Note on the Chen-Lin Result with the Li-Zhang Method

By Samy Skander Bahoura

Abstract. We give a new proof of the Chen-Lin result with the method of moving sphere in a work of Li-Zhang.

Introduction and Results

We set $\Delta = \partial_{11} + \partial_{22}$ the Laplace-Beltrami operator on $\mathbb{R}^2$.

On an open set $\Omega$ of $\mathbb{R}^2$ with a smooth boundary we consider the following problem:

\[
(P) \quad \begin{cases} 
-\Delta u = V(x)e^u \text{ in } \Omega, \\
0 < a \leq V(x) \leq b < +\infty.
\end{cases}
\]

The previous equation is called the Prescribed Scalar Curvature equation in relation with conformal change of metrics. The function $V$ is the prescribed curvature.

Here, we try to find some a priori estimates for sequences of the previous problem.

Equations of this type were studied by many authors, see [7, 8, 10, 12, 13, 17, 18, 21, 22, 25]. We can see in [8] different results for the solutions of those type of equations with or without boundary conditions and, with minimal conditions on $V$, for example we suppose $V \geq 0$ and $V \in L^p(\Omega)$ or $Ve^u \in L^p(\Omega)$ with $p \in [1, +\infty]$.

Among other results, we can see in [8] the following important theorem,

**Theorem A** (Brezis-Merle [8]). If $(u_i)_i$ and $(V_i)_i$ are two sequences of functions relatively to the problem $(P)$ with, $0 < a \leq V_i \leq b < +\infty$, then, for all compact set $K$ of $\Omega$,

\[
\sup_{K} u_i \leq c = c(a, b, m, K, \Omega) \text{ if } \inf_{\Omega} u_i \geq m.
\]

2010 Mathematics Subject Classification. 35J60, 35B45, 35B50.
A simple consequence of this theorem is that, if we assume \( u_i = 0 \) on \( \partial \Omega \), then the sequence \((u_i)_i\) is locally uniformly bounded. We can find in [8] an interior estimate if we assume \( a = 0 \), but we need an assumption on the integral of \( e^{u_i} \).

If we assume \( V \) with more regularity, we can have another type of estimates, \( \sup + \inf \). It was proved by Shafrir in [22] that, if \((u_i)_i, (V_i)_i\) are two sequences of functions solutions of the previous equation without assumption on the boundary and \( 0 < a \leq V_i \leq b < +\infty \), then we have the following interior estimate:

\[
C \left( \frac{a}{b} \right) \sup_K u_i + \inf_\Omega u_i \leq c = c(a, b, K, \Omega).
\]

We can see in [12] an explicit value of \( C \left( \frac{a}{b} \right) = \sqrt{\frac{a}{b}} \). In [22] Shafrir has used the Stokes formula and an isoperimetric inequality; see [6]. In [12] Chen and Lin have used the blow-up analysis combined with some geometric type inequality for the integral curvature.

Now, if we suppose \((V_i)_i\) uniformly Lipschitzian with \( A \) the Lipschitz constant, then, \( C(a/b) = 1 \) and \( c = c(a, b, A, K, \Omega) \); see Brezis-Li-Shafrir [7]. This result was extended for Hölderian sequences \((V_i)_i\) by Chen-Lin; see [12]. Also, we can see in [17] an extension of the Brezis-Li-Shafrir’s result to compact Riemann surface without boundary. We can see in [18] explicit form \((8\pi m, m \in \mathbb{N}^* \text{ exactly})\), for the numbers in front of the Dirac masses, when the solutions blow-up. Here, the notion of isolated blow-up point is used. Also, we can see in [13] and [25] refined estimates near the isolated blow-up points and the bubbling behavior of the blow-up sequences.

On an open set \( \Omega \) of \( \mathbb{R}^2 \) we consider the following equation:

\[
\begin{cases}
-\Delta u_i = V_i e^{u_i} & \text{on } \Omega, \\
0 < a \leq V_i(x) \leq b < +\infty, & x \in \Omega, \\
|V_i(x) - V_i(y)| \leq A|\!\!x - y\!\!|^s, & 0 < s \leq 1, \ x, y \in \Omega.
\end{cases}
\]

Among other results, we have in [12] the following Harnack type inequality,

**Theorem B** (Chen-Lin [12]). For all compact \( K \subset \Omega \) and all \( s \in ]0, 1] \) there is a constant \( c = c(a, b, A, s, K, \Omega) \) such that,

\[
\sup_K u_i + \inf_\Omega u_i \leq c \text{ for all } i.
\]
Here we try to prove the previous theorem by the moving-plane method and Li-Zhang method; see [19]. The method of moving-plane was developed by Gidas-Ni-Nirenberg; see [14]. We can see in [9] one of the applications of this method and, in particular, the classification of the solutions of some elliptic PDEs.

