UTMS 2002–1 January 8, 2002

Stringy Hodge numbers and p-adic Hodge theory

by

Tetsushi Ito

UNIVERSITY OF TOKYO

GRADUATE SCHOOL OF MATHEMATICAL SCIENCES

KOMABA, TOKYO, JAPAN

STRINGY HODGE NUMBERS AND P-ADIC HODGE THEORY

TETSUSHI ITO

Abstract. Stringy Hodge numbers are introduced by Batyrev for a mathematical formulation of mirror symmetry. However, since the stringy Hodge numbers of an algebraic variety are defined by choosing a resolution of singularities, the well-definedness is not clear from the definition. Batyrev proved the well-definedness by using the theory of motivic integration developed by Kontsevich, Denef-Loeser. The aim of this paper is to give an alternative proof of the well-definedness of stringy Hodge numbers based on arithmetic results such as p-adic integration and p-adic Hodge theory.

1. Introduction

First of all, we give a statement of the main theorem of this paper (Theorem 1.1, below). Let X be an irreducible normal algebraic variety over $\mathbb{C}$ with at worst log-terminal singularities. Let $\rho : Y \to X$ be a resolution of singularities such that the exceptional divisor $\mathrm{Exc}(\rho)$ is a normal crossing divisor whose irreducible components $D_1, \ldots, D_r$ are smooth. Let $K_Y = \rho^* K_X + \sum_{i=1}^r a_i D_i$ with $a_i \in \mathbb{Q}$, $a_i > -1$, $I := \{1, \ldots, r\}$, $D_J^\circ := \big(\bigcap_{j\in J} D_j\big)\backslash\big(\bigcup_{j\in I\backslash J} D_j\big)$ for a nonempty subset $J \subset I$, and $D_\emptyset^\circ := Y\backslash\mathrm{Exc}(\rho)$. We define the *stringy E-function* of X by the formula

$$E_{st}(X; u, v) := \sum_{J\subset I} E(D_J^\circ; u, v) \prod_{j\in J} \frac{uv - 1}{(uv)^{a_j+1} - 1},$$

where $E(D_J^\circ; u, v) := \sum_k (-1)^k \sum_{i,j} h^{i,j}(\mathrm{Gr}^W_{i+j} H^k_c(D_J^\circ, \mathbb{Q}))\, u^i v^j$ is the generating function of the Hodge numbers of D_J°. (for details, see §2)

Theorem 1.1 ([Ba2], Theorem 3.4)**.** *The stringy E-function $E_{st}(X; u, v)$ defined as above is independent of the choice of a resolution of singularities $\rho : Y \to X$.*

Stringy Hodge numbers are introduced by Batyrev for a mathematical formulation of mirror symmetry as follows ([Ba2]). Assume that $E_{st}(X; u, v)$ is a polynomial in u, v. We define the *stringy Hodge numbers* of X by the formula

$$E_{st}(X; u, v) = \sum_{i,j} (-1)^{i+j}\, h^{i,j}_{st}(X)\, u^i v^j.$$

Therefore, we have the well-definedness of the stringy Hodge numbers whenever they are defined.

Date: December 28, 2001.

Theorem 1.1 is firstly proved by Batyrev in [Ba2] by using the theory of motivic integration developed by Kontsevich, Denef-Loeser ([Kon],[DL2]). In this paper, we give an alternative proof based on arithmetic results such as p-adic integration and p-adic Hodge theory by generalizing ideas in [Ba1],[Wa],[It1],[It2].

Here we note a motivation of stringy Hodge numbers in mirror symmetry. A mathematical formulation of mirror symmetry claims a symmetry between Hodge numbers of mirror varieties called the topological mirror symmetry test ([Ba2],[M]). However, some examples discovered by physicists show that the mirror of a smooth variety is not necessarily smooth. In some cases, usual Hodge theory doesn't work well.

For example, physicists predict that the mirror of a Fermat quintic threefold

$$X : f(z) = z_0^5 + z_1^5 + z_2^5 + z_3^5 + z_4^5 = 0 \subset \mathbb{P}^4$$

is isomorphic to the quotient X/G, where $G \cong (\mathbb{Z}/5\mathbb{Z})^3 \subset PSL(5, \mathbb{C})$ is the maximal diagonal subgroup which acts trivially on $f(z)$. Since X/G has only quotient singularities, the mixed Hodge structures of X/G defined by Deligne are pure ([De5],[De6]). However, since $h^{2,1}(X) = 101 \neq 1 = h^{3-2,1}(X/G)$, the usual Hodge numbers of $X, X/G$ don't satisfy the *topological mirror symmetry test*

$$h^{i,j}(X) = h^{\dim X - i,\, j}(X/G) \quad \text{for all } i, j$$

in [M]. To overcome this difficulty, Batyrev introduced stringy Hodge numbers as in the beginning of this paper. Indeed, in the above example, it is known that the stringy Hodge numbers of $X, X/G$ satisfy the topological mirror symmetry test as above because $h_{st}^{2,1}(X) = h_{st}^{3-2,1}(X/G) = 101$ ([Ba2], Example 1.1). Note that $h_{st}^{i,j}(X) = h^{i,j}(X)$ since X is proper and smooth (Corollary 2.8). Today, several examples of stringy Hodge numbers are computed from the viewpoint of mirror symmetry ([BB],[BD]).

We also note that the well-definedness of stringy Hodge numbers is a nontrivial problem in algebraic geometry. Motivic integration is introduced by Kontsevich as a geometric analogue of p-adic integration ([Kon]). It is interesting that Theorem 1.1 can be proved by either motivic integration or p-adic integration.

Theorem 1.1 has some applications in birational geometry. Firstly, for an algebraic variety X over $\mathbb{C}$ with a crepant resolution $\rho : Y \to X$, we show that the Hodge numbers of Y are independent of the choice of a crepant resolution $\rho : Y \to X$ (Corollary 2.9). This is important in the study of McKay correspondences in dimension ≥ 3. Secondly, we show that birational smooth minimal models X, Y (e.g. Calabi-Yau manifolds) have equal Hodge numbers (Corollary 2.10, see also [Ba1],[Wa],[It1]). In dimension ≤ 3, this can be proved by the minimal model program ([KMM],[Ka],[Kol]). However, in dimension ≥ 4, it seems very difficult to show this if we use neither motivic integration nor p-adic integration.

This paper is a continuation of author's previous works in [It1],[It2]. Here we note the new ingredients of this paper. Basically, the main ideas are the same as before. However, to treat stringy Hodge numbers rather than usual Hodge numbers, we calculate some p-adic integration explicitly (Proposition 3.5). Furthermore, to treat combinations of cohomology groups of open varieties, we generalize arithmetic results to open varieties by the methods of Deligne ([De4],[De5]) and work on the level of a Grothendieck group of Galois representations rather than individual cohomology groups (§5).

Acknowledgments. The author is grateful to Takeshi Saito and Kazuya Kato for their advice and support. He would like to thank Victor V. Batyrev, François Loeser for helpful discussion on motivic integration and useful comment, Shinichi Mochizuki, Takeshi Tsuji for invaluable suggestion on p-adic Hodge theory. He would also like to thank Yujiro Kawamata, Daisuke Matsushita, Yasunari Nagai, Keiji Oguiso, Atsushi Takahashi, Hokuto Uehara for their interest in this work and encouragement. The author was supported by the Japan Society for the Promotion of Science Research Fellowships for Young Scientists.

2. Stringy Hodge numbers

In this section, we recall the definition of the stringy E-function and the stringy Hodge numbers as in [Ba2]. We also note some corollaries of Theorem 1.1.

For an algebraic variety X over $\mathbb{C}$, the cohomology groups with compact supports $H_c^k(X,\mathbb{Q})$ have canonical mixed Hodge structures by Deligne ([De5], [De6]). Let W be the weight filtration on $H_c^k(X,\mathbb{Q})$. Each graded quotient $\mathrm{Gr}_l^W H_c^k(X,\mathbb{Q})$ has a pure Hodge structure of weight l. Let $h^{i,j}(\mathrm{Gr}_{i+j}^W H_c^k(X,\mathbb{Q}))$ be the dimension of the (i,j)-th Hodge component of $\mathrm{Gr}_{i+j}^W H_c^k(X,\mathbb{Q})$ for each i,j.

Definition 2.1. We define the *E-function* of X as follows

$$E(X;u,v) := \sum_k (-1)^k \sum_{i,j} h^{i,j}(\mathrm{Gr}_{i+j}^W H_c^k(X,\mathbb{Q}))\, u^i v^j.$$

For a proper smooth variety X over $\mathbb{C}$,

$$E(X;u,v) = \sum_{i,j} (-1)^{i+j} h^{i,j}(X) u^i v^j,$$

where $h^{i,j}(X) = \dim H^j(X,\Omega_X^i)$ are the usual Hodge numbers of X.

Remark 2.2. E-function satisfies the following properties :

1. For $Z \subset X$, we have $E(X;u,v) = E(X \backslash Z;u,v) + E(Z;u,v)$.
2. For X,Y, we have $E(X \times Y;u,v) = E(X;u,v) \cdot E(Y;u,v)$.

These two properties imply that it is natural to consider E-function as a ring homomorphism from some Grothendieck group of algebraic varieties over $\mathbb{C}$ to $\mathbb{Z}[u,v]$ as in [DL1],[DL2],[DL3].

Let X be an irreducible normal algebraic variety over $\mathbb{C}$, and $\rho : Y \to X$ be a resolution of singularities such that the exceptional divisor $\mathrm{Exc}(\rho)$ is a simple normal crossing divisor. Recall that a normal crossing divisor is simple if its irreducible components are smooth. Let the irreducible components of $\mathrm{Exc}(\rho)$ be $D_1, \dots , D_r$.

Definition 2.3 ([Ba2], Definition 2.2)**.** X is said to have *at worst log-terminal singularities* if the following conditions are satisfied :

1. The canonical divisor K_X is a $\mathbb{Q}$-Cartier divisor. (i.e. X is $\mathbb{Q}$-Gorenstein)
2. We have
$$K_Y = \rho^* K_X + \sum_{i=1}^{r} a_i D_i$$
with $a_i > -1$. (Note that this condition is independent of the choice of a resolution $\rho : Y \to X$.)

Let X, Y be as above and X have at worst log-terminal singularities. Let $I := \{1, \dots , r\}$. For any subset $J \subset I$, we set
$$D_J := \begin{cases} \bigcap_{j\in J} D_j & J \neq \emptyset \\ Y & J = \emptyset \end{cases}, \qquad D_J^{\circ} := D_J \setminus \bigcup_{j\in I\setminus J} D_j.$$

Definition 2.4 ([Ba2], Definition 3.1)**.** We define the *stringy E-function* of X as follows
$$E_{st}(X; u, v) := \sum_{J\subset I} E(D_J^{\circ}; u, v) \prod_{j\in J} \frac{uv - 1}{(uv)^{a_j+1} - 1},$$
where $E(D_J^{\circ}; u, v)$ is the E-function of a smooth variety D_J° defined in the beginning of this section.

Remark 2.5. Since X has at worst log-terminal singularities, $a_i + 1 > 0$ and hence the denominator of $E_{st}(X; u, v)$ doesn't vanish (see also Remark 3.8). In general, $E_{st}(X; u, v)$ is an element of $\mathbb{Q}(u^{1/d}, v^{1/d}) \cap \mathbb{Z}[[u^{1/d}, v^{1/d}]]$, where d is the least common multiplier of the denominators of a_i.

