UTMS 2001–32 December 13, 2001

On crossed products of the Cuntz algebra $\mathcal{O}_\infty$ by quasi-free actions of abelian groups

by

Takeshi Katsura

UNIVERSITY OF TOKYO

GRADUATE SCHOOL OF MATHEMATICAL SCIENCES

KOMABA, TOKYO, JAPAN

On crossed products of the Cuntz algebra $\mathcal{O}_\infty$ by quasi-free actions of abelian groups

Takeshi KATSURA
Department of Mathematical Sciences
University of Tokyo, Komaba, Tokyo, 153-8914, JAPAN
e-mail: katsu@ms.u-tokyo.ac.jp

Abstract

We investigate the structures of crossed products of the Cuntz algebra $\mathcal{O}_\infty$ by quasi-free actions of abelian groups. We completely determine their ideal structures and compute the strong Connes spectra and K-groups.

1 Introduction

The crossed products of C^*-algebras give us plenty of interesting examples, and the structures of them have been examined by several authors. In [Ki], A. Kishimoto gave a necessary and sufficient condition that the crossed products by abelian groups become simple in terms of the strong Connes spectrum. For the case of the crossed products of Cuntz algebras by so-called quasi-free actions of abelian groups, he gave a condition for simplicity, which is easy to check. In [KK1] and [KK2], A. Kishimoto and A. Kumjian dealt with, among others, the crossed products of Cuntz algebras by quasi-free actions of the real group $\mathbb{R}$. In our previous papers [Ka1], [Ka2], we examined the structures of crossed products of Cuntz algebras $\mathcal{O}_n$ by quasi-free actions of arbitrary locally compact, second countable, abelian groups. The class of our algebras has many examples of simple stably projectionless C^*-algebras as well as AF-algebras and purely infinite C^*-algebras. In [Ka1], we completely determined the ideal structures of our algebras, and gave another proof of A. Kishimoto's result on the simplicity of them. We also gave a necessary and sufficient condition that our algebras become primitive, and computed the Connes spectra and K-groups of our algebras. In [Ka2], we proved that our algebras become AF-embeddable when actions satisfy certain conditions. To the best of the author's knowledge, this is the first case to have succeeded in embedding crossed products of purely infinite C^*-algebras into AF-algebras except trivial cases. We also gave a necessary and sufficient condition that our algebras become simple and purely infinite, and consequently our algebras are either purely infinite or AF-embeddable when they are simple.

In this paper, we deal with crossed products of the Cuntz algebra $\mathcal{O}_\infty$ by quasi-free actions of arbitrary locally compact, second countable, abelian groups. From section 3 to section 6, we completely determine the ideal structures of such algebras by using the technique developed in [Ka1]. We omit detailed computations if similar computations have been already done in [Ka1]. Readers are referred to [Ka1]. In the last section, we gather some results on crossed products of the Cuntz algebra $\mathcal{O}_\infty$. Among others, we give another proof of the determination of the simplicity of the crossed products done by A. Kishimoto, and we succeed in computing the strong Connes spectra of quasi-free actions on the Cuntz algebra $\mathcal{O}_\infty$.

The crossed products examined in this paper or in [Ka1], [Ka2], can be considered as continuous counterparts of Cuntz-Krieger algebras or graph algebras (cf. [D]). From this point of view, the crossed products of $\mathcal{O}_n$ can be considered as graph algebras of locally finite graphs, and the ones of $\mathcal{O}_\infty$ can be considered as graph algebras of graphs whose vertices emit and receive infinitely many edges. Recently the ideal structures of graph algebras, which is not necessarily locally finite, were deeply examined in [BHRS] and [HS]. Compared with row finite case, it is rather difficult to describe ideal structures of graph algebras which have vertices emitting infinitely many edges. This seems to be related to the difficulty of examination of the ideal structures of the crossed products of $\mathcal{O}_\infty$ compared with the ones of $\mathcal{O}_n$ done in [Ka1].

Acknowledgment. The author would like to thank to his advisor Yasuyuki Kawahigashi for his support and encouragement, to Masaki Izumi for various comments and many suggestions. He is also grateful to Iain Raeburn and Wojciech Szymański for stimulating discussions. This work was partially supported by Research Fellowship for Young Scientists of the Japan Society for the Promotion of Science.

2 Preliminaries

The Cuntz algebra $\mathcal{O}_\infty$ is the universal C^*-algebra generated by infinitely many isometries $S_1, S_2, \dots$ satisfying $S_i^* S_j = \delta_{i,j}$. For $n \in \mathbb{Z}_+ := \{1, 2, \dots\}$ and $k \in \mathbb{N} := \{0, 1, \dots\}$, we define the set $\mathcal{W}_n^{(k)}$ of words in $\{1, 2, \dots, n\}$ with length k by $\mathcal{W}_n^{(0)} = \{\emptyset\}$ and

$$\mathcal{W}_n^{(k)} = \big\{(i_1, i_2, \dots, i_k) \bigm| i_j \in \{1, 2, \dots, n\}\big\}$$

for $k \geq 1$. Set $\mathcal{W}_n = \bigcup_{k=0}^\infty \mathcal{W}_n^{(k)}$ and $\mathcal{W}_\infty = \bigcup_{n=1}^\infty \mathcal{W}_n$. For $\mu = (i_1, i_2, \dots, i_k) \in \mathcal{W}_\infty$, we denote its length k by $|\mu|$, and set $S_\mu = S_{i_1} S_{i_2} \cdots S_{i_k} \in \mathcal{O}_\infty$. Let G be a locally compact abelian group which satisfies the second axiom of countability and Γ be the dual group of G. We use $+$ for multiplicative operations of abelian groups except for $\mathbb{T}$, which is the group of the unit circle in the complex plane $\mathbb{C}$. The pairing of $t \in G$ and $\gamma \in \Gamma$ is denoted by $\langle\, t \,|\, \gamma\, \rangle \in \mathbb{T}$.

For $\omega = (\omega_1, \omega_2, \dots) \in \Gamma^\infty$, we define an action α^ω of abelian group G on $\mathcal{O}_\infty$ by $\alpha_t^\omega(S_i) = \langle\, t \,|\, \omega_i\, \rangle S_i$ for $i \in \mathbb{Z}_+$ and $t \in G$. The action $\alpha^\omega : G \curvearrowright \mathcal{O}_\infty$ becomes quasi-free (for a definition of quasi-free actions on Cuntz algebras, see [E]). However, there exist quasi-free actions of abelian group G on $\mathcal{O}_\infty$, which are not conjugate to α^ω for any $\omega \in \Gamma^\infty$ though we do not deal with such actions. The crossed product $\mathcal{O}_\infty \rtimes_{\alpha^\omega} G$ has a C^*-subalgebra $\mathbb{C}1 \rtimes_{\alpha^\omega} G$ which is isomorphic to $C_0(\Gamma)$. We consider $C_0(\Gamma)$ as a C^*-subalgebra of $\mathcal{O}_\infty \rtimes_{\alpha^\omega} G$. The Cuntz algebra $\mathcal{O}_\infty$ is naturally embedded into the multiplier algebra $M(\mathcal{O}_\infty \rtimes_{\alpha^\omega} G)$ of $\mathcal{O}_\infty \rtimes_{\alpha^\omega} G$. For each $\mu = (i_1, i_2, \dots, i_k) \in \mathcal{W}_\infty$, we define an element ω_μ of Γ by $\omega_\mu = \sum_{j=1}^k \omega_{i_j}$. For $\gamma_0 \in \Gamma$, we define a (reverse) shift automorphism $\sigma_{\gamma_0} : C_0(\Gamma) \to$

$C_0(\Gamma)$ by $(\sigma_{\gamma_0} f)(\gamma) = f(\gamma + \gamma_0)$ for $f \in C_0(\Gamma)$. Once noting that $\alpha_t^\omega(S_\mu) = \langle\, t \,|\, \omega_\mu \,\rangle S_\mu$ for $\mu \in \mathcal{W}_\infty$, one can easily verify that $f S_\mu = S_\mu \sigma_{\omega_\mu} f$ for any $f \in C_0(\Gamma) \subset \mathcal{O}_\infty \rtimes_{\alpha^\omega} G$. For a subset X of a C^*-algebra, we denote by $\operatorname{span} X$ the linear span of X, and by $\overline{\operatorname{span}}\, X$ its closure. We have $\mathcal{O}_\infty \rtimes_{\alpha^\omega} G = \overline{\operatorname{span}}\{S_\mu f S_\nu^* \mid \mu, \nu \in \mathcal{W}_\infty,\ f \in C_0(\Gamma)\}$.

We denote by $\mathbb{M}_k$ the C^*-algebra of $k \times k$ matrices for $k = 1, 2, \ldots$, and by $\mathbb{K}$ the C^*-algebra of compact operators of the infinite dimensional separable Hilbert space.

3 Gauge invariant ideals

In this section, we determine all the ideals which are globally invariant under the gauge action. Here an ideal means a closed two-sided ideal, and the gauge action $\beta : \mathbb{T} \curvearrowright \mathcal{O}_\infty \rtimes_{\alpha^\omega} G$ is defined by $\beta_t(S_\mu f S_\nu^*) = t^{|\mu| - |\nu|} S_\mu f S_\nu^*$ for $\mu, \nu \in \mathcal{W}_\infty,\ f \in C_0(\Gamma)$ and $t \in \mathbb{T}$.

For a positive integer n, we define a projection p_n by $p_n = 1 - \sum_{i=1}^n S_i S_i^*$. We set $p_0 = 1$. Since p_n commutes with $C_0(\Gamma)$, $p_n C_0(\Gamma)$ is a C^*-subalgebra of $\mathcal{O}_\infty \rtimes_{\alpha^\omega} G$, which is isomorphic to $C_0(\Gamma)$.

Definition 3.1 Let I be an ideal of the crossed product $\mathcal{O}_\infty \rtimes_{\alpha^\omega} G$. For each $n \in \mathbb{N}$, we define the closed subset $X_I^{(n)}$ of Γ by

$$X_I^{(n)} = \{\gamma \in \Gamma \mid f(\gamma) = 0 \text{ for all } f \in C_0(\Gamma) \text{ with } p_n f \in I\}.$$

Set $X_I = X_I^{(0)}$, $X_I^{(\infty)} = \bigcap_{n=1}^\infty X_I^{(n)}$, and denote by $\widetilde{X}_I$ the pair $(X_I, X_I^{(\infty)})$ of subsets of Γ.

In other words, $X_I^{(n)}$ is determined by $p_n C_0(\Gamma \setminus X_I^{(n)}) = I \cap p_n C_0(\Gamma)$. One can easily see that $X_{I_1 \cap I_2}^{(n)} = X_{I_1}^{(n)} \cup X_{I_2}^{(n)}$ for any $n \in \mathbb{N}$, hence $X_{I_1 \cap I_2} = X_{I_1} \cup X_{I_2}$, $X_{I_1 \cap I_2}^{(\infty)} = X_{I_1}^{(\infty)} \cup X_{I_2}^{(\infty)}$ and that $I_1 \subset I_2$ implies $X_{I_1}^{(n)} \supset X_{I_2}^{(n)}$ for any $n \in \mathbb{N}$, hence implies $X_{I_1} \supset X_{I_2}$, $X_{I_1}^{(\infty)} \supset X_{I_2}^{(\infty)}$. For $n \in \mathbb{N}$, the set $X_I^{(n)}$ can be described only in terms of X_I and $X_I^{(\infty)}$.

Lemma 3.2 *For an ideal I of $\mathcal{O}_\infty \rtimes_{\alpha^\omega} G$, we have*

$$X_I^{(n)} = X_I^{(\infty)} \cup \bigcup_{i=n+1}^\infty (X_I + \omega_i),$$

for any $n \in \mathbb{N}$.

Proof. Let γ be an element of X_I and i be a positive integer grater than n. Take $f \in C_0(\Gamma)$ with $p_n f \in I$. Since

$$S_i^* p_n f S_i = S_i^* f S_i = S_i^* S_i \sigma_{\omega_i} f = \sigma_{\omega_i} f,$$

we have $\sigma_{\omega_i} f \in I \cap C_0(\Gamma)$. Since $\gamma \in X_I$, we have $\sigma_{\omega_i} f(\gamma) = 0$. Hence $f(\gamma + \omega_i) = 0$ for any $f \in C_0(\Gamma)$ with $p_n f \in I$. It implies $\gamma + \omega_i \in X_I^{(n)}$. Thus $X_I^{(n)} \supset X_I + \omega_i$ for any $i > n$. For $n \le m$, we have $X_I^{(n)} \supset X_I^{(m)}$ because $p_n p_m = p_m$. Therefore $X_I^{(n)} \supset X_I^{(\infty)}$. Thus $X_I^{(n)} \supset X_I^{(\infty)} \cup \bigcup_{i=n+1}^\infty (X_I + \omega_i)$.

Conversely, take $\gamma \notin X_I^{(\infty)} \cup \bigcup_{i=n+1}^\infty (X_I + \omega_i)$. Since $\gamma \notin X_I^{(\infty)}$, we can find a positive integer m so that $\gamma \notin X_I^{(m)}$. When $m \le n$, we see that $\gamma \notin X_I^{(n)}$. We will show $\gamma \notin X_I^{(n)}$ in the case $m > n$. Since $\gamma \notin X_I^{(m)}$, there exists $f \in C_0(\Gamma)$ such that $p_m f \in I$ and $f(\gamma) \neq 0$. For each $i = n+1, n+2, \dots, m$, there exists $f_i \in C_0(\Gamma) \cap I$ such that $f_i(\gamma - \omega_i) \neq 0$ because $\gamma \notin X_I + \omega_i$. Set $g = f \prod_{i=n+1}^{m} \sigma_{-\omega_i} f_i$. We have $g(\gamma) \neq 0$ and

$$p_n g = p_m g + \sum_{i=n+1}^{m} S_i S_i^* g = p_m g + \sum_{i=n+1}^{m} S_i(\sigma_{\omega_i} g) S_i^* \in I.$$

Therefore $\gamma \notin X_I^{(n)}$. Thus we have $X_I^{(n)} = X_I^{(\infty)} \cup \bigcup_{i=n+1}^{\infty} (X_I + \omega_i)$. ∎

Definition 3.3 A subset X of Γ is called *ω-invariant* if X is a closed set with $X + \omega_i \subset X$ for any $i \in \mathbb{Z}_+$. For an ω-invariant set X, we define a closed set H_X by

$$H_X = \overline{X \setminus \bigcup_{i=1}^{\infty} (X + \omega_i)} \ \cup \ \bigcap_{n=1}^{\infty} \overline{\bigcup_{i=n}^{\infty} (X + \omega_i)}.$$

Note that H_X is a closed subset of X.

Definition 3.4 A pair $\widetilde{X} = (X, X^\infty)$ of subsets of Γ is called *ω-invariant* if X is an ω-invariant set, and X^∞ is a closed set satisfying $H_X \subset X^\infty \subset X$.

Proposition 3.5 *For any ideal I of the crossed product $\mathcal{O}_\infty \rtimes_{\alpha^\omega} G$, the pair $\widetilde{X}_I$ is ω-invariant.*

Proof. By Lemma 3.2, we have $X_I = X_I^{(\infty)} \cup \bigcup_{i=1}^{\infty} (X_I + \omega_i)$. From this, we see that X_I is ω-invariant and that $\overline{X_I \setminus \bigcup_{i=1}^{\infty} (X_I + \omega_i)} \subset X_I^{(\infty)} \subset X_I$. By Lemma 3.2, we have $\overline{\bigcup_{i=n}^{\infty} (X + \omega_i)} \subset \overline{X_I^{(n)}} = X_I^{(n)}$. Hence $\bigcap_{n=1}^{\infty} \overline{\bigcup_{i=n}^{\infty} (X + \omega_i)} \subset \bigcap_{n=1}^{\infty} X_I^{(n)} = X_I^{(\infty)}$. Therefore we get $H_X \subset X_I^{(\infty)} \subset X_I$. ∎

We will show that for an ω-invariant pair $\widetilde{X}$, there exists a gauge invariant ideal I such that $\widetilde{X}_I = \widetilde{X}$ (Proposition 3.9).

Lemma 3.6 *Let $\widetilde{X} = (X, X^{(\infty)})$ be an ω-invariant pair. For $n \in \mathbb{N}$, set $X^{(n)} = X^{(\infty)} \cup \bigcup_{i=n+1}^{\infty} (X + \omega_i)$. Then we have the following.*

(i) *$X^{(n)}$ is closed for all $n \in \mathbb{N}$.*

(ii) *$X = X^{(0)}$, $X^{(\infty)} = \bigcap_{n=1}^{\infty} X^{(n)}$.*

(iii) *For $0 \leq n < m$, $X^{(n)} = X^{(m)} \cup \bigcup_{i=n+1}^{m} (X + \omega_i)$.*

(iv) *For a positive integer n,*

$$X = \bigcup_{\mu \in \mathcal{W}_n} (X^{(n)} + \omega_\mu) \cup \bigcap_{k=1}^{\infty} \Bigg(\bigcup_{\mu \in \mathcal{W}_n^{(k)}} (X + \omega_\mu) \Bigg).$$

Proof.

