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Algebraic formulae for the q-inverse in a free group

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ALGEBRAIC FORMULAE FOR THE q-INVERSE IN A FREE GROUP

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Abstract. Let F_n be a free group with n fixed generators, which we assume are linearly ordered. In a previous paper [M], a curious mapping $I : F_n \to F_n$ was introduced pictorially. It is a "square root "of the inner automorphism of F_n induced by the "smallest "generator. In the present paper, two algebraic formulae will be given, by which one can compute the mapping I purely algebraically.

1. Introduction

Let F_n denote a free group with n fixed generators $x_1, x_2, \dots, x_n$, which are referred to as *the preferred generators*. We assume that these generators are linearly ordered:

$$x_1 < x_2 < \cdots < x_n.$$

In what follows the "smallest" generator x_1 will play a special role, and it will be denoted by a special letter q:

$$q = x_1.$$

The present paper is concerned with a curious mapping

$$I : F_n \to F_n \quad ,$$

which was introduced in [M] pictorially in connection with the conjugation formula for the mapping class group of a punctured sphere. In that paper, the mapping I was called the "quantum inverse", but here we will call it the *q-inverse* for simplicity. The purpose of this paper is to give two different algebraic formulae, each of which allows one to compute this mapping I purely algebraically.

The q-inverse I has several interesting properties:

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(1) For each element $W \in F_n$,

$$I(I(W)) = q^{-1}Wq.$$

In other words, I is a "square root" of the inner automorphism of F_n induced by q. In particular, I is a bijection.

(2) I is *stable*: Let F_m be another free group with preferred generators $y_1, y_2, \dots, y_m$ which are linearly ordered:

$$y_1 < y_2 < \cdots < y_m.$$

Let $h : F_n \to F_m$ be an embedding which preserves q and the order of the preferred generators. More precisely, let $h : F_n \to F_m$ be a homomorphism satisfying the following conditions:

$$\begin{aligned}
&h(x_1) = y_1, \\
&h(x_i) = y_{\sigma(i)},\ \sigma(i) \in \{2, \dots, m\}\ \ (i = 2, \dots, n), \quad \text{and} \\
&\sigma(i) < \sigma(j)\ \text{ for }\ i < j.
\end{aligned}$$

Then the following diagram commutes:

$$\begin{tikzcd}
F_n \arrow[r, "I"] \arrow[d, "h"'] & F_n \arrow[d, "h"] \\
F_m \arrow[r, "I"'] & F_m
\end{tikzcd}$$

(2′) Let E be a subset of $\{x_2, \dots, x_n\}$ and $\pi_E : F_n \to F_n/N(E)$ the projection to the quotient group by the normal subgroup $N(E)$ generated by E. Then the following diagram commutes:

$$\begin{tikzcd}
F_n \arrow[r, "I"] \arrow[d, "\pi_E"'] & F_n \arrow[d, "\pi_E"] \\
F_n/N(E) \arrow[r, "I"'] & F_n/N(E)
\end{tikzcd}$$

(3) Substitution $q = 1$ reduces $I(W)$ to the classical inverse W^{-1}. More precisely,

the following diagram commutes:

$$\begin{array}{ccc} F_n & \xrightarrow{I} & F_n \\ {\scriptstyle \pi_q}\downarrow & & \downarrow{\scriptstyle \pi_q} \\ F_n/N(q) & \xrightarrow[\text{inverse}]{} & F_n/N(q) \end{array}$$

where $N(q)$ is the normal subgroup generated by q, and $\pi_q : F_n \to F_n/N(q)$ is the quotient homomorphism.

(4) ("The law of conservation of energy") For each $W \in F_n$, we have $e_q(W) = e_q(I(W))$, where $e_q(W)$ is the sum of the exponents of q in the word expression of W.

