subset \mathbf{P}^n_{\mathbf{Q}}. \tag{1.4}$$

By Shirosaki [Sh98] $X_{\mathbf{C}}$ is hyperbolic for $e > 2$.

Main Theorem *Assume that $e > 2$. Then X defined by (1.4) satisfies the arithmetic finiteness property; that is, for an arbitrary number field K, the set $X(K)$ of K-rational points of X is finite.*

It is interesting to observe that there is an analogy not only in the result, but also in the way of the proof; similar analogues were found in [No97] and [NW99]. For general references on the present subject, cf. [La86], [La91], [No89], [Ko98], [Vo87].

2 Lemmas

We keep the notation in §1.

Lemma 2.1 *For every $0 \leqq k \leqq n$ there is a number $\lambda(n, k) \in \mathbf{N}$ such that*

$$P_n(0, \ldots, 0, w_k, 0, \ldots, 0) = \lambda(n, k) w_k^{d^n}.$$

Proof. Note first that for $w_j \in \mathbf{N}, 0 \leqq j \leqq n$,

$$P(w_0, \ldots, w_n) \in \mathbf{N}. \tag{2.2}$$

By definition we have

$$P_n(0, \ldots, 0, w_k, 0, \ldots, 0) = P_{n-1}(0, \ldots, 0, w_k^d, w_k^d, 0, \ldots, 0).$$

Inductively, we have

$$P_n(0, \ldots, 0, w_k, 0, \ldots, 0) = P_{n-l}(\mu(l, 0) w_k^{d^l}, \ldots, \mu(l, n-l) w_k^{d^l}),$$

where $\max\{k, n-k\} < l < n$ and $\mu(l, j) \in \mathbf{N}$. Set $\lambda(n, k) = P_{n-l}(\mu(l, 0), \ldots, \mu(l, n-l))$. Then by (2.2) we have that $\lambda(n, k) \in \mathbf{N}$ and

$$P_n(0, \ldots, 0, w_k, 0, \ldots, 0) = \lambda(n, k) w_k^{d^n}.$$

Q.E.D.

Lemma 2.3 *Let $F_j, 1 \leqq j \leqq m$, be holomorphic functions on $\mathbf{C}$, and let $d_j \in \mathbf{N}$. Assume the following:*

(i) $\sum_{j=1}^{m} F_j = 0$;

(ii) *the order of every zero of* F_j *is at least* d_j;

(iii) *the functions* $F_j, 1 \leqq j \leqq m-1$, *have no common zero and are linearly independent over* $\mathbf{C}$.

Then we have

$$\sum_{j=1}^{m} \frac{1}{d_j} \geqq \frac{1}{m-2}.$$

This is due to Cartan [Ca33] (cf., [MN96] for an application to the hyperbolicity problem). The next lemma is a slight modification of [Sh98], Theorem 4.2.

Lemma 2.4 *Let* $\alpha, \beta \in \mathbf{C}$ *and* $\alpha \neq 0$. *Then the curve* $C_{\alpha,\beta} \subset \mathbf{P}^2(\mathbf{C})$ *defined by*

$$C_{\alpha,\beta} = \{(w_0; w_1; w_2) \in \mathbf{P}^2(\mathbf{C}); P(w_0, w_1) = \alpha P(\beta w_1, w_2)\}$$

is hyperbolic for (i) $\beta \neq 0, e > 2$, *and for (ii)* $\beta = 0, e > 3$.

Proof. When $\beta = 1$, the lemma was proved by [Sh98], Theorem 4.2. If $\beta \neq 0$, then $P(\beta w_1, w_2) = \beta^d P(w_1, \beta^{-1} w_2)$. Thus it is reduced to the case of $\beta = 1$.

Let $\beta = 0$, and assume that $C_{\alpha,0}$ is not hyperbolic. Then there is a non-constant holomorphic mapping $f : \mathbf{C} \to C_{\alpha,0} \subset \mathbf{P}^2(\mathbf{C})$. Let $f = (f_0; f_1; f_2)$ be reduced representation of f. Then one has

$$f_0^d + f_1^d - \alpha f_2^d + f_0^e f_1^{d-e} = 0. \tag{2.5}$$

If one of $\{f_j\}_{j=0}^2$ vanishes identically, it follows from (2.5) that other f_j must be proportional; hence the mapping f is constant. This contradicts the hypothesis. Thus none of $\{f_j\}_{j=0}^2$ vanishes identically. If $f_j^d, 0 \leqq j \leqq 2$, are linearly dependent, there is a non-trivial relation,

$$c_0 f_0^d + c_1 f_1^d + c_2 f_2^d = 0, \qquad c_j \in \mathbf{C}. \tag{2.6}$$

If one of $\{c_j\}_{j=0}^2$, is zero, it follows from (2.6) and (2.5) that f is constant. This is absurd, and so $c_0 c_1 c_2 \neq 0$. By Lemma 2.3, $3/d \geqq 1$; this contradicts $d > 16$. Therefore $f_j^d, 0 \leqq j \leqq 2$, must be linearly independent. Then, Lemma 2.3 and (2.5) yields

$$\frac{3}{d} + \frac{1}{e} \geqq \frac{1}{2}.$$

It follows from (1.1) and $e > 3$ that

$$\begin{aligned} \frac{3}{d} + \frac{1}{e} - \frac{1}{2} &< \frac{3}{2e+8} + \frac{1}{e} - \frac{1}{2} \\ &= \frac{-e^2 + e + 8}{2(e+4)e} \\ &< 0. \end{aligned}$$

This is again a contradiction. *Q.E.D.*

By Faltings' Theorem [Fa83] and Lemma 2.4 we have

Lemma 2.7 *Let K be an arbitrary number field, and let $\alpha, \beta \in K$ with $\alpha \neq 0$. Assume that $\beta \neq 0$ and $e > 2$, or that $\beta = 0$ and $e > 3$. Then the set $C_{\alpha,\beta}(K)$ is finite.*

The following is an analogue of [Sh98], Theorem 4.3.

