On Non-Sensitive Homeomorphisms of the Boundary of a Proper Cocompact \( \text{CAT}(0) \) Space

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Abstract. We investigate the homeomorphism \( \tilde{f} \) of the boundary \( \partial X \) of a proper cocompact \( \text{CAT}(0) \) space \( X \) with \( |\partial X| > 2 \) induced by an isometry \( f \) of \( X \), and we study when the induced homeomorphism \( \tilde{f} \) of the boundary \( \partial X \) is non-expansive or non-sensitive.

1. Introduction

In this paper, we study non-expansive homeomorphisms and non-sensitive homeomorphisms of the boundary of a proper cocompact \( \text{CAT}(0) \) space. Definitions and basic properties of \( \text{CAT}(0) \) spaces and their boundaries are found in [1]. We introduce some basic of \( \text{CAT}(0) \) spaces and their boundaries in Section 2. For a proper \( \text{CAT}(0) \) space \( X \) and the boundary \( \partial X \) of \( X \), we can define a metric on the boundary \( \partial X \) as follows: We first fix a basepoint \( x_0 \in X \). Let \( \alpha, \beta \in \partial X \). There exist unique geodesic rays \( \xi_{x_0,\alpha} \) and \( \xi_{x_0,\beta} \) in \( X \) with \( \xi_{x_0,\alpha}(0) = \xi_{x_0,\beta}(0) = x_0 \), \( \xi_{x_0,\alpha}(\infty) = \alpha \) and \( \xi_{x_0,\beta}(\infty) = \beta \). Then the metric \( d_{\partial X}^{x_0}(\alpha, \beta) \) of \( \alpha \) and \( \beta \) on \( \partial X \) with respect to the basepoint \( x_0 \) is defined by

\[
d_{\partial X}^{x_0}(\alpha, \beta) = \sum_{i=1}^{\infty} \min\{d(\xi_{x_0,\alpha}(i), \xi_{x_0,\beta}(i)), \frac{1}{2^i}\}.
\]

The metric \( d_{\partial X}^{x_0} \) depends on the basepoint \( x_0 \) and the topology of \( \partial X \) does not depend on \( x_0 \).

An isometry \( f \) of a proper \( \text{CAT}(0) \) space \( X \) naturally induces the homeomorphism \( \tilde{f} \) of the boundary \( \partial X \) (cf. [1, p.264, Corollary II.8.9]). The purpose of this paper is to investigate when the homeomorphism \( \tilde{f} \) of the

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boundary $\partial X$ is non-expansive or non-sensitive. Here, in this paper, non-expansive homeomorphisms and non-sensitive homeomorphisms are defined as follows: A homeomorphism $g : Y \to Y$ of a metric space $(Y, d)$ is said to be non-expansive if for any $\epsilon > 0$ there exist $x, y \in Y$ with $x \neq y$ such that $d(g^i(x), g^i(y)) < \epsilon$ for any $i \in \mathbb{Z}$. Also a homeomorphism $g : Y \to Y$ is said to be non-sensitive if for any $\epsilon > 0$ there exist a point $x \in Y$ and a neighborhood $U$ of $x$ in $Y$ such that the diameter $\text{diam}(g^i(U)) < \epsilon$ for any $i \in \mathbb{Z}$. (We note that non-expansiveness and non-sensitiveness of a homeomorphism $g$ of a metric space $(Y, d)$ depends on the topology of $Y$ and does not depend on the metric $d$ of $Y$.) In dynamical systems and chaos theory, (non-)expansive homeomorphisms and (non-)sensitive homeomorphisms are important concepts. In this paper, we would like to obtain some information of homeomorphisms of boundaries of CAT(0) spaces by using a concept of the dynamical systems and the chaotic theory. We can find some recent research using a concept of the dynamical systems and the chaotic theory on minimality and scrambled sets of boundaries of CAT(0) groups and Coxeter groups in [7], [8], [9], [10], [11] and [13].

We introduce some remarks on isometries of CAT(0) spaces and induced homeomorphisms of boundaries in Section 3, and we show the following theorem in Sections 4–7.

**Theorem 1.1.** Let $X$ be a proper cocompact CAT(0) space with $|\partial X| > 2$. Suppose that $f : X \to X$ is an isometry and $\bar{f} : \partial X \to \partial X$ is the homeomorphism induced by $f$.

1. If $f$ is an elliptic isometry, then there exists a point $x'_0 \in X$ such that $\bar{f} : (X, d^\partial_{\partial X}) \to (X, d^\partial_{\partial X})$ is an isometry, and hence $\bar{f}$ is a non-expansive and non-sensitive homeomorphism of $\partial X$ with respect to any metric on the boundary $\partial X$.

