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Abstract

In this paper, we estimate the degree of symmetry and the semi-simple degree of symmetry of certain fiber bundles by virtue of the rigidity theorem with respect to the harmonic map due to Schoen and Yau. As a corollary of this estimate, we compute the degree of symmetry and the semi-simple degree of symmetry of certain product manifolds. In addition, by Albanese map, we estimate the degree of symmetry and the semi-simple degree of symmetry of a compact smooth manifold under some topological assumptions.

1 Introduction

Let M^n be a compact connected smooth n-manifold and $N(M^n)$ the degree of symmetry of M^n, that is, the maximum of the dimensions of the isometry groups of all possible Riemannian metrics on M^n. (All the manifolds of this paper are to be compact and smooth.) Of course, $N(M)$ is the maximum of the dimensions of the compact Lie groups which can act effectively and smoothly on M. The following is well known:

$$N(M^n) \leq n(n+1)/2. \tag{1}$$

In addition, if the equality holds, then M^n is diffeomorphic to the standard sphere S^n or the real projective space $\mathbf{R}P^n$. In [10] H. T. Ku, L. N. Mann, J. L. Sicks and J. C. Su obtained similar results on a product manifold $M^n = M_1^{n_1} \times M_2^{n_2}$ ($n \geq 19$) where M_i is a compact connected smooth manifold of dimension n_i: they showed that

$$N(M) \leq n_1(n_1+1)/2 + n_2(n_2+1)/2, \tag{2}$$

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and that if the equality holds, then M^n is a product of two spheres, two real projective spaces or a sphere and a real projective space. A preliminary lemma for the proof of Ku-Mann-Sicks-Su's results claims that if M^n ($n \geq 19$) is a compact connected smooth n-manifold which is not diffeomorphic to the complex projective space $\mathbf{C}P^m$ ($n = 2m$), then

$$N(M^n) \leq k(k+1)/2 + (n-k)(n-k+1)/2 \tag{3}$$

holds for each $k \in \mathbf{N}$ such that the k-th Betti number b_k of M is nonzero. Then we see that when a compact oriented smooth manifold M of dimension $4m \geq 20$ has nonzero signature $\sigma(M)$, then the following holds:

$$N(M) \leq N(\mathbf{C}P^{2m}) = 4m(m+1). \tag{4}$$

The equality in (4) was also showed by Ku-Mann-Sicks-Su in [10]. The semi-simple degree of symmetry $N_s(M)$ [4] is defined similarly, where we consider only actions of semi-simple compact Lie groups on M. It is clear that (1) also holds for the semi-simple degree of symmetry if the manifold has dimension ≥ 2. D. Burghelea and R. Schultz showed [4] $N_s(M^n) = 0$ if there exist $\alpha_1, \cdots, \alpha_n$ in $H^1(M^n; \mathbf{R})$ with $\alpha \cup \cdots \cup \alpha_n \neq 0$.

Ku-Mann-Sicks-Su's estimates (2) and (3) and Burghelea-Schultz's result on the degree of symmetry and the semi-simple degree of symmetry of a manifold mainly depend on its topological structure. On the other hand, when we consider if there exists a nontrivial S^1-action or S^3-*action* on a manifold, we meet some obstructions from its differential structure. Here a nontrivial S^3-*action* on a manifold means [11] that a Lie group S^3 or $SO(3)$ acts effectively and smoothly on it. Let us see some examples as follows:

A spin manifold is an oriented Riemannian manifold with a spin structure on its tangent bundle (cg [11]). A famous theorem of M. Atiyah and F. Hirzebruch [2] claims that a spin manifold has degree of symmetry 0 if the index of the Dirac operator on it, or equivalently, its $\hat{A}$-genus, is nonzero. Let X be a spin manifold of dimension $8q+1$ (resp. $8q+2$), Atiyah and Singer [3] showed that the real dimension (resp. complex dimension) (mod 2) of the space of harmonic spinors X can be identified with a certain KO-characteristic number $\alpha(X)$ of the spin-cobordism class of X. Let Θ_n be the group of homotopy n-spheres. This KO-characteristic number was shown by Milnor and Adams to give a nontrivial homomorphism $\alpha : \Theta_n \to \mathbf{Z}_2$ for $n = 8q+1$ or $8q+2$ (cf [1], [15]). Since for $n = 8q+1$ or $8q+2$, the homotopy n-spheres which bound spin manifolds form a subgroup $BSpin_n$ of index 2 in Θ_n, we see that Ker $\alpha = BSpin_n$. For the α-invariant is additive with respect to connected sums of manifolds, it is always possible to change the differentiable structure of a spin manifold X, in dimension $8q+1$ or $8q+2$, to make $\alpha(X)$ nonzero. It follows from Lawson and Yau [13] that if $\alpha(X)$ is nonzero, then there exists no nontrivial smooth effective S^3 action on X, or equivalently, the only compact, connected effective transformation groups on X are tori, from which the followings hold:

$$N(X) \leq \dim\ X, \quad N_s(M) = 0. \tag{5}$$

Definition 1.1 We call a manifold *significant* if and only if it is oriented and has nonzero signature. A manifold is said to be *$\hat{A}$-nontrivial* if and only if it is spin and has nonzero $\hat{A}$-genus. A manifold X is said to be *α-nontrivial* if and only if it is spin, of dimension $8q+1$ or $8q+2$, and $\alpha(X) \neq 0$, where q may be zero.

Definition 1.2 We call a manifold *S^3-trivial* if and only if there exists no smooth and effective S^3-action on it, or equivalently, its semisimple degree of symmetry is zero.

Remark 1.1 Both $\hat{A}$-nontrivial manifolds and α-nontrivial manifolds are S^3-trivial. Lawson and Yau [13] showed that if a compact manifold doesn't admit a Riemannian metric of positive scalar curvature, then it is S^3-trivial.

One of the purposes of this paper is to make some estimate for certain nontrivial compact fiber bundles and generalize partially Ku-Mann-Sicks-Su's estimates (2). In particular, we obtain a bundle version of (4) and the results by Atiyah-Hirzebruch and Lawson-Yau when taking the special fibers as in Definition 1.1 and 1.2.

Theorem 1.1 *Let V be a compact manifold which can be equipped with a real analytic metric of nonpositive curvature and E a compact smooth fiber bundle over V such that the fiber F of E is connected. Then the followings hold:*

$$N(E) \leq \dim\ F(\dim\ F+1)/2 + N(V), \quad N_s(E) \leq \dim\ F(\dim\ F+1)/2. \tag{6}$$

Particularly,
(i) *suppose E is oriented and F is a significant manifold of dimension ≥ 19. Then the following holds:*

$$N(E) \leq \dim\ F(\dim\ F+4)/4 + N(V). \tag{7}$$

(ii) *Suppose E is spin and F is an $\hat{A}$-nontrivial manifold. Then E is S^3-trivial and the following holds:*

$$N(E) \leq N(V). \tag{8}$$

(iii) *Suppose E is spin and F is an α-nontrivial manifold. Then E is S^3-trivial and the following holds:*

$$N(E) \leq \dim\ F + N(V). \tag{9}$$

(iv) *Suppose Σ^n is an exotic n-sphere which does not bound a spin manifold and V is spin. Then $\Sigma^n \times V$ is not diffeomorphic to $S^n \times V$.*

Remark 1.2 By a result in [12] we know the dimension of isometry group of V is rank of the center of $\pi_1(V)$. On the other hand from [5] we know that if a compact connected Lie group acting smoothly and effectively on a compact aspherical manifold A, then it is a torus of dimension $\leq$ rank of the center of $\pi_1(A)$. Combining these two results, we immediately see the degree of symmetry of V is equal to rank of the center of $\pi_1(V)$.

Remark 1.3 In Theorem 1.1, V cannot be replaced by an arbitrary compact manifold because the Hopf bundle $S^1 \to S^3 \to S^2$ forms a counterexample.

