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T-Duality Group for Open String Theory

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Abstract

We study T-duality for open strings on tori $\mathbb{T}^d$. The general boundary conditions for the open strings are constructed, and it is shown that T-duality group, which preserves the mass spectrum of closed strings, preserves also the mass spectrum of the open strings. The open strings are transformed to those with different boundary conditions by T-duality. We also discuss the T-duality for D-brane mass spectrum, and show that the D-branes and the open strings with both ends on them are transformed together consistently.

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1 Introduction

Noncommutative(NC) tori arise when compactifying matrix models on tori. When one fix the noncommutativity Θ of the base torus, there are still ambiguities for the choice of (NC) gauge theories on the NC torus, and the algebras of these gauge theories are Morita equivalent to each other (and Morita equivalent to the base NC torus)[1].

Morita equivalent tori are generated by $SL(2,\mathbb{Z})(\times SL(2,\mathbb{Z}))$ in the two-torus case, and it is proven that, in general d where d is the dimension of the torus, Morita equivalent NC $\mathbb{T}^d$ is generated by $SO(d,d|\mathbb{Z})$ [2] and conversely NC $\mathbb{T}^d$ generated by $SO(d,d|\mathbb{Z})$ is Morita equivalent[3, 4] (for irrational noncommutative parameter Θ^{ij}).

Originally, $SO(d,d|\mathbb{Z})$ is known as the T-duality group, which acts on the closed string background g, B so as to preserve the closed string mass spectrum[5]. For this reason, this correspondence between Morita equivalence and $SO(d,d|\mathbb{Z})$ is often said that 'Morita equivalence is equivalent to T-duality'. However physically, the relation between this $SO(d,d|\mathbb{Z})$ group (acting on noncommutative parameter Θ) and T-duality (acting on closed string background g, B) seems somehow not clear.

On the other hand, in [6], it is argued that in flat background g, B (on $\mathbb{R}^d$ or $\mathbb{T}^d$), the field theory can be described by either commutative or noncommutative representation. The commutative description is natural for closed string theory, and noncommutative one is natural for open string theory (by the argument in [6], it is natural when we construct the field theory from the OPE for the boundary of the string). Now on $\mathbb{T}^d$ in flat background, T-duality group acts on commutative field theory side and Morita equivalence ($SO(d,d|\mathbb{Z})$) is on noncommutative side, i.e on the open string picture.

Moreover for instance in [6], the transformation between the variables in the commutative picture (g_s, g, B) and those in the noncommutative picture (G_s, G, Θ) is given [1], and on $\mathbb{T}^d$ (for $|B/g| \to \infty$ limit), the $SO(d,d|\mathbb{Z})$ (T-duality) transformation on the noncommutative side is defined by the action of the T-duality group on (g_s, g, B) and using this transformation between (g_s, g, B) and (G_s, G, Θ). By this definition, the transformation between (g_s, g, B) and (G_s, G, Θ) is equivariant with the T-duality action on both sides [2], and for any commutative theory generated by T-duality, there is one equivalent noncommutative theory.

However, noncommutative theory is the theory on the open string picture. Therefore instead of the above indirect realization of $SO(d,d|\mathbb{Z})$ T-duality action on noncommutative space, we would like to realize Morita equivalence $SO(d,d|\mathbb{Z})$ on NC $\mathbb{T}^d$ directly from open string physics, like that the realization of the noncommutativity on tori was studied from open strings [8][9][10] [3] .

[1] Here g_s are the closed string coupling constant, G_s are the open string coupling constant, G is the open string metric, and Θ is the noncommutativity of the space.

[2] This compatibility is true also for finite $|B/g|$ with a little modification[7].

[3] In ref. [6], an interpretation on Morita equivalence from open strings is given in $|B/g| \to \infty$ limit.

This is the motivation of the present paper. Towards this propose, in this paper, in section 2, it is shown that on $\mathbb{T}^d$ T-duality group (for closed strings) preserves the mass spectrum of open strings with both ends on the same D$d'(\leq d)$-brane. For the simple case, by the element of the T-duality group which interchanges the winding modes and the momentum of the closed strings, boundary conditions of the open strings (Neumann/Dirichlet) are interchanged, which agrees with the results in [11][12]. The transformation for the oscillator modes preserving the mass spectrum is also derived.

In section 3, we discuss about the T-duality transformation of D-branes from the viewpoint of the D-brane mass. For instance by the action of the above element of the T-duality group, a Dd'-brane translates to another one whose dimension is different from the previous one, and this transformation is consistent with the above one for the open strings. The fact is true for any element of the T-duality group, and it is shown that by the action of T-duality group D-branes and open strings with both ends on them are transformed together (to other D-branes and open strings with both ends on them).

On two-tori, the T-duality group $SO(2,2|\mathbb{Z})$ decomposes to $SL(2,\mathbb{Z})\times SL(2,\mathbb{Z})$ and the arguments become very simple. We apply the above general arguments on two-tori and show that it agrees with some expected results in section 4.

The physical meaning of the above results, especially the relation to Morita equivalence is discussed in Conclusions and Discussions.

2 T-duality for open strings

Consider a bosonic open string on torus $\mathbb{T}^d$ in flat background g and B with its periodicity 2π for all d-direction. The action is

$$S=\int d\tau\int_0^{2\pi}d\sigma L=\frac{1}{4\pi}\int d\tau\int_0^{2\pi}d\sigma\Big[g_{\mu\nu}\partial_\alpha X^\mu\partial^\alpha X^\nu+\epsilon^{\alpha\beta}B_{\mu\nu}\partial_\alpha X^\mu\partial_\beta X^\nu\Big] \tag{1}$$

where μ,ν run the Euclidean space direction $1,\cdots,d$, α,β are τ or σ and its signature is Lorentzian : $(\tau,\sigma)=(+,-)$.

The canonical momentum from the action (1) is given by

$$P_\mu=\frac{\delta S}{\delta X^\mu}=\frac{1}{2\pi}\left(g_{\mu\nu}\dot{X}^\nu+B_{\mu\nu}\acute{X}^\nu\right)=\frac{1}{2\pi}p_\mu+\cdots \tag{2}$$

where $\dot{X}:=\frac{\partial X}{\partial\tau}$, $\acute{X}:=\frac{\partial X}{\partial\sigma}$, p_μ is the zero mode momentum, and $\cdots$ means the higher oscillator modes. The Hamiltonian is

$$H=\int_0^{2\pi}d\sigma\big[P_\mu X^\mu-L\big]=\frac{1}{4\pi}\int_0^{2\pi}d\sigma\big[\dot{X}^\mu g_{\mu\nu}\dot{X}^\nu+\acute{X}^\mu g_{\mu\nu}\acute{X}^\nu\big]\ . \tag{3}$$

We take a different approach and the results in the present paper are valid without taking the $|B/g|\to\infty$ limit.

Though in this section we will show the invariance of the open string mass spectrum under the T-duality including the oscillator parts, at first we concentrate on the zero mode parts of the open string. Then X is represented as

$$X^\mu = x^\mu + f^\mu_\tau \tau + f^\mu_\sigma \sigma \tag{4}$$

where f^μ_τ and f^μ_σ are constant but linearly dependent because of boundary conditions for open strings.

In the beginning, we consider the following standard boundary condition at $\sigma = 0, 2\pi$

$$\begin{array}{rcl} \text{Neumann b.c.} & : & g_{i\nu}\acute{X}^\nu + B_{i\nu}\dot{X}^\nu = 0 \\ \text{Dirichlet b.c.} & : & \dot{X}^a = 0 \ , \end{array} \tag{5}$$

where $\{i\} \cup \{a\} = \{1, \cdots, d\}$, i is for Neumann b.c., and a is for Dirichlet b.c. . As will be seen later, by T-duality transformation it transforms to other boundary conditions. Here we start with this standard one and derive the zero-mode open string mass systematically. Substituting eq.(4) into eq.(5) leads

$$g_{i\nu} f^\nu_\sigma + B_{i\nu} f^\nu_\tau = 0 \ , \qquad f^a_\tau = 0 \tag{6}$$

and from eq.(2) with eq.(4) one gets $p_\mu = g_{\mu\nu} f^\nu_\tau + B_{\mu\nu} f^\nu_\sigma$. Moreover on tori, the zero mode momentum p_μ for Neumann b.c. direction and f^μ_σ, which is the length of the open string over 2π, for Dirichlet b.c. direction are quantized in integer, i.e.

$$g_{i\nu} f^\nu_\tau + B_{i\nu} f^\nu_\sigma = p_i \in \mathbb{Z} \ , \qquad f^a_\tau \in \mathbb{Z} \tag{7}$$

for each i or a. Combining the boundary conditions (6) and the integral conditions (7) leads

$$\left[\begin{pmatrix} e & \\ & e \end{pmatrix} \begin{pmatrix} g & B \\ B & g \end{pmatrix} + \begin{pmatrix} \mathbf{1}-e & \\ & \mathbf{1}-e \end{pmatrix} \right] \begin{pmatrix} f_\tau \\ f_\sigma \end{pmatrix} = \begin{pmatrix} e \cdot p \\ (\mathbf{1}-e)\cdot m \end{pmatrix} \tag{8}$$

where $p \in \mathbb{Z}^d$ is the momentum on $\mathbb{T}^d$, $m \in \mathbb{Z}^d$ is the winding number ($\propto$ length)of the open string, and e is a projection from $\{1, \cdots, d\}$ to Neumann b.c. directions $\{i\}$, i.e. $e \in Mat(d, \mathbb{Z})$ is a diagonal matrix with diagonal entries 1 (for Neumann b.c.) or 0 (for Dirichlet b.c.). Furthermore act $T_e := \begin{pmatrix} e & \mathbf{1}-e \\ \mathbf{1}-e & e \end{pmatrix}$ on both sides of (8) and we gets

$$M \begin{pmatrix} f_\tau \\ f_\sigma \end{pmatrix} = \begin{pmatrix} q \\ 0 \end{pmatrix} \ , \qquad M = \begin{pmatrix} eg & eB + (\mathbf{1}-e) \\ eB + (\mathbf{1}-e) & eg \end{pmatrix} \tag{9}$$

for $q = e \cdot p + (\mathbf{1}-e) \cdot m$. Hereafter let $q \in \mathbb{Z}^d$ be the degree of freedom of the open string zero-modes. This M represents the open string with both ends on Dd'-brane for

$d' = \mathrm{rank}(e)$. Because M decides the type of the open strings, we will call M 'open string data (OSD)' (and the form of M is generalized later).