2. If the CAT(0) space $X$ is non-hyperbolic, then $\bar{f}$ is a non-expansive homeomorphism of $\partial X$.

3. If the CAT(0) space $X$ is hyperbolic, then $\bar{f}$ is a non-sensitive homeomorphism of $\partial X$.

4. $\bar{f}$ is a non-expansive homeomorphism of $\partial X$. 
Here we note that the boundary $\partial X$ of a proper cocompact CAT(0) space $X$ with $|\partial X| > 2$ has no isolated points (cf. [6]). Hence if $\bar{f}$ is a non-sensitive homeomorphism of the boundary $\partial X$, then $\bar{f}$ is a non-expansive homeomorphism of $\partial X$. Thus, in Theorem 1.1, the statements (2) and (3) implies (4).

We introduce sensitiveness of the induced homeomorphisms of the boundary with respect to neighborhoods of a point in Section 8, and we provide some remarks and questions in Section 9.

2. CAT(0) Spaces and Their Boundaries

We say that a metric space $(X, d)$ is a geodesic space if for each $x, y \in X$, there exists an isometric embedding $\xi : [0, d(x, y)] \to X$ such that $\xi(0) = x$ and $\xi(d(x, y)) = y$ (such $\xi$ is called a geodesic). Also a metric space $X$ is said to be proper if every closed metric ball is compact.

Let $X$ be a geodesic space and let $T$ be a geodesic triangle in $X$. A comparison triangle for $T$ is a geodesic triangle $\overline{T}$ in the Euclidean plane $\mathbb{R}^2$ with same edge lengths as $T$. Choose two points $x$ and $y$ in $T$. Let $\bar{x}$ and $\bar{y}$ denote the corresponding points in $\overline{T}$. Then the inequality

$$d(x, y) \leq d_{\mathbb{R}^2}(\bar{x}, \bar{y})$$

is called the CAT(0)-inequality, where $d_{\mathbb{R}^2}$ is the usual metric on $\mathbb{R}^2$. A geodesic space $X$ is called a CAT(0) space if the CAT(0)-inequality holds for all geodesic triangles $T$ and for all choices of two points $x$ and $y$ in $T$.

Let $X$ be a proper CAT(0) space and $x_0 \in X$. The boundary of $X$ with respect to $x_0$, denoted by $\partial_{x_0}X$, is defined as the set of all geodesic rays issuing from $x_0$. Then we define a topology on $X \cup \partial_{x_0}X$ by the following conditions:

1. $X$ is an open subspace of $X \cup \partial_{x_0}X$.

2. For $\alpha \in \partial_{x_0}X$ and $r, \epsilon > 0$, let

$$U_{x_0}(\alpha; r, \epsilon) = \{x \in X \cup \partial_{x_0}X \mid x \notin B(x_0, r), \ d(\alpha(r), \xi_x(r)) < \epsilon\},$$

where $\xi_x : [0, d(x_0, x)] \to X$ is the geodesic from $x_0$ to $x$ ($\xi_x = x$ if $x \in \partial_{x_0}X$). Then for each $\epsilon_0 > 0$, the set

$$\{U_{x_0}(\alpha; r, \epsilon_0) \mid r > 0\}$$
is a neighborhood basis for \( \alpha \) in \( X \cup \partial_{x_0}X \). This topology is called the \textit{cone topology} on \( X \cup \partial_{x_0}X \). It is known that \( X \cup \partial_{x_0}X \) is a metrizable compactification of \( X \) ([1]).

Let \( X \) be a proper CAT(0) space. Two geodesic rays \( \xi, \zeta : [0, \infty) \rightarrow X \) are said to be \textit{asymptotic} if there exists a constant \( N \) such that \( d(\xi(t), \zeta(t)) \leq N \) for any \( t \geq 0 \). It is known that for each geodesic ray \( \xi \) in \( X \) and each point \( x \in X \), there exists a unique geodesic ray \( \xi' \) issuing from \( x \) such that \( \xi \) and \( \xi' \) are asymptotic.

Let \( x_0 \) and \( x_1 \) be two points of a proper CAT(0) space \( X \). Then there exists a unique bijection \( \Phi : \partial_{x_0}X \rightarrow \partial_{x_1}X \) such that \( \xi \) and \( \Phi(\xi) \) are asymptotic for any \( \xi \in \partial_{x_0}X \). It is known that \( \Phi : \partial_{x_0}X \rightarrow \partial_{x_1}X \) is a homeomorphism ([1]).