Remark 1.4 Let T^2 be a two dimensional torus and K a Klein bottle. Then $N(T^2) = 2$ and $N(K) = 1$ hold. Therefore we see that the connectivity of fiber F is necessary for the first inequality in (6) in Theorem 1.1.

By the definition of degree of symmetry, it is easy to see that for a product manifold $M_1 \times M_2$, where M_i is a compact connected smooth manifold, the following holds:

$$N(M_1 \times M_2) \geq N(M_1) + N(M_2). \tag{10}$$

Combining (6), (7) and (8) with (10), we immediately obtain the following

Corollary 1.1 *Let V be a compact manifold which can be equipped with a real analytic metric of nonpositive curvature. Then the following holds:*

$$N(S^n \times V) = N(S^n) + N(V), \quad N_s(S^n \times V) = N_s(S^n). \tag{11}$$

Suppose V is oriented. Then the inequality

$$N(\mathbf{C}P^{2m} \times V) = N(\mathbf{C}P^{2m}) + N(V),$$

holds provided $4m \geq 20$. Moreover if V is spin and X is $\hat{A}$-nontrivial, then the following holds:

$$N(X \times V) = N(V).$$

Remark 1.5 Corollary 1.1 shows the estimates (6), (7) and (8) in Theorem 1.1 are sharp for bundles with fibers as sphere, significant manifold and $\hat{A}$-nontrivial manifold respectively.

From (3) we see that if M^n ($n \geq 19$) is a compact connected smooth n-manifold with nonzero first Betti number, then the following holds:

$$N(M^n) \leq n(n-1)/2 + 1.$$

The other of the purposes of this paper is to refine this inequality and Burghelea-Schultz's result:

Theorem 1.2 *Let M be an n-dimensional compact smooth manifold with nonzero first Betti number b_1.*
(i) *Suppose that there exists one dimensional real cohomology classes $\alpha_1, \cdots, \alpha_k$ such that $\alpha_1 \cup \cdots \cup \alpha_k$ does not vanish in $H^k(M; \mathbf{R})$. Then the followings hold:*

$$N(M) \leq (n-k+1)(n-k)/2 + k, \quad N_s(M) \leq (n-k+1)(n-k)/2.$$

(ii) *Suppose $b_1 \geq i = 1$ or 2. Then the followings hold:*

$$N(M) \leq (n-i+1)(n-i)/2 + i, \quad N_s(M) \leq (n-i+1)(n-i)/2.$$

Particularly, if $n - i = 1$, then $N_s(M) = 0$.
(iii) *Suppose $b_1 \geq 3$. Then the following holds:*

$$N(M) \leq \begin{cases} (n-2)(n-1)/2, & \text{if } n \geq 5, \\ 4, & \text{if } n = 4. \end{cases} \tag{12}$$

This paper is organized as follows. In Section 2, we prepare for the following sections. Particularly, if there exists a nontrivial harmonic map from a compact Riemannian manifold M to a compact manifold of nonpositive curvature, we can estimate the dimension of isometry group of M from above (cf Lemma 2.1 and Lemma 2.2). In Section 3, from the assumptions in Theorem 1.1, we show the nontriviality of the harmonic map homotopic the fibration map from E to V and prove Theorem 1.1 by the cobordism theory. In Section 4, we show the nontriviality of the Albanese map from M to a $b_1(M)$-dimensional flat torus and prove Theorem 1.2.

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2 Preliminaries

2.1 The isometry group of a Riemannian manifold and the harmonic map due to Schoen and Yau

For a compact Riemannian manifold M let $I(M)$, $I^0(M)$ be the isometry group of M and its identity component respectively. The following is known:

Proposition 2.1 (cf Theorem 4 in [19]) *Suppose M, N are compact real analytic Riemannian manifolds and N has nonpositive sectional curvatures. Suppose $h : M \to N$ is a surjective harmonic map and its induced map $h_* : \pi_1(M) \to \pi_1(N)$ is also surjective. Then the space of surjective harmonic maps homotopic to h is represented by $\{\beta \circ h | \beta \in I^0(N)\}$.*

We shall prove

Lemma 2.1 *Under the hypotheses of Proposition 2.1 it follows that*

$$\dim\ I^0(M) \leq (m-n+1)(m-n)/2 + \dim I^0(N),$$

where $m = \dim\ M$ *and* $n = \dim\ N$.