(i) Take $\gamma \in \overline{X^{(n)}}$ for a positive integer n. If $U \cap X^{(\infty)} \neq \emptyset$ for all neighborhood U of γ, then $\gamma \in X^{(\infty)} \subset X^{(n)}$ because $X^{(\infty)}$ is closed. Otherwise, we can find a positive integer i_U grater than n with $U \cap (X + \omega_{i_U}) \neq \emptyset$ for any neighborhood U of γ. If there exists i such that $i_U = i$ eventually, then $\gamma \in X + \omega_i \subset X^{(n)}$ because $X + \omega_i$ is closed. If there are no such i, then we can see that $\gamma \in \overline{\bigcup_{i=m}^{\infty}(X + \omega_i)}$ for any m with $m > n$. Hence $\gamma \in H_X \subset X^{(\infty)} \subset X^{(n)}$. Thus we have proved that $\gamma \in X^{(n)}$, from which it follows that $X^{(n)}$ is closed.

(ii) Since $X \setminus \bigcup_{i=1}^{\infty}(X + \omega_i) \subset X^{(\infty)} \subset X$, we have $X = X^{(0)}$. We see that

$$\bigcap_{n=1}^{\infty} X^{(n)} = \bigcap_{n=1}^{\infty}\Big(X^{(\infty)} \cup \bigcup_{i=n+1}^{\infty}(X + \omega_i)\Big) = X^{(\infty)} \cup \bigcap_{n=1}^{\infty}\Big(\bigcup_{i=n+1}^{\infty}(X + \omega_i)\Big).$$

Since $\bigcap_{n=1}^{\infty}\big(\bigcup_{i=n+1}^{\infty}(X + \omega_i)\big) \subset H_X \subset X^{(\infty)}$, we have $\bigcap_{n=1}^{\infty} X^{(n)} = X^{(\infty)}$.

(iii) It is obvious by the definition.

(iv) For a positive integer n, we have $X = X^{(n)} \cup \bigcup_{i=1}^{n}(X + \omega_i)$ by (iii). Recursively, we get $X = \bigcup_{m=0}^{k-1}\big(\bigcup_{\mu \in \mathcal{W}_n^{(m)}}(X^{(n)} + \omega_\mu)\big) \cup \bigcup_{\mu \in \mathcal{W}_n^{(k)}}(X + \omega_\mu)$ for any positive integer k. Hence $X = \bigcup_{\mu \in \mathcal{W}_n}(X^{(n)} + \omega_\mu) \cup \bigcap_{k=1}^{\infty}\big(\bigcup_{\mu \in \mathcal{W}_n^{(k)}}(X + \omega_\mu)\big)$. ∎

Definition 3.7 For an ω-invariant pair $\widetilde{X} = (X, X^{(\infty)})$, we define $I_{\widetilde{X}} \subset \mathcal{O}_\infty \rtimes_{\alpha^\omega} G$ by

$$I_{\widetilde{X}} = \overline{\mathrm{span}}\{S_\mu p_n f S_\nu^* \mid \mu, \nu \in \mathcal{W}_\infty,\ f \in C_0(\Gamma \setminus X^{(n)}),\ n \in \mathbb{N}\},$$

where $X^{(n)} = X^{(\infty)} \cup \bigcup_{i=n+1}^{\infty}(X + \omega_i)$.

Proposition 3.8 *For an ω-invariant pair $\widetilde{X} = (X, X^{(\infty)})$, the set $I_{\widetilde{X}}$ becomes a gauge invariant ideal of $\mathcal{O}_\infty \rtimes_{\alpha^\omega} G$.*

Proof. Clearly $I_{\widetilde{X}}$ is a $*$-invariant closed linear space, and is invariant under the gauge action β because $\beta_t(S_\mu p_n f S_\nu^*) = t^{|\mu|-|\nu|} S_\mu p_n f S_\nu^*$ for $t \in \mathbb{T}$. To prove that $I_{\widetilde{X}}$ is an ideal, it suffices to show that for any $\mu_1, \nu_1, \mu_2, \nu_2 \in \mathcal{W}_\infty$ and any $f \in C_0(\Gamma \setminus X^{(n)})$, $g \in C_0(\Gamma)$, the product xy of $x = S_{\mu_1} p_n f S_{\nu_1}^* \in I_{\widetilde{X}}$ and $y = S_{\mu_2} g S_{\nu_2}^* \in \mathcal{O}_\infty \rtimes_{\alpha^\omega} G$ is in $I_{\widetilde{X}}$. If $S_{\nu_1}^* S_{\mu_2} = 0$ or $S_{\nu_1}^* S_{\mu_2} = S_\mu^*$ for some $\mu \in \mathcal{W}_\infty$, then it is easy to see that $xy \in I_{\widetilde{X}}$. Otherwise $S_{\nu_1}^* S_{\mu_2} = S_\mu$ for some $\mu = (i_1, i_2, \ldots, i_k) \in \mathcal{W}_\infty$ with $\mu \neq \emptyset$. When $i_1 \leq n$, we have $p_n f S_\mu = p_n S_\mu \sigma_{\omega_\mu} f = 0$. Hence $xy = 0 \in I_{\widetilde{X}}$. When $i_1 > n$, we have $p_n f S_\mu = p_n S_\mu \sigma_{\omega_\mu} f = S_\mu \sigma_{\omega_\mu} f$. Now, $f \in C_0(\Gamma \setminus X^{(n)})$ implies $\sigma_{\omega_\mu} f \in C_0(\Gamma \setminus X)$ because $X + \omega_\mu \subset X + \omega_{i_1} \subset X^{(n)}$. Hence we have $xy \in I_{\widetilde{X}}$. It completes the proof. ∎

Proposition 3.9 *Let $\widetilde{X} = (X, X^{(\infty)})$ be an ω-invariant pair, and set $I = I_{\widetilde{X}}$. Then $\widetilde{X}_I = \widetilde{X}$.*

Proof. By the definition of I, we get $X_I^{(n)} \subset X^{(n)}$ for any $n \in \mathbb{N}$. We will first prove that $X_I = X$. To the contrary, assume that $X_I \subsetneqq X$. Then there exists $f \in I \cap C_0(\Gamma)$ such that $f(\gamma_0) = 1$ for some $\gamma_0 \in X$. Since $f \in I$, there exist $n_l \in \mathbb{N}$, $f_l \in C_0(\Gamma \setminus X^{(n_l)})$ and $\mu_l, \nu_l \in \mathcal{W}_\infty$ $(l = 1, 2, \dots, L)$ such that

$$\left\| f - \sum_{l=1}^{L} S_{\mu_l} p_{n_l} f_l S_{\nu_l}^* \right\| < \frac{1}{2}.$$

Take a positive integer n so large that $n_l \leq n$ and $\mu_l, \nu_l \in \mathcal{W}_n$ for $l = 1, 2, \dots, L$. For any $\mu_0 \in \mathcal{W}_n$, we have $p_n S_{\mu_0}^* f S_{\mu_0} p_n = p_n \sigma_{\omega_{\mu_0}} f$ and $\sigma_{\omega_{\mu_0}} f(\gamma_0 - \omega_{\mu_0}) = 1$. For l with $\mu_l = \nu_l = \mu_0$, we have $p_n S_{\mu_0}^* (S_{\mu_l} p_{n_l} f_l S_{\nu_l}^*) S_{\mu_0} p_n = p_n f_l$. For l with $\mu_l \nu = \nu_l \nu = \mu_0$ for some $\nu = (i_1, i_2, \dots, i_k) \in \mathcal{W}_n$ with $i_1 > n_l$, we have $p_n S_{\mu_0}^* (S_{\mu_l} p_{n_l} f_l S_{\nu_l}^*) S_{\mu_0} p_n = p_n \sigma_{\omega_\nu} f_l$. We have $\sigma_{\omega_\nu} f_l \in C_0(\Gamma \setminus X)$, because $X + \omega_\nu \subset X + \omega_{i_1} \subset X^{(n_l)}$. For other l, we have $p_n S_{\mu_0}^* (S_{\mu_l} p_{n_l} f_l S_{\nu_l}^*) S_{\mu_0} p_n = 0$. Hence we get

$$\left\| \sigma_{\omega_{\mu_0}} f - \sum_{l=1}^{L} g_l \right\| = \left\| p_n \left(\sigma_{\omega_{\mu_0}} f - \sum_{l=1}^{L} g_l \right) \right\| = \left\| p_n S_{\mu_0}^* \left(f - \sum_{l=1}^{L} S_{\mu_l} p_{n_l} f_l S_{\nu_l}^* \right) S_{\mu_0} p_n \right\| < \frac{1}{2},$$

where $g_l \in C_0(\Gamma \setminus X^{(n_l)})$ when $\mu_l = \nu_l = \mu_0$, and $g_l \in C_0(\Gamma \setminus X)$ when $\mu_l \nu = \nu_l \nu = \mu_0$ for some $\nu = (i_1, i_2, \dots, i_k) \in \mathcal{W}_n$ with $i_1 > n_l$, and $g_l = 0$ otherwise. To derive a contradiction, it suffices to find $\mu_0 \in \mathcal{W}_n$ such that $g_l(\gamma_0 - \omega_{\mu_0}) = 0$ for any l. By Lemma 3.6 (iv), we have either $\gamma_0 \in \bigcap_{m=1}^\infty \left(\bigcup_{\mu \in \mathcal{W}_n^{(m)}} (X + \omega_\mu) \right)$ or $\gamma_0 \in X^{(n)} + \omega_\mu$ for some $\mu \in \mathcal{W}_n$.

When $\gamma_0 \in \bigcap_{m=1}^\infty \left(\bigcup_{\mu \in \mathcal{W}_n^{(m)}} (X + \omega_\mu) \right)$, take $\mu_0 \in \mathcal{W}_n$ so that $|\mu_0| > |\mu_l|, |\nu_l|$ for $l = 1, 2, \dots, L$ and $\gamma_0 \in X + \omega_{\mu_0}$. Then $\mu_l = \nu_l = \mu_0$ never occurs. Hence $g_l \in C_0(\Gamma \setminus X)$ for any l. We get $g_l(\gamma_0 - \omega_{\mu_0}) = 0$ because $\gamma_0 - \omega_{\mu_0} \in X$. When $\gamma_0 \in X^{(n)} + \omega_\mu$ for some $\mu \in \mathcal{W}_n$, take $\mu_0 = \mu$. Since $\gamma_0 - \omega_{\mu_0} \in X^{(n)} \subset X^{(n_l)} \subset X$, we have $g_l(\gamma_0 - \omega_{\mu_0}) = 0$ either if $g_l \in C_0(\Gamma \setminus X^{(n_l)})$ or if $g_l \in C_0(\Gamma \setminus X)$. Hence $g_l(\gamma_0 - \omega_{\mu_0}) = 0$ for any l. Therefore we have $X_I = X$.

Next we will show that $X_I^{(n)} = X^{(n)}$ for a positive integer n. To derive a contradiction, assume that $X_I^{(n)} \subsetneq X^{(n)}$. Then there exists $f \in C_0(\Gamma)$ such that $p_n f \in I$ and $f(\gamma_0) = 1$ for some $\gamma_0 \in X^{(n)}$. Since $p_n f \in I$, there exist $n_l \in \mathbb{N}$, $f_l \in C_0(\Gamma \setminus X^{(n_l)})$ and $\mu_l, \nu_l \in \mathcal{W}_\infty$ $(l = 1, 2, \dots, L)$ such that

$$\left\| p_n f - \sum_{l=1}^{L} S_{\mu_l} p_{n_l} f_l S_{\nu_l}^* \right\| < \frac{1}{2}.$$

Take a positive integer m so large that $\mu_l, \nu_l \in \mathcal{W}_m$, $n_l \leq m$ for $l = 1, 2, \dots, L$ and $n \leq m$. By Lemma 3.6 (iii), we have $X^{(n)} = X^{(m)} \cup \bigcup_{i=n+1}^{m} (X + \omega_i)$. When $\gamma_0 \in X^{(m)}$, we have $f_l(\gamma_0) = 0$ for any l. On the other hand, we get $\| f - \sum_{\mu_l = \nu_l = \emptyset} f_l \| < 1/2$ because

$$p_m \left(p_n f - \sum_{l=1}^{L} S_{\mu_l} p_{n_l} f_l S_{\nu_l}^* \right) p_m = p_m f - \sum_{\mu_l = \nu_l = \emptyset} p_m f_l.$$

This is a contradiction. When $\gamma_0 \in X + \omega_i$ for some i with $n < i \leq m$, we have $\sigma_{\omega_i} f = S_i^* (p_n f) S_i \in I$ and $\sigma_{\omega_i} f(\gamma_0 - \omega_i) = 1$. This contradicts the fact that $X_I = X$. Therefore $X_I^{(n)} = X^{(n)}$ for a positive integer n. Hence $X_I^{(\infty)} = \bigcap_{n=1}^\infty X_I^{(n)} = \bigcap_{n=1}^\infty X^{(n)} = X^{(\infty)}$. We have shown that $\widetilde{X}_I = \widetilde{X}$. ∎

By Proposition 3.9, the map $I \mapsto \widetilde{X}_I$ from the set of gauge invariant ideals I of $\mathcal{O}_\infty \rtimes_{\alpha^\omega} G$ to the set of ω-invariant pairs is surjective. Now, we turn to showing that this map is injective (Proposition 3.15). To do so, we investigate the quotient $(\mathcal{O}_\infty \rtimes_{\alpha^\omega} G)/I$ of $\mathcal{O}_\infty \rtimes_{\alpha^\omega} G$ by an ideal I which is not $\mathcal{O}_\infty \rtimes_{\alpha^\omega} G$. Since $I \cap C_0(\Gamma) = C_0(\Gamma \setminus X_I)$, a C^*-subalgebra $C_0(\Gamma)/(I \cap C_0(\Gamma))$ of $(\mathcal{O}_\infty \rtimes_{\alpha^\omega} G)/I$ is isomorphic to $C_0(X_I)$. We will consider $C_0(X_I)$ as a C^*-subalgebra of $(\mathcal{O}_\infty \rtimes_{\alpha^\omega} G)/I$. We will use the same symbols $S_1, S_2, \ldots \in M((\mathcal{O}_\infty \rtimes_{\alpha^\omega} G)/I)$ as the ones in $M(\mathcal{O}_\infty \rtimes_{\alpha^\omega} G)$ for denoting the isometries of $\mathcal{O}_\infty$ which is naturally embedded into $M((\mathcal{O}_\infty \rtimes_{\alpha^\omega} G)/I)$. For an ω-invariant set X, we can define a $*$-homomorphism $\sigma_{\omega_\mu} : C_0(X) \to C_0(X)$ for $\mu \in \mathcal{W}_\infty$. This map σ_{ω_μ} is always surjective, but it is injective only in the case that $X \subset X + \omega_\mu$, which is equivalent to $X = X + \omega_\mu$. One can easily verify the following.

Lemma 3.10 *Let I be an ideal that is not $\mathcal{O}_\infty \rtimes_{\alpha^\omega} G$. For $\mu, \nu \in \mathcal{W}_\infty$ and $f \in C_0(X_I) \subset (\mathcal{O}_\infty \rtimes_{\alpha^\omega} G)/I$, the following hold.*

(i) *$S_\mu f S_\nu^* = 0$ if and only if $f = 0$.*

(ii) *For $n \in \mathbb{N}$, $p_n f = 0$ if and only if $f \in C_0(X_I \setminus X_I^{(n)})$.*

(iii) *$f S_\mu = S_\mu \sigma_{\omega_\mu} f$.*

(iv) *$(\mathcal{O}_\infty \rtimes_{\alpha^\omega} G)/I = \overline{\operatorname{span}}\{S_\mu f S_\nu^* \mid \mu, \nu \in \mathcal{W}_\infty,\ f \in C_0(X_I)\}$.*

We define a C^*-subalgebra of $(\mathcal{O}_\infty \rtimes_{\alpha^\omega} G)/I$, which corresponds to the AF-core for Cuntz algebras.

Definition 3.11 Let I be an ideal that is not $\mathcal{O}_\infty \rtimes_{\alpha^\omega} G$. We define C^*-subalgebras of $(\mathcal{O}_\infty \rtimes_{\alpha^\omega} G)/I$ by

$$\begin{aligned}
\mathcal{G}_I^{(n,k)} &= \operatorname{span}\{S_\mu f S_\nu^* \mid \mu, \nu \in \mathcal{W}_n^{(k)},\ f \in C_0(X_I)\},\\
\mathcal{F}_I^{(n,k)} &= \operatorname{span}\{S_\mu p_n f S_\nu^* \mid \mu, \nu \in \mathcal{W}_n^{(k)},\ f \in C_0(X_I)\},\\
\mathcal{F}_I^{(n)} &= \operatorname{span}\{S_\mu f S_\nu^* \mid \mu, \nu \in \mathcal{W}_n,\ 0 \le |\mu| = |\nu| \le n,\ f \in C_0(X_I)\},\\
\mathcal{F}_I &= \overline{\operatorname{span}}\{S_\mu f S_\nu^* \mid \mu, \nu \in \mathcal{W}_\infty,\ |\mu| = |\nu|,\ f \in C_0(X_I)\},
\end{aligned}$$

for $n \in \mathbb{Z}_+, 0 \le k \le n$.