Lemma 2.8 *Let $e > 2$, $n \geqq 2$, and let $(p_0; \dots ; p_{n-1}) \in \mathbf{P}^{n-1}(K)$ be a point such that at least two of $\{p_j\}_{j=0}^{n-1}$ are different to zero. Then there are only finitely many points $(w_0; \dots ; w_n) \in \mathbf{P}^n(K)$ such that*

$$(P(w_0, w_1); \dots ; P(w_{n-1}, w_n)) = (p_0; \dots ; p_{n-1}).$$

Proof. We use the induction on n. If $n = 2$, there is a number $\alpha \in K^* = K \setminus \{0\}$ such that $P(w_0, w_1) = \alpha P(w_1, w_2)$. There are only finitely many such $(w_0; w_1; w_2)$ by Lemma 2.7.

Assume that the statement holds up to $n - 1$. We consider the case of $n > 2$, and let $(w_0; \dots ; w_n) \in \mathbf{P}^n(K)$ be such points. If $P(w_{n-1}, w_n) = 0$, the induction hypothesis implies the finiteness of the number of points $(w_0; \dots ; w_{n-1})$. If $w_{n-1} = 0$, then $w_n = 0$; if $w_{n-1} \neq 0$, then $w_n \neq 0$ and the number of ratio w_n/w_{n-1} is at most d. Therefore the number of points $(w_0; \dots ; w_n)$ is finite.

Assume that $P(w_{n-1}, w_n) \neq 0$. Then there is a number $k < n$ such that

$$\begin{aligned} &P(w_{j-1}, w_j) = 0, \qquad k < j \leqq n - 1, \\ &P(w_{k-1}, w_k) \neq 0. \end{aligned} \tag{2.9}$$

There is a number $\alpha \in K^*$ such that

$$P(w_{k-1}, w_k) = \alpha P(w_{n-1}, w_n). \tag{2.10}$$

If $w_{n-1} = 0$, then $w_j = 0$ for every $k \leqq j \leqq n - 1$. Then (2.10) yields

$$w_{k-1}^d = \alpha w_n^d \neq 0.$$

Hence the number of points

$$(w_{k-1}; \dots ; w_n) = (w_{k-1}; 0; \dots ; 0; w_n)$$

is finite. If $w_{n-1} \neq 0$, then $w_j \neq 0$ for $k \leqq j \leqq n - 1$ by (2.9). Moreover, the number of ratios $w_{j-1}/w_j, k + 1 \leqq j \leqq n - 1$, is finite. Hence there are only finitely many $\beta \in K^*$ such that $w_{n-1} = \beta w_k$. It follows from (2.10) that

$$P(w_{k-1}, w_k) = \alpha P(\beta w_k, w_n), \qquad \alpha\beta \neq 0.$$

By Lemma 2.7 the number of points $(w_{k-1}; w_k; w_n)$ is finite. Hence the number of points $(w_{k-1}; \dots ; w_n)$ is finite. If there is a number $1 \leqq j < k$ with $P(w_{j-1}, w_j) \neq 0$, the induction hypothesis implies the finiteness of the number of points $(w_0; \dots ; w_{n-1})$. In all, there are only finitely many such points $(w_0; \dots ; w_n)$.

Assume that $P(w_{j-1}, w_j) = 0$ for all $1 \leqq j < k$. Then either $w_j = 0$ for all $1 \leqq j < k$, or $w_j \neq 0$ for all $1 \leqq j < k$ and the number of ratios w_{j-1}/w_j is at most d for every $1 \leqq j < k$. Henceforth there are only finitely many points $(w_0; \dots ; w_n)$. *Q.E.D.*

3 Proof of the main theorem

We use the induction on $n \geqq 2$. The case of $n = 2$ is done by Lemma 2.7. Assume the case of $n-1$ to be true. For n we recall the definition of X:

$$P_n(w_0, \ldots, w_n) = P_{n-1}(P(w_0, w_1), \ldots, P(w_{n-1}, w_n)) = 0.$$

It follows from Lemma 2.1 that either all $P(w_{j-1}, w_j) = 0, 1 \leqq j \leqq n$, or at least two of $P(w_{j-1}, w_j)$ are different to zero. In the first case, the finiteness of such points $(w_0; \ldots; w_n)$ is clear. In the latter case the induction hypothesis and Lemma 2.8 imply that there are only finitely many those points $(w_0; \ldots; w_n)$. Therefore, $X(K)$ is a finite set.

This completes the proof of the Main Theorem.

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