Definition 2.6. Assume that $E_{st}(X; u, v)$ is a polynomial in u, v. Then we define the *stringy Hodge numbers* of X by the formula
$$E_{st}(X; u, v) = \sum_{i,j} (-1)^{i+j}\, h_{st}^{i,j}(X)\, u^i v^j.$$

Conjecture 2.7. *Batyrev conjectured that if X is a projective algebraic variety over $\mathbb{C}$ with at worst Gorenstein canonical singularities (i.e. K_X is a Cartier divisor and $a_i \geq 0$) and $E_{st}(X; u, v)$ is a polynomial in u, v, then all stringy Hodge numbers $h_{st}^{i,j}(X)$ are nonnegative integers. (*[Ba2]*, Conjecture 3.10)*

Theorem 1.1 claims that $E_{st}(X; u, v)$ is independent of the choice of a resolution $\rho : Y \to X$. Once we know the well-definedness, we can prove some fundamental properties of $E_{st}(X; u, v)$ as in [Ba2].

Here we give some immediate corollaries of Theorem 1.1.

Corollary 2.8 ([Ba2], Corollary 3.6)**.** *If X is smooth, then we have $E_{st}(X; u, v) = E(X; u, v)$.*

Proof. This is clear because the identity map $\mathrm{id} : X \to X$ is a resolution of singularities. $\square$

Corollary 2.9 ([Ba2], Theorem 3.12)**.** *Let X be a projective algebraic variety over $\mathbb{C}$ which has a crepant resolution $\rho : Y \to X$ (i.e. $\rho : Y \to X$ is a resolution of singularities with $\rho^* K_X = K_Y$). Then the stringy Hodge numbers of X are equal to the Hodge numbers of Y :*

$$h^{i,j}_{st}(X) = h^{i,j}(Y) \quad \textit{for all } i, j.$$

In particular, the Hodge numbers of Y are independent of the choice of a crepant resolution $\rho : Y \to X$. Therefore, we can compute the Hodge numbers of a crepant resolution of X via any resolution of singularities which is not necessarily crepant.

Proof. This is clear because we have $E_{st}(X; u, v) = E(Y; u, v)$ by Definition 2.4. $\square$

Corollary 2.10 ([Ba1],[Ba2],[Wa],[It1])**.** *Let X, Y be projective smooth algebraic varieties over $\mathbb{C}$ whose canonical bundles are nef (i.e. X, Y are* minimal models*). Assume that X, Y are birational. Then X, Y have equal Hodge numbers:*

$$h^{i,j}(X) = h^{i,j}(Y) \quad \textit{for all } i, j.$$

Proof. Let $f : X \dashrightarrow Y$ be a birational map. Then we can find a smooth projective variety Z over $\mathbb{C}$ and birational morphisms $g : Z \to X$, $h : Z \to Y$ such that $f \circ g = h$ as birational maps and $g^* K_X = h^* K_Y$.

$$\begin{tikzcd} & Z \arrow[dl, "g"'] \arrow[dr, "h"] & \\ X \arrow[rr, dashed, "f"] & & Y \end{tikzcd}$$

This is a standard fact in birational geometry (for example, see [It1], Proposition 2.2). We consider $g : Z \to X$ (resp. $h : Z \to Y$) as a resolution of singularities of X (resp. Y) and calculate the stringy Hodge numbers of X (resp. Y). Since $g^* K_X = h^* K_Y$, we have

$$E(X; u, v) = E_{st}(X; u, v) = E_{st}(Y; u, v) = E(Y; u, v).$$

Hence we have the equality of the Hodge numbers of X, Y. $\square$

Remark 2.11. A weaker form of Corollary 2.10 (i.e. the equality of the Betti numbers) is firstly obtained by Batyrev and Wang by p-adic integration and the Weil conjecture ([Ba1],[Wa], see also Remark 3.2). The author obtained Corollary 2.10 by generalizing their arguments and using p-adic Hodge theory in [It1],[It2]. On the other hand, Kontsevich and Batyrev obtained Corollary 2.10 by motivic integration ([Ba2],[Kon], for motivic integration, see also [DL2]).

3. p-adic integration

In this section, we recall Weil's p-adic integration developed in [We] which is an important tool to count the number of rational points valued in a finite field.

3.1. **Setting.** Let p be a prime number and $\mathbb{Q}_p$ be the field of p-adic numbers. Let F be a finite extension of $\mathbb{Q}_p$, $R \subset F$ be the ring of integers in F, $m \subset R$ be the maximal ideal of R, $\mathbb{F}_q = R/m$ be the residue field of F with q elements, where q is a power of p. For an element $x \in F$, we define the *p-adic absolute value* $|x|_p$ by

$$|x|_p := \begin{cases} q^{-v(x)} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

where $v : F^{\times} \to \mathbb{Z}$ is the normalized discrete valuation of F.

Let $\mathfrak{X}$ be a smooth scheme over R of relative dimension n. We can compute the number of the $\mathbb{F}_q$-rational points $|\mathfrak{X}(\mathbb{F}_q)|$ by integrating certain p-adic measure on the set of R-rational points $\mathfrak{X}(R)$. We note that $\mathfrak{X}(R)$ is a compact and totally disconnected topological space with respect to its p-adic topology.

3.2. **p-adic integration of regular n-forms.** Let $\omega \in \Gamma(\mathfrak{X}, \Omega^n_{\mathfrak{X}/R})$ be a *regular n-form* on $\mathfrak{X}$, where $\Omega^n_{\mathfrak{X}/R}$ is the relative canonical bundle of $\mathfrak{X}/R$. We shall define the p-adic integration of ω on $\mathfrak{X}(R)$ as follows. Let $s \in \mathfrak{X}(R)$ be a R-rational point. Let $U \subset \mathfrak{X}(R)$ be a sufficiently small p-adic open neighborhood of s on which there exists a system of local p-adic coordinates $\{x_1, \dots, x_n\}$. Then $\{x_1, \dots, x_n\}$ defines a p-adic analytic map

$$x = (x_1, \dots, x_n) : U \longrightarrow R^n$$

which is a homeomorphism between U and a p-adic open set V of R^n. By using the above coordinates, ω is written as

$$\omega = f(x)\ dx_1 \wedge \cdots \wedge dx_n.$$

We consider $f(x)$ as a p-adic analytic function on V. Then we define the p-adic integration of ω on U by the equation

$$\int_U |\omega|_p := \int_V |f(x)|_p\ dx_1 \cdots dx_n,$$

where $|f(x)|_p$ is the p-adic absolute value of the value of f at $x \in V$ and $dx_1 \cdots dx_n$ is the Haar measure on R^n normalized by the condition

$$\int_{R^n} dx_1 \cdots dx_n = 1.$$

By patching them, we get the *p-adic integration of ω on $\mathfrak{X}(R)$*

$$\int_{\mathfrak{X}(R)} |\omega|_p.$$

3.3. **p-adic integration of gauge forms.** By definition, a *gauge form* ω on $\mathfrak{X}$ is a nowhere vanishing global section $\omega \in \Gamma(\mathfrak{X}, \Omega^n_{\mathfrak{X}/R})$. The most important property of p-adic integration is that the p-adic integration of a gauge form computes the number of $\mathbb{F}_q$-rational points.

Proposition 3.1 ([We], 2.2.5)**.** *Let $\mathfrak{X}$ be a smooth scheme over R of relative dimension n and ω be a gauge form on $\mathfrak{X}$. Then*

$$\int_{\mathfrak{X}(R)} |\omega|_p = \frac{|\mathfrak{X}(\mathbb{F}_q)|}{q^n}.$$

Proof. Let

$$\varphi : \mathfrak{X}(R) \longrightarrow \mathfrak{X}(\mathbb{F}_q)$$

be the reduction map. For $\bar{x} \in \mathfrak{X}(\mathbb{F}_q)$, $\varphi^{-1}(\bar{x})$ is a p-adic open set of $\mathfrak{X}(R)$. Therefore, it is enough to show

$$\int_{\varphi^{-1}(\bar{x})} |\omega|_p = \frac{1}{q^n}$$

Let $\{x_1, \ldots , x_n\} \subset \mathcal{O}_{\mathfrak{X},\bar{x}}$ be a regular system of parameters at $\bar{x}$. Then $\{x_1, \ldots , x_n\}$ defines a system of local p-adic coordinates on $\varphi^{-1}(\bar{x})$ and

$$x = (x_1, \ldots , x_n) : \varphi^{-1}(\bar{x}) \longrightarrow m^n \subset R^n$$

is a p-adic analytic homeomorphism. Let ω be written as $\omega = f(x)\, dx_1 \wedge \cdots \wedge dx_n$. Since ω is a gauge form, $f(x)$ is a p-adic unit for all $x \in \varphi^{-1}(\bar{x})$. Therefore $|f(x)|_p = 1$. Then we have

$$\int_{\varphi^{-1}(\bar{x})} |\omega|_p = \int_{m^n} dx_1 \cdots dx_n = \frac{1}{q^n}$$

since m^n is an index q^n subgroup of R^n. □

Remark 3.2. Batyrev used Proposition 3.1 and the Weil conjecture to show that birational Calabi-Yau manifolds have equal Betti numbers ([Ba1]). Batyrev's argument was generalized to smooth minimal models by Wang ([Wa]). The author generalized the results of Batyrev and Wang to the equality of Hodge numbers by using p-adic Hodge theory ([It1],[It2]). Interestingly, after Batyrev's work in [Ba1], Kontsevich proved that birational Calabi-Yau manifolds have equal Hodge numbers by introducing motivic integration as an analogue of p-adic integration for varieties over $\mathbb{C}$ ([Kon], see also [DL2]). Then Batyrev used motivic integration to prove Theorem 1.1 and its corollaries ([Ba2]).

3.4. **Computation of some p-adic integration.** Since we treat stringy Hodge numbers rather than usual Hodge numbers in this paper, we need to compute some p-adic integration slightly more general than Proposition 3.1.

Firstly, we generalize p-adic integration to an *r-pluricanonical form*

$$\omega \in \Gamma(\mathfrak{X}, (\Omega^n_{\mathfrak{X}/R})^{\otimes r})$$

for $r \in \mathbb{Z},\ r \geq 1$ as in [Wa]. As in the case of a regular n-form, locally in p-adic topology, ω is be written as

$$\omega = f(x)\ (dx_1 \wedge \cdots \wedge dx_n)^{\otimes r}$$

for a system of local p-adic coordinates $\{x_1, \ldots, x_n\}$. Then we define

$$\int_U |\omega|_p^{1/r} := \int_V |f(x)|_p^{1/r}\ dx_1 \cdots dx_n,$$

where U, V are the same as in the case of a regular n-form. By patching them, we get the *p-adic integration of an r-pluricanonical form ω on $\mathfrak{X}(R)$*

$$\int_{\mathfrak{X}(R)} |\omega|_p^{1/r}.$$

Remark 3.3. For an r-pluricanonical form ω, $\omega^{\otimes s}$ is an rs-pluricanonical form. Then the p-adic integrations of ω and $\omega^{\otimes s}$ are equal :

$$\int_{\mathfrak{X}(R)} |\omega|_p^{1/r} = \int_{\mathfrak{X}(R)} |\omega^{\otimes s}|_p^{1/rs}.$$

Before computing p-adic integration, we recall the definition of relative simple normal crossing divisors.