Once OSD M is given, the open strong zero-mode mass is obtained by substituting the OSD (9) into the Hamiltonian(3)

$$H_0 = \frac{1}{2}\begin{pmatrix} q^t & 0 \end{pmatrix} \mathcal{M}_o \begin{pmatrix} q \\ 0 \end{pmatrix} , \qquad \mathcal{M}_o := M^{t,-1} \begin{pmatrix} g & \mathbf{0} \\ \mathbf{0} & g \end{pmatrix} M^{-1} . \tag{10}$$

In particular case when $d' = d$ (all N b.c.), OSD (9) and its Hamiltonian (10) are

$$M = \begin{pmatrix} g & B \\ B & g \end{pmatrix} , \qquad H_0 = \frac{1}{2}\begin{pmatrix} q^t & 0 \end{pmatrix} \begin{pmatrix} G^{-1} & \\ & G^{-1} \end{pmatrix} \begin{pmatrix} q \\ 0 \end{pmatrix} \tag{11}$$

where $G := g - Bg^{-1}B$ is the open string metric defined in [6]. This implies that an open string on a Dd-brane [4] feels the open string metric. Conversely if $d' = 0$ (all D b.c.), then

$$M = \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{pmatrix} , \qquad H_0 = \frac{1}{2}\begin{pmatrix} q^t & 0 \end{pmatrix} \begin{pmatrix} g & \\ & g \end{pmatrix} \begin{pmatrix} q \\ 0 \end{pmatrix} . \tag{12}$$

B field does not affect the open string with its ends on the D0-brane, and the energy is proportional to its length [5].

Here we consider the action of T-duality. Let $E := g + B \in Mat(d, \mathbb{R})$. The symmetric part (resp. antisymmetric part) of the matrix $E \in Mat(d, \mathbb{R})$ is g (resp. B). T-duality acts on the closed string background E as

$$T(E) = (\mathcal{A}(E) + \mathcal{B})(\mathcal{C}(E) + \mathcal{D})^{-1} \tag{13}$$

where T is an elements of $O(d, d|\mathbb{Z})$ defined as

$$T^t \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{pmatrix} T = \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{pmatrix} , \qquad T = \begin{pmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{pmatrix} , \qquad \mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D} \in Mat(d, \mathbb{Z}) . \tag{14}$$

It is known that T-duality group $O(d, d|\mathbb{Z})$ is generated by following three type of generators [5, 2]

$$T_e := \begin{pmatrix} e & \mathbf{1} - e \\ \mathbf{1} - e & e \end{pmatrix} , \qquad T_A = \begin{pmatrix} A & \mathbf{0} \\ \mathbf{0} & A^{t-1} \end{pmatrix} , \qquad T_N = \begin{pmatrix} \mathbf{1} & N \\ \mathbf{0} & \mathbf{1} \end{pmatrix} \tag{15}$$

[4]Though noncommutativity or NC tori is not introduced in this paper, this Dd-brane is in fact on a NC torus (with metric G), i.e. this open string theory is defined on a NC torus.

[5]This open string theory can also be defined on a NC torus. The torus has metric g^{-1}, which is related to the metric G in eq.(11) by T-duality $T = T_{e=\mathbf{0}}$ defined below. The open string theories corresponding to eq.(11) are given by so-called Seiberg-Witten map (Dd $\to$ NC Dd) [6] and those corresponding to eq.(12) are given by compactifying matrix models (D0 $\to$ NC Dd) [1, 9, 10].

for $A \in GL(d,\mathbb{Z})$, $N^t = -N \in Mat(d,\mathbb{Z})$ and $e \in Mat(d,\mathbb{Z})$ is a projection defined previously below eq.(8).

We would like to find the transformation of OSD M preserving the mass H_0 under the T-duality action (13). In this paper we call such transformation as T-duality transformation of OSD M (or open strings). In order to find it, here we observe the action of these three generators.

1. $T_{e'}$ T_e's satisfy $T_{e'}T_e = T_{e''}$ for $e'' := e'e + (\mathbf{1}-e')(\mathbf{1}-e)$ where $e'' \in Mat(d,\mathbb{Z})$ is also the projection. Let $M[E,T_e]$ be the OSD charactarized by e with background $E = g + B$. Then it is natural to expect that the open string with OSD $M[E,T_e]$ is transformed to the one with OSD $M[T_{e'}(E), T_{e''}]$ by the action of T-duality $T_{e'}$. Actually it is. Let $\mathcal{M}_o^{-1}[E,T_e]$ be the inverse of $\mathcal{M}_o$ defined in eq.(3) with $M = M[E,T_e]$. Then the direct calculation shows that

$$\mathcal{M}_o^{-1}[E,T_e] = \mathcal{M}_o^{-1}[T_{e'}(E), T_{e''}]$$

is satisfied. (This equation will be shown also in eq. (20) where the arbitrary forms of OSD (19) which are compatible with the T-duality group are concerned.) Thus one can see that the mass of the open string with OSD $M[E,T_e]$ (on the Dd'-brane in background $E = g+B$) is equal to the mass of the another open string on the Dd''-brane in background $T_{e'}(E)$. In particular if $e' = e$ then $e'' = 1$, which means that OSD M' is transformed to the one of which the boundary condition is all-Neumann (11) in background $T_e(E)$, and in contrast in the case $e' = 1-e$ then $e'' = 0$ and the boundary condition becomes all-Dirichlet (12) (in background $T_{(1-e)}(E)$). Moreover when $g = \mathbf{1}$ and $B = 0$, the boundary condition part of these arguments reduce to those in [11] and the physical picture is the same as that in [12].

Anyway it was shown that the open string mass spectrum with OSD of the type $M[E,T_e]$ is invariant under the action of the subgroup $\{T_e'\}$.

2. T_A Next consider the action of T_A on the open string with OSD $M[E,T_e]$. Because E transforms to $T_A(E) = AEA^t$, if $M[E,T_e]$ is transformed as

$$M' = \left[\begin{pmatrix} eA^{-1} & \\ & eA^{-1}\end{pmatrix}\begin{pmatrix} T_A(g) & T_A(B) \\ T_A(B) & T_A(g)\end{pmatrix}\begin{pmatrix} A^{t,-1} & \\ & A^{t,-1}\end{pmatrix} + \begin{pmatrix} & 1-e \\ 1-e & \end{pmatrix}\right]\begin{pmatrix} A^t & \\ & A^t\end{pmatrix}, \tag{16}$$

then $\mathcal{M}_o^{-1}$, or equivalently the mass H_0 is preserved. Note that the A^t acting from right in eq.(16) came from rewriting $g^{-1} = A^t T(g^{-1}) A^{t,-1}$ in $\mathcal{M}_o^{-1}$.

Check the physical meaning of the transformed M'. $M'\begin{pmatrix} f'_\tau \\ f'_\sigma\end{pmatrix} = \begin{pmatrix} q \\ 0\end{pmatrix}$ leads the following equations

$$\begin{aligned} eA^{-1}(T(g)f'_\tau + T(B)f'_\sigma) = e\cdot q\ , &\quad eA^{-1}(T(B)f'_\tau + T(g)f'_\sigma) = 0 \\ (1-e)A^t f'_\sigma = (1-e)A^t q\ , &\quad (1-e)A^t f'_\tau = 0\ . \end{aligned} \tag{17}$$

These equations show that the open string with OSD M' is the one on the Dd'-brane which winds nontrivially on $\mathbb{T}^d$ due to A for $d' = \mathrm{rank}(e)$. We will write this M' as $M[T_A(E), T_A T_e]$.

3. T_N Furthermore we act T_N on the open string with OSD $M[T_A(E), T_A T_e]$. Here we rewrite $T_A(E)$ as E'. T_N acts on $E' = g' + B'$ as $T_N(g') = g'$ and $T_N(B') = B' + N$. Because T_N preserves g', rewriting $M[E', T_A T_e]$ as

$$M'' = \begin{pmatrix} eA^{-1} & \\ & eA^{-1} \end{pmatrix} \begin{pmatrix} T_N(g') & T_N(B') - N \\ T_N(B') - N & T_N(g') \end{pmatrix} + \begin{pmatrix} & 1-e \\ 1-e & \end{pmatrix} \begin{pmatrix} A^t & \\ & A^t \end{pmatrix} , \tag{18}$$

and the mass of the open string is invariant. The meaning of this consequence is clear. $-N$ is the $U(1)$ constant curvature F on $\mathbb{T}^d$. As is well-known, the curvature affects only on the direction of Neumann boundary condition, and the fact corresponds to the eA^{-1} in eq.(18). The action of T_N preserves the value of the pair $B + F$, and such a transformation is, on $\mathbb{R}^d$, often called as 'Λ-symmetry'. However on $\mathbb{T}^d$, the elements of F must be integer by the topological reason, and therefore the symmetry is discretized to be the group $\{T_N\}$.

From these three observations, it is natural to regard all the open strings in this system as those which are connected to the open string of all-Neumann b.c.(11) by any T-duality transformation. Moreover extending these three example of OSD (9)(16)(18), the general form of the OSD which are given by acting T on $M[T^{-1}(E), \mathbf{1}]$ (all-Neumann b.c.) can be expected to be the form

$$M[E,T] \begin{pmatrix} f_\tau \\ f_\sigma \end{pmatrix} = \begin{pmatrix} q \\ 0 \end{pmatrix} , \qquad M[E,T] := \begin{pmatrix} \bar{\mathcal{A}} g & \bar{\mathcal{A}} B + \bar{\mathcal{B}} \\ \bar{\mathcal{A}} B + \bar{\mathcal{B}} & \bar{\mathcal{A}} g \end{pmatrix} \tag{19}$$

for $T^{-1} =: \begin{pmatrix} \bar{\mathcal{A}} & \bar{\mathcal{B}} \\ \bar{\mathcal{C}} & \bar{\mathcal{D}} \end{pmatrix} = \begin{pmatrix} \mathcal{D}^t & \mathcal{B}^t \\ \mathcal{C}^t & \mathcal{A}^t \end{pmatrix}$ [6]. Of course $\mathcal{M}_o[E,T]$ and the mass H_0 are defined as the form in eq.(3). As will be shown below, the mass spectrum given by this OSD is invariant under the T-duality transformation. In this sense, the form of OSD seems to be almost unique. For further justification of this OSD(19), in the next section we will relate this to the T-duality transformation for D-branes and in the last section we will show that on two-tori this derives expected results.