Lemma 3.12 *Let I be an ideal that is not $\mathcal{O}_\infty \rtimes_{\alpha^\omega} G$. For $n \in \mathbb{Z}_+, 0 \le k \le n$, we have the following.*

(i) *$\mathcal{G}_I^{(n,k)} \cong \mathbb{M}_{n^k} \otimes C_0(X_I)$.*

(ii) *$\mathcal{F}_I^{(n,k)} \cong \mathbb{M}_{n^k} \otimes C_0(X_I^{(n)})$.*

(iii) *$\mathcal{F}_I^{(n)} \cong \bigoplus_{k=0}^{n-1} \mathcal{F}_I^{(n,k)} \oplus \mathcal{G}_I^{(n,n)}$.*

(iv) *$\bigcup_{n=1}^\infty \mathcal{F}_I^{(n)}$ is dense in $\mathcal{F}_I$.*

Proof.

(i) Since the set $\mathcal{W}_n^{(k)}$ has n^k elements, we may use $\{e_{\mu,\nu}\}_{\mu,\nu\in\mathcal{W}_n^{(k)}}$ for denoting the matrix units of $\mathbb{M}_{n^k}$. One can easily see that

$$\mathbb{M}_{n^k} \otimes C_0(X_I) \ni e_{\mu,\nu} \otimes f \mapsto S_\mu f S_\nu^* \in \mathcal{G}_I^{(n,k)}$$

gives us an isomorphism from $\mathbb{M}_{n^k} \otimes C_0(X_I)$ to $\mathcal{G}_I^{(n,k)}$.

(ii) We can define a surjective map from $\mathcal{G}_I^{(n,k)}$ to $\mathcal{F}_I^{(n,k)}$ by

$$\mathcal{G}_I^{(n,k)} \ni S_\mu f S_\nu^* \mapsto S_\mu p_n f S_\nu^* \in \mathcal{F}_I^{(n,k)}.$$

Its kernel is $\mathbb{M}_{n^k} \otimes C_0(X_I \setminus X_I^{(n)})$ under the isomorphism $\mathcal{G}_I^{(n,k)} \cong \mathbb{M}_{n^k} \otimes C_0(X_I)$ by Lemma 3.10 (ii). Hence we have $\mathcal{F}_I^{(n,k)} \cong \mathbb{M}_{n^k} \otimes C_0(X_I^{(n)})$.

(iii) It can be done just by computation.

(iv) Obvious by the definitions of $\mathcal{F}_I^{(n)}$ and $\mathcal{F}_I$. ∎

We will often identify $\mathcal{G}_I^{(n,n)}$ with $C_0(X_I, \mathbb{M}_{n^n})$. The following lemma essentially appeared in [C].

Lemma 3.13 *For $i = 1, 2$, let E_i be a conditional expectation from a C^*-algebra A_i onto a C^*-subalgebra B_i of A_i. Let $\varphi : A_1 \to A_2$ be a $*$-homomorphism with $\varphi \circ E_1 = E_2 \circ \varphi$. If the restriction of φ on B_1 is injective and E_1 is faithful, then φ is injective.*

For an ideal I which is invariant under the gauge action β, we can extend the gauge action on $\mathcal{O}_\infty \rtimes_{\alpha^\omega} G$ to one on $(\mathcal{O}_\infty \rtimes_{\alpha^\omega} G)/I$, which is also denoted by β. The following lemma is standard.

Lemma 3.14 *Let I be a gauge invariant ideal that is not $\mathcal{O}_\infty \rtimes_{\alpha^\omega} G$. Then,*

$$E_I : (\mathcal{O}_\infty \rtimes_{\alpha^\omega} G)/I \ni x \mapsto \int_{\mathbb{T}} \beta_t(x)dt \in (\mathcal{O}_\infty \rtimes_{\alpha^\omega} G)/I$$

is a faithful conditional expectation onto $\mathcal{F}_I$, where dt is the normalized Haar measure on $\mathbb{T}$.

Proposition 3.15 *For any gauge invariant ideal I, we have $I_{\widetilde{X}_I} = I$.*

Proof. When $I = \mathcal{O}_\infty \rtimes_{\alpha^\omega} G$, we have $X_I = X_I^{(\infty)} = \emptyset$. Thus $I_{\widetilde{X}_I} = \mathcal{O}_\infty \rtimes_{\alpha^\omega} G$. Let I be a gauge invariant ideal that is not $\mathcal{O}_\infty \rtimes_{\alpha^\omega} G$ and set $J = I_{\widetilde{X}_I}$. By the definition, $J \subset I$. Hence there exists a surjective $*$-homomorphism $\pi : (\mathcal{O}_\infty \rtimes_{\alpha^\omega} G)/J \to (\mathcal{O}_\infty \rtimes_{\alpha^\omega} G)/I$. By Proposition 3.9 and Lemma 3.12, the restriction of π on $\mathcal{F}_J^{(k)}$ is an isomorphism from $\mathcal{F}_J^{(k)}$ onto $\mathcal{F}_I^{(k)}$ and so the restriction of π on $\mathcal{F}_J$ is an isomorphism from $\mathcal{F}_J$ onto $\mathcal{F}_I$. By Lemma 3.14, there are faithful conditional expectations $E_J : (\mathcal{O}_\infty \rtimes_{\alpha^\omega} G)/J \to \mathcal{F}_J$ and $E_I : (\mathcal{O}_\infty \rtimes_{\alpha^\omega} G)/I \to \mathcal{F}_I$ with $E_I \circ \pi = \pi \circ E_J$. By Lemma 3.13, π is injective. Therefore $I_{\widetilde{X}_I} = I$. ∎

Theorem 3.16 *The maps $I \mapsto \widetilde{X}_I$ and $\widetilde{X} \mapsto I_{\widetilde{X}}$ induce a one-to-one correspondence between the set of gauge invariant ideals of $\mathcal{O}_\infty \rtimes_{\alpha^\omega} G$ and the set of ω-invariant pairs of subsets of Γ.*

Proof. Combine Proposition 3.9 and Proposition 3.15. ∎

4 Primeness for ω-invariant pairs

In this section, we give a necessary condition for an ideal to be primitive in terms of ω-invariant pairs. We will use it after in order to determine all primitive ideals.

An ideal of a C^*-algebra is called primitive if it is a kernel of some irreducible representation. A C^*-algebra is called primitive if 0 is a primitive ideal. When a C^*-algebra A is separable, an ideal I of A is primitive if and only if I is prime, i.e. for two ideals I_1, I_2 of A, $I_1 \cap I_2 \subset I$ implies either $I_1 \subset I$ or $I_2 \subset I$. We define primeness for ω-invariant pairs. For two ω-invariant pair $\widetilde{X}_1 = (X_1, X_1^{(\infty)})$, $\widetilde{X}_2 = (X_2, X_2^{(\infty)})$, we write $\widetilde{X}_1 \subset \widetilde{X}_2$ if $X_1 \subset X_2, X_1^{(\infty)} \subset X_2^{(\infty)}$ and denote by $\widetilde{X}_1 \cup \widetilde{X}_2$ the ω-invariant pair $(X_1 \cup X_2, X_1^{(\infty)} \cup X_2^{(\infty)})$.

Definition 4.1 An ω-invariant pair $\widetilde{X}$ is called *prime* if $\widetilde{X}_1 \cup \widetilde{X}_2 \supset \widetilde{X}$ implies either $\widetilde{X}_1 \supset \widetilde{X}$ or $\widetilde{X}_2 \supset \widetilde{X}$ for two ω-invariant pairs $\widetilde{X}_1$, $\widetilde{X}_2$.

Proposition 4.2 *If an ideal I of $\mathcal{O}_\infty \rtimes_{\alpha^\omega} G$ is primitive, then $\widetilde{X}_I$ is a prime ω-invariant pair.*

Proof. Let I be a primitive ideal of $\mathcal{O}_\infty \rtimes_{\alpha^\omega} G$. Take two ω-invariant pairs $\widetilde{X}_1$, $\widetilde{X}_2$ with $\widetilde{X}_1 \cup \widetilde{X}_2 \supset \widetilde{X}_I$. Set $I_1 = I_{\widetilde{X}_1}$ and $I_2 = I_{\widetilde{X}_2}$. Then

$$I_1 \cap I_2 = I_{\widetilde{X}_1 \cup \widetilde{X}_2} \subset I_{\widetilde{X}_I} \subset I.$$

Since I is prime, we have either $I_1 \subset I$ or $I_2 \subset I$. Hence we get either $\widetilde{X}_1 \supset \widetilde{X}_I$ or $\widetilde{X}_2 \supset \widetilde{X}_I$. Thus $\widetilde{X}_I$ is prime. ∎

In general, the converse of Proposition 4.2 is not true (see Corollary 5.4 and Proposition 6.24). The ideal I is prime if and only if the equality $I_1 \cap I_2 = I$ implies either $I_1 = I$ or $I_2 = I$ for two ideals I_1, I_2 (see the proof of (iii)⇒(iv) of Proposition 4.3). The following is the counterpart of this fact for prime ω-invariant pairs.

Proposition 4.3 *For an ω-invariant pair $\widetilde{X}$, the following are equivalent.*

(i) *$\widetilde{X}$ is prime.*

(ii) *For two ω-invariant pairs $\widetilde{X}_1$, $\widetilde{X}_2$, the equality $\widetilde{X}_1 \cup \widetilde{X}_2 = \widetilde{X}$ implies either $\widetilde{X}_1 = \widetilde{X}$ or $\widetilde{X}_2 = \widetilde{X}$.*

(iii) *For two gauge invariant ideals I_1, I_2 of $\mathcal{O}_\infty \rtimes_{\alpha^\omega} G$, the equality $I_1 \cap I_2 = I_{\widetilde{X}}$ implies either $I_1 = I_{\widetilde{X}}$ or $I_2 = I_{\widetilde{X}}$.*

(iv) *For two gauge invariant ideals I_1, I_2 of $\mathcal{O}_\infty \rtimes_{\alpha^\omega} G$, the inclusion $I_1 \cap I_2 \subset I_{\widetilde{X}}$ implies either $I_1 \subset I_{\widetilde{X}}$ or $I_2 \subset I_{\widetilde{X}}$.*

Proof. (i)⇒(ii): Take two ω-invariant pairs $\widetilde{X}_1$, $\widetilde{X}_2$ with $\widetilde{X}_1 \cup \widetilde{X}_2 = \widetilde{X}$. By (i), we have either $\widetilde{X}_1 \supset \widetilde{X}$ or $\widetilde{X}_2 \supset \widetilde{X}$. Hence we get either $\widetilde{X}_1 = \widetilde{X}$ or $\widetilde{X}_2 = \widetilde{X}$.

(ii)⇒(iii): Take two gauge invariant ideals I_1, I_2 with $I_1 \cap I_2 = I_{\widetilde{X}}$. We have $\widetilde{X}_{I_1} \cup \widetilde{X}_{I_2} = \widetilde{X}$. By (ii), we have either $\widetilde{X}_{I_1} = \widetilde{X}$ or $\widetilde{X}_{I_2} = \widetilde{X}$. By Proposition 3.15, we have either $I_1 = I_{\widetilde{X}}$ or $I_2 = I_{\widetilde{X}}$.

(iii)⇒(iv): Take two gauge invariant ideals I_1, I_2 with $I_1 \cap I_2 \subset I_{\widetilde{X}}$. Then we have

$$(I_1 + I_{\widetilde{X}}) \cap (I_2 + I_{\widetilde{X}}) = (I_1 \cap I_2) + I_{\widetilde{X}} = I_{\widetilde{X}}.$$

By (iii), either $I_1 + I_{\widetilde{X}} = I_{\widetilde{X}}$ or $I_2 + I_{\widetilde{X}} = I_{\widetilde{X}}$ holds. Hence we get either $I_1 \subset I_{\widetilde{X}}$ or $I_2 \subset I_{\widetilde{X}}$.

(iv)⇒(i): Similarly as the proof of Proposition 4.2. ∎

We will use the implication (ii)⇒(i) to determine which ω-invariant pair is prime. We also need a notion of primeness for ω-invariant sets.

Definition 4.4 An ω-invariant set X is called *prime* if $X_1 \cup X_2 \supset X$ implies either $X_1 \supset X$ or $X_2 \supset X$, for any ω-invariant sets X_1, X_2.

We set $\mathrm{sg}(\omega) = \{\omega_\mu \mid \mu \in \mathcal{W}_\infty\}$ which is the semigroup generated by $\omega_1, \omega_2, \ldots$ and denote by $\overline{\mathrm{sg}}(\omega)$ its closure. Note that a closed subset X of Γ is ω-invariant if and only if $X + \overline{\mathrm{sg}}(\omega) = X$. For any $\gamma \in \Gamma$, it is easy to see that the set $\gamma + \overline{\mathrm{sg}}(\omega)$ is a prime ω-invariant set. The following is a necessary and sufficient condition for an ω-invariant set to be prime, which can be considered as an analogue of maximal tails in [BHRS].

Proposition 4.5 *An ω-invariant set X of Γ is prime if and only if for any $\gamma_1, \gamma_2 \in X$ and any neighborhoods U_1, U_2 of γ_1, γ_2 respectively, there exist $\gamma \in X$ and $\mu_1, \mu_2 \in \mathcal{W}_\infty$ with $\gamma + \omega_{\mu_1} \in U_1$ and $\gamma + \omega_{\mu_2} \in U_2$.*

Proof. Suppose X is a prime ω-invariant set. Take $\gamma_1, \gamma_2 \in X$ and neighborhoods U_1, U_2 of γ_1, γ_2 respectively. Set $X_j = \Gamma \setminus \bigcup_{\mu \in \mathcal{W}_\infty} (U_j - \omega_\mu)$ for $j = 1, 2$. Then X_1 and X_2 are ω-invariant sets satisfying $X_1 \not\supset X$ and $X_2 \not\supset X$. Since X is prime, we have $X_1 \cup X_2 \not\supset X$. Hence there exists $\gamma \in X$ with $\gamma \notin X_1 \cup X_2$. By the definition of X_1 and X_2, there exist μ_1, μ_2 such that $\gamma + \omega_{\mu_1} \in U_1$ and $\gamma + \omega_{\mu_2} \in U_2$.