Let \( X \) be a proper CAT(0) space. The asymptotic relation is an equivalence relation on the set of all geodesic rays in \( X \). The \textit{boundary} of \( X \), denoted by \( \partial X \), is defined as the set of asymptotic equivalence classes of geodesic rays. The equivalence class of a geodesic ray \( \xi \) is denoted by \( \xi(\infty) \). For each \( x_0 \in X \) and each \( \alpha \in \partial X \), there exists a unique element \( \xi \in \partial_{x_0}X \) with \( \xi(\infty) = \alpha \). Thus we may identify \( \partial X \) with \( \partial_{x_0}X \) for each \( x_0 \in X \).

We can define the metric \( d^{\partial_0}_{\partial X} \) on the boundary \( \partial X \) as in Section 1. In this paper, we suppose that every CAT(0) space \( X \) has a fixed basepoint \( x_0 \) and \( d^{\partial_0}_{\partial X} \) is the metric on the boundary \( \partial X \) as in Section 1.

Let \( X \) be a non-compact proper cocompact CAT(0) space. (Here \( X \) is said to be \textit{cocompact} if there exists a compact subset \( K \) of \( X \) such that \( \text{Isom}(X) \cdot K = X \), where \( \text{Isom}(X) \) is the isometry group of \( X \).) Then \( X \) is \textit{almost geodesically complete} by [5, Corollary 3] (cf. [5] and [12]). Hence by the proof of [6, Theorem 3.1], we can obtain the following proposition.

\textbf{Proposition 2.1.} \textit{Let \( X \) be a proper cocompact CAT(0) space with }\(|\partial X| > 2\). \textit{Then every point of }\( \partial X \text{ is an accumulation point, i.e., } \partial X \text{ has no isolated points.}

\section{On Homeomorphisms of Boundaries Induced by Isometries of CAT(0) Spaces}

Let \( (X, d) \) be a metric space and let \( f : X \rightarrow X \) be an isometry of \( X \). Then the \textit{translation length} of \( f \) is defined as \( |f| := \inf\{d(x, f(x)) \mid x \in X\} \).
We also define the set \( \text{Min}(f) := \{ x \in X \mid d(x, f(x)) = |f| \} \). An isometry \( f \) of a metric space \( X \) is said to be \emph{semi-simple} if \( \text{Min}(f) \) is non-empty.

**Definition 3.1** (cf. [1, p.229]). Let \( f \) be an isometry of a metric space \( X \).

1. \( f \) is called \emph{elliptic} if \( f \) has a fixed-point (in this case, \( |f| = 0 \) and \( \text{Min}(f) \) is the fixed-points set of \( f \)).
2. \( f \) is called \emph{hyperbolic} if \( f \) is semi-simple and \( |f| > 0 \).
3. \( f \) is called \emph{parabolic} if \( f \) is not semi-simple, i.e., \( \text{Min}(f) \) is empty.

For a hyperbolic isometry of a CAT(0) space, the following remark is well-known (cf. [1, p.231, Theorem II.6.8]).

**Remark.** Let \( f \) be a hyperbolic isometry of a proper CAT(0) space \( X \). Then there exists a geodesic line \( \sigma : \mathbb{R} \to X \) such that \( f(\sigma(t)) = \sigma(t + |f|) \) for any \( t \in \mathbb{R} \). Such a geodesic line is called an \emph{axis} of \( f \). We note that \( \text{Im} \sigma \subset \text{Min}(f) \). It is known that the axes of \( f \) are parallel to each other and \( \text{Min}(f) \) is the union of the all axes. Hence \( \text{Min}(f) \) splits as \( \text{Min}(f) = Y \times \mathbb{R} \) for some \( Y \subset X \).

For an axis \( \sigma \) of \( f \), we define \( f^\infty := \sigma(\infty) \) and \( f^{-\infty} := \sigma(-\infty) \). Here the two points \( f^\infty \) and \( f^{-\infty} \) of the boundary \( \partial X \) are not dependent on the axis \( \sigma \). Also we note that for every point \( x \in X \), the sequence \( \{ f^i(x) \} \) converges to \( f^\infty \) as \( i \to \infty \) in \( X \cup \partial X \), and the sequence \( \{ f^{-i}(x) \} \) converges to \( f^{-\infty} \) as \( i \to -\infty \) in \( X \cup \partial X \).