In order to confirm that actually the mass spectrum for general open strings characterized by OSD(19) is compatible with the $O(d,d|\mathbb{Z})$ T-duality action, we check that $\mathcal{M}_o^{-1}$ is preserved under the action like as the above three examples. Define matrix $\mathcal{J} := \frac{1}{\sqrt{2}} \begin{pmatrix} \mathbf{1} & \mathbf{1} \\ \mathbf{1} & -\mathbf{1} \end{pmatrix} \in Mat(2d, \mathbb{Z})$ which satisfies $\mathcal{J}^2 = \mathbf{1}$ and $M[E,T]$ is rewritten as

[6]The second equality follows from the definition of $O(d,d|\mathbb{Z})$ (14).

$M[E,T]=\mathcal{J}\begin{pmatrix}\bar{\mathcal{A}}E+\bar{\mathcal{B}} & \\ & \bar{\mathcal{A}}E^t-\bar{\mathcal{B}}\end{pmatrix}\mathcal{J}$. Then one gets

$$\begin{aligned}\mathcal{M}_o^{-1}[E,T] &= \mathcal{J}\begin{pmatrix}\bar{\mathcal{A}}E+\bar{\mathcal{B}} & \\ & \bar{\mathcal{A}}E^t-\bar{\mathcal{B}}\end{pmatrix}\begin{pmatrix}g^{-1} & \\ & g^{-1}\end{pmatrix}\begin{pmatrix}\bar{\mathcal{A}}E+\bar{\mathcal{B}} & \\ & \bar{\mathcal{A}}E^t-\bar{\mathcal{B}}\end{pmatrix}^t\mathcal{J} \\ &= \begin{pmatrix}T^{-1}(Eg^{-1}E^t) & \\ & T^{-1}(Eg^{-1}E^t)\end{pmatrix}=\begin{pmatrix}T^{-1}(G) & \\ & T^{-1}(G)\end{pmatrix}\end{aligned}$$

where $T^{-1}(E)=(\bar{\mathcal{A}}E+\bar{\mathcal{B}})(\bar{\mathcal{C}}E+\bar{\mathcal{D}})^{-1}$, $T^{-1}(g^{-1})=(\bar{\mathcal{C}}E+\bar{\mathcal{D}})g^{-1}(\bar{\mathcal{C}}E+\bar{\mathcal{D}})^t$. In the second step we used the identity $(\bar{\mathcal{A}}E^t-\bar{\mathcal{B}})g^{-1}(\bar{\mathcal{A}}E^t-\bar{\mathcal{B}})^t=(\bar{\mathcal{A}}E+\bar{\mathcal{B}})g^{-1}(\bar{\mathcal{A}}E+\bar{\mathcal{B}})^t$.

Thus, it has been shown that the mass of open strings with any OSD of the form (19) is equal to the mass of the open string with all-Neumann boundary condition (11) in background $T^{-1}(E)$ i.e. $\mathcal{M}_o^{-1}[E,T]=\mathcal{M}_o^{-1}[T^{-1}E,\mathbf{1}]$. Of course, this result means that any open string with OSD $M[E,T]$ (any T) translate to that with another OSD $M[T'(E),T'T]$ in another T-dual background $T'(E)$ and the mass is preserved

$$\mathcal{M}_o[E,T]=\mathcal{M}_o[T'(E),T'T]\ . \tag{20}$$

Mention that a certain subset of T-duality group acts trivially on open strings, because the open strings have half degree of freedom compared with the closed strings.

In the last of this section, we will derive the T-duality transformation for the oscillator parts. The mode expansion of X is

$$X^\mu=x^\mu+f^\mu_\tau\tau+f^\mu_\sigma\sigma+\sum_{n\neq0}\frac{2e^{\frac{-in\tau}{2}}}{n}(ia_{(n)}\cos(\frac{n\sigma}{2})+b_{(n)}\sin(\frac{n\sigma}{2}))\ , \tag{21}$$

and substituting this into the Hamiltonian (3) leads

$$H=\frac{1}{2}\begin{pmatrix}q^t & 0\end{pmatrix}\mathcal{M}_o\begin{pmatrix}q\\0\end{pmatrix}+\frac{1}{2}\sum_{n\neq0}\begin{pmatrix}a^t_{(n)} & b^t_{(n)}\end{pmatrix}\begin{pmatrix}g^{-1} & \\ & g^{-1}\end{pmatrix}\begin{pmatrix}a_{(-n)}\\b_{(-n)}\end{pmatrix}\ .$$

For open strings, the $a_{(n)}$ and $b_{(n)}$ are linearly dependent because of the constraint from boundary conditions. The general form of boundary conditions are

$$(\bar{\mathcal{A}}B+\bar{\mathcal{B}})\dot{X}+\bar{\mathcal{A}}g\acute{X}\Big|_{\sigma=0,2\pi}=0\ ,\qquad T\in O(d,d|\mathbb{Z}) \tag{22}$$

or for each mode n, $(\bar{\mathcal{A}}B+\bar{\mathcal{B}})a_{(n)}+\bar{\mathcal{A}}gb_{(n)}=0$. In the same spirits of the OSD(19), let us define $q_{(n)}$ as

$$M[E,T]\begin{pmatrix}a_{(n)}\\b_{(n)}\end{pmatrix}=\begin{pmatrix}q_{(n)}\\0\end{pmatrix}\ ,\qquad M[E,T]:=\begin{pmatrix}\bar{\mathcal{A}}g & \bar{\mathcal{A}}B+\bar{\mathcal{B}}\\ \bar{\mathcal{A}}B+\bar{\mathcal{B}} & \bar{\mathcal{A}}g\end{pmatrix}\ , \tag{23}$$

and the Hamiltonian can be represented as the following form

$$H = \frac{1}{2}\begin{pmatrix} q^t & 0 \end{pmatrix} \mathcal{M}_o \begin{pmatrix} q \\ 0 \end{pmatrix} + \frac{1}{2}\sum_{n\neq 0}\begin{pmatrix} q^t_{(n)} & 0 \end{pmatrix} \mathcal{M}_o \begin{pmatrix} q_{(-n)} \\ 0 \end{pmatrix} .$$

As was already shown, when background E transforms to $T'(E)$, $\mathcal{M}_o$ is invariant if OSD $M[E,T]$ changes to $M[T'(E),T'T]$. Therefore the mass is invariant if each $q_{(n)}$ is preserved under the transformation T'. Thus the transformations for $a_{(n)}$ and $b_{(n)}$ are given by acting $M[E,T]^{-1}$ on both sides of eq.(23) similarly for the zero modes f_τ, f_σ in eq.(19).

3 T-duality for D-branes

In this section, we discuss the T-duality transformation for D-branes and show that the transformation agrees with the consequence in the previous section.

T-duality group acts on g_s as

$$T(g_s) = g_s \det\left(\mathcal{C}E + \mathcal{D}\right)^{-\frac{1}{2}} .$$

The mass of the Dd-brane is the constant term of the DBI-action

$$\mathbb{M}_D[E,\mathbf{1}] = \frac{1}{g_s\sqrt{\alpha'}}\sqrt{\det E} \ , \tag{24}$$

and which can be rewritten with the variable in background $T(E)$ as

$$\mathbb{M}_D[E,\mathbf{1}] = \frac{1}{T(g_s)\sqrt{\alpha'}}\sqrt{\det(\bar{\mathcal{A}}T(E)+\bar{\mathcal{B}})} =: \mathbb{M}_D[T(E),T] \ , \qquad T^{-1} =: \begin{pmatrix} \bar{\mathcal{A}} & \bar{\mathcal{B}} \\ \bar{\mathcal{C}} & \bar{\mathcal{D}} \end{pmatrix} . \tag{25}$$

This identities is considered as the T-duality transformation for the D-brane mass corresponding to that for the open string (19)(23) [7] . The right hand side of eq.(25) is regarded as the mass of some D-brane bound state in background $T(E)$ (and which implies that the D-brane mass spectrum is invariant under the T-duality).

In order to clarify the states represented by the right hand side in eq.(25), here let us consider as T in eq.(25) the three type of generators(15) or some compositions of those as discussed previously.

1. T_e Take T as T_e and the right hand side in (25) becomes

$$\frac{1}{g_s\sqrt{\alpha'}}\sqrt{\det(eE+(1-e))} = \frac{1}{g_s\sqrt{\alpha'}}\sqrt{\det{}_e(E)} = \mathbb{M}_D[T(E),T] \tag{26}$$

[7]This D-brane transformation is in fact compatible with that on NC tori. In two-tori case these arguments are given in [13].

where $\det_e(E)$ means the determinant for the submatrix of E restricted on the elements for the Neumann b.c. direction. This is exactly the mass of the Dd'-brane in background $T_e(E)$ for $d' = \mathrm{rank}(e) = \sharp\{i\}$. On this D$d'$-brane the open strings with OSD $M[T_e(E), T_e]$ live.

2. T_A Here replace the above $T_e(E), T_e(g_s)$ with E, g_s and consider acting T_A on eq.(26) in parallel with the previous arguments in section 2. The mass of the single Dd'-brane in background E is

$$\frac{1}{T_A(g_s)\sqrt{\alpha'}}\sqrt{\det(eA^{-1}T_A(E) + (1-e)A^t)} = \mathbb{M}_D[T_A(E), T_A T_e] \tag{27}$$

in background $T_A(E)$ [8] . Because of

$$\begin{aligned}\sqrt{\det(eA^{-1}T_A(E) + (1-e)A^t)} &= \sqrt{\det(eA^{-1}(T_A(E))A^{t,-1} + (1-e))} \\ &= \sqrt{\det_e(A^{-1}T_A(E)A^{t,-1})}\end{aligned}$$

and comparing with the OSD(16), one can see that this is the mass of the Dd'-brane which winds nontrivially on the torus with background $T_A(E)$ due to the transformation of A.