Conversely assume that for any $\gamma_1, \gamma_2 \in X$ and any neighborhoods U_1, U_2 of γ_1, γ_2 respectively, there exist $\gamma \in X$ and $\mu_1, \mu_2 \in \mathcal{W}_\infty$ with $\gamma + \omega_{\mu_1} \in U_1$ and $\gamma + \omega_{\mu_2} \in U_2$. Take ω-invariant sets X_1 and X_2 satisfying $X_1 \not\supset X$ and $X_2 \not\supset X$. There exist $\gamma_1, \gamma_2 \in X$ with $\gamma_1 \notin X_1$ and $\gamma_2 \notin X_2$. Hence there exist $\gamma \in X$ and $\mu_1, \mu_2 \in \mathcal{W}_\infty$ with $\gamma + \omega_{\mu_1} \notin X_1$ and $\gamma + \omega_{\mu_2} \notin X_2$. Since X_1 and X_2 are ω-invariant, we have $\gamma \notin X_1$ and $\gamma \notin X_2$. Therefore, $X_1 \cup X_2 \not\supset X$. Thus, X is prime. ∎

Lemma 4.6 *If an ω-invariant pair $\widetilde{X} = (X, X^{(\infty)})$ is prime, then $X^{(\infty)} = H_X$ or $X^{(\infty)} = H_X \cup \{\gamma\}$ for some $\gamma \notin H_X$.*

Proof. Let $\widetilde{X} = (X, X^{(\infty)})$ be a prime ω-invariant pair. To derive a contradiction, assume $X^{(\infty)} \setminus H_X$ has two points γ_1, γ_2. Take open sets $U_1 \ni \gamma_1$, $U_2 \ni \gamma_2$ with $U_1 \cap U_2 = \emptyset$, $U_1 \cap H_X = \emptyset$ and $U_2 \cap H_X = \emptyset$. Then $\widetilde{X}_i = (X, X^{(\infty)} \setminus U_i)$ $(i = 1, 2)$ are ω-invariant pairs satisfying $\widetilde{X} = \widetilde{X}_1 \cup \widetilde{X}_2$. However, we have $\widetilde{X} \not\subset \widetilde{X}_1$ and $\widetilde{X} \not\subset \widetilde{X}_2$. This contradicts the primeness of $\widetilde{X}$. ∎

Lemma 4.7 *An ω-invariant pair (X, H_X) is prime if and only if X is a prime ω-invariant set.*

Proof. Suppose that (X, H_X) is a prime ω-invariant pair. Take ω-invariant sets X_1, X_2 with $X \subset X_1 \cup X_2$. We have $(X, H_X) \subset (X_1, X_1) \cup (X_2, X_2)$. Since (X, H_X) is prime, either $(X, H_X) \subset (X_1, X_1)$ or $(X, H_X) \subset (X_2, X_2)$ holds. Therefore X is a prime ω-invariant set. Conversely assume that X is a prime ω-invariant set. Take two ω-invariant pairs $(X_1, X_1^{(\infty)})$, $(X_2, X_2^{(\infty)})$ with $(X_1, X_1^{(\infty)}) \cup (X_2, X_2^{(\infty)}) = (X, H_X)$. Since X is prime, either $X \subset X_1$ or $X \subset X_2$. We may assume $X \subset X_1$. Then $X = X_1$. Hence $H_X = H_{X_1} \subset X_1^{(\infty)} \subset H_X$. We get $(X_1, X_1^{(\infty)}) = (X, H_X)$. By Proposition 4.3, (X, H_X) is a prime ω-invariant pair. ∎

Lemma 4.8 *An ω-invariant pair $(X, H_X \cup \{\gamma\})$ is prime for some $\gamma \notin H_X$ if and only if $X = \gamma + \overline{\mathrm{sg}}(\omega)$.*

Proof. Suppose that an ω-invariant pair $(X, H_X \cup \{\gamma\})$ is prime. Then $(X, H_X \cup \{\gamma\}) \subset (X, H_X) \cup (\gamma + \overline{\mathrm{sg}}(\omega), \gamma + \overline{\mathrm{sg}}(\omega))$ implies $(X, H_X \cup \{\gamma\}) \subset (\gamma + \overline{\mathrm{sg}}(\omega), \gamma + \overline{\mathrm{sg}}(\omega))$ because $H_X \cup \{\gamma\} \not\subset H_X$. Hence $\gamma + \overline{\mathrm{sg}}(\omega) \subset X \subset \gamma + \overline{\mathrm{sg}}(\omega)$. Thus, we get $X = \gamma + \overline{\mathrm{sg}}(\omega)$. Conversely, assume $X = \gamma + \overline{\mathrm{sg}}(\omega)$. Take two ω-invariant pairs $(X_1, X_1^{(\infty)})$, $(X_2, X_2^{(\infty)})$ with $(X_1, X_1^{(\infty)}) \cup (X_2, X_2^{(\infty)}) = (X, H_X \cup \{\gamma\})$. We may assume $\gamma \in X_1^{(\infty)}$. Then we have $X = \gamma + \overline{\mathrm{sg}}(\omega) \subset X_1^{(\infty)} + \overline{\mathrm{sg}}(\omega) \subset X_1 \subset X$. Hence $X_1 = X$. We have $H_X \cup \{\gamma\} = H_{X_1} \cup \{\gamma\} \subset X_1^{(\infty)} \subset H_X \cup \{\gamma\}$. Therefore $(X_1, X_1^{(\infty)}) = (X, H_X \cup \{\gamma\})$. By Proposition 4.3, $(X, H_X \cup \{\gamma\})$ is a prime ω-invariant pair. ∎

Proposition 4.9 *An ω-invariant pair $(X, X^{(\infty)})$ is prime if and only if either X is prime and $X^{(\infty)} = H_X$ or $X = \gamma + \overline{\mathrm{sg}}(\omega)$ and $X^{(\infty)} = H_X \cup \{\gamma\}$ for some $\gamma \notin H_X$.*

Proof. Combine Lemma 4.6, Lemma 4.7 and Lemma 4.8. ∎

5 The ideal structure of $\mathcal{O}_\infty \rtimes_{\alpha^\omega} G$ (part 1)

In this section and the next section, we completely determine the ideal structure of $\mathcal{O}_\infty \rtimes_{\alpha^\omega} G$ (Theorem 5.3, Theorem 6.30). The ideal structure of $\mathcal{O}_\infty \rtimes_{\alpha^\omega} G$ depends on whether $\omega \in \Gamma^\infty$ satisfies the following condition:

Condition 5.1 For each $i \in \mathbb{Z}_+$, one of the following two conditions is satisfied:

(i) For any positive integer k, $k\omega_i \neq 0$.

(ii) There exists a sequence $\mu_1, \mu_2, \ldots$ in $\mathcal{W}_\infty$ such that $S_{\mu_k}^* S_i = 0$ for any k and $\lim_{k\to\infty} \omega_{\mu_k} = 0$.

This condition is an analogue of Condition (K) in the case of graph algebras [BHRS]. In this section, we deal with the case that ω satisfies Condition 5.1.

Proposition 5.2 *If ω satisfies Condition 5.1, then for an ideal I that is not $\mathcal{O}_\infty \rtimes_{\alpha^\omega} G$, there exists a unique conditional expectation E_I from $(\mathcal{O}_\infty \rtimes_{\alpha^\omega} G)/I$ onto $\mathcal{F}_I$ such that $E_I(S_\mu f S_\nu^*) = \delta_{|\mu|,|\nu|} S_\mu f S_\nu^*$ for $\mu, \nu \in \mathcal{W}_\infty$, $f \in C_0(X_I)$.*

Proof. Take $x = \sum_{l=1}^{L} S_{\mu_l} f_l S_{\nu_l}^* \in (\mathcal{O}_\infty \rtimes_{\alpha^\omega} G)/I$ where $\mu_l, \nu_l \in \mathcal{W}_\infty$ and $f_l \in C_0(X_I)$ for $l = 1, 2, \ldots, L$. Set $x_0 = \sum_{|\mu_l|=|\nu_l|} S_{\mu_l} f_l S_{\nu_l}^*$ and we will prove that $\|x_0\| \le \|x\|$. If we choose a positive integer n so that $|\mu_l|, |\nu_l| \le n$ and $\mu_l, \nu_l \in \mathcal{W}_n$ for $l = 1, 2, \ldots, L$, then $x_0 \in \mathcal{F}_I^{(n)}$. By Lemma 3.12, there exist $x_0^{(k)} \in \mathcal{F}_I^{(n,k)}$ $(0 \le k \le n-1)$ and $x_0^{(n)} \in \mathcal{G}_I^{(n,n)}$ such that $x_0 = \sum_{k=0}^{n} x_0^{(k)}$. We have $\|x_0\| = \max\{\|x_0^{(0)}\|, \ldots, \|x_0^{(n)}\|\}$.

First we consider the case that $\|x_0\| = \|x_0^{(k)}\|$ for some $k \le n-1$. If we set $q_k = \sum_{\mu \in \mathcal{W}_n^{(k)}} S_\mu p_n S_\mu^* \in M((\mathcal{O}_\infty \rtimes_{\alpha^\omega} G)/I)$, then q_k is a projection satisfying that $q_k S_{\mu_l} S_{\nu_l}^* q_k = 0$ if $|\mu_l| \ne |\nu_l|$. Hence $q_k x q_k = q_k x_0 q_k = x_0^{(k)}$. We get $\|x_0\| = \|x_0^{(k)}\| = \|q_k x q_k\| \le \|x\|$. Next we consider the case that $\|x_0\| = \|x_0^{(n)}\|$. Then there exists $\gamma_0 \in X_I$ such that $\|x_0^{(n)}\| = \|x_0^{(n)}(\gamma_0)\|$. By Lemma 3.6 (iv), we have

$$X_I = \bigcup_{\mu \in \mathcal{W}_n} (X_I^{(n)} + \omega_\mu) \cup \bigcap_{k=1}^{\infty} \Big(\bigcup_{\mu \in \mathcal{W}_n^{(k)}} (X_I + \omega_\mu) \Big).$$

When $\gamma_0 \in X_I^{(n)} + \omega_\mu$ for some $\mu \in \mathcal{W}_n$, set $u = \sum_{\nu \in \mathcal{W}_n^{(n)}} S_\nu S_\mu p_n S_\nu^* \in M((\mathcal{O}_\infty \rtimes_{\alpha^\omega} G)/I)$. Then u is a partial isometry. We have $u^* x u = u^* x_0 u = u^* x_0^{(n)} u = \pi_n(\sigma_{\omega_\mu}(x_0^{(n)}))$ where π_n is the natural surjection from $\mathcal{G}_I^{(n,n)}$ onto $\mathcal{F}_I^{(n,n)}$. Since $\gamma_0 - \omega_\mu \in X_I^{(n)}$, we have

$$\|\pi_n(\sigma_{\omega_\mu}(x_0^{(n)}))\| \ge \|\sigma_{\omega_\mu}(x_0^{(n)})(\gamma_0 - \omega_\mu)\| = \|x_0^{(n)}(\gamma_0)\| = \|x_0^{(n)}\| = \|x_0\|.$$

Therefore $\|x_0\| \le \|u^* x u\| \le \|x\|$.

When $\gamma_0 \in \bigcap_{k=1}^{\infty} \big(\bigcup_{\mu \in \mathcal{W}_n^{(k)}} (X_I + \omega_\mu) \big)$, we can find $i \in \{1, 2, \ldots, n\}$ such that $\gamma_0 - k\omega_i \in X_I$ for all $k \in \mathbb{N}$. Since ω satisfies Condition 5.1, either $k\omega_i \ne 0$ for any $k \in \mathbb{Z}_+$ or there exists a sequence $\{\mu_k\}_{k \in \mathbb{Z}_+} \subset \mathcal{W}_n$ with $\lim_{k\to\infty} \omega_{\mu_k} = 0$ and $S_{\mu_k}^* S_i = 0$ for any k. In the case that $k\omega_i \ne 0$ for any $k \in \mathbb{Z}_+$, we can find a neighborhood U of $\gamma_0 - n\omega_i \in X_I$ such that $U \cap (U + k\omega_i) = \emptyset$ for $k = 1, 2, \ldots, n$. Choose a function f with $0 \le f \le 1$ satisfying that the support of f is contained in U and $f(\gamma_0 - n\omega_i) = 1$. Set $u = \sum_{\mu \in \mathcal{W}_n^{(n)}} S_\mu S_i^n f^{1/2} S_\mu^* \in (\mathcal{O}_\infty \rtimes_{\alpha^\omega} G)/I$. Since

$$u^* u = \sum_{\mu,\nu \in \mathcal{W}_n^{(n)}} S_\mu f^{1/2} S_i^{*n} S_\mu^* S_\nu S_i^n f^{1/2} S_\nu^* = \sum_{\mu \in \mathcal{W}_n^{(n)}} S_\mu f S_\mu^*,$$

u^*u corresponds to $1 \otimes f$ under the isomorphism $\mathcal{G}_I^{(n,n)} \cong \mathbb{M}_{n^n} \otimes C_0(X_I)$. Thus we have $\|u^*u\| = \sup_{\gamma \in X_I} |f(\gamma)| = 1$, and so $\|u\| = 1$. A routine computation shows that $u^* x u = u^* x_0^{(n)} u = f \sigma_{n\omega_i} x_0^{(n)} \in C_0(X_I, \mathbb{M}_{n^n})$. Since $\gamma_0 - n\omega_i \in X_I$, we have

$$\|u^* x u\| \ge \|f(\gamma_0 - n\omega_i) \sigma_{n\omega_i} x_0^{(n)}(\gamma_0 - n\omega_i)\| = \|x_0^{(n)}(\gamma_0)\| = \|x_0\|.$$

Hence $\|x_0\| \le \|u^* x u\| \le \|x\|$. Finally, we consider the case that there exists a sequence $\{\mu_k\}_{k \in \mathbb{Z}_+} \subset \mathcal{W}_n$ with $\lim_{k\to\infty} \omega_{\mu_k} = 0$ and $S_{\mu_k}^* S_i = 0$ for any $k \in \mathbb{Z}_+$. For $k \in \mathbb{Z}_+$, define a partial isometry $u_k = \sum_{\mu \in \mathcal{W}_n^{(n)}} S_\mu S_i^n S_{\mu_k} S_\mu^* \in \mathcal{O}_\infty \subset M((\mathcal{O}_\infty \rtimes_{\alpha^\omega} G)/I)$. A routine computation shows that $u_k^* x u_k = u_k^* x_0^{(n)} u_k = \sigma_{n\omega_i + \omega_{\mu_k}} x_0^{(n)} \in C_0(X_I, M_{n^n})$. Since $\gamma_0 - n\omega_i \in X_I$, we have

$$\|u_k^* x u_k\| \ge \|\sigma_{n\omega_i + \omega_{\mu_k}} x_0^{(n)}(\gamma_0 - n\omega_i)\| = \|x_0^{(n)}(\gamma_0 + \omega_{\mu_k})\|.$$

Hence we have $\|x_0^{(n)}(\gamma_0+\omega_{\mu_k})\| \leq \|u_k^* x u_k\| \leq \|x\|$ for any $k \in \mathbb{Z}_+$. Therefore $\|x_0\| = \|x_0^{(n)}(\gamma_0)\| = \lim_{k\to\infty} \|x_0^{(n)}(\gamma_0+\omega_{\mu_k})\| \leq \|x\|$.

Hence the map

$$\begin{aligned}\operatorname{span}\{S_\mu f S_\nu^* \mid \mu,\nu \in \mathcal{W}_\infty,\ f \in C_0(X_I)\} \ni x \\ \mapsto\ x_0 \in \operatorname{span}\{S_\mu f S_\nu^* \mid \mu,\nu \in \mathcal{W}_\infty,\ |\mu|=|\nu|,\ f\in C_0(X_I)\}.\end{aligned}$$

is well-defined and norm-decreasing. The extension E_I of the map above is the desired conditional expectation onto $\mathcal{F}_I$. Uniqueness is easy to verify. ∎

By uniqueness, the conditional expectation E_I above coincides with the one in Lemma 3.14 when I is gauge invariant. Actually an ideal of $\mathcal{O}_\infty \rtimes_{\alpha^\omega} G$ is gauge invariant if there exists such a conditional expectation, as we see in the proof of the following theorem.

Theorem 5.3 *Suppose that ω satisfies Condition 5.1. Then for any ideal I we have $I_{\widetilde{X}_I} = I$, and so I is gauge invariant. Hence there is a one-to-one correspondence between the set of ideals of $\mathcal{O}_\infty \rtimes_{\alpha^\omega} G$ and the set of ω-invariant pairs of subsets of Γ.*

Proof. If $X_I = \emptyset$, then $I = \mathcal{O}_\infty \rtimes_{\alpha^\omega} G$ so $I_{\widetilde{X}_I} = I$. Let I be an ideal that is not $\mathcal{O}_\infty \rtimes_{\alpha^\omega} G$, and set $J = I_{\widetilde{X}_I}$. By the same way as in the proof of Proposition 3.15, we can find a surjective $*$-homomorphism $\pi : (\mathcal{O}_\infty \rtimes_{\alpha^\omega} G)/J \to (\mathcal{O}_\infty \rtimes_{\alpha^\omega} G)/I$ whose restriction on $\mathcal{F}_J$ is an isomorphism from $\mathcal{F}_J$ onto $\mathcal{F}_I$. By Proposition 5.2, there exists a conditional expectation $E_I : (\mathcal{O}_\infty \rtimes_{\alpha^\omega} G)/I \to \mathcal{F}_I$ satisfying $E_I \circ \pi = \pi \circ E_J$, where $E_J : (\mathcal{O}_\infty \rtimes_{\alpha^\omega} G)/I \to \mathcal{F}_J$ is a faithful conditional expectation defined in Lemma 3.14. By Lemma 3.13, π is injective. Therefore $I = I_{\widetilde{X}_I}$. The last part follows from Theorem 3.16. ∎

Corollary 5.4 *When ω satisfies Condition 5.1, an ideal I of $\mathcal{O}_\infty \rtimes_{\alpha^\omega} G$ is primitive if and only if the ω-invariant pair $\widetilde{X}_I$ is prime.*

Proof. It follows from Proposition 4.3 and Theorem 5.3. ∎

6 The ideal structure of $\mathcal{O}_\infty \rtimes_{\alpha^\omega} G$ (part 2)

In this section, we investigate the ideal structure of $\mathcal{O}_\infty \rtimes_{\alpha^\omega} G$ when ω does not satisfy Condition 5.1 i.e. there exists $i \in \mathbb{Z}_+$ such that $k\omega_i = 0$ for some positive integer k, and that there exist no sequences $\mu_1, \mu_2, \ldots$ in $\mathcal{W}_\infty$ such that $S_{\mu_k}^* S_i = 0$ for any k and $\lim_{k\to\infty} \omega_{\mu_k} = 0$. Note that such i is unique. Without loss of generality, we may assume $i = 1$. Let K be the smallest positive integer satisfying $K\omega_1 = 0$. Denote by Γ' the quotient of Γ by the subgroup generated by ω_1, which is isomorphic to $\mathbb{Z}/K\mathbb{Z}$. We denote by $[\gamma]$ and $[U]$ the images in Γ' of $\gamma \in \Gamma$ and $U \subset \Gamma$ respectively. We use the symbol $([\gamma], \theta)$ for denoting elements of $\Gamma' \times \mathbb{T}$. Define $A = \overline{\operatorname{span}}\{S_1^k f S_1^{*l} \mid f \in C_0(\Gamma), k, l \in \mathbb{N}\}$ which is a C^*-subalgebra of $\mathcal{O}_\infty \rtimes_{\alpha^\omega} G$. In [Ka1], we defined a C^*-algebra T_K and a continuous family of $*$-homomorphisms $\varphi_\gamma : A \to T_K$ for $\gamma \in \Gamma$. Note that $\varphi_\gamma(x) = 0$ if and only if $\varphi_{\gamma+\omega_1}(x) = 0$ for $x \in A$. We also defined $\psi_{\gamma,\theta} = \pi_\theta \circ \varphi_\gamma$ for $(\gamma, \theta) \in \Gamma \times \mathbb{T}$, where $\pi_\theta : T_K \to \mathbb{M}_K$ is a continuous family of $*$-homomorphisms.