Definition 3.4. Let $f : \mathfrak{X} \to S$ be a proper smooth morphism of schemes and $\mathfrak{D} = \bigcup_{i=1}^r \mathfrak{D}_i$ be a reduced divisor on $\mathfrak{X}$. $\mathfrak{D}$ is called a *relative simple normal crossing divisor* on $\mathfrak{X}$ over S if all $\mathfrak{D}_i$ are smooth over S and, for all $x \in \mathfrak{D}$, the completion of $\mathfrak{D} \hookrightarrow \mathfrak{X}$ at x is isomorphic to

$$\mathrm{Spec}\,(\widehat{\mathcal{O}}_{S,f(x)}[[x_1, \ldots, x_d]]/(x_1 \cdots x_s)) \hookrightarrow \mathrm{Spec}\,(\widehat{\mathcal{O}}_{S,f(x)}[[x_1, \ldots, x_d]])$$

for some s $(1 \leq s \leq d)$, where d is the relative dimension of f. Note that $\widehat{\mathcal{O}}_{\mathfrak{X},x}$ is isomorphic to $\widehat{\mathcal{O}}_{S,f(x)}[[x_1, \ldots, x_d]]$ because f is smooth of relative dimension d. In this case, for a nonempty subset $J \subset \{1, \ldots, r\}$, $\bigcap_{j \in J} \mathfrak{D}_j$ is smooth of relative dimension $d - |J|$ over S.

Now, we shall compute some p-adic integration. Let $\mathfrak{X}$ be a smooth scheme over R of relative dimension n and $\omega \in \Gamma(\mathfrak{X}, (\Omega^n_{\mathfrak{X}/R})^{\otimes r})$ be an r-pluricanonical form on $\mathfrak{X}$. Assume that

$$\mathrm{div}(\omega) = \sum_{i=1}^{s} a_i \mathfrak{D}_i$$

is a relative simple normal crossing divisor on $\mathfrak{X}$ over R. Let $I := \{1, \ldots, s\}$. For any subset $J \subset I$, we set

$$\mathfrak{D}_J := \begin{cases} \bigcap_{j \in J} \mathfrak{D}_j & J \neq \emptyset \\ \mathfrak{X} & J = \emptyset \end{cases}, \qquad \mathfrak{D}_J^{\circ} := \mathfrak{D}_J \setminus \bigcup_{j \in I \setminus J} \mathfrak{D}_j.$$

Proposition 3.5. *Let notation be as above. Then, we have*

$$\int_{\mathfrak{X}(R)} |\omega|_p^{1/r} = \frac{1}{q^n} \sum_{J\subset I} |\mathfrak{D}_J^\circ(\mathbb{F}_q)| \prod_{j\in J} \frac{q-1}{q^{(a_j/r)+1}-1}.$$

If $r=1$ and ω is a gauge form, Proposition 3.5 is nothing but Proposition 3.1.

Proof. The idea of proof is the same as in Proposition 3.1. Let

$$\varphi : \mathfrak{X}(R) \longrightarrow \mathfrak{X}(\mathbb{F}_q)$$

be the reduction map. For $\bar{x} \in \mathfrak{X}(\mathbb{F}_q)$, $\varphi^{-1}(\bar{x})$ is a p-adic open set of $\mathfrak{X}(R)$. Therefore, it is enough to show

$$\int_{\varphi^{-1}(\bar{x})} |\omega|_p^{1/r} = \frac{1}{q^n} \prod_{j\in I \text{ s.t. } \bar{x}\in\mathfrak{D}_j(\mathbb{F}_q)} \frac{q-1}{q^{(a_j/r)+1}-1}.$$

Let $\{j_1,\dots,j_m\} = \{j\in I \mid \bar{x}\in\mathfrak{D}_j(\mathbb{F}_q)\}$. Let $\{x_1,\dots,x_n\}\subset\mathcal{O}_{\mathfrak{X},\bar{x}}$ be a regular system of parameters at $\bar{x}$ such that $\mathfrak{D}_{j_i}$ is defined by $x_i=0$ at $\bar{x}$ for all $i=1,\dots,m$. Then $\{x_1,\dots,x_n\}$ defines a system of local p-adic coordinates on $\varphi^{-1}(\bar{x})$ and

$$x=(x_1,\dots,x_n) : \varphi^{-1}(\bar{x}) \longrightarrow m^n \subset R^n$$

is a p-adic analytic homeomorphism. Here ω is written as

$$\omega = f(x)\cdot x_1^{a_{j_1}}\cdots x_m^{a_{j_m}}(dx_1\wedge\cdots\wedge dx_n)^{\otimes r},$$

where $f(x)$ is a p-adic unit for all $x\in\varphi^{-1}(\bar{x})$. Hence we have

$$|f(x)\cdot x_1^{a_{j_1}}\cdots x_m^{a_{j_m}}|_p^{1/r} = |x_1|_p^{a_{j_1}/r}\cdots|x_m|_p^{a_{j_m}/r},$$

and

$$\int_{\varphi^{-1}(\bar{x})} |\omega|_p^{1/r} = \int_{m^n} |x_1|_p^{a_{j_1}/r}\cdots|x_m|_p^{a_{j_m}/r} dx_1\cdots dx_n.$$

Therefore, it is enough to prove the following lemma. □

Lemma 3.6. *For $k_1,\dots,k_n\in\mathbb{Q}$, $k_i>-1$, we have*

$$\int_{m^n} |x_1|_p^{k_1}\cdots|x_n|_p^{k_n} dx_1\cdots dx_n = \frac{1}{q^n}\prod_{i=1}^n \frac{q-1}{q^{k_i+1}-1}.$$

Proof. By iterated integration, we have.

$$\int_{m^n} |x_1|_p^{k_1}\cdots|x_n|_p^{k_n} dx_1\cdots dx_n = \left(\int_m |x_1|_p^{k_1}dx_1\right)\cdots\left(\int_m |x_n|_p^{k_n}dx_n\right)$$

Therefore, it is enough to prove

$$\int_m |x|_p^k\,dx = \frac{1}{q}\cdot\frac{q-1}{q^{k+1}-1}$$

for $k\in\mathbb{Q}$, $k>-1$.

We compute the above integration by dividing m as a disjoint union of open subsets as follows

$$m = \coprod_{i=1}^{\infty} m^i \backslash m^{i+1}.$$

For $x \in m^i \backslash m^{i+1}$, $|x|_p^k = q^{-ki}$. The volume of m^i is q^{-i} with respect to the normalized Haar measure on R since m^i is an index q^i subgroup of R. Therefore, we have

$$\begin{aligned}\int_m |x|_p^k \, dx &= \sum_{i=1}^{\infty} q^{-ik} \operatorname{vol}(m^i \backslash m^{i+1}) = \sum_{i=1}^{\infty} q^{-ik}(q^{-i} - q^{-(i+1)}) \\ &= (1 - q^{-1}) \sum_{i=1}^{\infty} (q^{-(k+1)})^i.\end{aligned}$$

Since $k > -1$, this infinite sum converges to

$$(1 - q^{-1}) \cdot \frac{q^{-(k+1)}}{1 - q^{-(k+1)}} = \frac{1}{q} \cdot \frac{q - 1}{q^{k+1} - 1}.$$

Hence we have Lemma 3.6 and the proof of Proposition 3.5 is completed. □

Remark 3.7. A curious reader may notice the similarity between the expression in Proposition 3.5 and Definition 2.4. This is the starting point of our proof of Theorem 1.1. However, to recover information of the Hodge numbers of an algebraic variety from the numbers of rational points valued in finite fields, we need some deep arithmetic results as in §4, §5.

Remark 3.8. As we easily see in the proof of Lemma 3.6, the p-adic integration

$$\int_{m^n} |x_1|_p^{k_1} \cdots |x_n|_p^{k_n} dx_1 \cdots dx_n$$

doesn't converge if $k_i \leq -1$ for some i. This is the reason why we assume singularities are at worst log-terminal in Theorem 1.1.

4. Local Galois representations

In this section, we recall some results on Galois representations over a p-adic field.

4.1. **Setting.** Let K be a number field. Let $\mathfrak{p}$ be a maximal ideal of $\mathcal{O}_K$. Let $K_\mathfrak{p}$ be a $\mathfrak{p}$-adic completion of K, $\mathcal{O}_{K_\mathfrak{p}}$ be the ring of integers of $K_\mathfrak{p}$, and $\mathbb{F}_q = \mathcal{O}_K/\mathfrak{p}$ be the residue field of $K_\mathfrak{p}$ with q elements.

We have an exact sequence

$$0 \longrightarrow I_{K_\mathfrak{p}} \longrightarrow \operatorname{Gal}(\overline{K_\mathfrak{p}}/K_\mathfrak{p}) \longrightarrow \operatorname{Gal}(\overline{\mathbb{F}_q}/\mathbb{F}_q) \longrightarrow 0,$$

where $I_{K_\mathfrak{p}}$ is called the *inertia group* at $\mathfrak{p}$. $\operatorname{Gal}(\overline{\mathbb{F}_q}/\mathbb{F}_q)$ is topologically generated by the q-th power Frobenius automorphism $x \mapsto x^q$ of $\overline{\mathbb{F}_q}$. The inverse of this automorphism is called the *geometric Frobenius element* at $\mathfrak{p}$ and denoted by $\mathrm{Frob}_{\mathfrak{p}}$.

Let X be a proper smooth variety over $K_{\mathfrak{p}}$, l be a prime number, and k be an integer. Then the absolute Galois group $\mathrm{Gal}(\overline{K_{\mathfrak{p}}}/K_{\mathfrak{p}})$ acts continuously on the l-adic étale cohomology group $H^k_{\text{ét}}(X_{\overline{K_{\mathfrak{p}}}}, \mathbb{Q}_l)$ of $X_{\overline{K_{\mathfrak{p}}}} = X \otimes_{K_{\mathfrak{p}}} \overline{K_{\mathfrak{p}}}$. In the followings, we recall some results on this $\mathrm{Gal}(\overline{K_{\mathfrak{p}}}/K_{\mathfrak{p}})$-representation in two cases.

4.2. **The Weil conjecture.** Firstly, we assume $\mathfrak{p} \notin S$ and $\mathfrak{p}$ doesn't divide l and there exists a proper smooth scheme $\mathfrak{X}$ over $\mathcal{O}_{K_{\mathfrak{p}}}$ such that $\mathfrak{X} \otimes_{\mathcal{O}_{K_{\mathfrak{p}}}} K_{\mathfrak{p}} = X$ ($\mathfrak{X}$ is called a *proper smooth model* of X over $\mathcal{O}_{K_{\mathfrak{p}}}$).

In this case, the action of $I_{K_{\mathfrak{p}}}$ on $H^k_{\text{ét}}(X_{\overline{K_{\mathfrak{p}}}}, \mathbb{Q}_l)$ is trivial (i.e. the action of $\mathrm{Gal}(\overline{K_{\mathfrak{p}}}/K_{\mathfrak{p}})$ is *unramified*) by the proper smooth base change theorem on étale cohomology. Therefore, the action of $\mathrm{Gal}(\overline{K_{\mathfrak{p}}}/K_{\mathfrak{p}})$ is determined by the action of $\mathrm{Frob}_{\mathfrak{p}}$.