Let \( f \) be an isometry of a proper CAT(0) space \( X \). For each geodesic ray \( \xi \) in \( X \), the map \( f \circ \xi \) is also a geodesic ray in \( X \) since \( f \) is an isometry of \( X \). We define the map \( \tilde{f} : \partial X \to \partial X \) by \( \tilde{f}([\xi]) := [f \circ \xi] \) for \([\xi] \in \partial X \) (where \([\xi]\) is the equivalence class of asymptotic relation of a geodesic ray \( \xi \) in \( X \)). Then it is known that \( \tilde{f} \) is a homeomorphism of the boundary \( \partial X \) (cf. [1, p.264, Corollary II.8.9]).

The purpose of this paper is to investigate the homeomorphism \( \tilde{f} \) of the boundary \( \partial X \) induced by an isometry \( f \) of \( X \).
4. On Homeomorphisms of Boundaries Induced by Elliptic Isometries of CAT(0) Spaces

In this section, we consider the homeomorphism $\bar{f}$ of the boundary $\partial X$ induced by an elliptic isometry $f$ of a proper cocompact CAT(0) space $X$.

We show the following theorem.

**Theorem 4.1.** Let $X$ be a proper cocompact CAT(0) space with $|\partial X| > 2$ and let $f : X \to X$ be an elliptic isometry. Then there exists a point $x_0' \in X$ such that $\bar{f}$ is an isometry of the metric space $(\partial X, d_{x_0'}^{r_0})$. Hence $\bar{f}$ is a non-expansive and non-sensitive homeomorphism of the boundary $\partial X$ with respect to any metric on the boundary $\partial X$.

**Proof.** Since $f$ is an elliptic isometry, there exists a fixed-point $x_0' \in X$ of $f$. Let $\alpha, \beta \in \partial X$ and let $\xi$ and $\zeta$ be the geodesic rays in $X$ such that $\xi(0) = \zeta(0) = x_0'$, $\xi(\infty) = \alpha$ and $\zeta(\infty) = \beta$. Then $f(x_0') = x_0'$, and $f \circ \xi$ and $f \circ \zeta$ are the geodesic rays issuing from $x_0'$ such that $f \circ \xi(\infty) = \bar{f}(\alpha)$ and $f \circ \zeta(\infty) = \bar{f}(\beta)$.

Now $d(f \circ \xi(t), f \circ \zeta(t)) = d(\xi(t), \zeta(t))$ for any $t \geq 0$ because $f$ is an isometry. Hence

$$d_{\partial X}^{r_0}(\bar{f}(\alpha), \bar{f}(\beta)) = \sum_{i=1}^{\infty} \min\{d(f \circ \xi(i), f \circ \zeta(i)), \frac{1}{2^i}\}$$

$$= \sum_{i=1}^{\infty} \min\{d(\xi(i), \zeta(i)), \frac{1}{2^i}\}$$

$$= d_{\partial X}^{r_0}(\alpha, \beta),$$

that is, $\bar{f}$ is an isometry of $(\partial X, d_{\partial X}^{r_0})$.

For any $\epsilon > 0$, we take a point $\alpha \in \partial X$ and $\epsilon/4$-neighborhood $U$ of $\alpha$ in $(\partial X, d_{\partial X}^{r_0})$. Then

$$\text{diam } \tilde{f}^i(U) = \text{diam } U < \epsilon$$

for any $i \in \mathbb{Z}$ because $\tilde{f}$ is an isometry of $(\partial X, d_{\partial X}^{r_0})$. Hence $\tilde{f}$ is a non-sensitive homeomorphism of $\partial X$. Here the non-sensitiveness of $\tilde{f}$ is not dependent on the metric $d_{\partial X}^{r_0}$. In particular, it is independent of the point $x_0'$. 
Since $X$ is a proper cocompact $\text{CAT}(0)$ space with $|\partial X| > 2$, every point of the boundary $\partial X$ is an accumulation point and $\partial X$ has no isolated points by Proposition 2.1. Thus $\bar{f}$ is also a non-expansive homeomorphism of $\partial X$. □

5. On Hyperbolic Spaces

In this section, we introduce hyperbolic $\text{CAT}(0)$ spaces.