Definition 6.1 For an ideal I of $\mathcal{O}_\infty \rtimes_{\alpha^\omega} G$, we define the closed subset Y_I of $\Gamma' \times \mathbb{T}$ by

$$Y_I = \{([\gamma], \theta) \in \Gamma' \times \mathbb{T} \mid \psi_{\gamma,\theta}(x) = 0 \text{ for all } x \in A \cap I\}.$$

We denote by $\widetilde{Y}_I$ the pair $(Y_I, X_I^{(\infty)})$ of a subset Y_I of $\Gamma' \times \mathbb{T}$ and a subset $X_I^{(\infty)}$ of Γ.

Definition 6.2 For a pair $\widetilde{Y} = (Y, X^{(\infty)})$ of a subset Y of $\Gamma' \times \mathbb{T}$ and a subset $X^{(\infty)}$ of Γ, we define subsets X and $X^{(n)}$ of Γ by

$$X = \{\gamma \in \Gamma \mid ([\gamma], \theta) \in Y \text{ for some } \theta \in \mathbb{T}\},$$
$$X^{(n)} = X^{(\infty)} \cup \bigcup_{i=n+1}^{\infty} (X + \omega_i).$$

With this notation, a pair $\widetilde{Y} = (Y, X^{(\infty)})$ is called *ω-invariant* if $(X, X^{(\infty)})$ is an ω-invariant pair of subsets of Γ and if Y is a closed set satisfying that $[X^{(1)}] \times \mathbb{T} \subset Y$.

Proposition 6.3 *For an ideal I of $\mathcal{O}_\infty \rtimes_{\alpha^\omega} G$, the pair $\widetilde{Y}_I$ is ω-invariant.*

Proof. By [Ka1, Proposition 5.15], we have

$$X_I = \{\gamma \in \Gamma \mid ([\gamma], \theta) \in Y_I \text{ for some } \theta \in \mathbb{T}\}.$$

By the argument in the proof of [Ka1, Lemma 5.21], we have

$$X_I^{(1)} = \{\gamma \in \Gamma \mid \varphi_\gamma(x) = 0 \text{ for any } x \in A \cap I\}.$$

Therefore $[X_I^{(1)}] \times \mathbb{T} \subset Y_I$. Thus the pair $\widetilde{Y}_I$ is ω-invariant. ∎

We get the ω-invariant pair $\widetilde{Y}_I$ from an ideal I of $\mathcal{O}_\infty \rtimes_{\alpha^\omega} G$. Conversely, from an ω-invariant pair $\widetilde{Y}$, we can construct the ideal $I_{\widetilde{Y}}$ of $\mathcal{O}_\infty \rtimes_{\alpha^\omega} G$.

Definition 6.4 For an ω-invariant pair $\widetilde{Y} = (Y, X^{(\infty)})$, we define $J_{\widetilde{Y}} \subset A$ and $I_{\widetilde{Y}} \subset \mathcal{O}_\infty \rtimes_{\alpha^\omega} G$ by

$$\begin{aligned} J_{\widetilde{Y}} &= \{x \in A \mid \psi_{\gamma,\theta}(x) = 0 \text{ for } ([\gamma], \theta) \in Y, \text{ and } \varphi_\gamma(x) = 0 \text{ for } \gamma \in X^{(1)}\}, \\ I_{\widetilde{Y}} &= \overline{\operatorname{span}}\big(\{S_\mu x S_\nu^* \mid \mu, \nu \in \mathcal{W}_\infty,\ x \in J_{\widetilde{Y}}\} \\ &\qquad \cup \{S_\mu p_n f S_\nu \mid \mu, \nu \in \mathcal{W}_\infty,\ n \in \mathbb{Z}_+,\ f \in C_0(\Gamma \setminus X^{(n)})\}\big), \end{aligned}$$

with the notation in Definition 6.2.

Proposition 6.5 *For an ω-invariant pair $\widetilde{Y}$, $I_{\widetilde{Y}}$ is an ideal of $\mathcal{O}_\infty \rtimes_{\alpha^\omega} G$.*

Proof. Once noting that $J_{\widetilde{Y}} \cap C_0(\Gamma) = C_0(\Gamma \setminus X)$ and $J_{\widetilde{Y}} \cap p_1 C_0(\Gamma) = p_1 C_0(\Gamma \setminus X^{(1)})$, we can prove that $I_{\widetilde{Y}}$ is an ideal in a similar way to Proposition 3.8 with the help of the computation in [Ka1, Proposition 5.20]. ∎

Lemma 6.6 *Let $\widetilde{Y} = (Y, X^{(\infty)})$ be an ω-invariant pair. For any $([\gamma], \theta) \notin Y$, there exists $x \in J_{\widetilde{Y}}$ such that $\psi_{\gamma,\theta}(x) \neq 0$.*

Proof. The proof goes exactly the same as in the proof of [Ka1, Lemma 5.22], once noting that $([\gamma], \theta) \notin Y$ implies $\gamma \notin X^{(1)}$. ∎

Proposition 6.7 *Let $\widetilde{Y} = (Y, X^{(\infty)})$ be an ω-invariant pair, and set $I = I_{\widetilde{Y}}$. Then we have $\widetilde{Y}_I = \widetilde{Y}$.*

Proof. By Lemma 6.6, we get $Y_I \subset Y$. To prove the other inclusion, it is sufficient to see that $\psi_{\gamma,\theta}(x) = 0$ for $([\gamma], \theta) \in Y$ and $x \in I \cap A$. Take $\varepsilon > 0$ arbitrarily. Since $x \in I$, there exist $\mu_l, \nu_l \in \mathcal{W}_\infty$, $x_l \in J_{\widetilde{Y}}$ for $l = 1, 2, \ldots, L$ and $\mu'_k, \nu'_k \in \mathcal{W}_\infty$, $n_k \in \mathbb{Z}_+$, $f_k \in C_0(\Gamma \setminus X^{(n_k)})$ for $k = 1, 2, \ldots, K$ such that

$$\left\| x - \sum_{l=1}^{L} S_{\mu_l} x_l S^*_{\nu_l} - \sum_{k=1}^{K} S_{\mu'_k} p_{n_k} f_k S^*_{\nu'_k} \right\| < \varepsilon.$$

Take a positive integer m such that $m \geq |\mu_l|, |\nu_l|$ for any l and $m > |\mu'_k|, |\nu'_k|$ for any k. Then, $\left\| S_1^{*m} x S_1^m - \sum_{l=1}^{L} x'_l \right\| < \varepsilon$ where $x'_l = S_1^{*m} S_{\mu_l} x_l S^*_{\nu_l} S_1^m$ for $l = 1, 2, \ldots, L$. Since $x'_l \in J_{\widetilde{Y}}$, we have $\|\psi_{\gamma,\theta}(S_1^{*m} x S_1^m)\| < \varepsilon$. Since $\psi_{\gamma,\theta}(S_1)$ is a unitary, we have $\|\psi_{\gamma,\theta}(x)\| < \varepsilon$ for arbitrary $\varepsilon > 0$. Hence, we have $\psi_{\gamma,\theta}(x) = 0$. Therefore we get $Y_I = Y$.

From $Y_I = Y$, we have $X_I = X$. By the definition of I, we see that $X_I^{(n)} \subset X^{(n)}$ for $n \in \mathbb{Z}_+$. To the contrary, assume that $X_I^{(n)} \subsetneq X^{(n)}$. Then there exists $f \in C_0(\Gamma)$ such that $p_n f \in I$ and $f(\gamma_0) = 1$ for some $\gamma_0 \in X^{(n)}$. Since $p_n f \in I$, there exist $\mu_l, \nu_l \in \mathcal{W}_\infty$, $x_l \in J_{\widetilde{Y}}$ for $l = 1, 2, \ldots, L$ and $\mu'_k, \nu'_k \in \mathcal{W}_\infty$, $n_k \in \mathbb{Z}_+$, $f_k \in C_0(\Gamma \setminus X^{(n_k)})$ for $k = 1, 2, \ldots, K$ such that

$$\left\| p_n f - \sum_{l=1}^{L} S_{\mu_l} x_l S^*_{\nu_l} - \sum_{k=1}^{K} S_{\mu'_k} p_{n_k} f_k S^*_{\nu'_k} \right\| < \frac{1}{2}.$$

Take a positive integer m so large that $\mu_l, \nu_l, \mu'_k, \nu'_k \in \mathcal{W}_m$, $n_k \leq m$ for any l, k and $n \leq m$. By Lemma 3.6 (iii), we have $X^{(n)} = X^{(m)} \cup \bigcup_{i=n+1}^{m} (X + \omega_i)$. We first consider the case that $\gamma_0 \in X^{(m)}$. By [Ka1, Lemma 5.4], there exists $g_l \in C_0(\Gamma \setminus X^1)$ with $p_1 x_l p_1 = p_1 g_l$ for any l. Hence we have $p_m x_l p_m = p_m p_1 x_l p_1 p_m = p_m g_l$ for any l. Since

$$p_m \left(p_n f - \sum_{l=1}^{L} S_{\mu_l} x_l S^*_{\nu_l} - \sum_{k=1}^{K} S_{\mu'_k} p_{n_k} f_k S^*_{\nu'_k} \right) p_m = p_m f - \sum_{\mu_l = \nu_l = \emptyset} p_m g_l - \sum_{\mu'_k = \nu'_k = \emptyset} p_m f_k,$$

we get $\|f - \sum_{\mu_l=\nu_l=\emptyset} g_l - \sum_{\mu'_k=\nu'_k=\emptyset} f_k\| < 1/2$. This contradicts the fact that $f(\gamma_0) = 1$, $g_l(\gamma_0) = 0$ and $f_k(\gamma_0) = 0$ for any l, k. When $\gamma_0 \in X + \omega_i$ for some i with $n < i \leq m$, we have $\sigma_{\omega_i} f = S_i^*(p_n f) S_i \in I$ and $\sigma_{\omega_i} f(\gamma_0 - \omega_i) = 1$. This contradicts the fact that $X_I = X$. Therefore $X_I^{(n)} = X^{(n)}$ for a positive integer n. Hence $X_I^{(\infty)} = \bigcap_{n=1}^{\infty} X_I^{(n)} = \bigcap_{n=1}^{\infty} X^{(n)} = X^{(\infty)}$. Thus we have $\widetilde{Y}_I = \widetilde{Y}$. ∎

Corollary 6.8 *For two ω-invariant pairs $\widetilde{Y}_1 = (Y_1, X_1^{(\infty)})$, $\widetilde{Y}_2 = (Y_2, X_2^{(\infty)})$, we have $I_{\widetilde{Y}_1} \subset I_{\widetilde{Y}_2}$ if and only if $Y_1 \supset Y_2$ and $X_1^{(\infty)} \supset X_2^{(\infty)}$.*

A relation between $I_{\widetilde{Y}}$ and $I_{\widetilde{X}}$ can be described as follows.

Proposition 6.9 *Let $\widetilde{Y} = (Y, X^{(\infty)})$ be an ω-invariant pair. For $t \in \mathbb{T}$, set $\widetilde{Y}_t = (Y_t, X^{(\infty)})$ where $Y_t = \{([\gamma], \theta) \in \Gamma' \times \mathbb{T} \mid ([\gamma], t\theta) \in Y\}$. Then $\widetilde{Y}_t$ is ω-invariant and $\beta_t(I_{\widetilde{Y}}) = I_{\widetilde{Y}_{tK}}$ where β is the gauge action. We also have $I_{\widetilde{X}} = \bigcap_{t\in\mathbb{T}} I_{\widetilde{Y}_t}$ where $\widetilde{X} = (X, X^{(\infty)})$ and $X = \{\gamma \in \Gamma \mid ([\gamma], \theta) \in Y$ for some $\theta \in \mathbb{T}\}$.*

Proof. See [Ka1, Proposition 5.24]. ∎

Proposition 6.10 *For an ω-invariant pair $\widetilde{X} = (X, X^{(\infty)})$ of subsets of Γ, the pair $\widetilde{Y} = ([X] \times \mathbb{T}, X^{(\infty)})$ is ω-invariant and $I_{\widetilde{Y}} = I_{\widetilde{X}}$.*

Proof. Obvious by Proposition 6.9. ∎

Now, we turn to showing that $I_{\widetilde{Y}_I} = I$ for any ideal I (Theorem 6.30). To see this, we examine the primitive ideal space of $\mathcal{O}_\infty \rtimes_{\alpha^\omega} G$. Set $\overline{\mathrm{sg}}_1(\omega) = \overline{\mathrm{sg}}(\omega) \setminus \{0, \omega_1, \ldots, (K-1)\omega_1\}$.

Lemma 6.11 *We have $\overline{\mathrm{sg}}_1(\omega) = \overline{\bigcup_{i=2}^\infty (\mathrm{sg}(\omega) + \omega_i)}$ and $\overline{\mathrm{sg}}_1(\omega)$ is an ω-invariant set.*

Proof. For $\gamma \in \overline{\mathrm{sg}}(\omega)$, we can find $\mu_k \in \mathcal{W}_\infty$ such that $\gamma = \lim_{k\to\infty} \omega_{\mu_k}$. If $\mu_k = (1, 1, \ldots, 1)$ for sufficiently large k, then $\gamma = m\omega_1$ for some $m \in \mathbb{N}$. Hence for $\gamma \in \overline{\mathrm{sg}}_1(\omega)$, we can find $\mu_k \in \mathcal{W}_\infty$ with $\omega_{\mu_k} \in \bigcup_{i=2}^\infty (\mathrm{sg}(\omega) + \omega_i)$ such that $\gamma = \lim_{k\to\infty} \omega_{\mu_k}$. Thus $\overline{\mathrm{sg}}_1(\omega) \subset \overline{\bigcup_{i=2}^\infty (\mathrm{sg}(\omega) + \omega_i)}$. To prove the other inclusion, suppose $m\omega_1 \in \overline{\bigcup_{i=2}^\infty (\mathrm{sg}(\omega) + \omega_i)}$ for some $0 \leq m < K$ and we will derive a contradiction. In this case, 0 is also in $\overline{\bigcup_{i=2}^\infty (\mathrm{sg}(\omega) + \omega_i)}$. Hence there exists a sequence $\{\mu_k\}$ in $\mathcal{W}_\infty$ with $S_{\mu_k}^* S_1 = 0$ such that $0 = \lim_{k\to\infty} \omega_{\mu_k}$. This contradicts the fact that ω does not satisfy Condition 5.1. Therefore $\overline{\mathrm{sg}}_1(\omega) = \overline{\bigcup_{i=2}^\infty (\mathrm{sg}(\omega) + \omega_i)}$. From this equality, it is easy to see that $\overline{\mathrm{sg}}_1(\omega)$ is an ω-invariant set. ∎

Corollary 6.12 *For any $\gamma_0 \in \Gamma$, there exists a compact neighborhood X of γ_0 satisfying that $X \cap (X + \gamma) = \emptyset$ for any $\gamma \in \overline{\mathrm{sg}}(\omega) \setminus \{0\}$.*

Proof. Since $\overline{\mathrm{sg}}(\omega) \setminus \{0\} = \overline{\mathrm{sg}}_1(\omega) \cup \{\omega_1, 2\omega_1, \ldots, (K-1)\omega_1\}$ is closed by Lemma 6.11, there exists a neighborhood U of 0 with $U \cap (\overline{\mathrm{sg}}(\omega) \setminus \{0\}) = \emptyset$. If we take a compact neighborhood V of 0 such that $V - V \subset U$, then $X = \gamma_0 + V$ becomes a desired compact neighborhood of γ_0. ∎

Lemma 6.13 *For an ω-invariant set X, we have $H_X = \bigcap_{n=1}^\infty \overline{\bigcup_{i=n}^\infty (X + \omega_i)}$. If two ω-invariant sets X_1 and X_2 satisfy $X_1 \subset X_2$, then $H_{X_1} \subset H_{X_2}$.*

Proof. The former part follows from $X = X + \omega_1$, and this implies the latter part. ∎

Proposition 6.14 *For any $\gamma \in \Gamma$, we have $\gamma \notin H_{\gamma + \overline{\mathrm{sg}}(\omega)}$.*

Proof. By Lemma 6.13, we have

$$H_{\gamma+\overline{\mathrm{sg}}(\omega)} = \bigcap_{n=1}^{\infty} \overline{\bigcup_{i=n}^{\infty}(\gamma+\overline{\mathrm{sg}}(\omega)+\omega_i)} \subset \overline{\bigcup_{i=2}^{\infty}(\gamma+\overline{\mathrm{sg}}(\omega)+\omega_i)} = \gamma+\overline{\mathrm{sg}}_1(\omega).$$

Hence $\gamma \notin H_{\gamma+\overline{\mathrm{sg}}(\omega)}$. ∎

For $\gamma \in \Gamma$, we set $P_\gamma = I_{\widetilde{X}}$ where $\widetilde{X} = (\gamma+\overline{\mathrm{sg}}(\omega), H_{\gamma+\overline{\mathrm{sg}}(\omega)} \cup \{\gamma\})$ which is a prime ω-invariant pair. We will show that P_γ is the unique primitive ideal satisfying that $\widetilde{X}_{P_\gamma} = (\gamma+\overline{\mathrm{sg}}(\omega), H_{\gamma+\overline{\mathrm{sg}}(\omega)} \cup \{\gamma\})$. To see this, we need the following lemma.