Here are some examples of computations, where y and z stand for any two preferred generators x_i and x_j such that $x_1(= q) < x_i(= y) < x_j(= z)$.

$$\begin{aligned}
&I(q^k) = q^k, \qquad \forall k \in \mathbb{Z},\\
&I(y^k) = q^k(q^{-1}y^{-1})^k, \qquad \forall k \in \mathbb{Z}\\
&I(yz) = z^{-1}y^{-1},\\
&I(zy) = qy^{-1}q^{-1}z^{-1}qyq^{-1}y^{-1},\\
&I(zyz^2) = q^2z^{-1}q^{-1}z^{-1}y^{-1}zqz^{-1}q^{-1}z^{-1}yzqzq^{-1}z^{-1}q^{-1}z^{-1}y^{-1}.
\end{aligned}$$

2. Two definitions of I

Let P be a set of $n+1$ interior points $p_0, p_1, \ldots, p_n$ of a 2-disk D arranged on a line in this order. By a *cord* on (D, P), we mean an embedded curve α in the interior of D such that $\alpha \cap P = \partial\alpha = \{p_i, p_j\}$ for some i, j with $i \neq j$. For a cord α on (D, P), let $\tau(\alpha)$ be a counter-clockwise $180°$-twist along α interchanging the end-points of α executed in a sufficiently small disk neighborhood of α. This defines a map

$$\tau : C(D, P) \to M(D, P)$$

from $C(D, P)$, the set of isotopy classes of cords on (D, P), to $M(D, P)$, the mapping class group of (D, P) relative to ∂D. We use the same symbol α for a cord α and its isotopy class $[\alpha]$ unless it makes any confusion.

The group $M(D, P)$ is identified with the braid group B_{n+1} so that each standard generator σ_i $(i = 0, \ldots, n-1)$ of B_{n+1} corresponds to the mapping class ζ_i of $\tau(\alpha_i)$, where α_i is the line segment between p_i and p_{i+1}.

Let $A_{i,j}$ $(i < j)$ be Artin's pure braid generator

$$\begin{aligned} A_{i,j} &= \sigma_i^{-1}\sigma_{i+1}^{-1}\ldots\sigma_{j-2}^{-1}\sigma_{j-1}^{2}\sigma_{j-2}\ldots\sigma_{i+1}\sigma_i \\ &= \sigma_{j-1}\sigma_{j-2}\ldots\sigma_{i+1}\sigma_i^{2}\sigma_{i+1}^{-1}\ldots\sigma_{j-2}^{-1}\sigma_{j-1}^{-1}. \end{aligned}$$

Artin [A] proved that the subgroup of $M(D,P)$ generated by $A_{0,j}$ $(j = 1,\ldots n)$ is a free group isomorphic to $\pi_1(D - \{p_1,\ldots,p_n\}, p_0)$. Setting $x_j = A_{0,j}$, we sometimes identify this subgroup of $M(D,P)$ with the free group F_n generated by x_j $(j = 1,\ldots n)$, see Figures 1 and 2.

$\boxed{Figure 1}$

$\boxed{Figure 2}$

We will adopt the convention that the mapping class group $M(D,P)$ acts on (D,P) from the *right*. Thus for a cord $\alpha \in C(D,P)$ and a mapping class $f \in M(D,P)$, the notation $(\alpha)f$ will denote the image of α under the action of f. For each $W \in F_n \subset M(D,P)$, let $\eta(W)$ be the cord (or its isotopy class) $(\alpha_0)W$ in this notation. This defines a map

$$\eta : F_n \to C(D,P)_{01},$$

where $C(D,P)_{01}$ is the subset of $C(D,P)$ consisting of the isotopy classes of cords α with $\partial\alpha = \{p_0, p_1\}$.

Proposition 2.1 ([M]). *The map $\eta : F_n \to C(D,P)_{01}$ induces a bijection*

$$\eta : \langle q\rangle\backslash F_n \to C(D,P)_{01}$$

from the right coset $\langle q\rangle\backslash F_n$ to $C(D,P)_{01}$.

Since $(\alpha_0)q = \alpha_0$ in $C(D,P)_{01}$, $\eta : \langle q\rangle\backslash F_n \to C(D,P)_{01}$ is well-defined. We will describe the inverse map: Consider mutually disjoint half lines labelled $x_1,\ldots,x_n$ starting from $p_1,\ldots,p_n$ as in Figure 3. Assume that a cord α intersects these half lines transversely and assign each intersection point the label of the half line. Trace the curve α from p_1 to p_0, and we obtain a word W by reading the labels on the intersection points, together with the signs of the intersections as their exponents. The word W as an element of $\langle q\rangle\backslash F_n$ is uniquely determined by the isotopy class of the cord α.

$$\boxed{Figure 3}$$

Using the bijection $\eta : \langle q\rangle\backslash F_n \to C(D,P)_{01}$, we define a map $I : F_n \to F_n$ as follows: Let W be an element of F_n and α a cord on (D,P) with $\eta(W) = \alpha$. Apply ζ_0 $(= \tau(\alpha_0))$ to the cord α, and we have another cord $\alpha' = (\alpha)\zeta_0$ with the same property $\partial\alpha' = \{p_0, p_1\}$. By the inverse map of η, the cord α' corresponds to an element $[W']$ of $\langle q\rangle\backslash F_n$. Take a unique representative W' of $[W']$ such that $e_q(W) = e_q(W')$.