Note that in our proof we do not need a classification result for some particular elliptic PDEs as showed in [7] and [12].

In a similar way we have in dimension $n \geq 3$, with different methods, some a priori estimates of the type $\sup \times \inf$ for equation of the type:

$$-\Delta u + \frac{n-2}{4(n-1)} R_g(x)u = V(x)u^{(n+2)/(n-2)} \quad \text{on} \ M,$$

where $R_g$ is the scalar curvature of a riemannian manifold $M$, and $V$ is a function. The operator $\Delta = \nabla^i(\nabla_i)$ is the Laplace-Beltrami operator on $M$.

When $V \equiv 1$ and $M$ compact, the previous equation is the Yamabe equation. T. Aubin and R. Scheon solved the Yamabe problem, see for example [1]. Also, we can have an idea on the Yamabe Problem in [15]. If $V$ is not a constant function, the previous equation is called a prescribing curvature equation, we have many existence results see also [1].

Now, if we look at the problem of a priori bound for the previous equation, we can see in [2], [4], [11], [16], some results concerning the $\sup \times \inf$ type of inequalities when the manifold $M$ is the sphere or more generality a locally conformally flat manifold.

For general manifolds $M$ of dimension $n \geq 3$ we have some Harnack type estimates; see for example [3, 5], [19] and [20], for equation of the type,

$$-\Delta u + h(x)u = V(x)u^{(n+2)/(n-2)} \quad \text{on} \ M.$$

Also, there are similar problems defined on complex manifolds for the Complex Monge-Ampere equation; see [23, 24]. They consider, on compact Kahler manifold $(M, g)$, the following equation

$$\begin{cases}
(\omega_g + \partial \bar{\partial} \phi)^n = e^{f-t\phi} \omega^n_g, \\
\omega_g + \partial \bar{\partial} \phi > 0 \quad \text{on} \ M
\end{cases}$$

And, they prove some estimates of type $\sup_M(\phi - \psi) + m \inf_M(\phi - \psi) \leq C(t)$ or $\sup_M(\phi - \psi) + m \inf_M(\phi - \psi) \geq C(t)$ under the positivity of the first Chern class of $M$. 

The function $\psi$ is a $C^2$ function such that
\[
\omega_g + \partial\bar\partial\psi \geq 0 \quad \text{and} \quad \int_M e^{f-t\psi} \omega_g^n = Vol_g(M),
\]

**New Proof of the Theorem B.**
We argue by contradiction and we want to prove that
\[
\exists R > 0, \text{ such that } 4 \log R + \sup_{B_R(0)} u + \inf_{B_{2R}(0)} u \leq c = c(a, b, A),
\]
Thus, by contradiction we can assume
\[
\exists (R_i), (u_i) : R_i \to 0, \quad 4 \log R_i + \sup_{B_{R_i}(0)} u_i + \inf_{B_{2R_i}(0)} u_i \to +\infty.
\]

**Step 1. The blow-up analysis**
For $x_0 \in \Omega$ we want to prove the theorem locally around $x_0$. We use the previous assertion with $x_0 = 0$. The classical blow-up analysis gives the existence of the sequence $(x_i)_i$ and a sequence of functions $(v_i)_i$ satisfying the following properties.

We set
\[
\sup_{B_{R_i}(0)} u_i = u_i(\bar{x}_i),
\]
\[
s_i(x) = 2 \log(R_i - |x - \bar{x}_i|) + u_i(x), \text{ and }
\]
\[
s_i(x_i) = \sup_{B_{R_i}(x_i)} s_i, \quad \sigma_i = \frac{1}{2}(R_i - |x_i - \bar{x}_i|).
\]
Also, we set
\[
v_i(x) = u_i[x_i + xe^{-u_i(x_i)/2}] - u_i(x_i), \quad \bar{V}_i(x) = V_i[x_i + xe^{-u_i(x_i)/2}],
\]
Then, with this classical selection process, we have
\[
2 \log M_i = u_i(x_i) \geq u_i(\bar{x}_i)
\]
\[
u_i(x) \leq C_1 u_i(x_i), \quad \forall x \in B(x_i, \sigma_i),
\]
where $C_1$ is a constant independent of $i$.