3. T_N Further acting T_N on eq.(27) yields

$$\frac{1}{g_s'\sqrt{\alpha'}}\sqrt{\det(eA^{-1}(T_N(E') - N) + (1-e)A^t)} = \mathbb{M}_D[T_N(E'), T_N T_A T_e]$$

for $E' = T_A(E)$ and $g_s' = T_A(g_s)$ $(= T_N(g_s'))$. This is also the expected results and is consistent with the corresponding OSD(18). On the Dd'-brane twisted on $\mathbb{T}^d$ by A, the line bundle over it is also twisted and the open strings with OSD $M'' = M[T_N(E'), T_N T_A T_e]$ are on it.

One has seen that both the open string with generic OSD and the D-branes on which the open string ends transform together consistently in the examples where the number of the D-brane of the highest dimension is one.

In general, the D-brane of the highest dimension can be more than one. We will see such an example in the $\mathbb{T}^2$ case.

4 T-duality on two-tori

$SO(d, d|\mathbb{Z})$, which is given by restricting the rank of $\mathbf{1} - e$ in $T_e \in O(d, d|\mathbb{Z})$ to even rank, is known as the group which arises in the issue of Morita equivalence on NC $\mathbb{T}^d$. Furthermore, when the dimension of the torus is two, it is well-known that it can be decomposed as $SO(2, 2|\mathbb{Z}) \simeq SL(2, \mathbb{Z}) \times SL(2, \mathbb{Z})$. One of two $SL(2, \mathbb{Z})$ groups is the

[8]The mass corresponds to the $\mathbb{M}_D[T(E), T]$ in eq.(25) with replacing E with $T_e^{-1}(E)$ and $T = T_A T_e$.

modular transformation of the target space $\mathbb{T}^2$. It corresponds to T_A ($A \in SL(2,\mathbb{Z})$) in $SO(2,2|\mathbb{Z})$ and preserves g_s and $\sqrt{\det(E)}$. The other $SL(2,\mathbb{Z})$ is discussed as the group generating Morita equivalent NC $\mathbb{T}^2$ [1, 7, 14] and here at first we discuss the action of this part. This $SL(2,\mathbb{Z})$ transformation can be embedded into $SO(2,2|\mathbb{Z})$ as follows

$$\begin{array}{rcl} SL(2,\mathbb{Z}) & \hookrightarrow & SO(2,2|\mathbb{Z}) \\ t_2^+ := \begin{pmatrix} a & b \\ c & d \end{pmatrix} & \mapsto & \begin{pmatrix} a\mathbf{1} & b\mathbf{J} \\ c(-\mathbf{J}) & d\mathbf{1} \end{pmatrix} =: t^+ \end{array} \tag{28}$$

for $\mathbf{J} := \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right) \in Mat(2,\mathbb{Z})$ [9].

The general OSD (19) and D-brane mass (25) are simplified as

$$\begin{gathered} \begin{pmatrix} d\cdot g & d\cdot B - b\mathbf{J} \\ d\cdot B - b\mathbf{J} & d\cdot g \end{pmatrix} \begin{pmatrix} f_\tau \\ f_\sigma \end{pmatrix} = \begin{pmatrix} q \\ 0 \end{pmatrix} , \qquad q \in \mathbb{Z}^2 , \\ \mathbb{M}_D[E, t^+] = \frac{1}{g_s\sqrt{\alpha'}} \sqrt{\det(dE - b\mathbf{J})} . \end{gathered} \tag{29}$$

This is the system of the $(d,-b)$ D2-D0 bound state in background E. When $d \geq 1$, the system is described by rank d gauge theory with the constant curvature.

The OSD generated by all the elements of $SO(2,2|\mathbb{Z})$ are given by acting $t^- \in \{T_A | A \in SL(2,\mathbb{Z})\}$ on eq.(29).

One more interesting example is the system derived by acting T_e with $\mathrm{rank}(e) = 1$ on the above D2-D0 system. The system is transformed to the system of only a single D1-brane, and the arguments reduce to those discussed in [15]. Let us discuss the connection finally.

The OSD for this system is $M[E, T_e t^- t^+]$ for $e = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ (fixed), $t^- = T_A$ with any $A = \left(\begin{smallmatrix} p & q \\ r & s \end{smallmatrix}\right) \in SL(2,\mathbb{Z})$ and any t^+ of $t_2^+ = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2,\mathbb{Z})$ in eq.(28). Explicitly, with a little calculation,

$$M[E, T_e t^- t^+] = \begin{pmatrix} A^{-1}e(t^+)^{-1}g & A^{-1}e(t^+)^{-1}B + A^{-1}(1-e)(t^+)^t \\ A^{-1}e(t^+)^{-1}B + A^{-1}(1-e)(t^+)^t & A^{-1}e(t^+)^{-1}g \end{pmatrix} . \tag{30}$$

[9] This $SL(2,\mathbb{Z}) \subset SO(2,2|\mathbb{Z})$ is essentially generated by $\{T_e\}$ and $\{T_N\}$ in the following sense. When $d = 2$, the subgroup $\{T_e\}$ (resp. $\{T_N\}$) is generated by $\left(\begin{smallmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{smallmatrix}\right)$ (resp. $\left(\begin{smallmatrix} \mathbf{1} & \mathbf{J} \\ \mathbf{0} & \mathbf{1} \end{smallmatrix}\right)$). On the other hand, it is known that $SL(2,\mathbb{Z})$ is generated by $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$, which are embedded into $SO(2,2|\mathbb{Z})$ as $\left(\begin{smallmatrix} 0 & \mathbf{J} \\ \mathbf{J} & 0 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} \mathbf{1} & \mathbf{J} \\ 0 & \mathbf{1} \end{smallmatrix}\right)$, respectively. $\left(\begin{smallmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 & \mathbf{J} \\ \mathbf{J} & 0 \end{smallmatrix}\right)$ are related by the modification of $T_{A=\mathbf{J}} := \left(\begin{smallmatrix} \mathbf{J} & 0 \\ 0 & \mathbf{J} \end{smallmatrix}\right) \in \{T_A\}$ as $\left(\begin{smallmatrix} \mathbf{0} & \mathbf{J} \\ \mathbf{J} & \mathbf{0} \end{smallmatrix}\right) = T_{A=\mathbf{J}} \left(\begin{smallmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{smallmatrix}\right)$.

Note that any element of $SO(2,2|\mathbb{Z})$ can be written as the form t^-t^+ for some $t^- \in \{T_A\}$ and $t^+ \in \{\left(\begin{smallmatrix} 0 & \mathbf{J} \\ \mathbf{J} & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} \mathbf{1} & \mathbf{J} \\ 0 & \mathbf{1} \end{smallmatrix}\right)\}$.

Acting A in both sides of $M[E, T_e t^- t^+] \begin{pmatrix} f_\tau \\ f_\sigma \end{pmatrix} = \begin{pmatrix} q \\ 0 \end{pmatrix}$ leads

$$M[E, t^+ T_e] \begin{pmatrix} f_\tau \\ f_\sigma \end{pmatrix} = \begin{pmatrix} A \cdot q \\ 0 \end{pmatrix} \ . \tag{31}$$

This shows that the number of the D1-brane is exactly one. Moreover one can read that $A \in SL(2, \mathbb{Z})$ acts as the automorphism for the zero-modes of the open strings and that t^+ twists the D1-brane on $\mathbb{T}^2$, as argued in [15]. The previous general arguments (19)(25) guarantee that the mass spectrum for the open strings and D1-brane are preserved under the two $SL(2, \mathbb{Z})$.

5 Conclusions and Discussions

We studied the T-duality group for open string theory on $\mathbb{T}^d$ with flat but general background $E = g + B$. We constructed the generic boundary conditions which are expected from the existence of $O(d, d|\mathbb{Z})$ T-duality group, and showed that the mass spectrum of the open strings with those boundary conditions is invariant under the T-duality group. Furthermore, by discussing the D-brane mass spectrum which is invariant under the T-duality, we derived the T-duality transformation for D-branes. They should be the boundary of the open strings, and showed that actually the D-branes and the open strings on them transform together consistently.

Physically, from the viewpoints of field theories on the target space $\mathbb{T}^d$, the open string mass spectrum $\frac{1}{2} \left(q^t \ 0 \right) \mathcal{M}_o \left(\begin{smallmatrix} q \\ 0 \end{smallmatrix} \right)$ corresponds to the kinetic term for field theories on NC $\mathbb{T}^d$. The open string zero-modes q are, even if the degree of freedom are the length of open strings on the commutative $\mathbb{T}^d$, translated into the momentum of the fields on the NC $\mathbb{T}^d$. This picture seems to give us the realization of the Morita equivalence from the open string physics [10]. The mass of the D-branes on $\mathbb{T}^d$ are also translated as that on the NC $\mathbb{T}^d$, which is the constant (leading) term of the DBI-action.

Precisely, when saying Morita equivalence, we have to study the noncommutativity, which is not discussed explicitly in the present paper. It is one approach to the issue along this paper to study the interaction terms for noncommutative field theory by following the arguments in [9, 10].

It is also interesting to comment about the Chan-Paton degree of freedom. T-duality is originally the duality for closed string theory. When applying this T-duality to open string theory, the gauge bundles emerge. They are generally twisted and the rank are generally greater than one, though only the $U(1)$ parts of the gauge group is concerned.

[10]Precisely, when connecting the T-duality transformation on commutative $\mathbb{T}^d$ with that on NC $\mathbb{T}^d$ with finite $|B/g|$, we have to introduce background Φ on NC $\mathbb{T}^d$ [2, 6, 7, 14, 13].

This relation between closed string theory and open string theory is intriguing. Moreover, the T-duality for such pairs of submanifolds in $\mathbb{T}^d$ and the $U(1)$-bundle discussed in the present paper may be applied to other compactified manifolds.