Definition. For each $W \in F_n$, there exists a unique element $W' \in F_n$ such that $(\eta(W))\zeta_0 = \eta(W')$ and $e_q(W) = e_q(W')$. Then $I(W)$ is defined to be W'.

For example, a word $W = x_2x_4$ is represented by a cord illustrated in Figure 3. Applying a disk twist ζ_0 to this cord, we have a cord as in Figure 4, which represents $[x_4^{-1}x_2^{-1}]$ of $\langle q\rangle\backslash F_n$. Thus $I(x_2x_4) = x_4^{-1}x_2^{-1}$. It is easily seen that $I(x_i) = x_i^{-1}$ for each i with $2 \le i \le n$; nevertheless $I(x_1) = x_1$.

$$\boxed{Figure 4}$$

In terms of the mapping class group $M(D,P)$, this definition of I is interpreted as follows: For an element $W \in F_n \subset M(D,P)$, the conjugate $W^{-1}\zeta_0 W$ is a disk twist $\tau(\alpha)$, where $\alpha = (\alpha_0)W$. If we consider a further conjugation $\zeta_0^{-1}W^{-1}\zeta_0 W\zeta_0$, this is again a disk twist $\tau(\alpha')$, where $\alpha' = (\alpha)\zeta_0$. Proposition 2.1 implies that there is a unique element $W' \in F_n$ such that $\eta(W') = \alpha'$ and $e_q(W') = e_q(W)$, which is $I(W)$. It is obvious that

$$\begin{aligned}
\zeta_0^{-1}W^{-1}\zeta_0 W\zeta_0 &= \zeta_0^{-1}\tau(\alpha)\zeta_0 \\
&= \tau((\alpha)\zeta_0) \\
&= \tau(\alpha') \\
&= {W'}^{-1}\zeta_0 W'.
\end{aligned}$$

The second definition of I. For each element $W \in F_n$, there exists uniquely an element $W' \in F_n$ such that $\zeta_0^{-1}W^{-1}\zeta_0 W\zeta_0 = {W'}^{-1}\zeta_0 W'$ in $M(D,P)$ and $e_q(W) = e_q(W')$. Then $I(W)$ is defined to be W'.

3. The first algebraic formula

In what follows, we work in the braid group B_{n+1} instead of $M(D,P)$. Identify the free group F_n generated by x_j $(j=1,\dots,n)$ with the subgroup of B_{n+1} generated by $A_{0,j}$ $(j=1,\dots,n)$ so that $x_j = A_{0,j}$ (see Figure 1). We denote by σ the standard generator σ_0 of B_{n+1}. By a^b, we mean the conjugate $b^{-1}ab$.

We can further interpret the definition of $I : F_n \to F_n$ in terms of B_{n+1} :

The third definition of I. For each element $W \in F_n$, there exists a unique element $W' \in F_n$ such that $\sigma^{W\sigma} = \sigma^{W'}$ and $e_q(W) = e_q(W')$. Then $I(W)$ is defined to be W'.

For an element $W \in F_n \subset B_{n+1}$, let

$$B_W : B_{n+1} \to B_{n+1}$$

denote the inner-automorphism induced by W^σ, namely

$$B_W(b) = b^{W^\sigma}.$$

Since W^σ is a pure braid and F_n is closed under conjugation by a pure braid, $B_W(V)$ belongs to F_n for any $V \in F_n$. Thus restricting B_W to F_n, we have a homomorphism

$$B_W : F_n \to F_n.$$

It is obvious that $B_{W_1W_2}(V) = B_{W_2}(B_{W_1}(V))$. Thus we have a "right representation"

$$B : F_n \to \mathrm{Aut}(F_n), \qquad W \mapsto B_W.$$

To calculate $B_W(V)$ for $W, V \in F_n$ it will be sufficient to calculate $B_W(V)$ in the case when W and V are preferred generators of F_n and their inverses. By direct calculations, we have the following results:

Lemma 3.1.

$$B_{x_j}(x_i^\epsilon) = B^{-1}_{x_j^{-1}}(x_i^\epsilon) = \begin{cases} x_j q^\epsilon x_j^{-1}, & i = 1, \\ q^{-1} x_i^\epsilon q, & j = 1, \\ x_i^\epsilon, & 1 < i < j, \\ x_i q x_i^\epsilon q^{-1} x_i^{-1}, & 1 < i = j, \\ (x_j q x_j^{-1} q^{-1}) x_i^\epsilon (q x_j q^{-1} x_j^{-1}), & 1 < j < i, \end{cases}$$

and

$$B_{x_j^{-1}}(x_i^\epsilon) = B_{x_j}^{-1}(x_i^\epsilon) = \begin{cases} q^{-1}x_j^{-1}q^\epsilon x_j q, & i=1, \\ q x_i^\epsilon q^{-1}, & j=1, \\ x_i^\epsilon, & 1<i<j, \\ q^{-1}x_i^\epsilon q, & 1<i=j, \\ (q^{-1}x_j^{-1}qx_j)x_i^\epsilon(x_j^{-1}q^{-1}x_jq), & 1<j<i. \end{cases}$$

Corollary 3.2. *The homomorphism $B_W : F_n \to F_n$ preserves e_q; namely*

$$e_q(B_W(V)) = e_q(V) \quad \textit{for any } V \in F_n.$$

Theorem 3.3. *$I : F_n \to F_n$ is a crossed anti-homomorphism twisted by the right representation $B : F_n \to \mathrm{Aut}(F_n)$:*

$$I(W_1W_2) = I(W_2)B_{W_2}(I(W_1)).$$

Proof.

$$\begin{aligned} \sigma^{(W_1W_2)\sigma} &= \sigma^{W_1\sigma W_2^\sigma} \\ &= \sigma^{I(W_1)W_2^\sigma} \\ &= \sigma^{W_2^\sigma B_{W_2}(I(W_1))} \\ &= \sigma^{I(W_2)B_{W_2}(I(W_1))} \end{aligned}$$

Since $e_q(I(W_2)B_{W_2}(I(W_1))) = e_q(I(W_2)) + e_0(I(W_1)) = e_q(W_1W_2)$, we have $I(W_1W_2) = I(W_2)B_{W_2}(I(W_1))$. $\square$

This theorem, together with $I(q) = q$ and $I(x_i) = x_i^{-1}$, $(i = 2, \dots, n)$, allows us to compute $I(W)$ algebraically. In the next section we give another algebraic formula for the q-inverse, which seems more naturally fitted to the q-inverse.

4. The second algebraic formula

For each j $(j = 1, \dots, n)$, we put

$$\widehat{x_j} = \begin{cases} 1 & \text{for } j=1, \\ A_{0,j}A_{1,j} \quad (= x_j\sigma x_j\sigma^{-1} = \sigma^{-1}x_j\sigma x_j) & \text{for } j>1. \end{cases}$$

$$\boxed{Figure5}$$

For an element W of $F_n \subset B_{n+1}$, we denote by $\widehat{W}$ the element of B_{n+1} that is obtained from W by replacing each letter x_j by $\widehat{x_j}$. In particular, $\widehat{x_j^{-1}} = (\widehat{x_j})^{-1}$. We note that if $\pi_q(W) = \pi_q(W')$ then $\widehat{W} = \widehat{W'}$.