By the Lefschetz trace formula for étale cohomology, we have

$$|\mathfrak{X}(\mathbb{F}_q)| = \sum_k (-1)^k \operatorname{Tr}(\mathrm{Frob}_{\mathfrak{p}}; H^k_{\text{ét}}(X_{\overline{K_{\mathfrak{p}}}}, \mathbb{Q}_l)).$$

Furthermore, the characteristic polynomial $P_k(t) = \det(1 - t \cdot \mathrm{Frob}_{\mathfrak{p}}; H^k_{\text{ét}}(X_{\overline{K_{\mathfrak{p}}}}, \mathbb{Q}_l))$ has integer coefficients and all complex absolute values of all conjugates of the roots of $P_k(t)$ are equal to $q^{-k/2}$ by the Weil conjecture proved by Deligne ([De2],[De3]).

4.3. ***p*-adic Hodge theory.** Secondly, we assume $\mathfrak{p}$ divides l. Let $p = l$ in this subsection to avoid confusion Here no assumption is needed for a model of X over $\mathcal{O}_{K_{\mathfrak{p}}}$. (see Remark 4.1)

In this case, the action of the inertia group $I_{K_{\mathfrak{p}}}$ on $H^k_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p)$ is highly nontrivial.

Let $\mathbb{C}_p$ be a p-adic completion of $\overline{K_{\mathfrak{p}}}$ on which $\mathrm{Gal}(\overline{K_{\mathfrak{p}}}/K_{\mathfrak{p}})$ acts continuously.

We recall the *Tate twists*. Let $\mathbb{Q}_p(0) = \mathbb{Q}_p$, $\mathbb{Q}_p(1) = \left(\varprojlim \mu_{p^n}\right) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. For $n \geq 1$, let $\mathbb{Q}_p(n) = \mathbb{Q}_p(1)^{\otimes n}$, $\mathbb{Q}_p(-n) = \mathrm{Hom}(\mathbb{Q}_p(n), \mathbb{Q}_p)$. Moreover, for a $\mathrm{Gal}(\overline{K_{\mathfrak{p}}}/K_{\mathfrak{p}})$-representation V over $\mathbb{Q}_p$, we define $V(n) = V \otimes_{\mathbb{Q}_p} \mathbb{Q}_p(n)$ on which $\mathrm{Gal}(\overline{K_{\mathfrak{p}}}/K_{\mathfrak{p}})$ acts diagonally.

p-adic Hodge theory claims the *Hodge-Tate decomposition* of X as follows :

$$\bigoplus_{i,j \text{ s.t. } i+j=k} H^j(X, \Omega^i_X) \otimes_K \mathbb{C}_p(-i) \cong H^k_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathbb{C}_p.$$

This is an isomorphism of $\mathrm{Gal}(\overline{K_\mathfrak{p}}/K_\mathfrak{p})$-representations, where $\mathrm{Gal}(\overline{K_\mathfrak{p}}/K_\mathfrak{p})$ acts on $H^i(X,\Omega^j_X)$ trivially and the right hand side diagonally. This is an p-adic analogue of the usual Hodge decomposition over $\mathbb{C}$.

As a corollary, we recover the Hodge numbers of X from its p-adic Galois representations as follows :

$$\dim_{K_\mathfrak{p}} H^j(X,\Omega^i_X) = \dim_{K_\mathfrak{p}} (H^{i+j}_{\text{ét}}(X_{\overline{K_\mathfrak{p}}},\mathbb{Q}_p)\otimes_{\mathbb{Q}_p}\mathbb{C}_p(i))^{\mathrm{Gal}(\overline{K_\mathfrak{p}}/K_\mathfrak{p})},$$

since $(\mathbb{C}_p)^{\mathrm{Gal}(\overline{K_\mathfrak{p}}/K_\mathfrak{p})} = K_\mathfrak{p}$ and $(\mathbb{C}_p(i))^{\mathrm{Gal}(\overline{K_\mathfrak{p}}/K_\mathfrak{p})} = 0$ for all $i \neq 0$ (see [Ta], Theorem 2).

Remark 4.1. A proof of Hodge-Tate decomposition was given by by Faltings ([Fa]). Tsuji gave another proof by reducing to the semi-stable reduction case by de Jong's alteration ([Ts]). However, in this paper, we don't need the full version of the Hodge-Tate decomposition. For example, the result of Fontaine-Messing is enough for us ([FM]). They proved the Hodge-Tate decomposition when $K_\mathfrak{p}$ is unramified over $\mathbb{Q}_p$, $\dim X < p$, and X has a proper smooth model over $\mathcal{O}_{K_\mathfrak{p}}$.

Remark 4.2. Moreover, by Lemma 4.3 below, we see that the semisimplification of the p-adic Galois representation determines the Hodge numbers by the same formula :

$$\dim_{K_\mathfrak{p}} H^j(X,\Omega^i_X) = \dim_{K_\mathfrak{p}} (H^{i+j}_{\text{ét}}(X_{\overline{K_\mathfrak{p}}},\mathbb{Q}_p)^{ss}\otimes_{\mathbb{Q}_p}\mathbb{C}_p(i))^{\mathrm{Gal}(\overline{K_\mathfrak{p}}/K_\mathfrak{p})},$$

where ss denotes the semisimplification as a $\mathrm{Gal}(\overline{K_\mathfrak{p}}/K_\mathfrak{p})$-representation. This is a simple but important observation to consider the Hodge-Tate decomposition on the level of Grothendieck groups of Galois representations in §5.

Lemma 4.3 ([It1], Corollary 6.3)**.** *Let $0 \to V_1 \to V_2 \to V_3 \to 0$ be an exact sequence of finite dimensional $\mathrm{Gal}(\overline{K_\mathfrak{p}}/K_\mathfrak{p})$-representations over $\mathbb{Q}_p$. We define $h^n(V_i) := \dim(V_i\otimes_{\mathbb{Q}_p}\mathbb{C}_p(n))^{\mathrm{Gal}(\overline{K_\mathfrak{p}}/K_\mathfrak{p})}$ for $i = 1,2,3$ and an integer n. Assume that $\dim V_2 = \sum_n h^n(V_2)$. Then, we have $\dim V_1 = \sum_n h^n(V_1)$, $\dim V_3 = \sum_n h^n(V_3)$ and $h^n(V_2) = h^n(V_1) + h^n(V_3)$ for all n.*

Proof. This lemma seems well-known for specialists. However, we write the proof for reader's convenience. In general, we have an inequality $\sum_n h^n(V_i) \leq \dim V_i$ for $i = 1,3$ (for example, see [Fo]). We shall prove these inequalities are in fact equalities. Since the functor taking $\mathrm{Gal}(\overline{K_\mathfrak{p}}/K_\mathfrak{p})$-invariant is left exact,

$$\begin{aligned} 0 \longrightarrow (V_1\otimes_{\mathbb{Q}_p}\mathbb{C}_p(n))^{\mathrm{Gal}(\overline{K_\mathfrak{p}}/K_\mathfrak{p})} &\longrightarrow (V_2\otimes_{\mathbb{Q}_p}\mathbb{C}_p(n))^{\mathrm{Gal}(\overline{K_\mathfrak{p}}/K_\mathfrak{p})} \\ &\longrightarrow (V_3\otimes_{\mathbb{Q}_p}\mathbb{C}_p(n))^{\mathrm{Gal}(\overline{K_\mathfrak{p}}/K_\mathfrak{p})} \end{aligned}$$

is exact. Therefore $h^n(V_2) \leq h^n(V_1) + h^n(V_3)$ for all n. Then we have

$$\begin{aligned} \dim V_2 &= \sum_n h^n(V_2) \leq \sum_n h^n(V_1) + \sum_n h^n(V_3) \leq \dim V_1 + \dim V_3 \\ &= \dim V_2 \end{aligned}$$

and hence Lemma 4.3. $\square$

5. Global Galois representations

In this section, we recall some results on Galois representations over a number field.

5.1. **An application of the Chebotarev density theorem.**

The following proposition is very important to work on the level of a Grothendieck group of Galois representations over a number field. This is an application of the Chebotarev density theorem in algebraic number theory.

Proposition 5.1 ([Se], I.2.3)**.** *Let K be a number field and l be a prime number. Let V, V' be two continuous l-adic $\mathrm{Gal}(\overline{K}/K)$-representations such that they are unramified outside a finite set S of maximal ideals of $\mathcal{O}_K$ and satisfy*

$$\mathrm{Tr}(\mathrm{Frob}_{\mathfrak{p}}; V) = \mathrm{Tr}(\mathrm{Frob}_{\mathfrak{p}}; V') \qquad \textit{for all} \quad \mathfrak{p} \notin S.$$

Then V and V' have the same semisimplifications as $\mathrm{Gal}(\overline{K}/K)$-representations.

Proof. We only sketch the proof (for details, see [Se]). By the representation theory of a group over a field of characteristic 0 (see, for example, Bourbaki, *Algèbre*, Ch. 8, §12, n° 1, Prop 3.), the semisimplification of a $\mathrm{Gal}(\overline{K}/K)$-representation is determined by the traces of all elements in $\mathrm{Gal}(\overline{K}/K)$. Roughly speaking, the Chebotarev density theorem claims the set of conjugates of $\mathrm{Frob}_{\mathfrak{p}}$ for $\mathfrak{p} \notin S$ is dense in $\mathrm{Gal}(\overline{K}/K)$. Since V and V' are continuous representations, the equality of the traces of all $\mathrm{Frob}_{\mathfrak{p}}$ for $\mathfrak{p} \notin S$ implies the equality of traces of all elements in $\mathrm{Gal}(\overline{K}/K)$. Hence we have Proposition 5.1. □

5.2. **Some Grothendieck groups of Galois representations.**

Let K be a number field. Let S be a finite set of maximal ideals of $\mathcal{O}_K$. We fix a prime number $l = p$ and a maximal ideal $\mathfrak{p}$ dividing p. For every maximal ideal $\mathfrak{q}$ of $\mathcal{O}_K$, we fix an inclusion $\overline{K} \hookrightarrow \overline{K_{\mathfrak{q}}}$. Then we consider $\mathrm{Gal}(\overline{K_{\mathfrak{q}}}/K_{\mathfrak{q}}) \subset \mathrm{Gal}(\overline{K}/K)$ for all $\mathfrak{q}$.

Definition 5.2. Let $K(l, S, \mathfrak{p})$ be an abelian group generated by $\mathrm{Gal}(\overline{K}/K)$-representations V satisfying the following conditions modulo an equivalence relation $\sim$ generated by $[V_1] + [V_3] \sim [V_2]$ for an exact sequence $0 \to V_1 \to V_2 \to V_3 \to 0$:

1. (*unramifiedness* outside S) For $\mathfrak{q} \notin S$, $I_{K_{\mathfrak{q}}}$ acts on V trivially.
2. (*weight filtration* outside S) There exists a unique increasing filtration W on V indexed by integers satisfying the following conditions :
   (a) $W_k V = 0$ for $k << 0$, $W_k V = V$ for $k >> 0$.
   (b) For every integer k and $\mathfrak{q} \notin S$, the characteristic polynomial $P_k(t) = \det(1 - t \cdot \mathrm{Frob}_{\mathfrak{q}}; \mathrm{Gr}_k^W V)$ has integer coefficients and all complex absolute values of all conjugates of the roots of $P_k(t)$ are equal to $|\mathcal{O}_K/\mathfrak{q}|^{-k/2}$.

   W is called the *weight filtration* on V.

3. (*Hodge-Tate decomposition* at $\mathfrak{p}$) For integers i, j, we define

$$h^{i,j}_{\mathfrak{p}}(V) = \dim_{K_{\mathfrak{p}}}(\mathrm{Gr}^W_{i+j} V \otimes_{\mathbb{Q}_p} \mathbb{C}_p(i))^{\mathrm{Gal}(\overline{K_{\mathfrak{p}}}/K_{\mathfrak{p}})}.$$

Then these numbers satisfy $\sum_{i,j} h^{i,j}_{\mathfrak{p}}(V) = \dim_{\mathbb{Q}_p} V$.