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T-Duality Group for Open String Theory

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Abstract

We study T-duality for open strings on tori $\mathbb{T}^d$. The general boundary conditions for the open strings are constructed, and it is shown that T-duality group, which preserves the mass spectrum of closed strings, preserves also the mass spectrum of the open strings. The open strings are transformed to those with different boundary conditions by T-duality. We also discuss the T-duality for D-brane mass spectrum, and show that the D-branes and the open strings with both ends on them are transformed together consistently.

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1 Introduction

Noncommutative(NC) tori arise when compactifying matrix models on tori. When one fix the noncommutativity Θ of the base torus, there are still ambiguities for the choice of (NC) gauge theories on the NC torus, and the algebras of these gauge theories are Morita equivalent to each other (and Morita equivalent to the base NC torus)[1].

Morita equivalent tori are generated by $SL(2,\mathbb{Z})(\times SL(2,\mathbb{Z}))$ in the two-torus case, and it is proven that, in general d where d is the dimension of the torus, Morita equivalent NC $\mathbb{T}^d$ is generated by $SO(d,d|\mathbb{Z})$ [2] and conversely NC $\mathbb{T}^d$ generated by $SO(d,d|\mathbb{Z})$ is Morita equivalent[3, 4] (for irrational noncommutative parameter Θ^{ij}).

Originally, $SO(d,d|\mathbb{Z})$ is known as the T-duality group, which acts on the closed string background g, B so as to preserve the closed string mass spectrum[5]. For this reason, this correspondence between Morita equivalence and $SO(d,d|\mathbb{Z})$ is often said that 'Morita equivalence is equivalent to T-duality'. However physically, the relation between this $SO(d,d|\mathbb{Z})$ group (acting on noncommutative parameter Θ) and T-duality (acting on closed string background g, B) seems somehow not clear.

On the other hand, in [6], it is argued that in flat background g, B (on $\mathbb{R}^d$ or $\mathbb{T}^d$), the field theory can be described by either commutative or noncommutative representation. The commutative description is natural for closed string theory, and noncommutative one is natural for open string theory (by the argument in [6], it is natural when we construct the field theory from the OPE for the boundary of the string). Now on $\mathbb{T}^d$ in flat background, T-duality group acts on commutative field theory side and Morita equivalence ($SO(d,d|\mathbb{Z})$) is on noncommutative side, i.e on the open string picture.

Moreover for instance in [6], the transformation between the variables in the commutative picture (g_s, g, B) and those in the noncommutative picture (G_s, G, Θ) is given [1], and on $\mathbb{T}^d$ (for $|B/g| \to \infty$ limit), the $SO(d,d|\mathbb{Z})$ (T-duality) transformation on the noncommutative side is defined by the action of the T-duality group on (g_s, g, B) and using this transformation between (g_s, g, B) and (G_s, G, Θ). By this definition, the transformation between (g_s, g, B) and (G_s, G, Θ) is equivariant with the T-duality action on both sides [2], and for any commutative theory generated by T-duality, there is one equivalent noncommutative theory.

However, noncommutative theory is the theory on the open string picture. Therefore instead of the above indirect realization of $SO(d,d|\mathbb{Z})$ T-duality action on noncommutative space, we would like to realize Morita equivalence $SO(d,d|\mathbb{Z})$ on NC $\mathbb{T}^d$ directly from open string physics, like that the realization of the noncommutativity on tori was studied from open strings [8][9][10] [3] .

[1]Here g_s are the closed string coupling constant, G_s are the open string coupling constant, G is the open string metric, and Θ is the noncommutativity of the space.

[2]This compatibility is true also for finite $|B/g|$ with a little modification[7].

[3]In ref. [6], an interpretation on Morita equivalence from open strings is given in $|B/g| \to \infty$ limit.

This is the motivation of the present paper. Towards this propose, in this paper, in section 2, it is shown that on $\mathbb{T}^d$ T-duality group (for closed strings) preserves the mass spectrum of open strings with both ends on the same D$d'(\leq d)$-brane. For the simple case, by the element of the T-duality group which interchanges the winding modes and the momentum of the closed strings, boundary conditions of the open strings (Neumann/Dirichlet) are interchanged, which agrees with the results in [11][12]. The transformation for the oscillator modes preserving the mass spectrum is also derived.

In section 3, we discuss about the T-duality transformation of D-branes from the viewpoint of the D-brane mass. For instance by the action of the above element of the T-duality group, a Dd'-brane translates to another one whose dimension is different from the previous one, and this transformation is consistent with the above one for the open strings. The fact is true for any element of the T-duality group, and it is shown that by the action of T-duality group D-branes and open strings with both ends on them are transformed together (to other D-branes and open strings with both ends on them).

On two-tori, the T-duality group $SO(2,2|\mathbb{Z})$ decomposes to $SL(2,\mathbb{Z})\times SL(2,\mathbb{Z})$ and the arguments become very simple. We apply the above general arguments on two-tori and show that it agrees with some expected results in section 4.

The physical meaning of the above results, especially the relation to Morita equivalence is discussed in Conclusions and Discussions.

2 T-duality for open strings

Consider a bosonic open string on torus $\mathbb{T}^d$ in flat background g and B with its periodicity 2π for all d-direction. The action is

$$S=\int d\tau\int_0^{2\pi}d\sigma L=\frac{1}{4\pi}\int d\tau\int_0^{2\pi}d\sigma\Big[g_{\mu\nu}\partial_\alpha X^\mu\partial^\alpha X^\nu+\epsilon^{\alpha\beta}B_{\mu\nu}\partial_\alpha X^\mu\partial_\beta X^\nu\Big] \tag{1}$$

where μ,ν run the Euclidean space direction $1,\cdots,d$, α,β are τ or σ and its signature is Lorentzian : $(\tau,\sigma)=(+,-)$.

The canonical momentum from the action (1) is given by

$$P_\mu=\frac{\delta S}{\delta X^\mu}=\frac{1}{2\pi}\left(g_{\mu\nu}\dot{X}^\nu+B_{\mu\nu}\acute{X}^\nu\right)=\frac{1}{2\pi}p_\mu+\cdots \tag{2}$$

where $\dot{X}:=\frac{\partial X}{\partial\tau}$, $\acute{X}:=\frac{\partial X}{\partial\sigma}$, p_μ is the zero mode momentum, and $\cdots$ means the higher oscillator modes. The Hamiltonian is

$$H=\int_0^{2\pi}d\sigma\big[P_\mu X^\mu-L\big]=\frac{1}{4\pi}\int_0^{2\pi}d\sigma\big[\dot{X}^\mu g_{\mu\nu}\dot{X}^\nu+\acute{X}^\mu g_{\mu\nu}\acute{X}^\nu\big]\ . \tag{3}$$

We take a different approach and the results in the present paper are valid without taking the $|B/g|\to\infty$ limit.

Though in this section we will show the invariance of the open string mass spectrum under the T-duality including the oscillator parts, at first we concentrate on the zero mode parts of the open string. Then X is represented as

$$X^\mu = x^\mu + f_\tau^\mu \tau + f_\sigma^\mu \sigma \tag{4}$$

where f_τ^μ and f_σ^μ are constant but linearly dependent because of boundary conditions for open strings.

In the beginning, we consider the following standard boundary condition at $\sigma = 0, 2\pi$

$$\begin{array}{rcl} \text{Neumann b.c.} & : & g_{i\nu}\acute{X}^\nu + B_{i\nu}\dot{X}^\nu = 0 \\ \text{Dirichlet b.c.} & : & \dot{X}^a = 0 \ , \end{array} \tag{5}$$

where $\{i\} \cup \{a\} = \{1, \cdots, d\}$, i is for Neumann b.c., and a is for Dirichlet b.c. . As will be seen later, by T-duality transformation it transforms to other boundary conditions. Here we start with this standard one and derive the zero-mode open string mass systematically. Substituting eq.(4) into eq.(5) leads

$$g_{i\nu} f_\sigma^\nu + B_{i\nu} f_\tau^\nu = 0 \ , \qquad f_\tau^a = 0 \tag{6}$$

and from eq.(2) with eq.(4) one gets $p_\mu = g_{\mu\nu} f_\tau^\nu + B_{\mu\nu} f_\sigma^\nu$. Moreover on tori, the zero mode momentum p_μ for Neumann b.c. direction and f_σ^μ, which is the length of the open string over 2π, for Dirichlet b.c. direction are quantized in integer, i.e.

$$g_{i\nu} f_\tau^\nu + B_{i\nu} f_\sigma^\nu = p_i \in \mathbb{Z} \ , \qquad f_\tau^a \in \mathbb{Z} \tag{7}$$

for each i or a. Combining the boundary conditions (6) and the integral conditions (7) leads

$$\left[\begin{pmatrix} e & \\ & e \end{pmatrix} \begin{pmatrix} g & B \\ B & g \end{pmatrix} + \begin{pmatrix} \mathbf{1} - e & \\ & \mathbf{1} - e \end{pmatrix} \right] \begin{pmatrix} f_\tau \\ f_\sigma \end{pmatrix} = \begin{pmatrix} e \cdot p \\ (\mathbf{1} - e) \cdot m \end{pmatrix} \tag{8}$$

where $p \in \mathbb{Z}^d$ is the momentum on $\mathbb{T}^d$, $m \in \mathbb{Z}^d$ is the winding number ($\propto$ length)of the open string, and e is a projection from $\{1, \cdots, d\}$ to Neumann b.c. directions $\{i\}$, i.e. $e \in Mat(d, \mathbb{Z})$ is a diagonal matrix with diagonal entries 1 (for Neumann b.c.) or 0 (for Dirichlet b.c.). Furthermore act $T_e := \begin{pmatrix} e & \mathbf{1} - e \\ \mathbf{1} - e & e \end{pmatrix}$ on both sides of (8) and we gets

$$M \begin{pmatrix} f_\tau \\ f_\sigma \end{pmatrix} = \begin{pmatrix} q \\ 0 \end{pmatrix} \ , \qquad M = \begin{pmatrix} eg & eB + (\mathbf{1} - e) \\ eB + (\mathbf{1} - e) & eg \end{pmatrix} \tag{9}$$

for $q = e \cdot p + (\mathbf{1} - e) \cdot m$. Hereafter let $q \in \mathbb{Z}^d$ be the degree of freedom of the open string zero-modes. This M represents the open string with both ends on Dd'-brane for

$d' = \text{rank}(e)$. Because M decides the type of the open strings, we will call M 'open string data (OSD)' (and the form of M is generalized later).