Lemma 6.15 *Let I be an ideal of $\mathcal{O}_\infty \rtimes_{\alpha^\omega} G$ with $X_I = \overline{X_I^{(\infty)} + \mathrm{sg}(\omega)}$. Then $I = I_{\widetilde{X}_I}$.*

Proof. By the argument in the proof of Proposition 5.2 and Theorem 5.3, it suffices to show that $\|x_0\| \leq \|x\|$ for $x = \sum_{l=1}^{L} S_{\mu_l} f_l S_{\nu_l}^* \in (\mathcal{O}_\infty \rtimes_{\alpha^\omega} G)/I$ and $x_0 = \sum_{|\mu_l|=|\nu_l|} S_{\mu_l} f_l S_{\nu_l}^*$. If we choose a positive integer n so that $|\mu_l|, |\nu_l| \leq n$ and $\mu_l, \nu_l \in \mathcal{W}_n$ for any l, then $x_0 \in \mathcal{F}_I^{(n)}$. We can find $x_0^{(k)} \in \mathcal{F}_I^{(n,k)}$ $(0 \leq k \leq n-1)$ and $x_0^{(n)} \in \mathcal{G}_I^{(n,n)}$ such that $x_0 = \sum_{k=0}^{n} x_0^{(k)}$. We have $\|x_0\| = \max\{\|x_0^{(0)}\|, \ldots, \|x_0^{(n)}\|\}$. In the case that $\|x_0\| = \|x_0^{(k)}\|$ for some $k \leq n-1$, we can prove $\|x_0\| \leq \|x\|$ in a similar way to the proof of Proposition 5.2. In the case that $\|x_0\| = \|x_0^{(n)}\|$, there exists $\gamma_0 \in X_I$ such that $\|x_0\| = \|x_0^{(n)}(\gamma_0)\|$. Since $X_I = \overline{X_I^{(\infty)} + \mathrm{sg}(\omega)}$, there exist a sequence $\mu_1, \mu_2, \ldots \in \mathcal{W}_\infty$ and a sequence $\gamma_1, \gamma_2, \ldots, \in X_I^{(\infty)}$ such that $\gamma_0 = \lim_{k\to\infty}(\gamma_k + \omega_{\mu_k})$. We can find sequences $\mu'_1, \mu'_2, \ldots \in \mathcal{W}_\infty$ and $\nu_1, \nu_2, \ldots \in \mathcal{W}_n$ such that $\omega_{\mu_k} = \omega_{\mu'_k} + \omega_{\nu_k}$ and none of $1, 2, \ldots, n$ appears in the word μ'_k for any k. For $k \in \mathbb{Z}_+$, define a partial isometry $u_k = \sum_{\mu \in \mathcal{W}_n^{(n)}} S_\mu S_{\nu_k} p_n S_\mu^*$. We have $u_k^* x u_k = u_k^* x_0 u_k = u_k^* x_0^{(n)} u_k = \pi_n(\sigma_{\omega_{\nu_k}} x_0^{(n)})$, where π_n is the natural surjection from $\mathcal{G}_I^{(n,n)}$ onto $\mathcal{F}_I^{(n,n)}$. Since $\gamma_k \in X_I^{(\infty)}$, we have $\gamma_k + \omega_{\mu'_k} \in X_I^{(n)}$. Hence

$$\|\pi_n(\sigma_{\omega_{\nu_k}} x_0^{(n)})\| \geq \|\sigma_{\omega_{\nu_k}} x_0^{(n)}(\gamma_k + \omega_{\mu'_k})\| = \|x_0^{(n)}(\gamma_k + \omega_{\mu'_k} + \omega_{\nu_k})\|.$$

Therefore we get

$$\|x_0\| = \|x_0^{(n)}(\gamma_0)\| = \lim_{k\to\infty} \|x_0^{(n)}(\gamma_k + \omega_{\mu'_k} + \omega_{\nu_k})\| \leq \|x\|.$$

We are done. ∎

Proposition 6.16 *For any $\gamma \in \Gamma$, the ideal P_γ is the unique primitive ideal satisfying that $\widetilde{X}_{P_\gamma} = (\gamma+\overline{\mathrm{sg}}(\omega), H_{\gamma+\overline{\mathrm{sg}}(\omega)} \cup \{\gamma\})$.*

Proof. To prove that P_γ is primitive, it suffices to show that it is prime because $\mathcal{O}_\infty \rtimes_{\alpha^\omega} G$ is separable. Let I_1, I_2 be ideals of $\mathcal{O}_\infty \rtimes_{\alpha^\omega} G$ with $I_1 \cap I_2 = P_\gamma$. Then we get $\widetilde{X}_{I_1} \cup \widetilde{X}_{I_2} = \widetilde{X}_{P_\gamma}$. Since $\widetilde{X}_{P_\gamma}$ is a prime ω-invariant pair, we have either $\widetilde{X}_{I_1} = \widetilde{X}_{P_\gamma}$ or $\widetilde{X}_{I_2} = \widetilde{X}_{P_\gamma}$. By Lemma 6.15, we have either $I_1 = P_\gamma$ or $I_2 = P_\gamma$. Therefore P_γ is primitive. The uniqueness follows from Lemma 6.15. ∎

We denote by $\mathit{\Delta}$ the set of prime ω-invariant sets which are not of the form $\gamma+\overline{\mathrm{sg}}(\omega)$. For $X \in \mathit{\Delta}$, we denote by P_X the ideal $I_{\widetilde{X}}$ for $\widetilde{X} = (X, H_X)$ which is a prime ω-invariant pair. We will show that for any $X \in \mathit{\Delta}$, P_X is the unique primitive ideal satisfying $\widetilde{X}_{P_X} = (X, H_X)$.

Lemma 6.17 *Let $X \in \Delta$ and $\gamma \in X$. Then there exist a sequence $\mu_1, \mu_2, \ldots$ in $\mathcal{W}_\infty$ and a sequence $\gamma_1, \gamma_2, \ldots$ in X such that $S_{\mu_k}^* S_1 = 0$ for any k and $\gamma = \lim_{k\to\infty}(\gamma_k + \omega_{\mu_k})$.*

Proof. Since $X \in \Delta$, there exists $\gamma' \in X \setminus (\gamma + \overline{\mathrm{sg}}(\omega))$. Since X is prime, Proposition 4.5 gives us two sequences $\mu_1, \mu_2, \ldots, \nu_1, \nu_2, \ldots$ in $\mathcal{W}_\infty$ and a sequence $\gamma_1, \gamma_2, \ldots$ in X with $\gamma = \lim_{k\to\infty}(\gamma_k + \omega_{\mu_k})$ and $\gamma' = \lim_{k\to\infty}(\gamma_k + \omega_{\nu_k})$. We will show that we can choose such μ_k satisfying $S_{\mu_k}^* S_1 = 0$. If not so, then $\mu_k = (1, 1, \ldots, 1)$ for sufficiently large k. This implies $\gamma' = \lim_{k\to\infty}(\gamma - |\mu_k|\omega_1 + \omega_{\nu_k})$ which contradicts the fact that $\gamma' \notin \gamma + \overline{\mathrm{sg}}(\omega)$. Therefore we can find desired sequences. ∎

Lemma 6.18 *If an ideal I of $\mathcal{O}_\infty \rtimes_{\alpha^\omega} G$ satisfies $X_I \in \Delta$, then $I = I_{\widetilde{X}_I}$.*

Proof. Similarly as the proof of Lemma 6.15, it suffices to show that $\|x_0\| \leq \|x\|$ for $x = \sum_{l=1}^{L} S_{\mu_l} f_l S_{\nu_l}^* \in (\mathcal{O}_\infty \rtimes_{\alpha^\omega} G)/I$ and $x_0 = \sum_{|\mu_l|=|\nu_l|} S_{\mu_l} f_l S_{\nu_l}^* \in \mathcal{F}_I^{(n)}$. We can find $x_0^{(k)} \in \mathcal{F}_I^{(n,k)}$ $(0 \leq k \leq n-1)$ and $x_0^{(n)} \in \mathcal{G}_I^{(n,n)}$ such that $x_0 = \sum_{k=0}^{n} x_0^{(k)}$. We have $\|x_0\| = \max\{\|x_0^{(0)}\|, \ldots, \|x_0^{(n)}\|\}$. In the case that $\|x_0\| = \|x_0^{(k)}\|$ for some $k \leq n-1$, we can prove $\|x_0\| \leq \|x\|$ in a similar way to the proof of Proposition 5.2. In the case that $\|x_0\| = \|x_0^{(n)}\|$, there exists $\gamma_0 \in X_I$ such that $\|x_0\| = \|x_0^{(n)}(\gamma_0)\|$. By Lemma 6.17, we have a sequence $\mu_1, \mu_2, \ldots$ in $\mathcal{W}_\infty$ and a sequence $\gamma_1, \gamma_2, \ldots$ in X_I such that $S_{\mu_k}^* S_1 = 0$ and $\gamma = \lim_{k\to\infty}(\gamma_k + \omega_{\mu_k})$. For $k \in \mathbb{Z}_+$, set a partial isometry $u_k = \sum_{\mu \in \mathcal{W}_n^{(n)}} S_\mu S_1^{Kn} S_{\mu_k} S_\mu^*$. We have $u_k^* x u_k = u_k^* x_0 u_k = u_k^* x_0^{(n)} u_k = \sigma_{\omega_{\mu_k}} x_0^{(n)}$. Since $\gamma_k \in X_I$, we have

$$\|u_k^* x u_k\| \geq \|\sigma_{\omega_{\mu_k}} x_0^{(n)}(\gamma_k)\| = \|x_0^{(n)}(\gamma_k + \omega_{\mu_k})\|.$$

Therefore we get

$$\|x_0\| = \|x_0^{(n)}(\gamma)\| = \lim_{k\to\infty} \|x_0^{(n)}(\gamma_k + \omega_{\mu_k})\| \leq \|x\|.$$

We are done. ∎

Proposition 6.19 *For $X \in \Delta$, the ideal P_X is the unique primitive ideal satisfying $\widetilde{X}_{P_X} = (X, H_X)$.*

Proof. With the help of Lemma 6.18, the proof goes similarly as the one in Proposition 6.16. ∎

By Proposition 4.9, the remaining candidates for primitive ideals are ideals P satisfying $\widetilde{X}_P = (\gamma_0 + \overline{\mathrm{sg}}(\omega), H_{\gamma_0 + \overline{\mathrm{sg}}(\omega)})$ for some $\gamma_0 \in \Gamma$. We will determine such primitive ideals.

Definition 6.20 For $([\gamma], \theta) \in \Gamma' \times \mathbb{T}$, we set $Y_{([\gamma],\theta)} = \{([\gamma], \theta)\} \cup \big([\gamma + \overline{\mathrm{sg}}_1(\omega)] \times \mathbb{T}\big)$. Then $\widetilde{Y} = (Y_{([\gamma],\theta)}, H_{\gamma + \overline{\mathrm{sg}}(\omega)})$ is an ω-invariant pair. We write $P_{([\gamma],\theta)}$ for denoting $I_{\widetilde{Y}}$.

We can show that $P_{([\gamma],\theta)}$ is a primitive ideal for any $([\gamma], \theta) \in \Gamma' \times \mathbb{T}$ by using the technique in [Ka1]. To do so, we need Proposition 6.22, which will also be used to determine the topology of primitive ideal space of $\mathcal{O}_\infty \rtimes_{\alpha^\omega} G$.

Lemma 6.21 *For an ω-invariant set X, the pair $\widetilde{X} = (X, X)$ is ω-invariant and we have*

$$I_{\widetilde{X}} = \overline{\mathrm{span}}\{S_\mu f S_\nu^* \mid \mu, \nu \in \mathcal{W}_\infty,\ f \in C_0(\Gamma \setminus X)\}.$$

Proof. Clearly, $\widetilde{X} = (X, X)$ is ω-invariant. Set $I = \overline{\mathrm{span}}\{S_\mu f S_\nu^* \mid \mu, \nu \in \mathcal{W}_\infty,\ f \in C_0(\Gamma \setminus X)\}$. In a similar way to the proof of Proposition 3.8, we can see that I is a gauge-invariant ideal of $\mathcal{O}_\infty \rtimes_{\alpha^\omega} G$. We also see that $X_I^{(n)} = X$ for any $n \in \mathbb{N}$ by arguing as in the proof of Proposition 3.9. Hence $I_{\widetilde{X}} = I$ by Theorem 3.16. ∎

Proposition 6.22 *Let X be a compact subset of Γ such that $X \cap (X + \gamma) = \emptyset$ for any $\gamma \in \overline{\mathrm{sg}}(\omega) \setminus \{0\}$, and set $X_1 = X + \overline{\mathrm{sg}}(\omega)$ and $X_2 = X + \overline{\mathrm{sg}}_1(\omega)$. Then we have that $\widetilde{X}_0 = (X_1, X_1)$, $\widetilde{X}_1 = (X_1, X_2)$ and $\widetilde{X}_2 = (X_2, X_2)$ are ω-invariant pairs, and that*

$$I_{\widetilde{X}_2}/I_{\widetilde{X}_1} \cong \mathbb{K} \otimes C(X \times \mathbb{T}), \qquad I_{\widetilde{X}_1}/I_{\widetilde{X}_0} \cong \mathbb{K} \otimes C(X_1 \setminus X_2).$$

Proof. Since X is compact and $\overline{\mathrm{sg}}(\omega)$ is closed, $X_1 = X + \overline{\mathrm{sg}}(\omega)$ becomes closed. The same reason shows that X_2 is closed. By Lemma 6.11, both X_1 and X_2 are ω-invariant and $X_2 = \overline{\bigcup_{i=2}^\infty (X_1 + \omega_i)}$. Therefore $\widetilde{X}_0, \widetilde{X}_1, \widetilde{X}_2$ are ω-invariant pairs. Since $I_{\widetilde{X}_1} \cap p_1 C_0(\Gamma) = p_1 C_0(\Gamma \setminus X_2)$, we have $p_1 f = 0$ for any $f \in C_0(X_1 \setminus X_2) \subset I_{\widetilde{X}_2}/I_{\widetilde{X}_1}$. Note that $X_1 \setminus X_2$ is a disjoint union of compact sets $X, X + \omega_1, \ldots, X + (K-1)\omega_1$. For $f \in C(X + m\omega_1) \subset I_{\widetilde{X}_2}/I_{\widetilde{X}_1}$ with $0 < m < K$, we have $\sigma_{m\omega_1} f \in C(X)$ and

$$\begin{aligned} S_1^m \sigma_{m\omega_1} f S_1^{*m} &= S_1^{m-1} S_1 S_1^* \sigma_{(m-1)\omega_1} f S_1^{*m-1} = S_1^{m-1} \sigma_{(m-1)\omega_1} f S_1^{*m-1} \\ &= \cdots = f. \end{aligned}$$

Hence, we have $I_{\widetilde{X}_2}/I_{\widetilde{X}_1} = \overline{\mathrm{span}}\{S_\mu f S_\nu^* \mid \mu, \nu \in \mathcal{W}_\infty, f \in C(X)\}$ by Lemma 6.21. Set $\mathcal{W}_\infty^+ = \mathcal{W}_\infty \setminus \{\mu 1^K \in \mathcal{W}_\infty \mid \mu \in \mathcal{W}_\infty\}$ and denote by χ the characteristic function of X. Then $\{S_\mu \chi S_\nu^*\}_{\mu,\nu \in \mathcal{W}_\infty^+}$ satisfies the relation of matrix units and $\sum_{\mu \in \mathcal{W}_\infty^+} S_\mu \chi S_\mu^* = 1$ (strictly). Hence we have $I_{\widetilde{X}_2}/I_{\widetilde{X}_1} \cong \mathbb{K} \otimes B$ where $B = \chi (I_{\widetilde{X}_2}/I_{\widetilde{X}_1}) \chi$. We have

$$B = \overline{\mathrm{span}}\{\chi S_\mu f S_\nu^* \chi \mid \mu, \nu \in \mathcal{W}_\infty, f \in C(X)\} = \overline{\mathrm{span}}\{(S_1^K)^m f \mid m \in \mathbb{Z}, f \in C(X)\}.$$

Since B is generated by $C(X)$ and a unitary $S_1^K \chi$ which commute with each other and since B is globally invariant under the gauge action, we have $B \cong C(X \times \mathbb{T})$. Therefore we get $I_{\widetilde{X}_2}/I_{\widetilde{X}_1} \cong \mathbb{K} \otimes C(X \times \mathbb{T})$.