$K(l, S, \mathfrak{p})$ is called the *Grothendieck group* of l-adic $\mathrm{Gal}(\overline{K}/K)$-representations which is unramified outside S, has weight filtration, and has Hodge-Tate decomposition at $\mathfrak{p}$. Let $[V]$ denote the class of V in $K(l, S, \mathfrak{p})$. An element in $K(l, S, \mathfrak{p})$ is called a *virtual* $\mathrm{Gal}(\overline{K}/K)$*-representation*.

Similarly, for an integer k, we define $K(l, S, \mathfrak{p}, k)$ as a subgroup of $K(l, S, \mathfrak{p})$ generated by V satisfying $\mathrm{Gr}^W_k V = V$. $K(l, S, \mathfrak{p}, k)$ is called the Grothendieck group of l-adic $\mathrm{Gal}(\overline{K}/K)$-representations which is unramified outside S, has *weight* k, and has Hodge-Tate decomposition at $\mathfrak{p}$.

By the Jordan-Hölder theorem, $K(l, S, \mathfrak{p})$ is a free abelian group generated by simple $\mathrm{Gal}(\overline{K}/K)$-representations in $K(l, S, \mathfrak{p})$. Since $[V] = \sum_k [\mathrm{Gr}^W_k V]$ in $K(l, S, \mathfrak{p})$, a simple $\mathrm{Gal}(\overline{K}/K)$-representation has only one weight. Therefore we have a direct sum decomposition as follows :

$$K(l, S, \mathfrak{p}) = \bigoplus_{k \in \mathbb{Z}} K(l, S, \mathfrak{p}, k).$$

We define a ring structure on $K(l, S, \mathfrak{p})$ by extending the tensor product $[V_1] \cdot [V_2] = [V_1 \otimes V_2]$. Then $K(l, S, \mathfrak{p})$ has a structure of a graded ring by the direct sum decomposition as above.

Definition 5.3. For a $\mathrm{Gal}(\overline{K}/K)$-representation V in $K(l, S, \mathfrak{p})$, we define the *p-adic E-function* of V as follows

$$E_{\mathfrak{p}}(V; u, v) = \sum_{i,j} h^{i,j}_{\mathfrak{p}}(V)\, u^i v^j.$$

Remark 5.4. It is easy to see that p-adic E-function satisfies the following properties (see Remark 4.2) :

1. For an exact sequence $0 \to V_1 \to V_2 \to V_3 \to 0$, we have $E_{\mathfrak{p}}(V_1; u, v) + E_{\mathfrak{p}}(V_3; u, v) = E_{\mathfrak{p}}(V_2; u, v)$.
2. For V_1, V_2, we have $E_{\mathfrak{p}}(V_1 \otimes V_2; u, v) = E_{\mathfrak{p}}(V_1; u, v) \cdot E_{\mathfrak{p}}(V_2; u, v)$.

Therefore, we can extend p-adic E-function to a ring homomorphism

$$E_{\mathfrak{p}} : K(l, S, \mathfrak{p}) \to \mathbb{Z}[u, v].$$

5.3. **A variant — Galois represenations with fractional weight filtration.** Let d be an integer. Here we introduce a variant of $K(l, S, \mathfrak{p})$ whose weight filtration is indexed by elements of $\frac{1}{d}\mathbb{Z}$ instead of $\mathbb{Z}$. This generalization is necessary to treat the $\mathbb{Q}$-Gorenstein case in the proof of Theorem 1.1.

Firstly, we introduce the *fractional Tate twists* $\mathbb{Q}_p(a)$ ($a \in \frac{1}{d}\mathbb{Z}$) as follows (for usual Tate twists, see §4.3). Let L be a field and p be a prime number. $\mathbb{Q}_p(1)$ is a one dimensional $\mathrm{Gal}(\overline{L}/L)$-representation

$$\rho : \mathrm{Gal}(\overline{L}/L) \to \mathrm{GL}(\mathbb{Q}_p(1)) \cong \mathrm{GL}(1, \mathbb{Q}_p) = \mathbb{Q}_p^\times$$

whose image is contained in $\mathbb{Z}_p^\times$. There exist open subgroups $U \subset \mathbb{Z}_p^\times$, $V \subset \mathbb{Z}_p$ on which $\log : U \to V$ and $\exp : V \to U$ converge. Therefore, if we replace L by its finite extension,

$$\rho_{1/d} : \mathrm{Gal}(\overline{L}/L) \ni \sigma \mapsto \exp\left(\frac{1}{d}\log(\sigma)\right) \in \mathbb{Q}_p^\times$$

is a one dimensional $\mathrm{Gal}(\overline{L}/L)$-representation denoted by $\mathbb{Q}_p(\frac{1}{d})$. Then $\mathbb{Q}_p(\frac{1}{d})^{\otimes d} \cong \mathbb{Q}_p(1)$. For $n \in \mathbb{Z},\ n \geq 1$, we define $\mathbb{Q}_p(\frac{n}{d}) = \mathbb{Q}_p(\frac{1}{d})^{\otimes n},\ \mathbb{Q}_p(-\frac{n}{d}) = \mathrm{Hom}(\mathbb{Q}_p(\frac{n}{d}), \mathbb{Q}_p)$. If L is a finite extension of $\mathbb{Q}_p$, we can similarly define $\mathbb{C}_p(a)$ ($a \in \frac{1}{d}\mathbb{Z}$) as in §4.3.

We define $K(l, S, \mathfrak{p})_{1/d}$ as follows. Let notation be the same as in §5.2. $K(l, S, \mathfrak{p})_{1/d}$ is an abelian group generated by $\mathrm{Gal}(\overline{K}/K)$-representations V satisfying the following conditions modulo an equivalence relation $\sim$ as in Definition 5.2 :

1. The conditions 1, 2 in Definition 5.2 but we allow k is an element of $\frac{1}{d}\mathbb{Z}$ instead of $\mathbb{Z}$.
2. Let L be a finite extension of $K_\mathfrak{p}$ such that $\mathbb{Q}_p(\frac{1}{d})$ exists as a $\mathrm{Gal}(\overline{L}/L)$-representation. For $i, j \in \frac{1}{d}\mathbb{Z}$, we define
$$h_\mathfrak{p}^{i,j}(V) = \dim_L(\mathrm{Gr}^W_{i+j} V \otimes_{\mathbb{Q}_p} \mathbb{C}_p(i))^{\mathrm{Gal}(\overline{L}/L)}.$$
Then these numbers satisfy $\sum_{i,j \in \frac{1}{d}\mathbb{Z}} h_\mathfrak{p}^{i,j}(V) = \dim_{\mathbb{Q}_p} V$. It is easy to see that this condition is independent of the choice of L.

Similarly, we can define $K(l, S, \mathfrak{p}, k)_{1/d}$ as in §5.2. We have a direct sum decomposition as follows :

$$K(l, S, \mathfrak{p})_{1/d} = \bigoplus_{k \in \frac{1}{d}\mathbb{Z}} K(l, S, \mathfrak{p}, k)_{1/d}.$$

$K(l, S, \mathfrak{p})_{1/d}$ has a ring structure. Moreover, we can define p-adic E-function $E_\mathfrak{p}(V; u, v) \in \mathbb{Z}[u^{1/d}, v^{1/d}]$ for a $\mathrm{Gal}(\overline{K}/K)$-representation V in $K(l, S, \mathfrak{p})_{1/d}$. We can extend this to a ring homomorphism

$$E_\mathfrak{p} : K(l, S, \mathfrak{p})_{1/d} \to \mathbb{Z}[u^{1/d}, v^{1/d}].$$

Example 5.5. Assume that $\mathbb{Q}_p(\frac{1}{d})$ exists as a $\mathrm{Gal}(\overline{K}/K)$-representation. Note that this is satisfied if we replace K by its finite extension. For $n \in \mathbb{Z}$, $\mathbb{Q}_p(\frac{n}{d})$ is in $K(l, S, \mathfrak{p}, \frac{-2n}{d})_{1/d}$ such that $E_\mathfrak{p}(\mathbb{Q}_p(\frac{n}{d}); u, v) = u^{-n/d}v^{-n/d}$.

6. Conclusion — the number of $\mathbb{F}_q$-rational points, Galois representations, Hodge numbers

We combine the results in §4 and §5. Let notation be the same as in §5.2.

Let X be a proper smooth variety over K which has a proper smooth model $\mathfrak{X}$ over $(\operatorname{Spec} \mathcal{O}_K)\backslash S$. Then $H^k_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_l)$ is a $\operatorname{Gal}(\overline{K}/K)$-representation in $K(l, S, \mathfrak{p}, k)$ by the Weil conjecture and p-adic Hodge theory (§4). Let $X_{\overline{K}} = X \otimes_K \overline{K}$ and $X_{\mathbb{C}} = X \otimes_K \mathbb{C}$. We define a virtual representation

$$[H^*_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_l)] := \sum_k (-1)^k [H^k_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_l)]$$

as an element in $K(l, S, \mathfrak{p})$. Then we have the equality of two E-functions

$$E(X_{\mathbb{C}}; u, v) = E_{\mathfrak{p}}([H^*_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_l)]; u, v)$$

by comparing the Hodge decomposition of $X_{\mathbb{C}}$ and the Hodge-Tate decomposition of $X_{K_{\mathfrak{p}}} = X \otimes_K K_{\mathfrak{p}}$.

6.1. The proper smooth case.

By combining results in §4 and §5, we have the following results which connects the number of rational points and the Hodge numbers.

Proposition 6.1. *Let X be a proper smooth variety over K which has a proper smooth model $\mathfrak{X}$ over $(\operatorname{Spec} \mathcal{O}_K)\backslash S$. Then we have*

$$|\mathfrak{X}(\mathcal{O}_K/\mathfrak{p})| = \operatorname{Tr}(\operatorname{Frob}_{\mathfrak{p}}; [H^*_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_l)]) \qquad \textit{for all} \quad \mathfrak{p} \notin S.$$

Proof. This follows from the Lefschetz trace formula for étale cohomology as in §4.2. □

Corollary 6.2. *Let X (resp. Y) be a proper smooth varieties over K which has a proper smooth model $\mathfrak{X}$ (resp. $\mathfrak{Y}$) over $(\operatorname{Spec} \mathcal{O}_K)\backslash S$. If $|\mathfrak{X}(\mathcal{O}_K/\mathfrak{p})| = |\mathfrak{Y}(\mathcal{O}_K/\mathfrak{p})|$ for all $\mathfrak{p} \notin S$, then $[H^*_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_l)] = [H^*_{\text{ét}}(Y_{\overline{K}}, \mathbb{Q}_l)]$ in $K(l, S, \mathfrak{p})$. Therefore, we have*

$$E(X_{\mathbb{C}}; u, v) = E_{\mathfrak{p}}([H^*_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_l)]; u, v) = E_{\mathfrak{p}}([H^*_{\text{ét}}(Y_{\overline{K}}, \mathbb{Q}_l)]; u, v) = E(Y_{\mathbb{C}}; u, v).$$

Namely, the Hodge numbers of $X_{\mathbb{C}}$ and $Y_{\mathbb{C}}$ are equal.