For an element $W \in F_n$, let

$$C_W : B_{n+1} \to B_{n+1}$$

denote the inner-automorphism induced by $\widehat{W}$, namely,

$$C_W(b) = \widehat{W}^{-1} b \widehat{W}.$$

Since $\widehat{W}$ is a pure braid, $C_W(V)$ belongs to F_n for any $V \in F_n$. Thus restricting C_W to F_n, we have a homomorphism

$$C_W : F_n \to F_n.$$

It is obvious that $C_{W_1 W_2}(V) = C_{W_2}(C_{W_1}(V))$. Thus we have a right representation

$$C : F_n \to \mathrm{Aut}(F_n), \qquad W \mapsto C_W.$$

By direct calculations, we have the following results:

Lemma 4.1.

$$C_{x_j}(x_i^\epsilon) = C^{-1}_{x_j^{-1}}(x_i^\epsilon) = \begin{cases} x_i^\epsilon, & i = 1 \text{ or } j = 1, \\ x_j^{-1} x_i^\epsilon x_j, & 1 < i < j, \\ q x_i^\epsilon q^{-1}, & 1 < i = j, \\ (q x_j^{-1} q^{-1}) x_i^\epsilon (q x_j q^{-1}), & 1 < j < i, \end{cases}$$

and

$$C_{x_j^{-1}}(x_i^\epsilon) = C^{-1}_{x_j}(x_i^\epsilon) = \begin{cases} x_i^\epsilon, & i = 1 \text{ or } j = 1, \\ (q^{-1} x_j q) x_i^\epsilon (q^{-1} x_j^{-1} q), & 1 < i < j, \\ q^{-1} x_i^\epsilon q, & 1 < i = j, \\ x_j x_i^\epsilon x_j^{-1}, & 1 < j < i. \end{cases}$$

Corollary 4.2. *Let $e_q : F_n \to \mathbf{Z}$ and $\pi_q : F_n \to F_n/N(q)$ be as before.*

(1) *The homomorphism $C_W : F_n \to F_n$ preserves e_q; namely*

$$e_q(C_W(V)) = e_q(V) \quad \text{for any } V \in F_n.$$

(2) *If $\pi_q(W) = \pi_q(W')$, then $C_W = C_{W'}$.*

(3) $\pi_q(C_W(V)) = \pi_q(W^{-1}VW)$.

Theorem 4.3. *$I : F_n \to F_n$ is a crossed homomorphism twisted by the representation $C : F_n \to \mathrm{Aut}(F_n)$:*

$$I(W_1W_2) = C_{W_2}(I(W_1))I(W_2).$$

Proof. It is obvious from Figure 5 that for any $W \in F_n$, $\widehat{W}$ commutes with σ in B_{n+1}. Thus we have

$$\begin{aligned}
\sigma^{(W_1W_2)\sigma} &= \sigma^{W_1\sigma\sigma^{-1}\widehat{W_2}\widehat{W_2}^{-1}W_2\sigma} \\
&= \sigma^{W_1\sigma\widehat{W_2}\sigma^{-1}\widehat{W_2}^{-1}W_2\sigma} \\
&= \sigma^{I(W_1)\widehat{W_2}\sigma^{-1}\widehat{W_2}^{-1}W_2\sigma} \\
&= \sigma^{\widehat{W_2}^{-1}I(W_1)\widehat{W_2}\sigma^{-1}\widehat{W_2}^{-1}W_2\sigma} \\
&= \sigma^{C_{W_2}(I(W_1))\sigma^{-1}\widehat{W_2}^{-1}W_2\sigma}.
\end{aligned}$$

Assertion 4.4. *$\sigma^{-1}\widehat{W}^{-1}W\sigma = I(W)$ for any $W \in F_n$.*

If this assertion is proved, then since $e_q(C_{W_2}(I(W_1))I(W_2)) = e_q(W_1W_2)$, we have $I(W_1W_2) = C_{W_2}(I(W_1))I(W_2)$.

Now we prove Assertion 4.4. We have for any $W \in F_n$,

$$\begin{aligned}
\sigma^{W\sigma} &= \sigma^{\widehat{W}^{-1}W\sigma} \\
&= \sigma^{\sigma^{-1}\widehat{W}^{-1}W\sigma}.
\end{aligned}$$

If $\sigma^{-1}\widehat{W}^{-1}W\sigma$ belongs to F_n, then since $e_q(\sigma^{-1}\widehat{W}^{-1}W\sigma) = e_q(W)$, Assertion 4.4 holds.