We first introduce a definition of hyperbolic spaces. A geodesic space $X$ is called a hyperbolic space, if there exists a number $\delta \geq 0$ such that every geodesic triangle in $X$ is “$\delta$-thin”. Here “$\delta$-thin” is defined as follows: Let $x, y, z \in X$ and let $\triangle := \triangle xyz$ be a geodesic triangle in $X$. There exist unique non-negative numbers $a, b, c$ such that

$$d(x, y) = a + b, \ d(y, z) = b + c, \ d(z, x) = c + a.$$  

Then we can consider the metric tree $T_\triangle$ that has three vertices of valence one, one vertex of valence three, and edges of length $a$, $b$ and $c$. Let $o$ be the vertex of valence three in $T_\triangle$ and let $v_x, v_y, v_z$ be the vertices of $T_\triangle$ such that $d(o, v_x) = a, \ d(o, v_y) = b$ and $d(o, v_z) = c$. Then the map $\{x, y, z\} \rightarrow \{v_x, v_y, v_z\}$ extends uniquely to a map $f : \triangle \rightarrow T_\triangle$ whose restriction to each side of $\triangle$ is an isometry. For some $\delta \geq 0$, the geodesic triangle $\triangle$ is said to be $\delta$-thin if $d(p, q) \leq \delta$ for each points $p, q \in \triangle$ with $f(p) = f(q)$.

It is known that a geodesic space $X$ is hyperbolic if and only if there exists a number $\delta \geq 0$ such that every geodesic triangle in $X$ is “$\delta$-slim”. Here a geodesic triangle is said to be $\delta$-slim if each of its sides is contained in the $\delta$-neighborhood of the union of the other two sides.

For a proper hyperbolic space $X$, we can define the boundary $\partial X$ of $X$, and if the space $X$ is hyperbolic and $\text{CAT}(0)$, then these “boundaries” coincide.

Details and basic properties of hyperbolic spaces and their boundaries are found in [1], [2], [3] and [4].

It is known when a proper cocompact $\text{CAT}(0)$ space is hyperbolic.

**Theorem 5.1 ([1, p.400, Theorem III.H.1.5]).** A proper cocompact $\text{CAT}(0)$ space $X$ is hyperbolic if and only if it does not contain a subspace which is isometric to the flat plane $\mathbb{R}^2$. 
6. On Non-Hyperbolic CAT(0) Spaces

In this section, we consider the homeomorphism $\bar{f}$ of the boundary $\partial X$ induced by an isometry $f$ of a proper cocompact non-hyperbolic CAT(0) space $X$.

We obtain the following theorem from Theorem 5.1 and the proof of [11, Theorem 4.3].

**Theorem 6.1.** Let $X$ be a proper cocompact non-hyperbolic CAT(0) space with $|\partial X| > 2$ and let $f : X \to X$ be an isometry of $X$ (need not to be semi-simple). Then the induced homeomorphism $\bar{f} : \partial X \to \partial X$ is non-expansive.

**Proof.** Since $X$ is not hyperbolic, $X$ contains some subspace $Z$ which is isometric to the flat plane $\mathbb{R}^2$ by Theorem 5.1. To prove that the homeomorphism $\bar{f}$ of the boundary $\partial X$ is non-expansive, we show that for any $\epsilon > 0$, there exist $\alpha, \beta \in \partial Z \subset \partial X$ with $\alpha \neq \beta$ such that

$$d_{\partial X}^{\mathbb{R}^2}(\bar{f}^i(\alpha), \bar{f}^i(\beta)) < \epsilon$$

for any $i \in \mathbb{Z}$. Here the proof of [11, Theorem 4.3] implies that for any $\epsilon > 0$, we can take $\alpha, \beta \in \partial Z$ with $\alpha \neq \beta$ as the angle $\angle(\alpha, \beta)$ is small enough in $Z$ and

$$d_{\partial X}^{\mathbb{R}^2}(\bar{g}(\alpha), \bar{g}(\beta)) < \epsilon$$

for any isometry $g$ of $X$ and the induced homeomorphism $\bar{g}$ of $\partial X$. Therefore $\bar{f}$ is a non-expansive homeomorphism of the boundary $\partial X$. □

7. On Hyperbolic CAT(0) Spaces

In this section, we investigate the homeomorphism $\bar{f}$ of the boundary $\partial X$ induced by an isometry $f$ of a proper cocompact hyperbolic CAT(0) space $X$.

For a parabolic isometry of a hyperbolic space, the following remark is known.

**Remark.** Let $f$ be a parabolic isometry of a proper hyperbolic space $X$. Then $f$ induces a homeomorphism $\bar{f}$ of the boundary $\partial X$, and there exists a unique fixed-point $\alpha_0$ of $\bar{f}$ on $\partial X$. Here, in this paper, we define
\( f^\infty := \alpha_0 \) and \( f^{-\infty} := \alpha_0 \). We note that for every point \( x \in X \), the sequence \( \{f^i(x)\}_i \) converges to \( f^\infty = \alpha_0 \) as \( i \to \infty \) in \( X \cup \partial X \), and the sequence \( \{f^i(x)\}_i \) converges to \( f^{-\infty} = \alpha_0 \) as \( i \to -\infty \) in \( X \cup \partial X \).