Proof. By Proposition 6.1, we have

$$\operatorname{Tr}(\operatorname{Frob}_{\mathfrak{p}}; [H^*_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_l)]) = \operatorname{Tr}(\operatorname{Frob}_{\mathfrak{p}}; [H^*_{\text{ét}}(Y_{\overline{K}}, \mathbb{Q}_l)]) \qquad \text{for all} \quad \mathfrak{p} \notin S.$$

Hence we have $[H^*_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_l)] = [H^*_{\text{ét}}(Y_{\overline{K}}, \mathbb{Q}_l)]$ in $K(l, S, \mathfrak{p})$ by Proposition 5.1. □

6.2. **A generalization — the open smooth case.** Next we generalize Proposition 6.1 to open smooth varieties by the methods of Deligne in [De4],[De5].

Let X be a smooth variety over K of dimension n which is not necessarily proper. Assume that there exists a proper smooth variety $\overline{X} \supset X$ over K such that $\overline{X} \backslash X = \bigcup_{i=1}^{r} D_i$ is a simple normal crossing divisor on $\overline{X}$. Let $I = \{1, \dots, r\}$, $D_J = \bigcap_{j \in J} D_j$ for a nonempty subset $J \subset I$, and $D_{\emptyset} = \overline{X}$. We define

$$[H^*_{c,\text{ét}}(X_{\overline{K}}, \mathbb{Q}_l)] := \sum_k (-1)^k [H^k_{c,\text{ét}}(X_{\overline{K}}, \mathbb{Q}_l)],$$

where $H^k_{c,\text{ét}}$ denotes étale cohomology with compact supports.

Lemma 6.3. *Let X be as above. Then we have*

$$E(X_{\mathbb{C}}; u, v) = \sum_{J \subset I} (-1)^{|J|} E((D_J)_{\mathbb{C}}; u, v) \quad \text{in } \mathbb{Z}[u, v],$$
$$[H^*_{c,\text{ét}}(X_{\overline{K}}, \mathbb{Q}_l)] = \sum_{J \subset I} (-1)^{|J|} [H^*_{\text{ét}}((D_J)_{\overline{K}}, \mathbb{Q}_l)] \quad \text{in } K(l, S, \mathfrak{p}).$$

Furthermore, we have the equality of two E-functions for X:

$$E(X_{\mathbb{C}}; u, v) = E_{\mathfrak{p}}([H^*_{c,\text{ét}}(X_{\overline{K}}, \mathbb{Q}_l)]; u, v).$$

Proof. Since D_J is a proper smooth variety over K, we have $E((D_J)_{\mathbb{C}}; u, v) = E_{\mathfrak{p}}([H^*_{\text{ét}}((D_J)_{\overline{K}}, \mathbb{Q}_l)]; u, v)$. Therefore, the second assertion immediately follows from the first assertion.

We only prove the first assertion for $E(X_{\mathbb{C}}; u, v)$ since we can prove the case of $[H^*_{c,\text{ét}}(X_{\overline{K}}, \mathbb{Q}_l)]$ by the same way. The Leray spectral sequence for the inclusion $X_{\mathbb{C}} \hookrightarrow \overline{X}_{\mathbb{C}}$ induces a spectral sequence

$$E_2^{i,j} = \bigoplus_{J \subset I \text{ s.t. } |J| = j} H^i((D_J)_{\mathbb{C}}, \mathbb{Q})(-j) \Rightarrow H^{i+j}(X_{\mathbb{C}}, \mathbb{Q}),$$

which defines the canonical mixed Hodge structure on $H^k(X_{\mathbb{C}}, \mathbb{Q})$ ([De4],[De5], 3.2). For a finite dimensional $\mathbb{Q}$-vector space V with mixed Hodge structure, we define $E(V; u, v) = \sum_{i,j} h^{i,j}(\mathrm{Gr}^W_{i+j} V)\, u^i v^j$. Note that

$$E(X_{\mathbb{C}}; u, v) = \sum_{k=1}^{2n} (-1)^k E(H^k_c(X_{\mathbb{C}}, \mathbb{Q}); u, v)$$

by definition. By the above spectral sequence, we have

$$\sum_{i,j} (-1)^{i+j} E(E_2^{i,j}; u, v) = \sum_{k=1}^{2n} (-1)^k E(H^k(X_{\mathbb{C}}, \mathbb{Q}); u, v).$$

By Poincaré duality, $H^k(X_{\mathbb{C}}, \mathbb{Q})$ is dual to $H^{2n-k}_c(X_{\mathbb{C}}, \mathbb{Q})(n)$. Then we have

$$E(H^k(X_{\mathbb{C}}, \mathbb{Q}); u, v) = (uv)^n E(H^{2n-k}_c(X_{\mathbb{C}}, \mathbb{Q}); u^{-1}, v^{-1}).$$

On the other hand, since D_J is a proper smooth variety of dimension $n - |J|$, $H^i((D_J)_{\mathbb{C}}, \mathbb{Q})(-|J|)$ is dual to $H^{2n-2|J|-i}((D_J)_{\mathbb{C}}, \mathbb{Q})(n)$ by Poincaré duality. Hence we have

$$E(E_2^{i,j}; u, v) = \sum_{|J|=j} (uv)^n E(H^{2n-2|J|-i}((D_J)_{\mathbb{C}}, \mathbb{Q}); u^{-1}, v^{-1}).$$

By combining them, we have Lemma 6.3. $\square$

Next we consider the numbers of rational points valued in finite fields. For open smooth varieties, we have the following generalization of Proposition 6.1

Proposition 6.4. *Let X be a smooth variety over K. Assume that there exist a proper smooth scheme $\overline{\mathfrak{X}}$ over $(\operatorname{Spec} \mathcal{O}_K)\backslash S$ and an open subscheme $\mathfrak{X} \subset \overline{\mathfrak{X}}$ whose generic fiber is X such that $\overline{\mathfrak{X}}\backslash\mathfrak{X} = \bigcup_{i=1}^r \mathfrak{D}_i$ is a relative simple normal crossing divisor on $\overline{\mathfrak{X}}$ over $(\operatorname{Spec} \mathcal{O}_K)\backslash S$ (see Definition 3.4). Then we have*

$$|\mathfrak{X}(\mathcal{O}_K/\mathfrak{p})| = \operatorname{Tr}(\operatorname{Frob}_{\mathfrak{p}}; [H^*_{c,\text{ét}}(X_{\overline{K}}, \mathbb{Q}_l)]) \qquad \textit{for all} \quad \mathfrak{p} \notin S.$$

Proof. Let $I = \{1, \ldots, r\}$, $\mathfrak{D}_J = \bigcap_{j\in J} \mathfrak{D}_j$ for a nonempty subset $J \subset I$, and $\mathfrak{D}_\emptyset = \overline{\mathfrak{X}}$. Then, by inclusion-exclusion principle, we have

$$|\mathfrak{X}(\mathcal{O}_K/\mathfrak{p})| = \sum_{J\subset I} (-1)^{|J|} |\mathfrak{D}_J(\mathcal{O}_K/\mathfrak{p})| \qquad \text{for all} \quad \mathfrak{p} \notin S.$$

Since $\mathfrak{D}_J$ is proper and smooth over $(\operatorname{Spec} \mathcal{O}_K)\backslash S$, we have

$$|\mathfrak{D}_J(\mathcal{O}_K/\mathfrak{p})| = \operatorname{Tr}(\operatorname{Frob}_{\mathfrak{p}}; [H^*_{\text{ét}}((D_J)_{\overline{K}}, \mathbb{Q}_l)])$$

by Proposition 6.1. On the other hand, we have

$$\operatorname{Tr}(\operatorname{Frob}_{\mathfrak{p}}; [H^*_{c,\text{ét}}(X_{\overline{K}}, \mathbb{Q}_l)]) = \sum_{J\subset I} (-1)^{|J|} \operatorname{Tr}(\operatorname{Frob}_{\mathfrak{p}}; [H^*_{\text{ét}}((D_J)_{\overline{K}}, \mathbb{Q}_l)])$$

by Lemma 6.3. By combining them, we have Proposition 6.4. $\square$

We have the following generalization of Corollary 6.2.

Corollary 6.5. *Let X (resp. Y) be a proper smooth variety over K satisfying the assumptions in Proposition 6.4. Let $\mathfrak{X}$ (resp. $\mathfrak{Y}$) be a proper smooth model of X (resp. Y) over $(\operatorname{Spec} \mathcal{O}_K)\backslash S$. If $|\mathfrak{X}(\mathcal{O}_K/\mathfrak{p})| = |\mathfrak{Y}(\mathcal{O}_K/\mathfrak{p})|$ for all $\mathfrak{p} \notin S$, then $[H^*_{c,\text{ét}}(X_{\overline{K}}, \mathbb{Q}_l)] = [H^*_{c,\text{ét}}(Y_{\overline{K}}, \mathbb{Q}_l)]$ in $K(l, S, \mathfrak{p})$. Therefore we have*

$$E(X_{\mathbb{C}}; u, v) = E_{\mathfrak{p}}([H^*_{c,\text{ét}}(X_{\overline{K}}, \mathbb{Q}_l)]; u, v) = E_{\mathfrak{p}}([H^*_{c,\text{ét}}(Y_{\overline{K}}, \mathbb{Q}_l)]; u, v) = E(Y_{\mathbb{C}}; u, v).$$

Proof. The proof is the same as Corollary 6.2. By Proposition 6.4, we have

$$\operatorname{Tr}(\operatorname{Frob}_{\mathfrak{p}}; [H^*_{c,\text{ét}}(X_{\overline{K}}, \mathbb{Q}_l)]) = \operatorname{Tr}(\operatorname{Frob}_{\mathfrak{p}}; [H^*_{c,\text{ét}}(Y_{\overline{K}}, \mathbb{Q}_l)]) \qquad \text{for all} \quad \mathfrak{p} \notin S.$$

Hence we have $[H^*_{c,\text{ét}}(X_{\overline{K}}, \mathbb{Q}_l)] = [H^*_{c,\text{ét}}(Y_{\overline{K}}, \mathbb{Q}_l)]$ in $K(l, S, \mathfrak{p})$ by Proposition 5.1. The equality of two E-functions follows from Lemma 6.3. $\square$

7. Proof of the main theorem

Lemma 7.1. *Let $f : \mathfrak{X} \to T$ be a proper smooth morphism of schemes of characteristic 0 and $\mathfrak{D} = \bigcup_{i=1}^{r} \mathfrak{D}_i$ be a relative simple normal crossing divisor on $\mathfrak{X}$ over T (see Definition 3.4). Assume that T is connected. Then, all fibers of $\mathfrak{X} \backslash \mathfrak{D} \to T$ have the same E-functions defined in Definition 2.1.*

Proof. For a nonempty subset $J \subset \{1, \dots, r\}$, $\mathfrak{D}_J = \bigcap_{j \in J} \mathfrak{D}_j$ is proper and smooth over T. Hence by a theorem of Deligne ([De1], 5.5), the Hodge numbers of all fibers of $\mathfrak{D}_J \to T$ are the same. On the other hand, the E-function of a fiber of $\mathfrak{X} \backslash \mathfrak{D} \to T$ can be computed from the Hodge numbers of a fiber of $\mathfrak{D}_J \to T$ by Lemma 6.3. Therefore, we have Lemma 7.1. $\square$

Proof of Theorem 1.1. Let $\rho : Y \to X$ and $\rho' : Y' \to X$ be as in Theorem 1.1. Let n be the dimension of X. To avoid confusion, here $E_{st}(X; u, v)_\rho$ (resp. $E_{st}(X; u, v)_{\rho'}$) denotes the stringy E-function of X defined by $\rho : Y \to X$ (resp. $\rho' : Y' \to X$) as in Definition 2.4. We shall prove the equality $E_{st}(X; u, v)_\rho = E_{st}(X; u, v)_{\rho'}$.