Proof. Taking an element $\alpha \in I^0(M)$, we obtain a surjective harmonic map $h \circ \alpha$ homotopic to h. By Proposition 2.1, we see that there exists $\rho(\alpha) \in I^0(N)$ such that $h \circ \alpha = \rho(\alpha) \circ h$. We see that $\rho : I^0(M) \to I^0(N)$ is a homomorphism. The proof is completed if we show that Ker ρ, which acts smoothly and effectively on M, has dimension not greater than $(m-n+1)(m-n)/2$. By Theorem 4.27 in [9], we see that the sum of the principal orbits of Ker ρ action is an open dense subset of M. Since h is a surjective smooth map, we see by Sard's theorem that the preimage of the regular value set of h is a nonempty open set in M. Therefore there is a point $p \in M$ so that (Ker $\rho)(p)$ is a principal orbit and the fiber $h^{-1}(h(p))$ has dimension $m-n$. The equation $h \circ \beta = h$ for all $\beta \in$ Ker ρ shows that (Ker $\rho)(p)$ is a submanifold of the fiber $h^{-1}[h(p)]$ and has dimension$\leq m-n$. Since Ker ρ acts effectively on its principal orbit, Ker ρ has dimension $\leq (m-n+1)(m-n)/2$.

2.2 The isometry group of a Riemannian manifold and the Albanese map

For a compact oriented Riemannian manifold M with nonzero first Betti number $b_1(M)$, let $\mathcal{H}$ be the real vector space of all harmonic 1-forms on M and ν the natural projecion from the universal covering $\tilde{M}$ of M. For $x_0 \in \tilde{M}$, set $p_0 = \nu(x_0)$. We define a smooth map $\tilde{a} : \tilde{M} \to \mathcal{H}^*$ from $\tilde{M}$ to the dual space $\mathcal{H}^*$ of $\mathcal{H}$ by a line integral

$$\tilde{a}(x)(\omega) = \int_{x_0}^{x} \nu^*\omega.$$

For $\sigma \in \pi_1(M)$

$$\tilde{a}(\sigma x) = \tilde{a}(x) + \psi(\sigma)$$

holds, where $\psi(\sigma)(\omega) = \int_{x_0}^{\sigma x_0} \nu^*\omega$, so that ψ is a homomorphism from $\pi_1(M)$ into $\mathcal{H}^*$ as an additive group. It is a fact that $\Delta = \psi(\pi_1(M))$ is a lattice in the vector space $\mathcal{H}^*$, and clearly this vector space has a natural Euclidean metric from the global inner prduct of forms on M. With the quotient metric, we call the torus $A(M) = \mathcal{H}^*/\Delta$ the *Albanese torus* of Riemannian manifold M. By the above relation between $\tilde{a}$ and ψ, we obtain a map $a : M \to A(M)$ satisfying $\tilde{a}(x) \in a \circ \nu(x)$ for any $x \in \tilde{M}$. We call the map a the *Albanese map*. From the very construction of a, we see that the map it induces on fundamental groups

$$a_* : \pi_1(M) \to \pi_1(A(M))$$

is surjective and that a^* maps the space of harmonic 1-forms on $A(M)$ isomorphically onto $\mathcal{H}$. By Corollary 1 in [17], the Albanese map is harmonic. We shall prove

Lemma 2.2 *Let M be an n-dimensional oriented compact Riemannian manifold and $a : M \to A(M)$ its Albanese map. Let da denote the differential of a and set*

$$r_a := \max\{\text{rank } da(p) | p \in M\}.$$

Then dim $I^0(M) \leq (n - r_a + 1)(n - r_a)/2 + r_a$.

