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The BV space in variational and evolution problems

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An advantage of the space of functions of bounded variation, $BV(\Omega)$, over the Sobolev spaces is that it is closed under the multiplication by the characteristic functions of sets with smooth boundary of finite surface measure. In other words, the BV space contains (some) functions with jump discontinuities. This fact is exploited in the construction of the Rudin-Osher-Fatemi (ROF) algorithm for noise removal and edge detection. This algorithm leads to the following variational problem,

$$\min\{E[u] + \lambda \int_\Omega (u-f)^2\,dx\}, \tag{1}$$

where $f \in L^2(\Omega)$ is given and

$$E[u] = \int_\Omega |\nabla u|\,dx. \tag{2}$$

We notice that E is well-defined over the Sobolev space $W^{1,1}(\Omega)$, however this space is not closed under the weak convergence, because it is easy to show a uniformly bounded sequence of L^1 functions converging weakly to a delta function. This is why E has to be considered on $BV(\Omega)$. This also prompts us to present, during the course, the **basic facts about** BV: embedding theorems, how any BV-function may be approximated by smooth ones, the co-area formula. If time permits, we will mention Anzellotti's integration by parts formula.

The importance of the ROF algorithm encourages us to explore the area of the **calculus of variations**. We will study the least gradient problem, which is as a special version of the Free Material Design,

$$\min\{E[u]:\ u\in BV(\Omega),\ \gamma u = f\},$$

where $\Omega \subset \mathbb{R}^n$ is a bounded convex region with Lipschitz boundary and γ is the trace operator. We will discuss the lower semicontinuity of E as well as more general functionals like,

$$F[u] = \int_\Omega W(\nabla u)\,dx,$$

where W is convex, it grows linearly at infinity and $u \in W^{1,1}(\Omega)$. We also touch the question of functionals defined over measures.

One may notice that the ROF functional (1) is a time semi-discretization of the total variation flow,

$$u_t = -\frac{\delta}{\delta u} E[u], \tag{3}$$

where E is given by (2). In the one-dimensional case eq. (3) takes the following form,

$$u_t = (\operatorname{sgn} u_x)_x, \qquad u(x,0) = u_0(x), \tag{4}$$

where subscripts t, x denote partial derivatives. We notice that since E is finite on $BV(\Omega)$, then we can take initial conditions from this space. We will discuss different notions of solutions to (4) and **study properties of solutions to (4)**.

We will also present the notion of Γ-**convergence** of functionals. Roughly speaking, it implies that if $\Gamma - \lim_{\epsilon\to 0^+} F_\epsilon = F_0$, then minimizers of F_ϵ converge to minimizers of F_0. Due to this observation we can find solutions to elliptic problems, which are solutions to Euler-Lagrange equations for F_ϵ, provided that we know minimizers of F_0. This approach works when F_0 is simple. We will illustrate this idea with the well-known Kohn-Sternberg result, when F_0 is a set perimeter, i.e. $F_0(E) = \int_\Omega |D\chi_E|$. This method is effective when Ω is not convex.

Finally, we would like to present a more modern application of BV space to obtain a priori estimates sufficient to prove **stabilization of solutions** to an evolution equation.