By the definition,

$$I_{\widetilde{X}_1}/I_{\widetilde{X}_0} = \overline{\mathrm{span}}\{S_\mu p_n f S_\nu^* \mid \mu, \nu \in \mathcal{W}_\infty, n \geq 1,\ f \in C(X_1 \setminus X_2)\}.$$

For $f \in C(X_1 \setminus X_2) \subset I_{\widetilde{X}_1}/I_{\widetilde{X}_0}$ and $i \geq 2$, we have $S_i S_i^* f = S_i \sigma_{\omega_i} f S_i^* = 0$. Hence $p_n f = p_1 f$ for any $n \geq 1$ and any $f \in C(X_1 \setminus X_2)$. Thus $I_{X_1,X_2}/I_{X_1,X_1} = \overline{\mathrm{span}}\{S_\mu p_2 f S_\nu^* \mid \mu, \nu \in \mathcal{W}_\infty,\ f \in C(X_1 \setminus X_2)\}$. We can show that $\{S_\mu p_2 \chi' S_\nu^*\}_{\mu,\nu \in \mathcal{W}_\infty}$ satisfies the relation of matrix units and $\sum_{\mu \in \mathcal{W}_\infty} S_\mu p_2 \chi' S_\mu^* = 1$ (strictly), where χ' is the characteristic function of $X_1 \setminus X_2$. Hence we have $I_{\widetilde{X}_1}/I_{\widetilde{X}_0} \cong \mathbb{K} \otimes B'$ where

$$B' = p_2 \chi' (I_{\widetilde{X}_1}/I_{\widetilde{X}_0}) p_2 \chi' = \overline{\mathrm{span}}\{p_2 f \mid f \in C(X_1 \setminus X_2)\} \cong C(X_1 \setminus X_2).$$

Therefore we get $I_{\widetilde{X}_1}/I_{\widetilde{X}_0} \cong \mathbb{K} \otimes C(X_1 \setminus X_2)$. ∎

With the help of Proposition 6.22, we have the following proposition by exactly the same argument as the proof of [Ka1, Proposition 5.41].

Proposition 6.23 *For $\gamma_0 \in \Gamma$, the set of all primitive ideals P satisfying $\widetilde{X}_P = (\gamma_0 + \overline{\mathrm{sg}}(\omega), H_{\gamma_0+\overline{\mathrm{sg}}(\omega)})$ is $\{P_{([\gamma_0],\theta)} \mid \theta \in \mathbb{T}\}$.*

Now, we can describe the primitive ideal space $\mathrm{Prim}(\mathcal{O}_\infty \rtimes_{\alpha^\omega} G)$ of $\mathcal{O}_\infty \rtimes_{\alpha^\omega} G$ as follows.

Proposition 6.24 *We have* $\mathrm{Prim}(\mathcal{O}_\infty \rtimes_{\alpha^\omega} G) = \{P_z \mid z \in (\Gamma' \times \mathbb{T}) \sqcup \Gamma \sqcup \Delta\}$*, where $\sqcup$ means a disjoint union.*

The primitive ideal space $\mathrm{Prim}(\mathcal{O}_\infty \rtimes_{\alpha^\omega} G)$ is a topological space whose closed sets are given by $\{P \in \mathrm{Prim}(\mathcal{O}_\infty \rtimes_{\alpha^\omega} G) \mid I \subset P\}$ for ideals I. We will investigate which subset of $(\Gamma' \times \mathbb{T}) \sqcup \Gamma \sqcup \Delta$ corresponds to a closed subset of $\mathrm{Prim}(\mathcal{O}_\infty \rtimes_{\alpha^\omega} G)$. By Corollary 6.8, the following is easy to verify.

Lemma 6.25 *Let $\widetilde{Y} = (Y, X^{(\infty)})$ be an ω-invariant set.*

(i) *For $([\gamma],\theta) \in \Gamma' \times \mathbb{T}$, we have $I_{\widetilde{Y}} \subset P_{([\gamma],\theta)}$ if and only if $([\gamma],\theta) \in Y$.*

(ii) *For $\gamma \in \Gamma$, we have $I_{\widetilde{Y}} \subset P_\gamma$ if and only if $\gamma \in X^{(\infty)}$.*

(iii) *For $X \in \Delta$, we have $I_{\widetilde{Y}} \subset P_X$ if and only if $[X] \times \mathbb{T} \subset Y$.*

Lemma 6.26 *Let X be a compact subset of Γ such that $X \cap (X + \gamma) = \emptyset$ for any $\gamma \in \overline{\mathrm{sg}}(\omega) \setminus \{0\}$, and set $X_1 = X + \overline{\mathrm{sg}}(\omega)$ and $X_2 = X + \overline{\mathrm{sg}}_1(\omega)$, which are ω-invariant sets. If $X_0 \in \Delta$ satisfies $X_1 \supset X_0$, then $X_2 \supset X_0$.*

Proof. To the contrary, assume $X_0 \in \Delta$ satisfies $X_1 \supset X_0$ and $X_2 \not\supset X_0$. Then $X_0 \cap X \neq \emptyset$ and $(X_0 \cap X) + \overline{\mathrm{sg}}(\omega)$ is an ω-invariant set satisfying $(X_0 \cap X) + \overline{\mathrm{sg}}(\omega) \subset X_0$. Since $((X_0 \cap X) + \overline{\mathrm{sg}}(\omega)) \cup X_2 \supset X_0$ and X_0 is prime, we have $(X_0 \cap X) + \overline{\mathrm{sg}}(\omega) \supset X_0$. Hence $X_0 = (X_0 \cap X) + \overline{\mathrm{sg}}(\omega)$. If $X_0 \cap X$ has two points γ_1, γ_2, then we can take open sets U_1, U_2 such that $\gamma_1 \in U_1$, $\gamma_2 \in U_2$, $U_1 \cap U_2 = \emptyset$. Two ω-invariant sets $X_1' = (X_0 \cap X \setminus U_1) + \overline{\mathrm{sg}}(\omega)$, $X_2' = (X_0 \cap X \setminus U_2) + \overline{\mathrm{sg}}(\omega)$ satisfies $X_1' \not\supset X_0$, $X_2' \not\supset X_0$ and $X_1' \cup X_2' = X_0$. This contradicts the primeness of X_0. Hence $X_0 \cap X$ is just a point. However, this contradicts the fact that $X_0 \in \Delta$. Therefore $X_2 \supset X_0$ when $X_0 \in \Delta$ satisfies $X_1 \supset X_0$. ∎

Lemma 6.27 *Let $\widetilde{Y}_\lambda = (Y_\lambda, X_\lambda^{(\infty)})$ be an ω-invariant pair for each $\lambda \in \Lambda$. Set $I = \bigcap_{\lambda \in \Lambda} I_{\widetilde{Y}_\lambda}$. Then $Y_I = \overline{\bigcup_{\lambda \in \Lambda} Y_\lambda}$.*

Proof. For any $\lambda \in \Lambda$, we have $Y_I \supset Y_\lambda$ because $I \subset I_{\widetilde{Y}_\lambda}$. Hence we get $Y_I \supset \overline{\bigcup_{\lambda \in \Lambda} Y_\lambda}$. Take $([\gamma_0],\theta_0) \notin \overline{\bigcup_{\lambda \in \Lambda} Y_\lambda}$. Then there exists a neighborhood U of $([\gamma_0],\theta_0)$ satisfying $U \cap \overline{\bigcup_{\lambda \in \Lambda} Y_\lambda} = \emptyset$. By the same argument as in the proof of [Ka1, Lemma 5.22], we can find $x_0 \in A$ such that $\psi_{([\gamma_0],\theta_0)}(x_0) \neq 0$ and $\psi_{([\gamma],\theta)}(x_0) = 0$ if $([\gamma],\theta) \notin U$ and $\varphi_\gamma(x_0) = 0$ if $([\gamma] \times \mathbb{T}) \cap U = \emptyset$. Therefore we have $x_0 \in I$, and it implies that $([\gamma_0],\theta_0) \notin Y_I$. Thus $Y_I = \overline{\bigcup_{\lambda \in \Lambda} Y_\lambda}$. ∎

Lemma 6.28 *For any $X \in \Delta$, we have $P_X = \bigcap_{z \in [X] \times \mathbb{T}} P_z$.*

Proof. By Lemma 6.25, we have $P_X \subset \bigcap_{z\in[X]\times\mathbb{T}} P_z$. By Lemma 6.27, we have $Y_{\bigcap_{z\in[X]\times\mathbb{T}} P_z} = [X]\times\mathbb{T}$. Hence we have $\bigcap_{z\in[X]\times\mathbb{T}} P_z \subset P_X$ by Lemma 6.25. Thus $P_X = \bigcap_{z\in[X]\times\mathbb{T}} P_z$. ∎

In the proof of the following proposition, we use the fact that the subset $\{P \in \mathrm{Prim}(\mathcal{O}_\infty \rtimes_{\alpha^\omega} G) \mid I_1 \subset P, I_2 \not\subset P\}$ of $\mathrm{Prim}(\mathcal{O}_\infty \rtimes_{\alpha^\omega} G)$ is homeomorphic to $\mathrm{Prim}(I_2/I_1)$, for two ideals I_1, I_2 of $\mathcal{O}_\infty \rtimes_{\alpha^\omega} G$ with $I_1 \subset I_2$.

Proposition 6.29 *Let $Z = Y \sqcup X^{(\infty)} \sqcup \Lambda$ be a subset of $(\Gamma'\times\mathbb{T})\sqcup\Gamma\sqcup\Delta$. The set $P_Z = \{P_z \mid z \in Z\}$ is closed in $\mathrm{Prim}(\mathcal{O}_\infty \rtimes_{\alpha^\omega} G)$ if and only if $(Y, X^{(\infty)})$ is an ω-invariant set and $\Lambda = \{X \in \Delta \mid [X]\times\mathbb{T} \subset Y\}$.*

Proof. Let us take a subset $Z = Y \sqcup X^{(\infty)} \sqcup \Lambda$ of $(\Gamma'\times\mathbb{T})\sqcup\Gamma\sqcup\Delta$ satisfying that $(Y, X^{(\infty)})$ is an ω-invariant set and $\Lambda = \{X \in \Delta \mid [X]\times\mathbb{T} \subset Y\}$. Then the set $P_Z = \{P_z \mid z \in Z\}$ coincides with the closed subset defined by the ideal $I_{\widetilde{Y}}$ by Lemma 6.25.

Conversely, assume P_Z is closed, that is, there exists an ideal I of $\mathcal{O}_\infty \rtimes_{\alpha^\omega} G$ with $Z = \{z \in Y \sqcup X^{(\infty)} \sqcup \Lambda \mid I \subset P_z\}$. We first show that Y and $X^{(\infty)}$ is closed. Take $\gamma_0 \in \Gamma$ arbitrarily. By Corollary 6.12, there exists a compact neighborhood X of γ_0 such that $X \cap (X+\gamma) = \emptyset$ for any $\gamma \in \overline{\mathrm{sg}}(\omega)\setminus\{0\}$. Set $\widetilde{X}_0 = (X_1, X_1)$, $\widetilde{X}_1 = (X_1, X_2)$ and $\widetilde{X}_2 = (X_2, X_2)$ where $X_1 = X + \overline{\mathrm{sg}}(\omega)$ and $X_2 = X + \overline{\mathrm{sg}}_1(\omega)$. Note that $X \ni \gamma \mapsto [\gamma] \in [X_1 \setminus X_2]$ is a homeomorphism. By Lemma 6.25 and Lemma 6.26, we have

$$\begin{aligned} &\big\{z \in (\Gamma'\times\mathbb{T})\sqcup\Gamma\sqcup\Delta \mid I_{\widetilde{X}_1} \subset P_z, I_{\widetilde{X}_2} \not\subset P_z\big\} = [X_1\setminus X_2]\times\mathbb{T} \subset \Gamma'\times\mathbb{T},\\ &\big\{z \in (\Gamma'\times\mathbb{T})\sqcup\Gamma\sqcup\Delta \mid I_{\widetilde{X}_0} \subset P_z, I_{\widetilde{X}_1} \not\subset P_z\big\} = X_1\setminus X_2 \subset \Gamma. \end{aligned}$$

By Proposition 6.22, the map $[X_1\setminus X_2]\times\mathbb{T} \ni z \mapsto P_z \in \mathrm{Prim}(\mathcal{O}_\infty \rtimes_{\alpha^\omega} G)$ is a homeomorphism from $[X_1\setminus X_2]\times\mathbb{T}$, whose topology is the relative topology of $\Gamma'\times\mathbb{T}$, to the subset $\{P \in \mathrm{Prim}(\mathcal{O}_\infty \rtimes_{\alpha^\omega} G) \mid I_{\widetilde{X}_1} \subset P, I_{\widetilde{X}_2} \not\subset P\}$ of $\mathrm{Prim}(\mathcal{O}_\infty \rtimes_{\alpha^\omega} G)$. The set $Y \cap ([X_1\setminus X_2]\times\mathbb{T}) \subset \Gamma'\times\mathbb{T}$ is closed in $[X_1\setminus X_2]\times\mathbb{T}$ because P_Y is closed. Hence, the subset Y is closed in $\Gamma'\times\mathbb{T}$. Similarly $X^{(\infty)}$ is closed in Γ. Set $X = \{\gamma \in \Gamma \mid ([\gamma],\theta) \in Y \text{ for some } \theta \in \mathbb{T}\}$, which is closed because Y is closed. Set $J = \bigcap_{([\gamma],\theta)\in Y} P_{([\gamma],\theta)}$. We have $I \subset J$. By Lemma 6.27, we have $Y_J = Y$. Hence $H_X \subset X_J^{(\infty)}$. We have $J \subset P_\gamma$ for any $\gamma \in H_X$ by Lemma 6.25. Therefore $H_X \subset X^{(\infty)}$. We have $([\gamma+\omega_i],\theta') \in Y$ for any $([\gamma],\theta) \in Y$, any $i \geq 2$ and any $\theta' \in \mathbb{T}$ because $P_{([\gamma],\theta)} \subset P_{([\gamma+\omega_i],\theta')}$. Hence we get $[X+\omega_i]\times\mathbb{T} \subset Y$. We also have $[X^{(\infty)}]\times\mathbb{T} \subset Y$ because $P_\gamma \subset P_{([\gamma],\theta)}$ for any $([\gamma],\theta) \in \Gamma'\times\mathbb{T}$. Therefore we have proved that $(Y, X^{(\infty)})$ is an ω-invariant set. Finally, we have $\Lambda = \{X \in \Delta \mid [X]\times\mathbb{T} \subset Y\}$ by Lemma 6.28. It completes the proof. ∎

By the proposition above, we get the following.

Theorem 6.30 *When ω does not satisfy Condition 5.1, there is a one-to-one correspondence between the set of ideals of $\mathcal{O}_\infty \rtimes_{\alpha^\omega} G$ and the set of ω-invariant pairs of subsets of $\Gamma'\times\mathbb{T}$ and subsets of Γ. Hence for any ideal I of $\mathcal{O}_\infty \rtimes_{\alpha^\omega} G$, we have $I = I_{\widetilde{Y}_I}$.*

Proof. There is a one-to-one correspondence between the set of ideals of $\mathcal{O}_\infty \rtimes_{\alpha^\omega} G$ and the closed subset of $\mathrm{Prim}(\mathcal{O}_\infty \rtimes_{\alpha^\omega} G)$. By Proposition 6.29, the closed subset of $\mathrm{Prim}(\mathcal{O}_\infty \rtimes_{\alpha^\omega} G)$ corresponds bijectively to the set of ω-invariant pairs. ∎

7 More about $\mathcal{O}_\infty \rtimes_{\alpha^\omega} G$

In this section, we gather some general results on $\mathcal{O}_\infty \rtimes_{\alpha^\omega} G$. First we compute the strong Connes spectrum of the action $\alpha^\omega : G \curvearrowright \mathcal{O}_\infty$. We need the following lemma.