Proof. We have only to consider the dimension of $I^0(M)$, the identity component of $I(M)$. For any $\gamma \in I^0(M)$, $a \circ \gamma$ is also a harmonic mapping from M to the Albanese torus $A(M)$ and homotopic to a. Hence there is a translation $\rho(\gamma)$ of the torus $A(M)$ such that

$$a \circ \gamma = \rho(\gamma) \circ a .$$

Then we have a homomorphism $\rho : I^0(M) \to T^{b_1}$, where the torus T^{b_1} is the translation group of Albanese torus $A(M)$. Let U be a non-empty open set of M such that the followings hold:
(a) on U the rank of da is euqal to r_a;
(b) for any point $p \in U$, $a^{-1}(a(p))$ is a $(n - r_a)$-dimensional submanifold of M.
Similarly to the proof of Lemma 2.1, we can see that there is a point $p \in U$ such that (Ker $\rho)(p)$ is a principal orbit and the fiber $a^{-1}(a(p))$ has dimension $n - r$. The equation $a \circ \beta = a$ for all $\beta \in$ Ker ρ shows that (Ker $\rho)(p)$ is a subset of the fiber $a^{-1}[a(p)]$ and has dimension $\leq n - r$. Since Ker ρ acts effectively on its principal orbit, Ker ρ has dimension $\leq \frac{1}{2}(n - r + 1)(n - r)$. As a subgroup of the translation group of $A(M)$, Im ρ acts freely on the image of a so that Im ρ has dimension $\leq \dim a(M) = r$. The proof is completed.

3 Proof of Theorem 1.1

We firstly prove a topological result on fiber bundles.

Proposition 3.1 *Let $p_0 : E \to B$ be a fiber bundle over a compact connected smooth manifold B such that the fiber of E is also connected. Suppose $p_1 : E \to B$ is a map homotopic to p_0. Then p_1 is surjective.*

Proof. The proof is an application of the Serre spectral sequence. Suppose there exists a point $x \in B$ such that the image of $p_1 : E \to B$ lies in the space $B' := B - \{x\}$. Then the composition of $p_1 : E \to B$ with the inclusion $i : B' \to B$ is homotopic to the projection map p_0 of the fiber bundle $p_0 : E \to B$. It is known that there exists a fibration (in the sense of Serre) $p_2 : E' \to B'$ and a map $f : E' \to E$ such that f is a homotopy equivalence and the compostion $p_1 \circ f$ is homotopic to p_2. Let $F,\ F'$ and $\mathcal{H}_a^*,\ \mathcal{H}_b^*$ be the homotopy fibers and the

Serre local systems of the fiber bundles $p_0 : E \to B$ and $p_2 : E' \to B'$ respectively. Then for these two fiber bundles we have two spectral sequences

$$(\mathrm{a}): \quad E_2^{p,q} = H^p(B; \mathcal{H}_a^q) \to H^*(E)$$

and

$$(\mathrm{b}): \quad E_2^{p,q} = H^p(B'; \mathcal{H}_b^q) \to H^*(E')$$

repectively, where we use cohomology groups with coefficients $\mathbf{Z}/2\mathbf{Z}$. Since $f : E' \to E$ is a fiber bundle map (in the sense of homotopy) over the map $i : B' \to B$, we also have a natural map between the spectral sequences $\phi : (\mathrm{a}) \to (\mathrm{b})$. For $n = \dim\ B$ and $k = \dim\ F$, we compare the $E_\infty^{n,k}$-terms of (a) and (b) by the map ϕ. The $E_\infty^{n,k}$-term of (a) is $\mathbf{Z}/2\mathbf{Z}$, which is naturally isomorphic to $H^{n+k}(E)$. The $E_\infty^{n,k}$-term of (b) is 0, whose proof is put in the next paragragh. Hence the map ϕ on the $E_\infty^{n,k}$-term is 0, which implies that the natural map $f_* : H^{n+k}(E) \to H^{n+k}(E')$ is trivial. This contradicts that $f : E \to E'$ is a homotopy equivalence.