Once OSD M is given, the open strong zero-mode mass is obtained by substituting the OSD (9) into the Hamiltonian(3)

$$H_0 = \frac{1}{2}\begin{pmatrix} q^t & 0 \end{pmatrix} \mathcal{M}_o \begin{pmatrix} q \\ 0 \end{pmatrix} , \qquad \mathcal{M}_o := M^{t,-1}\begin{pmatrix} g & \mathbf{0} \\ \mathbf{0} & g \end{pmatrix} M^{-1} . \tag{10}$$

In particular case when $d' = d$ (all N b.c.), OSD (9) and its Hamiltonian (10) are

$$M = \begin{pmatrix} g & B \\ B & g \end{pmatrix} , \qquad H_0 = \frac{1}{2}\begin{pmatrix} q^t & 0 \end{pmatrix} \begin{pmatrix} G^{-1} & \\ & G^{-1} \end{pmatrix} \begin{pmatrix} q \\ 0 \end{pmatrix} \tag{11}$$

where $G := g - Bg^{-1}B$ is the open string metric defined in [6]. This implies that an open string on a Dd-brane [4] feels the open string metric. Conversely if $d' = 0$ (all D b.c.), then

$$M = \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{pmatrix} , \qquad H_0 = \frac{1}{2}\begin{pmatrix} q^t & 0 \end{pmatrix} \begin{pmatrix} g & \\ & g \end{pmatrix} \begin{pmatrix} q \\ 0 \end{pmatrix} . \tag{12}$$

B field does not affect the open string with its ends on the D0-brane, and the energy is proportional to its length [5].

Here we consider the action of T-duality. Let $E := g + B \in Mat(d, \mathbb{R})$. The symmetric part (resp. antisymmetric part) of the matrix $E \in Mat(d, \mathbb{R})$ is g (resp. B). T-duality acts on the closed string background E as

$$T(E) = (\mathcal{A}(E) + \mathcal{B})(\mathcal{C}(E) + \mathcal{D})^{-1} \tag{13}$$

where T is an elements of $O(d, d|\mathbb{Z})$ defined as

$$T^t \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{pmatrix} T = \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{pmatrix} , \qquad T = \begin{pmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{pmatrix} , \qquad \mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D} \in Mat(d, \mathbb{Z}) . \tag{14}$$

It is known that T-duality group $O(d, d|\mathbb{Z})$ is generated by following three type of generators [5, 2]

$$T_e := \begin{pmatrix} e & \mathbf{1} - e \\ \mathbf{1} - e & e \end{pmatrix} , \qquad T_A = \begin{pmatrix} A & \mathbf{0} \\ \mathbf{0} & A^{t-1} \end{pmatrix} , \qquad T_N = \begin{pmatrix} \mathbf{1} & N \\ \mathbf{0} & \mathbf{1} \end{pmatrix} \tag{15}$$

[4]Though noncommutativity or NC tori is not introduced in this paper, this Dd-brane is in fact on a NC torus (with metric G), i.e. this open string theory is defined on a NC torus.

[5]This open string theory can also be defined on a NC torus. The torus has metric g^{-1}, which is related to the metric G in eq.(11) by T-duality $T = T_{e=\mathbf{0}}$ defined below. The open string theories corresponding to eq.(11) are given by so-called Seiberg-Witten map (Dd $\to$ NC Dd) [6] and those corresponding to eq.(12) are given by compactifying matrix models (D0 $\to$ NC Dd) [1, 9, 10].

for $A \in GL(d,\mathbb{Z})$, $N^t = -N \in Mat(d,\mathbb{Z})$ and $e \in Mat(d,\mathbb{Z})$ is a projection defined previously below eq.(8).

We would like to find the transformation of OSD M preserving the mass H_0 under the T-duality action (13). In this paper we call such transformation as T-duality transformation of OSD M (or open strings). In order to find it, here we observe the action of these three generators.

1. $T_{e'}$ T_e's satisfy $T_{e'}T_e = T_{e''}$ for $e'' := e'e + (\mathbf{1}-e')(\mathbf{1}-e)$ where $e'' \in Mat(d,\mathbb{Z})$ is also the projection. Let $M[E,T_e]$ be the OSD charactarized by e with background $E = g + B$. Then it is natural to expect that the open string with OSD $M[E,T_e]$ is transformed to the one with OSD $M[T_{e'}(E),T_{e''}]$ by the action of T-duality $T_{e'}$. Actually it is. Let $\mathcal{M}_o^{-1}[E,T_e]$ be the inverse of $\mathcal{M}_o$ defined in eq.(3) with $M = M[E,T_e]$. Then the direct calculation shows that

$$\mathcal{M}_o^{-1}[E,T_e] = \mathcal{M}_o^{-1}[T_{e'}(E),T_{e''}]$$

is satisfied. (This equation will be shown also in eq. (20) where the arbitrary forms of OSD (19) which are compatible with the T-duality group are concerned.) Thus one can see that the mass of the open string with OSD $M[E,T_e]$ (on the Dd'-brane in background $E = g+B$) is equal to the mass of the another open string on the Dd''-brane in background $T_{e'}(E)$. In particular if $e' = e$ then $e'' = 1$, which means that OSD M' is transformed to the one of which the boundary condition is all-Neumann (11) in background $T_e(E)$, and in contrast in the case $e' = 1-e$ then $e'' = 0$ and the boundary condition becomes all-Dirichlet (12) (in background $T_{(1-e)}(E)$). Moreover when $g = \mathbf{1}$ and $B = 0$, the boundary condition part of these arguments reduce to those in [11] and the physical picture is the same as that in [12].

Anyway it was shown that the open string mass spectrum with OSD of the type $M[E,T_e]$ is invariant under the action of the subgroup $\{T_e'\}$.

2. T_A Next consider the action of T_A on the open string with OSD $M[E,T_e]$. Because E transforms to $T_A(E) = AEA^t$, if $M[E,T_e]$ is transformed as

$$M' = \left[\begin{pmatrix} eA^{-1} & \\ & eA^{-1}\end{pmatrix}\begin{pmatrix} T_A(g) & T_A(B) \\ T_A(B) & T_A(g)\end{pmatrix}\begin{pmatrix} A^{t,-1} & \\ & A^{t,-1}\end{pmatrix} + \begin{pmatrix} & 1-e \\ 1-e & \end{pmatrix}\right]\begin{pmatrix} A^t & \\ & A^t\end{pmatrix}, \tag{16}$$

then $\mathcal{M}_o^{-1}$, or equivalently the mass H_0 is preserved. Note that the A^t acting from right in eq.(16) came from rewriting $g^{-1} = A^tT(g^{-1})A^{t,-1}$ in $\mathcal{M}_o^{-1}$.

Check the physical meaning of the transformed M'. $M'\begin{pmatrix} f'_\tau \\ f'_\sigma\end{pmatrix} = \begin{pmatrix} q \\ 0\end{pmatrix}$ leads the following equations

$$\begin{aligned} eA^{-1}(T(g)f'_\tau + T(B)f'_\sigma) = e\cdot q\ , \quad eA^{-1}(T(B)f'_\tau + T(g)f'_\sigma) = 0 \\ (1-e)A^tf'_\sigma = (1-e)A^tq\ , \quad (1-e)A^tf'_\tau = 0\ . \end{aligned} \tag{17}$$

These equations show that the open string with OSD M' is the one on the Dd'-brane which winds nontrivially on $\mathbb{T}^d$ due to A for $d' = \mathrm{rank}(e)$. We will write this M' as $M[T_A(E), T_A T_e]$.

3. T_N Furthermore we act T_N on the open string with OSD $M[T_A(E), T_A T_e]$. Here we rewrite $T_A(E)$ as E'. T_N acts on $E' = g' + B'$ as $T_N(g') = g'$ and $T_N(B') = B' + N$. Because T_N preserves g', rewriting $M[E', T_A T_e]$ as

$$M'' = \begin{pmatrix} eA^{-1} & \\ & eA^{-1} \end{pmatrix} \begin{pmatrix} T_N(g') & T_N(B') - N \\ T_N(B') - N & T_N(g') \end{pmatrix} + \begin{pmatrix} & 1-e \\ 1-e & \end{pmatrix} \begin{pmatrix} A^t & \\ & A^t \end{pmatrix} , \tag{18}$$

and the mass of the open string is invariant. The meaning of this consequence is clear. $-N$ is the $U(1)$ constant curvature F on $\mathbb{T}^d$. As is well-known, the curvature affects only on the direction of Neumann boundary condition, and the fact corresponds to the eA^{-1} in eq.(18). The action of T_N preserves the value of the pair $B + F$, and such a transformation is, on $\mathbb{R}^d$, often called as 'Λ-symmetry'. However on $\mathbb{T}^d$, the elements of F must be integer by the topological reason, and therefore the symmetry is discretized to be the group $\{T_N\}$.

From these three observations, it is natural to regard all the open strings in this system as those which are connected to the open string of all-Neumann b.c.(11) by any T-duality transformation. Moreover extending these three example of OSD (9)(16)(18), the general form of the OSD which are given by acting T on $M[T^{-1}(E), \mathbf{1}]$ (all-Neumann b.c.) can be expected to be the form

$$M[E,T] \begin{pmatrix} f_\tau \\ f_\sigma \end{pmatrix} = \begin{pmatrix} q \\ 0 \end{pmatrix} , \qquad M[E,T] := \begin{pmatrix} \bar{\mathcal{A}} g & \bar{\mathcal{A}} B + \bar{\mathcal{B}} \\ \bar{\mathcal{A}} B + \bar{\mathcal{B}} & \bar{\mathcal{A}} g \end{pmatrix} \tag{19}$$

for $T^{-1} =: \begin{pmatrix} \bar{\mathcal{A}} & \bar{\mathcal{B}} \\ \bar{\mathcal{C}} & \bar{\mathcal{D}} \end{pmatrix} = \begin{pmatrix} \mathcal{D}^t & \mathcal{B}^t \\ \mathcal{C}^t & \mathcal{A}^t \end{pmatrix}$ [6]. Of course $\mathcal{M}_o[E,T]$ and the mass H_0 are defined as the form in eq.(3). As will be shown below, the mass spectrum given by this OSD is invariant under the T-duality transformation. In this sense, the form of OSD seems to be almost unique. For further justification of this OSD(19), in the next section we will relate this to the T-duality transformation for D-branes and in the last section we will show that on two-tori this derives expected results.