For a hyperbolic or parabolic isometry \( f \) of a proper hyperbolic space \( X \), we define \( \text{Fix}(\bar{f}) \) as the fixed-point set of the induced homeomorphism \( \bar{f} \) of the boundary \( \partial X \).

We obtain the following lemma from [3, Theorems 8.16 and 8.17] and [4, 8.1.F and 8.1.G].

**Lemma 7.1.** Let \( X \) be a proper hyperbolic \( \text{CAT}(0) \) space and let \( f: X \to X \) be a hyperbolic isometry or a parabolic isometry.

1. For any \( \alpha \in \partial X \setminus \text{Fix}(\bar{f}) \), the sequence \( \{\bar{f}^i(\alpha)\}_i \) converges to \( f^\infty \) as \( i \to \infty \) and converges to \( f^{-\infty} \) as \( i \to -\infty \) in \( \partial X \).

2. For any compact subset \( K \) of \( \partial X \setminus \text{Fix}(\bar{f}) \) and any neighborhood \( U^+ \) (resp. \( U^- \)) of \( f^\infty \) (resp. \( f^{-\infty} \)), there exists a number \( n \in \mathbb{N} \) such that \( \bar{f}^n(K) \subset U^+ \) (resp. \( \bar{f}^{-n}(K) \subset U^- \)).

Using Lemma 7.1, we show the following theorem.

**Theorem 7.2.** Let \( X \) be a proper cocompact hyperbolic \( \text{CAT}(0) \) space with \( |\partial X| > 2 \) and let \( f: X \to X \) be an isometry of \( X \). Then the induced homeomorphism \( \bar{f}: \partial X \to \partial X \) is non-sensitive.

**Proof.** The isometry \( f \) is either elliptic, hyperbolic or parabolic. If \( f \) is an elliptic isometry of \( X \), then the induced homeomorphism \( \bar{f} \) of \( \partial X \) is non-sensitive by Theorem 4.1. We suppose that \( f \) is a hyperbolic isometry or a parabolic isometry of \( X \).

Let \( \epsilon > 0 \) and let \( \alpha \in \partial X \setminus \text{Fix}(\bar{f}) \). Then we can take a sufficiently small closed neighborhood \( U_0 \) of \( \alpha \) in \( \partial X \) such that

\[
U_0 \cap \text{Fix}(\bar{f}) = \emptyset \quad \text{and} \quad \text{diam } U_0 < \epsilon.
\]

Here, by Lemma 7.1 (2), we obtain that

\[
\text{diam } \bar{f}^i(U_0) \to 0 \quad \text{as } i \to \infty \quad \text{and} \quad \text{diam } \bar{f}^i(U_0) \to 0 \quad \text{as } i \to -\infty.
\]
Hence the set
\[ A_0 = \{ i \in \mathbb{Z} \mid \text{diam} \, \tilde{f}^i(U_0) \geq \epsilon \} \]
is finite.

If \( A_0 \) is empty, then \( \text{diam} \, \tilde{f}^i(U_0) < \epsilon \) for any \( i \in \mathbb{Z} \), i.e., \( \tilde{f} \) is non-sensitive.

We suppose that \( A_0 \) is non-empty. Let \( i_0 \in A_0 \). Then \( \text{diam} \, \tilde{f}^{i_0}(U_0) \geq \epsilon \). Here we note that \( \tilde{f}^{i_0}(U_0) \) is a neighborhood of \( \tilde{f}^{i_0}(\alpha) \). Then we can take a small closed neighborhood \( V_1 \) of \( \tilde{f}^{i_0}(\alpha) \) such that \( V_1 \subset \tilde{f}^{i_0}(U_0) \) and \( \text{diam} \, V_1 < \epsilon \). Let \( U_1 := \tilde{f}^{-i_0}(V_1) \). Then \( U_1 \) is a closed neighborhood of \( \alpha \), \( U_1 \nsubseteq U_0 \) and \( \text{diam} \, U_1 \leq \text{diam} \, U_0 < \epsilon \). Here we consider the set
\[ A_1 = \{ i \in \mathbb{Z} \mid \text{diam} \, \tilde{f}^i(U_1) \geq \epsilon \} \].

We note that \( A_1 \nsubseteq A_0 \) because \( U_1 \nsubseteq U_0 \) and \( i_0 \in A_0 \setminus A_1 \).

If \( A_1 \) is empty, then \( \text{diam} \, \tilde{f}^i(U_1) < \epsilon \) for any \( i \in \mathbb{Z} \), i.e., \( \tilde{f} \) is non-sensitive.