Step 1. Let $f : Y \dashrightarrow Y'$ be a birational map between Y and Y' over X. Let Z be a resolution of singularities of the closure of the graph of f such that the exceptional divisor of $Z \to X$ is a simple normal crossing divisor. Let $\tau : Z \to Y$ and $\tau' : Z \to Y'$ be a natural morphism. Then we have the following commutative diagram.

$$\begin{tikzcd}
& Z \arrow[dl, "\tau"'] \arrow[dr, "\tau'"] & \\
Y \arrow[rr, dashed, "f"] \arrow[dr, "\rho"'] & & Y' \arrow[dl, "\rho'"] \\
& X &
\end{tikzcd}$$

If we prove $E_{st}(X; u, v)_\rho = E_{st}(X; u, v)_{\rho \circ \tau}$ and $E_{st}(X; u, v)_{\rho'} = E_{st}(X; u, v)_{\rho' \circ \tau'}$, then we have Theorem 1.1 since $\rho \circ \tau = \rho' \circ \tau'$. In the followings, we only prove $E_{st}(X; u, v)_\rho = E_{st}(X; u, v)_{\rho \circ \tau}$ since we can similarly prove $E_{st}(X; u, v)_{\rho'} = E_{st}(X; u, v)_{\rho' \circ \tau'}$.

Step 2. We shall show that we may assume everything is defined over a number field. Since X, Y, Z, ρ, τ are defined over a subfield K' of $\mathbb{C}$ which is finitely generated over $\mathbb{Q}$. Therefore, there exist an irreducible variety T over a number field K such that the function field of T is K'. Furthermore, there exist a proper scheme $\widetilde{X}$, proper smooth schemes $\widetilde{Y}, \widetilde{Z}$, and proper birational morphisms $\tilde{\rho} : \widetilde{Y} \to \widetilde{X}$, $\tilde{\tau} : \widetilde{Z} \to \widetilde{Y}$ over T such that $\widetilde{X} \times_T \mathbb{C} = X$, $\widetilde{Y} \times_T \mathbb{C} = Y$, $\widetilde{Z} \times_T \mathbb{C} =$

Z, $\tilde{\rho} \times_T \mathbb{C} = \rho$, $\tilde{\tau} \times_T \mathbb{C} = \tau$. We write

$$K_Y = \rho^* K_X + \sum_{i=1}^{r} a_i D_i, \qquad K_Z = (\rho \circ \tau)^* K_X + \sum_{j=1}^{s} b_j E_j$$

with $a_i \in \mathbb{Q}$, $a_i > -1$, $b_j \in \mathbb{Q}$, $b_j > -1$. Let d be a positive integer such that $(K_X)^{\otimes d}$ is a Cartier divisor on X. Then a_i, b_j are elements of $\frac{1}{d}\mathbb{Z}$.

By replacing T by its Zariski open subset, $(K_X)^{\otimes d}$ extends to a Cartier divisor $(\Omega^n_{\widetilde{X}/T})^{\otimes d}$ on $\widetilde{X}$ over T, and we can write

$$\Omega^n_{\widetilde{Y}/T} = \tilde{\rho}^* \Omega^n_{\widetilde{X}/T} + \sum_{i=1}^{r} a_i \widetilde{D}_i, \qquad \Omega^n_{\widetilde{Z}/T} = (\tilde{\rho} \circ \tilde{\tau})^* \Omega^n_{\widetilde{X}/T} + \sum_{j=1}^{s} b_j \widetilde{E}_j,$$

where $\widetilde{D} = \bigcup_{i=1}^{r} \widetilde{D}_i$ (resp. $\widetilde{E} = \bigcup_{j=1}^{s} \widetilde{E}_j$) is a relative simple normal crossing divisor on $\widetilde{Y}$ (resp. $\widetilde{Z}$) over T (see Definition 3.4). By replacing K by its finite extension, there exists a K-rational point $t \in T(K)$. Let $\widetilde{X}_t, \widetilde{Y}_t, \widetilde{Z}_t, \tilde{\rho}_t, \tilde{\tau}_t$ be the fibers at t. Then $E_{st}(X; u, v)_\rho = E_{st}(\widetilde{X}_t; u, v)_{\tilde{\rho}_t}$ and $E_{st}(X; u, v)_{\rho \circ \tau} = E_{st}(\widetilde{X}_t; u, v)_{\tilde{\rho}_t \circ \tilde{\tau}_t}$ by Lemma 7.1. By replacing X, Y, Z, ρ, τ by $\widetilde{X}_t, \widetilde{Y}_t, \widetilde{Z}_t, \tilde{\rho}_t, \tilde{\tau}_t$, we may assume X, Y, Z, ρ, τ are defined over a number field K.

Step 3. By the same argument as above, there exist a finite set S of maximal ideals of $\mathcal{O}_K$, a proper scheme $\mathfrak{X}$, proper smooth schemes $\mathfrak{Y}, \mathfrak{Z}$, and proper birational morphisms $\bar{\rho} : \mathfrak{Y} \to \mathfrak{X}$, $\bar{\tau} : \mathfrak{Z} \to \mathfrak{Y}$ over $\mathfrak{T} = (\operatorname{Spec} \mathcal{O}_K) \backslash S$ such that generic fibers of $\mathfrak{X}, \mathfrak{Y}, \mathfrak{Z}, \bar{\rho}, \bar{\tau}$ are X, Y, Z, ρ, τ. By enlarging S, $(K_X)^{\otimes d}$ extends to a Cartier divisor $(\Omega^n_{\mathfrak{X}/\mathfrak{T}})^{\otimes d}$ on $\mathfrak{X}$ over $\mathfrak{T}$ and we can write

$$\Omega^n_{\mathfrak{Y}/\mathfrak{T}} = \bar{\rho}^* \Omega^n_{\mathfrak{X}/\mathfrak{T}} + \sum_{i=1}^{r} a_i \mathfrak{D}_i, \qquad \Omega^n_{\mathfrak{Z}/\mathfrak{T}} = (\bar{\rho} \circ \bar{\tau})^* \Omega^n_{\mathfrak{X}/\mathfrak{T}} + \sum_{j=1}^{s} b_j \mathfrak{E}_j,$$

where $\mathfrak{D} = \bigcup_{i=1}^{r} \mathfrak{D}_i$ (resp. $\mathfrak{E} = \bigcup_{j=1}^{s} \mathfrak{E}_j$) is a relative simple normal crossing divisor on $\mathfrak{Y}$ (resp. $\mathfrak{Z}$) over $\mathfrak{T}$.

Step 4. Here we compute p-adic integration. Take a maximal ideal $\mathfrak{q} \notin S$. Let $K_\mathfrak{q}$ be a $\mathfrak{q}$-adic completion of K and $\mathcal{O}_{K_\mathfrak{q}}$ be the ring of integers of $K_\mathfrak{q}$. Let $q = |\mathcal{O}_K/\mathfrak{q}|$ be the number of elements of the residue field.

Let $\mathfrak{U}_1, \ldots, \mathfrak{U}_k$ be a finite open covering of $\mathfrak{X}$ over $\mathfrak{T}$ such that $(\Omega^n_{\mathfrak{X}/\mathfrak{T}})^{\otimes d}$ is a trivial line bundle on each $\mathfrak{U}_i$ $(1 \leq i \leq k)$. Let ω_i be a nowhere vanishing section of $(\Omega^n_{\mathfrak{X}/\mathfrak{T}})^{\otimes d}$ on $\mathfrak{U}_i$.

Then, by Proposition 3.5, we can compute the p-adic integration of $\bar{\rho}^* \omega_i$ on $\bar{\rho}^{-1}\mathfrak{U}_i(\mathcal{O}_{K_\mathfrak{q}})$ as follows :

$$\int_{\bar{\rho}^{-1}\mathfrak{U}_i(\mathcal{O}_{K_\mathfrak{q}})} |\bar{\rho}^* \omega_i|_p^{1/d} = \frac{1}{q^n} \sum_{J \subset \{1, \ldots, r\}} |(\mathfrak{D}_J^\circ \cap \bar{\rho}^{-1}\mathfrak{U}_i)(\mathbb{F}_q)| \prod_{j \in J} \frac{q-1}{q^{a_j+1} - 1},$$

where $\mathfrak{D}_J^\circ$ are the same as in Proposition 3.5. Note that $\operatorname{div}(\bar{\rho}^*\omega_i) = \sum_{i=1}^r da_i\mathfrak{D}_i$ since ω_i is a nowhere vanishing section of $(\Omega^n_{\mathfrak{X}/\mathfrak{T}})^{\otimes d}$.

Similarly, for $(\bar{\rho}\circ\bar{\tau})^*\omega_i$, we have

$$\int_{(\bar{\rho}\circ\bar{\tau})^{-1}\mathfrak{U}_i(\mathcal{O}_{K_\mathfrak{q}})} |(\bar{\rho}\circ\bar{\tau})^*\omega_i|_p^{1/d} = \frac{1}{q^n}\sum_{J'\subset\{1,\dots,s\}} |(\mathfrak{E}_{J'}^\circ \cap (\bar{\rho}\circ\bar{\tau})^{-1}\mathfrak{U}_i)(\mathbb{F}_q)| \prod_{j'\in J'} \frac{q-1}{q^{b_{j'}+1}-1},$$

where $\mathfrak{E}_{J'}^\circ$ is the same as above.

On the other hand, by the change-of-variable formula for p-adic integration, we have

$$\int_{\bar{\rho}^{-1}\mathfrak{U}_i(\mathcal{O}_{K_\mathfrak{q}})} |\bar{\rho}^*\omega_i|_p^{1/m} = \int_{(\bar{\rho}\circ\bar{\tau})^{-1}\mathfrak{U}_i(\mathcal{O}_{K_\mathfrak{q}})} |(\bar{\rho}\circ\bar{\tau})^*\omega_i|_p^{1/m}.$$

Since the same is true for all finite intersections of $\mathfrak{U}_i$, by inclusion-exclusion principle, we conclude

$$\frac{1}{q^n}\sum_{J\subset\{1,\dots,r\}} |\mathfrak{D}_J^\circ(\mathbb{F}_q)| \prod_{j\in J}\frac{q-1}{q^{a_j+1}-1} = \frac{1}{q^n}\sum_{J'\subset\{1,\dots,s\}} |\mathfrak{E}_{J'}^\circ(\mathbb{F}_q)| \prod_{j'\in J'}\frac{q-1}{q^{b_{j'}+1}-1}.$$

Note that the above argument works for every $\mathfrak{q}\notin S$.

Step 5. Fix a prime number $l = p$ and a maximal ideal $\mathfrak{p}$ of $\mathcal{O}_K$ dividing p. By enlarging S, we may assume that S contains all maximal ideals of $\mathcal{O}_K$ dividing p. We shall work on the level of the Grothendieck group $K(l,S,\mathfrak{p})_{1/d}$ of $\operatorname{Gal}(\overline{K}/K)$-representations introduced in §5.