In order to confirm that actually the mass spectrum for general open strings characterized by OSD(19) is compatible with the $O(d,d|\mathbb{Z})$ T-duality action, we check that $\mathcal{M}_o^{-1}$ is preserved under the action like as the above three examples. Define matrix $\mathcal{J} := \frac{1}{\sqrt{2}} \begin{pmatrix} \mathbf{1} & \mathbf{1} \\ \mathbf{1} & -\mathbf{1} \end{pmatrix} \in Mat(2d, \mathbb{Z})$ which satisfies $\mathcal{J}^2 = \mathbf{1}$ and $M[E,T]$ is rewritten as

[6] The second equality follows from the definition of $O(d,d|\mathbb{Z})$ (14).

$M[E,T] = \mathcal{J}\begin{pmatrix} \bar{\mathcal{A}}E+\bar{\mathcal{B}} & \\ & \bar{\mathcal{A}}E^t-\bar{\mathcal{B}}\end{pmatrix}\mathcal{J}$. Then one gets

$$\begin{aligned}\mathcal{M}_o^{-1}[E,T] &= \mathcal{J}\begin{pmatrix} \bar{\mathcal{A}}E+\bar{\mathcal{B}} & \\ & \bar{\mathcal{A}}E^t-\bar{\mathcal{B}}\end{pmatrix}\begin{pmatrix} g^{-1} & \\ & g^{-1}\end{pmatrix}\begin{pmatrix} \bar{\mathcal{A}}E+\bar{\mathcal{B}} & \\ & \bar{\mathcal{A}}E^t-\bar{\mathcal{B}}\end{pmatrix}^t\mathcal{J} \\ &= \begin{pmatrix} T^{-1}(Eg^{-1}E^t) & \\ & T^{-1}(Eg^{-1}E^t)\end{pmatrix} = \begin{pmatrix} T^{-1}(G) & \\ & T^{-1}(G)\end{pmatrix}\end{aligned}$$

where $T^{-1}(E) = (\bar{\mathcal{A}}E+\bar{\mathcal{B}})(\bar{\mathcal{C}}E+\bar{\mathcal{D}})^{-1}$, $T^{-1}(g^{-1}) = (\bar{\mathcal{C}}E+\bar{\mathcal{D}})g^{-1}(\bar{\mathcal{C}}E+\bar{\mathcal{D}})^t$. In the second step we used the identity $(\bar{\mathcal{A}}E^t-\bar{\mathcal{B}})g^{-1}(\bar{\mathcal{A}}E^t-\bar{\mathcal{B}})^t = (\bar{\mathcal{A}}E+\bar{\mathcal{B}})g^{-1}(\bar{\mathcal{A}}E+\bar{\mathcal{B}})^t$.

Thus, it has been shown that the mass of open strings with any OSD of the form (19) is equal to the mass of the open string with all-Neumann boundary condition (11) in background $T^{-1}(E)$ i.e. $\mathcal{M}_o^{-1}[E,T] = \mathcal{M}_o^{-1}[T^{-1}E,\mathbf{1}]$. Of course, this result means that any open string with OSD $M[E,T]$ (any T) translate to that with another OSD $M[T'(E),T'T]$ in another T-dual background $T'(E)$ and the mass is preserved

$$\mathcal{M}_o[E,T] = \mathcal{M}_o[T'(E),T'T]\ . \tag{20}$$

Mention that a certain subset of T-duality group acts trivially on open strings, because the open strings have half degree of freedom compared with the closed strings.

In the last of this section, we will derive the T-duality transformation for the oscillator parts. The mode expansion of X is

$$X^\mu = x^\mu + f^\mu_\tau\tau + f^\mu_\sigma\sigma + \sum_{n\neq 0}\frac{2e^{\frac{-in\tau}{2}}}{n}(ia_{(n)}\cos(\frac{n\sigma}{2}) + b_{(n)}\sin(\frac{n\sigma}{2}))\ , \tag{21}$$

and substituting this into the Hamiltonian (3) leads

$$H = \frac{1}{2}\begin{pmatrix} q^t & 0\end{pmatrix}\mathcal{M}_o\begin{pmatrix} q \\ 0\end{pmatrix} + \frac{1}{2}\sum_{n\neq 0}\begin{pmatrix} a^t_{(n)} & b^t_{(n)}\end{pmatrix}\begin{pmatrix} g^{-1} & \\ & g^{-1}\end{pmatrix}\begin{pmatrix} a_{(-n)} \\ b_{(-n)}\end{pmatrix}\ .$$

For open strings, the $a_{(n)}$ and $b_{(n)}$ are linearly dependent because of the constraint from boundary conditions. The general form of boundary conditions are

$$(\bar{\mathcal{A}}B+\bar{\mathcal{B}})\dot{X} + \bar{\mathcal{A}}g\acute{X}\Big|_{\sigma=0,2\pi} = 0\ , \qquad T\in O(d,d|\mathbb{Z}) \tag{22}$$

or for each mode n, $(\bar{\mathcal{A}}B+\bar{\mathcal{B}})a_{(n)} + \bar{\mathcal{A}}gb_{(n)} = 0$. In the same spirits of the OSD(19), let us define $q_{(n)}$ as

$$M[E,T]\begin{pmatrix} a_{(n)} \\ b_{(n)}\end{pmatrix} = \begin{pmatrix} q_{(n)} \\ 0\end{pmatrix}\ , \qquad M[E,T] := \begin{pmatrix} \bar{\mathcal{A}}g & \bar{\mathcal{A}}B+\bar{\mathcal{B}} \\ \bar{\mathcal{A}}B+\bar{\mathcal{B}} & \bar{\mathcal{A}}g\end{pmatrix}\ , \tag{23}$$

and the Hamiltonian can be represented as the following form

$$H = \frac{1}{2}\begin{pmatrix} q^t & 0 \end{pmatrix} \mathcal{M}_o \begin{pmatrix} q \\ 0 \end{pmatrix} + \frac{1}{2}\sum_{n\neq 0} \begin{pmatrix} q^t_{(n)} & 0 \end{pmatrix} \mathcal{M}_o \begin{pmatrix} q_{(-n)} \\ 0 \end{pmatrix} .$$

As was already shown, when background E transforms to $T'(E)$, $\mathcal{M}_o$ is invariant if OSD $M[E,T]$ changes to $M[T'(E),T'T]$. Therefore the mass is invariant if each $q_{(n)}$ is preserved under the transformation T'. Thus the transformations for $a_{(n)}$ and $b_{(n)}$ are given by acting $M[E,T]^{-1}$ on both sides of eq.(23) similarly for the zero modes f_τ, f_σ in eq.(19).

3 T-duality for D-branes

In this section, we discuss the T-duality transformation for D-branes and show that the transformation agrees with the consequence in the previous section.

T-duality group acts on g_s as

$$T(g_s) = g_s \det(\mathcal{C}E + \mathcal{D})^{-\frac{1}{2}} .$$

The mass of the Dd-brane is the constant term of the DBI-action

$$\mathbb{M}_D[E,\mathbf{1}] = \frac{1}{g_s\sqrt{\alpha'}}\sqrt{\det E} , \tag{24}$$

and which can be rewritten with the variable in background $T(E)$ as

$$\mathbb{M}_D[E,\mathbf{1}] = \frac{1}{T(g_s)\sqrt{\alpha'}}\sqrt{\det(\bar{\mathcal{A}}T(E)+\bar{\mathcal{B}})} =: \mathbb{M}_D[T(E),T] , \qquad T^{-1} =: \begin{pmatrix} \bar{\mathcal{A}} & \bar{\mathcal{B}} \\ \bar{\mathcal{C}} & \bar{\mathcal{D}} \end{pmatrix} . \tag{25}$$

This identities is considered as the T-duality transformation for the D-brane mass corresponding to that for the open string (19)(23) [7] . The right hand side of eq.(25) is regarded as the mass of some D-brane bound state in background $T(E)$ (and which implies that the D-brane mass spectrum is invariant under the T-duality).

In order to clarify the states represented by the right hand side in eq.(25), here let us consider as T in eq.(25) the three type of generators(15) or some compositions of those as discussed previously.

1. T_e Take T as T_e and the right hand side in (25) becomes

$$\frac{1}{g_s\sqrt{\alpha'}}\sqrt{\det(eE+(1-e))} = \frac{1}{g_s\sqrt{\alpha'}}\sqrt{\det{}_e(E)} = \mathbb{M}_D[T(E),T] \tag{26}$$

[7]This D-brane transformation is in fact compatible with that on NC tori. In two-tori case these arguments are given in [13].

where $\det_e(E)$ means the determinant for the submatrix of E restricted on the elements for the Neumann b.c. direction. This is exactly the mass of the Dd'-brane in background $T_e(E)$ for $d' = \mathrm{rank}(e) = \sharp\{i\}$. On this D$d'$-brane the open strings with OSD $M[T_e(E), T_e]$ live.

2. T_A Here replace the above $T_e(E), T_e(g_s)$ with E, g_s and consider acting T_A on eq.(26) in parallel with the previous arguments in section 2. The mass of the single Dd'-brane in background E is

$$\frac{1}{T_A(g_s)\sqrt{\alpha'}}\sqrt{\det(eA^{-1}T_A(E) + (1-e)A^t)} = \mathbb{M}_D[T_A(E), T_A T_e] \tag{27}$$

in background $T_A(E)$ [8]. Because of

$$\begin{aligned}\sqrt{\det(eA^{-1}T_A(E) + (1-e)A^t)} &= \sqrt{\det(eA^{-1}(T_A(E))A^{t,-1} + (1-e))} \\ &= \sqrt{\det_e(A^{-1}T_A(E)A^{t,-1})}\end{aligned}$$

and comparing with the OSD(16), one can see that this is the mass of the Dd'-brane which winds nontrivially on the torus with background $T_A(E)$ due to the transformation of A.