If \( A_1 \) is non-empty, then we take \( i_1 \in A_1 \) and by the same argument as above, we obtain a small closed neighborhood \( V_2 \) of \( \tilde{f}^{i_1}(\alpha) \) and \( U_2 = \tilde{f}^{-i_1}(V_1) \) as \( U_2 \) is a closed neighborhood of \( \alpha \), \( U_2 \nsubseteq U_1 \) and \( \text{diam} \, U_2 \leq \text{diam} \, U_1 \leq \text{diam} \, U_0 < \epsilon \). Also we consider the set
\[ A_2 = \{ i \in \mathbb{Z} \mid \text{diam} \, \tilde{f}^i(U_2) \geq \epsilon \} \].

Here \( A_2 \nsubseteq A_1 \nsubseteq A_0 \).

By iterating this argument, we obtain a sequence
\[ A_k \nsubseteq \cdots \nsubseteq A_2 \nsubseteq A_1 \nsubseteq A_0 \].

Here there exists a number \( k \) such that \( A_k \) is empty since \( A_0 \) is a finite set. Then \( \text{diam} \, \tilde{f}^i(U_k) < \epsilon \) for any \( i \in \mathbb{Z} \).

Therefore \( \tilde{f} \) is a non-sensitive homeomorphism of the boundary \( \partial X \). □

8. On Sensitiveness of the Induced Homeomorphisms with Respect to Neighborhoods of a Point of the Boundary

In this section, we investigate sensitiveness of the homeomorphisms of the boundary induced by an isometry of a proper cocompact CAT(0) space with respect to neighborhoods of a point of the boundary.

In this paper, a homeomorphism \( g : Y \to Y \) is said to be sensitive with respect to neighborhoods of a point \( y \) of \( Y \) if there exists a number \( \epsilon > 0 \) such
that for any neighborhood $U$ of $y$ in $Y$, the diameter $\text{diam} \, g^i(U) \geq \epsilon$ for some $i \in \mathbb{Z}$. Also a homeomorphism $g : Y \rightarrow Y$ is said to be non-sensitive with respect to neighborhoods of a point $y$ of $Y$ if for any $\epsilon > 0$ there exist a neighborhood $U$ of $y$ in $Y$ such that $\text{diam} \, g^i(U) < \epsilon$ for any $i \in \mathbb{Z}$.

We obtain the following theorem from the arguments in Sections 4–7.

**Theorem 8.1.** Let $X$ be a proper cocompact CAT(0) space with $|\partial X| > 2$. Suppose that $f : X \rightarrow X$ is an isometry and $\bar{f} : \partial X \rightarrow \partial X$ is the homeomorphism induced by $f$.

1. If $f$ is an elliptic isometry, then $\bar{f}$ is non-sensitive with respect to neighborhoods of any point of the boundary $\partial X$.

2. If the CAT(0) space $X$ is hyperbolic and $f$ is a hyperbolic isometry or a parabolic isometry, then $\bar{f}$ is non-sensitive with respect to neighborhoods of any point of $\partial X \setminus \text{Fix}(\bar{f})$.

3. If the CAT(0) space $X$ is hyperbolic and $f$ is a hyperbolic isometry or a parabolic isometry, then $\bar{f}$ is sensitive with respect to neighborhoods of the points $f^\infty$ and $f^{-\infty}$.

**Proof.** Theorem 4.1 implies that (1) holds and the proof of Theorem 7.2 implies that (2) holds.

We show that (3) holds. We suppose that $X$ is hyperbolic and $f$ is a hyperbolic isometry. For any neighborhood $U$ of $f^{-\infty}$ in the boundary $\partial X$, there exists $\alpha \in U$ with $\alpha \neq f^{-\infty}$ since $\partial X$ has no isolated points. Then the sequence $\{\bar{f}^i(\alpha)\}_i$ converges to $f^\infty$ as $i \rightarrow \infty$ by Lemma 7.1 (1). Also $\bar{f}^i(f^{-\infty}) = f^{-\infty}$ for any $i \in \mathbb{Z}$. Hence $\text{diam} \, \bar{f}^i(U) \geq d_{\partial X}^{\infty}(\bar{f}^i(f^{-\infty}), \bar{f}^i(\alpha)) = d_{\partial X}^{\infty}(f^{-\infty}, \bar{f}^i(\alpha))$, where $d_{\partial X}^{\infty}(f^{-\infty}, \bar{f}^i(\alpha))$ converges to $d_{\partial X}^{\infty}(f^{-\infty}, f^\infty)$ as $i \rightarrow \infty$. Therefore $\bar{f}$ is sensitive with respect to neighborhoods of the point $f^{-\infty}$. We also obtain that $\bar{f}$ is sensitive with respect to neighborhoods of the point $f^\infty$ by the same argument.