We rewrite the conclusion of Step 4 in the following form by multipling $q^n\cdot\prod_{j=1}^r(q^{a_j+1}-1)\cdot\prod_{j'=1}^s(q^{b_{j'}+1}-1)$ on both sides :

$$\begin{aligned}&\prod_{j'=1}^s(q^{b_{j'}+1}-1)\sum_{J\subset\{1,\dots,r\}}\Big(|\mathfrak{D}_J^\circ(\mathbb{F}_q)|\prod_{j\in J}(q-1)\prod_{j\notin J}(q^{a_j+1}-1)\Big)\\ &\qquad=\prod_{j=1}^r(q^{a_j+1}-1)\sum_{J'\subset\{1,\dots,s\}}\Big(|\mathfrak{E}_{J'}^\circ(\mathbb{F}_q)|\prod_{j'\in J'}(q-1)\prod_{j'\notin J'}(q^{b_{j'}+1}-1)\Big).\end{aligned}$$

By replacing K by its finite extension, we may assume that $\mathbb{Q}_l(\frac{1}{d})$ exists as a $\operatorname{Gal}(\overline{K}/K)$-representation (see Example 5.5). Recall that the image of $\operatorname{Frob}_\mathfrak{q}$ in $\operatorname{Gal}(\overline{\mathbb{F}_q}/\mathbb{F}_q)$ is the inverse of the q-th power automorphism $x\mapsto x^q$ of $\overline{\mathbb{F}_q}$. Therefore, for $m\in\frac{1}{d}\mathbb{Z}$, $\operatorname{Tr}(\operatorname{Frob}_\mathfrak{q};\mathbb{Q}_l(m)) = q^{-m}$.

Hence, we have the following equality in $K(l,S,\mathfrak{p})_{1/d}$:

$$
\begin{aligned}
&\prod_{j'=1}^{s}([\mathbb{Q}_l(-b_{j'}-1)]-1)\sum_{J\subset\{1,\dots,r\}}\Big([H^*_{c,\text{ét}}((D^\circ_J)_{\overline{K}},\mathbb{Q}_l)]\prod_{j\in J}([\mathbb{Q}_l(-1)]-1)\\
&\qquad\qquad\cdot\prod_{j\notin J}([\mathbb{Q}_l(-a_j-1)]-1)\Big)\\
&=\prod_{j=1}^{r}([\mathbb{Q}_l(-a_j-1)]-1)\sum_{J'\subset\{1,\dots,s\}}\Big([H^*_{c,\text{ét}}((E^\circ_{J'})_{\overline{K}},\mathbb{Q}_l)]\prod_{j'\in J'}([\mathbb{Q}_l(-1)]-1)\\
&\qquad\qquad\cdot\prod_{j'\notin J'}([\mathbb{Q}_l(-b_{j'}-1)]-1)\Big),
\end{aligned}
$$

since the traces of $\mathrm{Frob}_{\mathfrak{q}}$ on the both sides are equal for all $\mathfrak{q}\notin S$ (see Proposition 6.4 and Proposition 5.1). Note that $1\in K(l,S,\mathfrak{p})_{1/d}$ denotes the class of the trivial $\mathrm{Gal}(\overline{K}/K)$-representation.

Since $E_{\mathfrak{p}}(\mathbb{Q}_l(m);u,v)=u^{-m}v^{-m}$ for $m\in\frac{1}{d}\mathbb{Z}$ by Example 5.5, we have

$$
\begin{aligned}
&\prod_{j'=1}^{s}((uv)^{b_{j'}+1}-1)\sum_{J\subset\{1,\dots,r\}}\Big(E((D^\circ_J)_{\mathbb{C}};u,v)\prod_{j\in J}(uv-1)\prod_{j\notin J}((uv)^{a_j+1}-1)\Big)\\
&=\prod_{j=1}^{r}((uv)^{a_j+1}-1)\sum_{J'\subset\{1,\dots,s\}}\Big(E((E^\circ_{J'})_{\mathbb{C}};u,v)\prod_{j'\in J'}(uv-1)\prod_{j'\notin J'}((uv)^{b_{j'}+1}-1)\Big).
\end{aligned}
$$

by Lemma 6.3. By Definition 2.4, this proves $E_{st}(X_{\mathbb{C}};u,v)_\rho=E_{st}(X_{\mathbb{C}};u,v)_{\rho\circ\tau}$ and hence Theorem 1.1. □

Remark 7.2. If we take an appropriate $\mathfrak{p}$ in Step 5, we can use the result of Fontaine-Messing (Remark 4.1, [FM]). Therefore, we don't need the full version of the Hodge-Tate decomposition for Theorem 1.1.

References

[Ba1] V. V. Batyrev, *Birational Calabi-Yau n-folds have equal Betti numbers*, in *New trends in algebraic geometry (Warwick, 1996)*, 1–11, Cambridge Univ. Press, Cambridge, 1999.

[Ba2] V. V. Batyrev, *Stringy Hodge numbers of varieties with Gorenstein canonical singularities*, in *Integrable systems and algebraic geometry (Kobe/Kyoto, 1997)*, 1–32, World Sci. Publishing, River Edge, NJ, 1998.

[BB] V. V. Batyrev, L. A. Borisov, *Mirror duality and string-theoretic Hodge numbers* Invent. Math. **126** (1996), no. 1, 183–203.

[BD] V. V. Batyrev, D. I. Dais, *Strong McKay correspondence, string-theoretic Hodge numbers and mirror symmetry*, Topology **35** (1996), no. 4, 901–929.

[De1] P. Deligne, *Théorème de Lefschetz et critères de dégénérescence de suites spectrales*, Inst. Hautes Études Sci. Publ. Math. No. 35, (1968), 259–278.

[De2] P. Deligne, *La conjecture de Weil I*, Inst. Hautes Études Sci. Publ. Math. No. 43, (1974), 273–307.

[De3] P. Deligne, *La conjecture de Weil II*, Inst. Hautes Études Sci. Publ. Math. No. 52, (1980), 137–252.

[De4] P. Deligne, *Théorie de Hodge I*, in *Actes du Congrès International des Mathématiciens (Nice, 1970), Tome 1*, 425–430, Gauthier-Villars, Paris, 1971.

[De5] P. Deligne, *Théorie de Hodge II*, Inst. Hautes Études Sci. Publ. Math. No. 40, (1971), 5–57.

[De6] P. Deligne, *Théorie de Hodge III*, Inst. Hautes Études Sci. Publ. Math. No. 44, (1974), 5–77.

[DL1] J. Denef, F. Loeser, *Motivic Igusa zeta functions*, J. Algebraic Geom. **7** (1998), no. 3, 505–537.

[DL2] J. Denef, F. Loeser, *Germs of arcs on singular algebraic varieties and motivic integration*, Invent. Math. **135** (1999), no. 1, 201–232.

[DL3] J. Denef, F. Loeser, *Definable sets, motives and p-adic integrals*, J. Amer. Math. Soc. **14** (2001), no. 2, 429–469.

[Fa] G. Faltings, *p-adic Hodge theory*, J. Amer. Math. Soc. **1** (1988), no. 1, 255–299.

[Fo] J.-M. Fontaine, *Le corps des périodes p-adiques*, With an appendix by Pierre Colmez, *Périodes p-adiques* (Bures-sur-Yvette, 1988), Astérisque No. 223, (1994), 59–111.

[FM] J.-M. Fontaine, W. Messing, *p-adic periods and p-adic étale cohomology*, in *Current trends in arithmetical algebraic geometry (Arcata, Calif., 1985)*, 179–207, Contemp. Math., 67, Amer. Math. Soc., Providence, RI, 1987.

[It1] T. Ito, *A note on Hodge numbers of Calabi-Yau manifolds*, Master's thesis, University of Tokyo, March 2001.

[It2] T. Ito, *Birational smooth minimal models have equal Hodge numbers in all dimensions*, preprint

[KMM] Y. Kawamata, K. Matsuda and K. Matsuki, *Introduction to the minimal model problem*, in *Algebraic geometry, Sendai, 1985*, 283–360, Adv. Stud. Pure Math., 10, North-Holland, Amsterdam.

[Ka] Y. Kawamata, *Crepant blowing-up of* 3*-dimensional canonical singularities and its application to degenerations of surfaces*, Ann. of Math. (2) **127** (1988), no. 1, 93–163.

[Kol] J. Kollár, *Flops*, Nagoya Math. J. **113** (1989), 15–36.

[Kon] M. Kontsevich, Lecture at Orsay (December 7, 1995).

[M] D. R. Morrison, *The geometry underlying mirror symmetry*, in *New trends in algebraic geometry (Warwick, 1996)*, 283–310, Cambridge Univ. Press, Cambridge, 1999.

[Se] J.-P. Serre, *Abelian l-adic representations and elliptic curves*, W. A. Benjamin, Inc., New York, 1968.

[Ta] J. T. Tate, *p-divisible groups*, in *Proc. Conf. Local Fields (Driebergen, 1966)*, 158–183, Springer, Berlin, 1967.

[Ts] T. Tsuji, *p-adic étale cohomology and crystalline cohomology in the semi-stable reduction case*, Invent. Math. **137** (1999), no. 2, 233–411.

[Wa] C.-L. Wang, *On the topology of birational minimal models*, J. Differential Geom. **50** (1998), no. 1, 129–146

[We] A. Weil, *Adeles and algebraic groups*, Birkhäuser, Boston, Mass., 1982.

Department of Mathematical Sciences, University of Tokyo, 3-8-1 Komaba, Meguro, Tokyo 153-8914, Japan

E-mail address: itote2@ms.u-tokyo.ac.jp

UTMS

2001–24 Noguchi, junjiro: *Some results in view of Nevanlinna theory.*

2001–25 Fabien Trihan: *Image directe supérieure et unipotence.*

2001–26 Takeshi Saito: *Weight spectral sequences and independence of* ℓ.

2001–27 Takeshi Saito: *Log smooth extension of family of curves and semi-stable reduction.*

2001–28 Takeshi Katsura: *AF-embeddability of crossed products of Cuntz algebras.*

2001–29 Toshio Oshima: *Annihilators of generalized Verma modules of the scalar type for classical Lie algebras.*

2001–30 Kim Sungwhan and Masahiro Yamamoto: *Uniqueness in identification of the support of a source term in an elliptic equation.*

2001–31 Tetsuhiro Moriyama: *The mapping class group action on the homology of the configuration spaces of surfaces.*

2001–32 Takeshi Katsura: *On crossed products of the Cuntz algebra* $\mathcal{O}_\infty$ *by quasi-free actions of abelian groups.*

2001–33 Yuichi Sugiki: *The category of cosheaves and Laplace transforms.*

2001–34 Hiroshige Kajiura: *Homotopy algebra morphism and geometry of classical string field theories.*

2002–1 Tetsushi Ito: *Stringy Hodge numbers and p-adic Hodge theory.*

The Graduate School of Mathematical Sciences was established in the University of Tokyo in April, 1992. Formerly there were two departments of mathematics in the University of Tokyo: one in the Faculty of Science and the other in the College of Arts and Sciences. All faculty members of these two departments have moved to the new graduate school, as well as several members of the Department of Pure and Applied Sciences in the College of Arts and Sciences. In January, 1993, the preprint series of the former two departments of mathematics were unified as the Preprint Series of the Graduate School of Mathematical Sciences, The University of Tokyo. For the information about the preprint series, please write to the preprint series office.

ADDRESS:
Graduate School of Mathematical Sciences, The University of Tokyo
3–8–1 Komaba Meguro-ku, Tokyo 153, JAPAN
TEL +81-3-5465-7001 FAX +81-3-5465-7012