3. T_N Further acting T_N on eq.(27) yields

$$\frac{1}{g_s'\sqrt{\alpha'}}\sqrt{\det(eA^{-1}(T_N(E') - N) + (1-e)A^t)} = \mathbb{M}_D[T_N(E'), T_N T_A T_e]$$

for $E' = T_A(E)$ and $g_s' = T_A(g_s)$ $(= T_N(g_s'))$. This is also the expected results and is consistent with the corresponding OSD(18). On the Dd'-brane twisted on $\mathbb{T}^d$ by A, the line bundle over it is also twisted and the open strings with OSD $M'' = M[T_N(E'), T_N T_A T_e]$ are on it.

One has seen that both the open string with generic OSD and the D-branes on which the open string ends transform together consistently in the examples where the number of the D-brane of the highest dimension is one.

In general, the D-brane of the highest dimension can be more than one. We will see such an example in the $\mathbb{T}^2$ case.

4 T-duality on two-tori

$SO(d,d|\mathbb{Z})$, which is given by restricting the rank of $\mathbf{1} - e$ in $T_e \in O(d,d|\mathbb{Z})$ to even rank, is known as the group which arises in the issue of Morita equivalence on NC $\mathbb{T}^d$. Furthermore, when the dimension of the torus is two, it is well-known that it can be decomposed as $SO(2,2|\mathbb{Z}) \simeq SL(2,\mathbb{Z}) \times SL(2,\mathbb{Z})$. One of two $SL(2,\mathbb{Z})$ groups is the

[8] The mass corresponds to the $\mathbb{M}_D[T(E), T]$ in eq.(25) with replacing E with $T_e^{-1}(E)$ and $T = T_A T_e$.

modular transformation of the target space $\mathbb{T}^2$. It corresponds to T_A ($A \in SL(2,\mathbb{Z})$) in $SO(2,2|\mathbb{Z})$ and preserves g_s and $\sqrt{\det(E)}$. The other $SL(2,\mathbb{Z})$ is discussed as the group generating Morita equivalent NC $\mathbb{T}^2$ [1, 7, 14] and here at first we discuss the action of this part. This $SL(2,\mathbb{Z})$ transformation can be embedded into $SO(2,2|\mathbb{Z})$ as follows

$$\begin{array}{ccc} SL(2,\mathbb{Z}) & \hookrightarrow & SO(2,2|\mathbb{Z}) \\ t_2^+ := \begin{pmatrix} a & b \\ c & d \end{pmatrix} & \mapsto & \begin{pmatrix} a\mathbf{1} & b\mathbf{J} \\ c(-\mathbf{J}) & d\mathbf{1} \end{pmatrix} =: t^+ \end{array} \tag{28}$$

for $\mathbf{J} := \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right) \in Mat(2,\mathbb{Z})$ [9].

The general OSD (19) and D-brane mass (25) are simplified as

$$\begin{aligned} &\begin{pmatrix} d\cdot g & d\cdot B - b\mathbf{J} \\ d\cdot B - b\mathbf{J} & d\cdot g \end{pmatrix} \begin{pmatrix} f_\tau \\ f_\sigma \end{pmatrix} = \begin{pmatrix} q \\ 0 \end{pmatrix} , \qquad q \in \mathbb{Z}^2 , \\ &\mathbb{M}_D[E, t^+] = \frac{1}{g_s\sqrt{\alpha'}} \sqrt{\det(dE - b\mathbf{J})} \ . \end{aligned} \tag{29}$$

This is the system of the $(d, -b)$ D2-D0 bound state in background E. When $d \geq 1$, the system is described by rank d gauge theory with the constant curvature.

The OSD generated by all the elements of $SO(2,2|\mathbb{Z})$ are given by acting $t^- \in \{T_A | A \in SL(2,\mathbb{Z})\}$ on eq.(29).

One more interesting example is the system derived by acting T_e with $\mathrm{rank}(e) = 1$ on the above D2-D0 system. The system is transformed to the system of only a single D1-brane, and the arguments reduce to those discussed in [15]. Let us discuss the connection finally.

The OSD for this system is $M[E, T_e t^- t^+]$ for $e = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ (fixed), $t^- = T_A$ with any $A = \left(\begin{smallmatrix} p & q \\ r & s \end{smallmatrix}\right) \in SL(2,\mathbb{Z})$ and any t^+ of $t_2^+ = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2,\mathbb{Z})$ in eq.(28). Explicitly, with a little calculation,

$$M[E, T_e t^- t^+] = \begin{pmatrix} A^{-1}e(t^+)^{-1}g & A^{-1}e(t^+)^{-1}B + A^{-1}(1-e)(t^+)^t \\ A^{-1}e(t^+)^{-1}B + A^{-1}(1-e)(t^+)^t & A^{-1}e(t^+)^{-1}g \end{pmatrix} . \tag{30}$$

[9] This $SL(2,\mathbb{Z}) \subset SO(2,2|\mathbb{Z})$ is essentially generated by $\{T_e\}$ and $\{T_N\}$ in the following sense. When $d = 2$, the subgroup $\{T_e\}$ (resp. $\{T_N\}$) is generated by $\left(\begin{smallmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{smallmatrix}\right)$ (resp. $\left(\begin{smallmatrix} \mathbf{1} & \mathbf{J} \\ \mathbf{0} & \mathbf{1} \end{smallmatrix}\right)$). On the other hand, it is known that $SL(2,\mathbb{Z})$ is generated by $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$, which are embedded into $SO(2,2|\mathbb{Z})$ as $\left(\begin{smallmatrix} 0 & \mathbf{J} \\ \mathbf{J} & 0 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} \mathbf{1} & \mathbf{J} \\ 0 & \mathbf{1} \end{smallmatrix}\right)$, respectively. $\left(\begin{smallmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 & \mathbf{J} \\ \mathbf{J} & 0 \end{smallmatrix}\right)$ are related by the modification of $T_{A=\mathbf{J}} := \left(\begin{smallmatrix} \mathbf{J} & 0 \\ 0 & \mathbf{J} \end{smallmatrix}\right) \in \{T_A\}$ as $\left(\begin{smallmatrix} \mathbf{0} & \mathbf{J} \\ \mathbf{J} & \mathbf{0} \end{smallmatrix}\right) = T_{A=\mathbf{J}} \left(\begin{smallmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{smallmatrix}\right)$.

Note that any element of $SO(2,2|\mathbb{Z})$ can be written as the form $t^- t^+$ for some $t^- \in \{T_A\}$ and $t^+ \in \{\left(\begin{smallmatrix} 0 & \mathbf{J} \\ \mathbf{J} & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} \mathbf{1} & \mathbf{J} \\ 0 & \mathbf{1} \end{smallmatrix}\right)\}$.

Acting A in both sides of $M[E, T_e t^- t^+] \begin{pmatrix} f_\tau \\ f_\sigma \end{pmatrix} = \begin{pmatrix} q \\ 0 \end{pmatrix}$ leads

$$M[E, t^+ T_e] \begin{pmatrix} f_\tau \\ f_\sigma \end{pmatrix} = \begin{pmatrix} A \cdot q \\ 0 \end{pmatrix} . \tag{31}$$

This shows that the number of the D1-brane is exactly one. Moreover one can read that $A \in SL(2, \mathbb{Z})$ acts as the automorphism for the zero-modes of the open strings and that t^+ twists the D1-brane on $\mathbb{T}^2$, as argued in [15]. The previous general arguments (19)(25) guarantee that the mass spectrum for the open strings and D1-brane are preserved under the two $SL(2, \mathbb{Z})$.

5 Conclusions and Discussions

We studied the T-duality group for open string theory on $\mathbb{T}^d$ with flat but general background $E = g + B$. We constructed the generic boundary conditions which are expected from the existence of $O(d, d|\mathbb{Z})$ T-duality group, and showed that the mass spectrum of the open strings with those boundary conditions is invariant under the T-duality group. Furthermore, by discussing the D-brane mass spectrum which is invariant under the T-duality, we derived the T-duality transformation for D-branes. They should be the boundary of the open strings, and showed that actually the D-branes and the open strings on them transform together consistently.

Physically, from the viewpoints of field theories on the target space $\mathbb{T}^d$, the open string mass spectrum $\frac{1}{2} \left(q^t \; 0 \right) \mathcal{M}_o \left(\begin{smallmatrix} q \\ 0 \end{smallmatrix} \right)$ corresponds to the kinetic term for field theories on NC $\mathbb{T}^d$. The open string zero-modes q are, even if the degree of freedom are the length of open strings on the commutative $\mathbb{T}^d$, translated into the momentum of the fields on the NC $\mathbb{T}^d$. This picture seems to give us the realization of the Morita equivalence from the open string physics [10]. The mass of the D-branes on $\mathbb{T}^d$ are also translated as that on the NC $\mathbb{T}^d$, which is the constant (leading) term of the DBI-action.

Precisely, when saying Morita equivalence, we have to study the noncommutativity, which is not discussed explicitly in the present paper. It is one approach to the issue along this paper to study the interaction terms for noncommutative field theory by following the arguments in [9, 10].

It is also interesting to comment about the Chan-Paton degree of freedom. T-duality is originally the duality for closed string theory. When applying this T-duality to open string theory, the gauge bundles emerge. They are generally twisted and the rank are generally greater than one, though only the $U(1)$ parts of the gauge group is concerned.

[10] Precisely, when connecting the T-duality transformation on commutative $\mathbb{T}^d$ with that on NC $\mathbb{T}^d$ with finite $|B/g|$, we have to introduce background Φ on NC $\mathbb{T}^d$ [2, 6, 7, 14, 13].

This relation between closed string theory and open string theory is intriguing. Moreover, the T-duality for such pairs of submanifolds in $\mathbb{T}^d$ and the $U(1)$-bundle discussed in the present paper may be applied to other compactified manifolds.

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