Now we prove that the $E_\infty^{n,k}$-term of (b) is 0. We have only to show that the $E_2^{n,k}$-term of (b) is 0, which is just a special case of the equality

$$H^n(B'; \mathcal{S}) = 0$$

for any local system $\mathcal{S}$ with coefficients $\mathbf{Z}/2\mathbf{Z}$ over B'. Since B is a smooth compact connected manifold of dimension n, we claim that B has a cell structure with only one n-cell. In fact if taking a decomposition of B by polyhedra such that the number of the n-cells is minimum, then we see that the number of the n-cell is one. Otherwise, since B is connected, then there exists two n-cells which have a common $(n-1)$-dimensional cell. Deleting the common $(n-1)$-cell, we obtain another decomposition of B by polyhedra with less number of n-cells. Contradiciton! It is clear that $B' = B - \{x\}$ has the homotopy type of the $(n-1)$-skeleton B^{n-1} of this decomposition. Calculating $H^*(B^{n-1}; \mathcal{S})$ by this cell decomposition, we immediately see $H^n(B^{n-1}; \mathcal{S}) = 0$ and complete the proof.

Proof of Theorem 1.1: We remark that if a Lie group acts smoothly on a manifold, then there exists an analytic structure on this manifold such that the Lie group acts on it analytically. Hence the degree of symmetry of E is also equal to the maximum of the dimensions of the isometry groups of all real analytic Riemannian metrics on E.

For the proof of the first inequality in (6), we have only to show that for any real analytic Riemannian metric on E, the inequality

$$\dim\ I^0(E) \leq \dim\ F(\dim\ F + 1)/2 + N(V)$$

holds. The projection $p : E \to V$ of the fiber bundle induces a surjective map from $\pi_1(E)$ to $\pi_1(V)$ since the fiber F is connected. Using an well-known result by J. Eells and J. Sampson, we see that there exist harmonic maps homotopic to p. By Proposition 3.1, we see that each of them is surjective. Combining Remark 1.2 and Lemma 2.1, we obtain the above inequality. For the proof of the second inequality in (6), we have only to show that for a semi-simple compact Lie group G which acts isometrically on the analytic Riemannian manifold E, the following estimate holds:

$$\dim\ G \leq \dim\ F(\dim\ F+1)/2. \tag{13}$$

Since any Lie group homomorphism from G to a torus is trivial, it is followed that

$$\dim\ G \leq \dim\ \mathrm{Ker}\ \rho,$$

where $\rho : G \to I^0(V)$ is the homomporhism constructed in the proof of Lemma 2.1. we obtain (13) by the estimate of $\dim\ \mathrm{Ker}\ \rho$ in the proof of Lemma 2.1.

For the proof of (i), (ii) and (iii) of Theorem 1.1, we also have only to prove corresponding results for the isometry group with respect to any analytic Riemannian metric on E. Let $P = p_t : E \times [0,\ 1] \to V$ be the smooth homotopy between the projection map $p = p_0$ and a harmonic map $p_1 : E \to V$. By Proposition 3.1, we see that $p_t : E \to B$ is surjective for any $t \in [0,\ 1]$. Using the harmonic map p_1, by the same way in the proof of Lemma 2.1, we can construct a homomorphism $\rho : I^0(E) \to I^0(V)$ and find a point $x \in E$ such that the followings hold:

(a) $(\mathrm{Ker}\ \rho)(x)$ is a principal orbit for the transformation group $\mathrm{Ker}\ \rho$ on E;
(b) $p_1(x)$ is the regular value of the homotopy map P and its preimage $\mathcal{F}$ by P is a non-empty submanifold of $E \times [0,\ 1]$;
(c) $\mathcal{F}$ has boundary as the disjoint union of $p_0^{-1}(p_0(x)) \cong F$ and $p_1^{-1}(p_1(x)) = F'$;
(d) $(\mathrm{Ker}\ \rho)(x) \subset F'$ and the group $\mathrm{Ker}\ \rho$ acts effectively on the manifold F'.