Lemma 7.1 *For any $\omega \in \Gamma^\infty$, we have $\{0\} \cup H_{\overline{\mathrm{sg}}(\omega)} = \{0\} \cup \bigcap_{n=1}^\infty \overline{\bigcup_{i=n}^\infty (\mathrm{sg}(\omega) + \omega_i)}$.*

Proof. It suffices to show that

$$\overline{\mathrm{sg}}(\omega) \setminus \Big(\{0\} \cup \bigcup_{i=1}^{\infty} (\overline{\mathrm{sg}}(\omega) + \omega_i)\Big) \subset \overline{\bigcup_{i=n+1}^{\infty} (\mathrm{sg}(\omega) + \omega_i)}$$

for any $n \in \mathbb{Z}_+$. Take $\gamma \in \overline{\mathrm{sg}}(\omega) \setminus \big(\{0\} \cup \bigcup_{i=1}^\infty (\overline{\mathrm{sg}}(\omega) + \omega_i)\big)$ and $n \in \mathbb{Z}_+$. Since $\gamma \in \overline{\mathrm{sg}}(\omega)$, there exists a sequence $\{\mu_k\} \subset \mathcal{W}_\infty$ such that $\gamma = \lim_{k\to\infty} \omega_{\mu_k}$. We will show that we can find an integer grater than n in the word μ_k for infinitely many k, from which it follows that $\gamma \in \overline{\bigcup_{i=n+1}^\infty (\mathrm{sg}(\omega) + \omega_i)}$. To the contrary, assume that $\mu_k \in \mathcal{W}_n$ for sufficiently large k. Then there exists $i \in \{1, 2, \ldots, n\}$ which appears in μ_k eventually. We have $\gamma - \omega_i = \lim_{k\to\infty} (\omega_{\mu_k} - \omega_i) \in \overline{\mathrm{sg}}(\omega)$. This contradicts the fact that $\gamma \notin \overline{\mathrm{sg}}(\omega) + \omega_i$. Hence $\{0\} \cup H_{\overline{\mathrm{sg}}(\omega)} = \{0\} \cup \bigcap_{n=1}^\infty \overline{\bigcup_{i=n}^\infty (\mathrm{sg}(\omega) + \omega_i)}$. ∎

Proposition 7.2 *The strong Connes spectrum $\widetilde{\Gamma}(\alpha^\omega)$ of the action α^ω is $\{0\} \cup H_{\overline{\mathrm{sg}}(\omega)}$.*

Proof. By [Ki, Lemma 3.4], we have

$$\widetilde{\Gamma}(\alpha^\omega) = \{\gamma \in \Gamma \mid \widehat{\alpha^\omega}_\gamma(I) \subset I, \text{for any ideal } I \text{ of } \mathcal{O}_\infty \rtimes_{\alpha^\omega} G\},$$

where $\widehat{\alpha^\omega} : \Gamma \curvearrowright \mathcal{O}_\infty \rtimes_{\alpha^\omega} G$ is the dual action of α^ω. For an ω-invariant pair $\widetilde{X} = (X, X^{(\infty)})$ and $\gamma \in \Gamma$, we see that $\widehat{\alpha^\omega}_\gamma(I_{\widetilde{X}}) = I_{\widetilde{X}-\gamma}$ where $\widetilde{X} - \gamma = (X - \gamma, X^{(\infty)} - \gamma)$. Hence $\widehat{\alpha^\omega}_\gamma(I_{\widetilde{X}}) \subset I_{\widetilde{X}}$ is equivalent to say that $X+\gamma \subset X$ and $X^{(\infty)}+\gamma \subset X^{(\infty)}$ for an ω-invariant pair $\widetilde{X} = (X, X^{(\infty)})$ and $\gamma \in \Gamma$. Considering the case that $\widetilde{X} = (\overline{\mathrm{sg}}(\omega), \{0\} \cup H_{\overline{\mathrm{sg}}(\omega)})$, we have $(\{0\} \cup H_{\overline{\mathrm{sg}}(\omega)}) + \gamma \subset \{0\} \cup H_{\overline{\mathrm{sg}}(\omega)}$ for $\gamma \in \widetilde{\Gamma}(\alpha^\omega)$. Hence $\widetilde{\Gamma}(\alpha^\omega) \subset \{0\} \cup H_{\overline{\mathrm{sg}}(\omega)}$. Let $(X, X^{(\infty)})$ be an ω-invariant pair. For $\gamma \in X$, we get

$$\gamma + \bigcap_{n=1}^{\infty} \overline{\bigcup_{i=n}^{\infty} (\mathrm{sg}(\omega) + \omega_i)} = \bigcap_{n=1}^{\infty} \overline{\bigcup_{i=n}^{\infty} (\gamma + \mathrm{sg}(\omega) + \omega_i)} \subset \bigcap_{n=1}^{\infty} \overline{\bigcup_{i=n}^{\infty} (X + \omega_i)} \subset H_X \subset X^{(\infty)}.$$

By Lemma 7.1, we have $X^{(\infty)} + (\{0\} \cup H_{\overline{\mathrm{sg}}(\omega)}) \subset X^{(\infty)}$. Since $\{0\} \cup H_{\overline{\mathrm{sg}}(\omega)} \subset \overline{\mathrm{sg}}(\omega)$, we have $X + (\{0\} \cup H_{\overline{\mathrm{sg}}(\omega)}) \subset X$. Hence when ω satisfies Condition 5.1, we have $\widetilde{\Gamma}(\alpha^\omega) \supset \{0\} \cup H_{\overline{\mathrm{sg}}(\omega)}$ by Theorem 5.3, and so $\widetilde{\Gamma}(\alpha^\omega) = \{0\} \cup H_{\overline{\mathrm{sg}}(\omega)}$. Next we consider the case that ω does not satisfy Condition 5.1. For an ω-invariant pair $(Y, X^{(\infty)})$, we have $X + (H_{\overline{\mathrm{sg}}(\omega)} \setminus \{0\}) \subset X^{(\infty)}$ by the former part of this proof, where $X = \{\gamma \in \Gamma \mid ([\gamma], \theta) \in Y \text{ for some } \theta \in \mathbb{T}\}$. Hence for any $([\gamma_0], \theta_0) \in Y$ and $\gamma \in H_{\overline{\mathrm{sg}}(\omega)} \setminus \{0\}$, we have $\gamma_0 + \gamma \in X^{(\infty)}$ because $\gamma_0 \in X$. Since $[X^{(\infty)}] \times \mathbb{T} \subset Y$, we have $([\gamma_0 + \gamma], \theta_0) \in Y$. Therefore we also have $\{0\} \cup H_{\overline{\mathrm{sg}}(\omega)} \subset \widetilde{\Gamma}(\alpha^\omega)$ by Theorem 6.30. Thus $\widetilde{\Gamma}(\alpha^\omega) = \{0\} \cup H_{\overline{\mathrm{sg}}(\omega)}$. ∎

Next we give necessary and sufficient conditions for $\omega \in \Gamma^\infty$ that the crossed product $\mathcal{O}_\infty \rtimes_{\alpha^\omega} G$ becomes simple or primitive.

Lemma 7.3 *Let I be an ideal of the crossed product $\mathcal{O}_\infty \rtimes_{\alpha^\omega} G$. Then $I = 0$ if and only if $X_I = \Gamma$.*

Proof. The "only if" part is trivial. One can easily prove the "if" part by the same arguments as in the proofs of Proposition 5.2 and Theorem 5.3. ∎

Proposition 7.4 *For $\omega \in \Gamma^\infty$, the following are equivalent:*

(i) *The crossed product $\mathcal{O}_\infty \rtimes_{\alpha^\omega} G$ is simple.*

(ii) *There are no ω-invariants sets other than Γ and $\emptyset$.*

(iii) $\Gamma = \overline{\mathrm{sg}}(\omega)$.

If $\mathcal{O}_\infty \rtimes_{\alpha^\omega} G$ is simple, then it is purely infinite.

Proof. The equivalence between (i) and (ii) follows from Lemma 7.3. (ii) implies (iii) because $\overline{\mathrm{sg}}(\omega)$ is ω-invariant. (iii) implies (ii) because $X = X + \overline{\mathrm{sg}}(\omega)$ if X is ω-invariant.

For the last statement, see [Ka2, Proposition 5.2]. ∎

The equivalence between (i) and (iii) was already proved by A. Kishimoto [Ki] by using strong Connes spectrum. Note that the strong Connes spectrum $\widetilde{\Gamma}(\alpha^\omega)$ is equal to Γ if and only if $\overline{\mathrm{sg}}(\omega) = \Gamma$ by Proposition 7.2.

Proposition 7.5 *The following conditions for $\omega \in \Gamma^\infty$ are equivalent:*

(i) *The crossed product $\mathcal{O}_\infty \rtimes_{\alpha^\omega} G$ is primitive.*

(ii) *Γ is a prime ω-invariant set.*

(iii) *The closed group generated by $\omega_1, \omega_2, \ldots$ is equal to Γ.*

Proof. (i)⇒(ii): This follows from Proposition 4.2.

(ii)⇒(i): It suffices to show that 0 is prime. Let I_1, I_2 be ideals of $\mathcal{O}_\infty \rtimes_{\alpha^\omega} G$ with $I_1 \cap I_2 = 0$. We have $X_{I_1} \cup X_{I_2} = X_{I_1 \cap I_2} = \Gamma$. Since Γ is prime, either $X_{I_1} \supset \Gamma$ or $X_{I_2} \supset \Gamma$. If $X_{I_1} \supset \Gamma$ hence $X_{I_1} = \Gamma$, then $I_1 = 0$ by Lemma 7.3. Similarly if $X_{I_2} \supset \Gamma$, then $I_2 = 0$. Thus 0 is prime and so $\mathcal{O}_\infty \rtimes_{\alpha^\omega} G$ is a primitive C^*-algebra.

(ii) $\Longleftrightarrow$ (iii): This follows from Proposition 4.5. ∎

One can prove the equivalence between (i) and (iii) in the above theorem by characterization of primitivity of crossed products in terms of the Connes spectrum due to D. Olesen and G. K. Pedersen [OP] and the computation of the Connes spectrum of our actions α^ω due to A. Kishimoto [Ki].

Proposition 7.6 *The crossed product $\mathcal{O}_\infty \rtimes_{\alpha^\omega} G$ is isomorphic to the Cuntz-Pimsner algebra $\mathcal{O}_E$ of $C_0(\Gamma)$-bimodule $E = C_0(\Gamma)^\infty$, whose left module structure is given by*

$$f \cdot (f_1, f_2, \ldots, f_n, \ldots) = (\sigma_{\omega_1}(f) f_1, \sigma_{\omega_2}(f) f_2, \ldots, \sigma_{\omega_n}(f) f_n, \ldots) \in E$$

for $f \in C_0(\Gamma)$ and $(f_1, f_2, \ldots, f_n, \ldots) \in E$.

Proof. The inclusion $C_0(\Gamma) \hookrightarrow \mathcal{O}_\infty \rtimes_{\alpha^\omega} G$ and $E \ni (0, \dots, 0, f_n, 0 \dots) \mapsto S_n f_n \in \mathcal{O}_\infty \rtimes_{\alpha^\omega} G$ satisfies the conditions in [Pi, Theorem 3.12]. Hence there exists a $*$-homomorphism $\varphi : \mathcal{O}_E \to \mathcal{O}_\infty \rtimes_{\alpha^\omega} G$ which is surjective since $\mathcal{O}_\infty \rtimes_{\alpha^\omega} G$ is generated by $\{S_n f \mid n \in \mathbb{Z}_+,\ f \in C_0(\Gamma)\}$. One can show that φ is injective by using Lemma 3.13. Thus $\mathcal{O}_\infty \rtimes_{\alpha^\omega} G$ is isomorphic to $\mathcal{O}_E$. ∎

Corollary 7.7 *The inclusion $C_0(\Gamma) \hookrightarrow \mathcal{O}_\infty \rtimes_{\alpha^\omega} G$ is a KK-equivalence. Hence for $i = 0, 1$, we have $K_i(\mathcal{O}_\infty \rtimes_{\alpha^\omega} G) = K_i(C_0(\Gamma))$.*

Proof. See [Pi, Corollary 4.5]. ∎

Proposition 7.8 *If $\omega \in \Gamma^\infty$ satisfies $-\omega_i \notin \overline{\{\omega_\mu \mid \mu \in \mathcal{W}_n\}}$ for any $i, n \in \mathbb{Z}_+$, then the crossed product $\mathcal{O}_\infty \rtimes_{\alpha^\omega} G$ is AF-embeddable.*

Proof. See [Ka2, Proposition 5.1]. ∎

References

[BHRS] Bates, T.; Hong, J.; Raeburn, I.; Szymański, W. *The ideal structure of the C*-algebras of infinite graphs.* Preprint.

[C] Cuntz, J. *Simple C^*-algebras generated by isometries.* Comm. Math. Phys. **57** (1977), no. 2, 173–185.

[D] Deaconu, V. *Continuous graphs and C*-algebras.* Operator theoretical methods, 137–149, Theta Found., Bucharest, 2000.

[E] Evans, D. E. *On O_n*. Publ. Res. Inst. Math. Sci. **16** (1980), no. 3, 915–927.

[HS] Hong, J. H.; Szymanski, W. *The primitive ideal space of the C^*-algebras of infinite graphs.* Preprint.

[Ka1] Katsura, T. *The ideal structures of crossed products of Cuntz algebras by quasi-free actions of abelian groups.* Preprint.

[Ka2] Katsura, T. *AF-embeddability of crossed products of Cuntz algebras.* Preprint.

[Ki] Kishimoto, A. *Simple crossed products of C^*-algebras by locally compact abelian groups.* Yokohama Math. J. **28** (1980), no. 1-2, 69–85.

[KK1] Kishimoto, A.; Kumjian, A. *Simple stably projectionless C^*-algebras arising as crossed products.* Canad. J. Math. **48** (1996), no. 5, 980–996.

[KK2] Kishimoto, A.; Kumjian, A. *Crossed products of Cuntz algebras by quasi-free automorphisms.* Operator algebras and their applications, 173–192, Fields Inst. Commun., **13**, Amer. Math. Soc., Providence, RI, 1997.

[OP] Olesen, D.; Pedersen, G. K. *Applications of the Connes spectrum to C^*-dynamical systems.* J. Funct. Anal. **30** (1978), no. 2, 179–197.

[Pi] Pimsner, M. V. *A class of C^*-algebras generalizing both Cuntz-Krieger algebras and crossed products by Z.* Free probability theory, 189–212, Fields Inst. Commun., **12**, Amer. Math. Soc., Providence, RI, 1997.

UTMS

2001–21 Keiko Kawamuro: *An induction for bimodules arising from subfactors.*

2001–22 Yasuyuki Kawahigashi: *Generalized Longo-Rehren subfactors and* α*-induction.*

2001–23 Takeshi Katsura: *The ideal structures of crossed products of Cuntz algebras by quasi-free actions of abelian groups.*

2001–24 Noguchi, junjiro: *Some results in view of Nevanlinna theory.*

2001–25 Fabien Trihan: *Image directe supérieure et unipotence.*

2001–26 Takeshi Saito: *Weight spectral sequences and independence of* ℓ*.*

2001–27 Takeshi Saito: *Log smooth extension of family of curves and semi-stable reduction.*

2001–28 Takeshi Katsura: *AF-embeddability of crossed products of Cuntz algebras.*

2001–29 Toshio Oshima: *Annihilators of generalized Verma modules of the scalar type for classical Lie algebras.*

2001–30 Kim Sungwhan and Masahiro Yamamoto: *Uniqueness in identification of the support of a source term in an elliptic equation.*

2001–31 Tetsuhiro Moriyama: *The mapping class group action on the homology of the configuration spaces of surfaces.*

2001–32 Takeshi Katsura: *On crossed products of the Cuntz algebra* $\mathcal{O}_\infty$ *by quasi-free actions of abelian groups.*

The Graduate School of Mathematical Sciences was established in the University of Tokyo in April, 1992. Formerly there were two departments of mathematics in the University of Tokyo: one in the Faculty of Science and the other in the College of Arts and Sciences. All faculty members of these two departments have moved to the new graduate school, as well as several members of the Department of Pure and Applied Sciences in the College of Arts and Sciences. In January, 1993, the preprint series of the former two departments of mathematics were unified as the Preprint Series of the Graduate School of Mathematical Sciences, The University of Tokyo. For the information about the preprint series, please write to the preprint series office.

ADDRESS:
Graduate School of Mathematical Sciences, The University of Tokyo
3–8–1 Komaba Meguro-ku, Tokyo 153, JAPAN
TEL +81-3-5465-7001 FAX +81-3-5465-7012