We prove that $\sigma^{-1}\widehat{V}^{-1}V\sigma \in F_n$ for any $V \in F_n$ by induction on the length of V. If V is a generator or its inverse, it is directly seen that $\sigma^{-1}\widehat{V}^{-1}V\sigma \in F_n$. If

$V = V_1V_2$, then we have

$$\begin{aligned}\sigma^{-1}\widehat{V}^{-1}V\sigma &= \sigma^{-1}\widehat{V_1V_2}^{-1}V_1V_2\sigma \\ &= \sigma^{-1}\widehat{V_2}^{-1}\widehat{V_1}^{-1}V_1V_2\sigma \\ &= \sigma^{-1}\widehat{V_2}^{-1}\sigma\sigma^{-1}\widehat{V_1}^{-1}V_1\sigma\sigma^{-1}V_2\sigma \\ &= \sigma^{-1}\widehat{V_2}^{-1}\sigma z_1\sigma^{-1}V_2\sigma \\ &= \sigma^{-1}\widehat{V_2}^{-1}\sigma\sigma^{-1}V_2\sigma {z_1}^{\sigma^{-1}V_2\sigma} \\ &= \sigma^{-1}\widehat{V_2}^{-1}V_2\sigma {z_1}^{\sigma^{-1}V_2\sigma} \\ &= z_2{z_1}^{\sigma^{-1}V_2\sigma},\end{aligned}$$

where $z_i = \sigma^{-1}\widehat{V_i}^{-1}V_i\sigma$ $(i = 1, 2)$. By induction hypothesis, z_1 and z_2 belong to F_n. Since $\sigma^{-1}V_2\sigma$ is a pure braid, ${z_1}^{\sigma^{-1}V_2\sigma}$ belongs to F_n. Hence $\sigma^{-1}\widehat{V}^{-1}V\sigma \in F_n$. $\square$

If $W = 1$, then by definition $I(W) = 1$. If W is a generator of F_n or its inverse, then by a direct calculation we have

$$I(x_i^{\epsilon}) = \begin{cases} q^{\epsilon} & \text{if } i = 1, \\ x_i^{-1} & \text{if } i \neq 1,\ \epsilon = +1, \\ q^{-1}x_iq & \text{if } i \neq 1,\ \epsilon = -1. \end{cases}$$

If W is $x_{i_1}^{\epsilon_1}\dots x_{i_m}^{\epsilon_m}$, then $I(W)$ is calculated by

$$I(W) = \prod_{j=1}^{m} C_{W_j}(I(x_{i_j}^{\epsilon_j})),$$

where $W_j = x_{i_{j+1}}^{\epsilon_{j+1}}\dots x_{i_m}^{\epsilon_m}$ for $j = 1, \dots, m-1$, and $W_m = 1$. This is the most efficient formula known to the authors that fits to computer programing.

Proposition 4.5.

$$I(W_1q^kW_2) = I(W_1W_2)I(W_2)^{-1}q^kI(W_2).$$

In particular,

$$I(q^kW) = q^kI(W), \text{ and } I(Wq^k) = I(W)q^k.$$

This proposition follows from Theorem 4.3. The detailed proof will be left to the reader.

Finally we give a relation among I, B_W and C_W.

Proposition 4.6.

$$C_W(V) = I(W)B_W(V)I(W)^{-1}.$$

Proof. By Theorems 3.3 and 4.3, we have $I(W_1W_2) = I(W_2)B_{W_2}(I(W_1)) = C_{W_2}(I(W_1))I(W_2)$ for any W_1 and W_2. Hence we have the relation. $\square$

Proposition 4.7. *C_W commutes with I; namely, $I(C_W(V)) = C_W(I(V))$.*

Proof. Note that $\widehat{W}$ ($W \in F_n$) commutes with σ. Therefore we see that

$$\begin{aligned}
\sigma^{I(C_W(V))} &= \sigma^{C_W(V)\sigma} \\
&= \sigma^{\widehat{W}^{-1}V\widehat{W}\sigma} \\
&= \sigma^{V\sigma\widehat{W}} \\
&= \sigma^{I(V)\widehat{W}} \\
&= \sigma^{\widehat{W}^{-1}I(V)\widehat{W}} \\
&= \sigma^{C_W(I(V))}.
\end{aligned}$$

Since I and C_W preserve e_q, we have $I(C_W(V)) = C_W(I(V))$. $\square$

Proposition 4.8.

$$C_{I(W)} = C_{W^{-1}}$$

Proof. Since $\pi_q(I(W)) = \pi_q(W^{-1})$ (Property (3)), it is a direct consequence of Corollary 4.2(2). $\square$

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