Since the normal bundle of $\mathcal{F}$ in $E \times [0,\ 1]$ is trivial, $\mathcal{F}$ is oriented if E is oriented and it is spin if E is spin. It implies that F and F' are oriented cobordant if E is oriented and they are spin cobordant if E is spin. Since signature, $\hat{A}$-genus and KO-characteristic number are invariants of oriented cobordism, spin cobordism respectively, if F is significant, $\hat{A}$-nontrivial or α-nontrivial, so is F'. Although F' may be not connected, we see that there exists one component F^* of F' which is significant, $\hat{A}$-nontrivial or α-nontrivial. For $\mathrm{Ker}\ \rho$ acts effectively on F^*, if F^* is significant, we can estimate its dimension by (4) and then obtain (i) of Theorem 1.1. If F^* is $\hat{A}$-nontrivial, by Atiyah-Hirzebruch's theorem we see that $\mathrm{Ker}\ \rho$ is trivial and (ii) of Theorem 1.1 follows. The proof of (iii) of Theorem 1.1 is completed by Lawson-Yau's result (cf (5)).

Finally taking $\Sigma^n \times V$ as a trivial bundle over V in the above argument, since the fiber $F = \Sigma^n$ is spin cobordant to F', we see that F' does not bound spin manifold. Since $\mathrm{Ker}\ \rho$ acts effectively on F', it follows by (1) that

$$\dim\ \mathrm{Ker}\ \rho \leq N(F') < N(S^n) = n(n+1)/2.$$

Hence we see that $N(\Sigma^n \times V) < N(S^n \times V)$, which completes the proof of (iv).

4 Proof of Theorem 1.2

In order to prove Theorem 1.2, we need lemmas.

Lemma 4.1 *Let $\pi : M' \to M$ be a finite covering between compact smooth manifolds. Then we have $N(M) \leq N(M')$.*

Proof. We can assume that $\pi : \tilde{M} \to M$ is a Riemannian covering. It is enough to show dim $I(M) \leq$ dim $I(M')$. Since $I(M)$ and $I(M')$ are Lie groups of finite dimension, we only need to compare the dimensions of their Lie algebras. Given a Killing vector field V on M, the pullback of V by π is also a Killing field on M' so that the Lie algebra of $I(M)$ is a subalgebra of that of $I(M')$.

Let $\pi : \tilde{X} \to X$ be an n-sheeted covering space defined by an action of group Γ on $\tilde{X}$. Then (cf [8], Proposition 3H.1) with coefficients in a field F whose characteristic is 0 or a prime not dividing n, the map $\pi^* : H^k(X; F) \to H^k(\tilde{X}; F)$ is injective with image the subgroup $H^k(\tilde{X}; F)^\Gamma$ consisting of classes α such that $\gamma^*(\alpha) = \alpha$ for all $\gamma \in \Gamma$. In particular, we see

Lemma 4.2 *Let M be a non-orientable compact manifold, $\pi : M' \to M$ its orientable double covering. Then*
(1) $b_1(M) \leq b_1(M')$;
(2) *If there exist k one dimensional real cohomology classes $\alpha_1, \cdots, \alpha_k$ of M such that $\alpha_1 \cup \cdots \cup \alpha_k$ is not trivial in $H^k(M; \mathbf{R})$, there also exist k one dimensional real cohomoly classes $\beta_1, \cdots, \beta_k$ whose cup product does not vanish in $H^k(M'; \mathbf{R})$.*

Lemma 4.3 *Let M be an n-dimensional oriented compact Riemannian manifold with nonzero first Betti number b_1. Let $a : M \to A(M)$ be its Albanese map.*
(1) *Suppose there exist k integral one dimensional real cohomology classes $\alpha_1, \cdots, \alpha_k$ such that $\alpha_1 \cup \cdots \cup \alpha_k$ does not vanish in $H^k(M; \mathbf{R})$. Then $r_a \geq k$ holds.*
(2) *Suppose $b_1 \geq r = 1$ or 2, then $r_a \geq r$ holds.*