We suppose that $X$ is hyperbolic and $f$ is a parabolic isometry. Let $\alpha \in \partial X \setminus \{f^\infty\}$ and let $\epsilon_0 = d_{\partial X}^{\infty}(\alpha, f^\infty)$. Then for any neighborhood $U$ of
\[ f^\infty = f^{-\infty} \text{ in the boundary } \partial X, \text{ there exists a number } i_0 \in \mathbb{N} \text{ such that } \bar{f}^{i_0}(\alpha) \in U \text{ by Lemma } 7.1 \text{ (1). Hence } \alpha \in \bar{f}^{-i_0}(U) \text{ and } \]

\[ \text{diam } \bar{f}^{-i_0}(U) \geq d_{\partial X}^{\infty}(\alpha, f^\infty) = \epsilon_0. \]

Therefore \( \bar{f} \) is sensitive with respect to neighborhoods of the point \( f^\infty = f^{-\infty} \). □

9. Remarks

We introduce an example of an isometry of a proper cocompact CAT(0) space which is not hyperbolic.

**Example 9.1.** Let \( G = (\mathbb{Z} \times \mathbb{Z}) * \mathbb{Z} \) and let \( X \) be a proper CAT(0) space on which \( G \) acts properly and cocompactly by isometries. Here we denote \( G = \langle \{a, b, c\} \mid ab = ba \rangle \), i.e., \( G = (\langle a \rangle \times \langle b \rangle) * \langle c \rangle \). Also, for example, we can suppose that \( X \) is the CAT(0) complex whose 1-skeleton is the Cayley graph of \( G \) with respect to the generating set \( \{a, b, c\} \). Then we consider the hyperbolic isometry \( f := a \) of \( X \).

We first note that if \( Z \) is the flat plane in \( X \) on which \( \langle a \rangle \times \langle b \rangle \) acts, then \( \bar{f}(\alpha) = \alpha \) for any \( \alpha \in \partial Z \). In particular, \( \bar{f}(b^\infty) = b^\infty \).

Next, we note that the sequence \( \{\bar{f}^i(c^\infty)\}_i \) converges to \( a^\infty \) as \( i \to \infty \) and converges to \( a^{-\infty} \) as \( i \to -\infty \). Also, in fact, for any \( \alpha \in \partial X \setminus \partial Z \), the sequence \( \{\bar{f}^i(\alpha)\}_i \) converges to \( a^\infty \) as \( i \to \infty \) and converges to \( a^{-\infty} \) as \( i \to -\infty \).

For any neighborhood \( U \) of \( b^\infty \) in \( \partial X \), there exists \( \alpha \in U \setminus \partial Z \) and the sequence \( \{\bar{f}^i(\alpha)\}_i \) converges to \( a^\infty \) as \( i \to \infty \). Here \( \bar{f}^i(b^\infty) = b^\infty \) for any \( i \in \mathbb{Z} \). Hence we obtain that \( \bar{f} \) is sensitive with respect to neighborhoods of the point \( b^\infty \).

On the other hand, for any small neighborhood \( U \) of \( c^\infty \) in \( \partial X \) with \( U \cap \partial Z = \emptyset \),

\[ \text{diam } \bar{f}^i(U) \to 0 \text{ as } i \to \infty \text{ and } \]

\[ \text{diam } \bar{f}^i(U) \to 0 \text{ as } i \to -\infty. \]

Hence we obtain that \( \bar{f} \) is non-sensitive with respect to neighborhoods of the point \( c^\infty \).
Thus there exist points $\beta, \gamma \in \partial X$ such that $\bar{f}$ is sensitive with respect to neighborhoods of the point $\beta$ and $\bar{f}$ is non-sensitive with respect to neighborhoods of the point $\gamma$.

On a hyperbolic isometry of a proper cocompact CAT(0) space which is not hyperbolic, Theorem 6.1 implies that the induced homeomorphism of the boundary is non-expansive. On the other hand, we do not know whether the induced homeomorphism of the boundary is non-sensitive.

The author has the following question.

**Question 9.2.** Let $X$ be a proper cocompact non-hyperbolic CAT(0) space with $|\partial X| > 2$ and let $f : X \to X$ be a hyperbolic isometry or a parabolic isometry of $X$. Then is it the case that the induced homeomorphism $\bar{f} : \partial X \to \partial X$ is non-sensitive?

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