Proof. (1) By the assumption, the Albanese map a of M induces a non-trivial homomorphism $a^* : H^k(A(M); \mathbf{R}) \to H^k(M; \mathbf{R})$, which implies that the rank of da is at least k at some point $p \in M$.
(2) In case of $r = 1$ this statement is obvious. When $b_1 \geq 2$, it is implied by the general unique continuation property of harmonic mappings (cf Theorem 3 in [18]). In fact, if the maximal rank of da is 1, a maps M onto a closed geodesic of $A(M)$ since $a : M \to A(M)$ is harmonic. This contradicts surjectivity of the homomorphism $a_* : \pi_1(M) \to \pi_1(A(M)) \cong \mathbf{Z}^{b_1}$.

Finally we arrive at the proof of Theorem 1.2.

Proof of Theorem 1.2: By Lemma 4.1 together with Lemma 4.2, we may assume M is an oriented Riemannian manifold and let $a : M \to a(M)$ denote its Albanese map. We omit the proof of the estimates for semi-simple degree of symmetry here since it is similar to that of the second inequality in (6.

From Lemma 2.2 together with Lemma 4.3, we see that one of the upper bounds of dim $I(M)$ is

$$\max\{\frac{1}{2}(n-j+1)(n-j)+j | j=k,k+1,\cdots,n\},$$

which is equal to $(n-k+1)(n-k)/2+k$. Hence we obtain (i) of Theorem 1.2.

By Lemma 2.2 and Lemma 4.3, we obtain (ii) of Theorem 1.2.

For the proof of (iii) of Theorem 1.2, We have only to consider the analytic Riemannian metric on M. Since the first Betti number b_1 is not less than 3, we see by Lemma 4.3 that $r_a \geq 2$ holds. If $r_a \geq 3$, then from Lemma 2.2 we know

$$\dim\ I(M) \leq \frac{1}{2}(n-2)(n-3)+3.$$

Suppose $r = 2$. We recall the homomorphism ρ from $I^0(M)$ to the translation group T^{b_1} of $A(M)$ constructed in the proof of Lemma 2.2. We claim that the homomorphism ρ is trivial so that

$$\dim\ I^0(M) = \dim\ \mathrm{Ker}\,\rho \leq \frac{1}{2}(n-1)(n-2).$$

Otherwise, there is a translation group S^1 acting freely and isometrically on the image of a. Since both M and $A(M)$ are real analytic, a theorem of Morrey [16] shows that the harmonic mapping a is in fact real analytic. By well-known theorems in real analytic geometry [14] we know that both M and $A(M)$ can be triangulated so that $a(M)$ is a 2-dimensional compact connected simplicial subcomplex of $A(M)$. We write the orbit space of the free and isometric S^1 actions on $A(M)$ and $a(M)$ by $A(M)/S^1$ and $a(M)/S^1$ respectively, in which the former is in fact also a flat torus of dimension $b_1 - 1$. Since the natural projection map $\pi : A(M) \to A(M)/S^1$ is totally geodesic, we see that by a result in [6] the composition map $\pi \circ a : M \to A(M)/S^1$ is a harmonic map, whose image is $a(M)/S^1$, the orbit space of the free S^1 action on the two dimensional simplicial subcomplex $a(M)$ of $A(M)$. Hence $a(M)/S^1$, the image of $\pi \circ a$ in $A(M)/S^1$ has dimensional 1 so that the differential of harmonic map $\pi \circ a$ has rank ≤ 1 at any point of M. By Theorem 3 in [18], we see that $\pi \circ a$ maps M onto a closed geodesic of $A(M)/S^1$, which means that $a(M)$ is a 2-dimensional torus. This contradicts the surjectivity of the homomorphism $a_* : \pi_1(M) \to \pi_1(A(M)) \cong \mathbf{Z}^{b_1} (b_1 \geq 3)$. Hence we obtain

$$\dim\ I(M) \leq \max\{\frac{1}{2}(n-3)(n-2)+3,\ \frac{1}{2}(n-1)(n-2)\},$$

which implies (iii) of Theorem 1.2.

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