Also,

$$u_i(x_i) + \min_{\partial B(x_i,R_i)} u_i + 4 \log R_i \geq u_i(\bar{x}_i) + \min_{B(0,2R_i)} u_i + 4 \log R_i \to +\infty,$$

and

$$\lim_{i \to +\infty} R_i e^{u_i(x_i)/2} = \lim_{i \to +\infty} \sigma_i e^{u_i(x_i)/2} = +\infty.$$

Finally, we have

$$\begin{cases}
\Delta v_i + \bar{V}_i e^{v_i} = 0 \text{ for } |y| \leq R_iM_i, \\
v_i(0) = 0, \\
v_i(y) \leq C_1 \text{ for } |y| \leq \sigma_iM_i, \\
\lim_{i \to +\infty} \min_{|y|=2R_iM_i} (v_i(y) + 4 \log |y|) = +\infty.
\end{cases}$$

Because of the classical elliptic estimates and the classical Harnack inequality, we can prove the uniform convergence on each compact of $\mathbb{R}^2$

$$v_i \to v \text{ when } v \text{ is a solution on } \mathbb{R}^2 \text{ of }$$

$$\begin{cases}
\Delta v + V(0)e^v = 0 \text{ in } \mathbb{R}^2, \\
v(0) = 0, \ 0 < v \leq C_1.
\end{cases}$$

with $V(0) = \lim_{i \to +\infty} V_i(x_i)$ and $0 < a \leq V(0) \leq b < +\infty$.

**Step 2.** The moving-plane method

Here we use the Kelvin transform and the Li-Zhang’s method.

For $0 < \lambda < \lambda_1$ we define

$$\Sigma_\lambda = B(0,R_iM_i) - B(0,\lambda).$$

First, we set

$$\bar{v}_i^\lambda = v_i^\lambda - 4 \log |x| + 4 \log \lambda = v_i \left( \frac{\lambda^2 x}{|x|^2} \right) + 4 \log \frac{\lambda}{|x|},$$

$$x^\lambda = \frac{\lambda^2 x}{|x|^2} \text{ and } \bar{V}_i^\lambda = \bar{V}_i \left( \frac{\lambda^2 x}{|x|^2} \right),$$

$$M_i = e^{u_i(x_i)/2}.$$
We want to compare $v_i$ and $\bar{v}_i^\lambda$, we set

$$w_\lambda = v_i - \bar{v}_i^\lambda.$$  

Then

$$-\Delta \bar{v}_i^\lambda = \bar{V}_i^\lambda e^{\bar{v}_i^\lambda},$$

$$-\Delta (v_i - \bar{v}_i^\lambda) = \bar{V}_i(e^{v_i} - e^{\bar{v}_i^\lambda}) + (\bar{V}_i - \bar{V}_i^\lambda) e^{\bar{v}_i^\lambda},$$

We have the following estimate

$$|\bar{V}_i - \bar{V}_i^\lambda| \leq AM^{-s}|x|^s|1 - \frac{\lambda^2}{|x|^2}|^s. \Box$$

The auxiliary function:

We take an auxiliary function $h_\lambda$.

Because $v_i(x^\lambda) \leq C(\lambda_1) < +\infty$, we have

$$h_\lambda = C_1 M_i^{-s}\lambda^2(\log(\lambda/|x|)) + C_2 M_i^{-s}\lambda^{2+s}[1 - (\frac{\lambda}{|x|})^{2-s}], \ |x| > \lambda,$$

with $C_1, C_2 = C_1, C_2(s, \lambda_1) > 0$

$$h_\lambda = M_i^{-s}\lambda^2(1 - \lambda/|x|)(C_1 \frac{\log(\lambda/|x|)}{1 - \lambda/|x|} + C_2'),$$

with $C_2' = C_2'(s, \lambda_1) > 0$. We can choose $C_1$ big enough to have $h_\lambda < 0$.

**Lemma 1.** There is an $\lambda_{i,0} > 0$ small enough, such that, for $0 < \lambda \leq \lambda_{i,0}$, we have

$$w_\lambda + h_\lambda > 0.$$

**Proof of the Lemma 1.**

We set

$$f(r, \theta) = v_i(r\theta) + 2\log r,$$

then

$$\frac{\partial f}{\partial r}(r, \theta) = <\nabla v_i(r\theta)|\theta > + \frac{2}{r},$$
According to the blow-up analysis,

\[ \exists r_0 > 0, C > 0, |\nabla v_i(r\theta)| \theta > | \leq C, \text{ for } 0 \leq r < r_0. \]

Then

\[ \exists r_0 > 0, C' > 0, \frac{\partial f}{\partial r}(r, \theta) > \frac{C'}{r}, 0 < r < r_0. \]

**Case 1.** If \( 0 < \lambda < |y| < r_0 \)

\[ w_\lambda(y) + h_\lambda(y) = v_i(y) - v_i^\lambda(y) + h_\lambda(y) > C(\log |y| - \log |y^\lambda|) + h_\lambda(y), \]

by the definition of \( h_\lambda \), we have, for \( C, C_0 > 0 \) and \( 0 < \lambda \leq |y| < r_0, \)

\[ w_\lambda(y) + h_\lambda(y) > (|y| - \lambda)[C \log \frac{|y|}{|y^\lambda|} - \lambda^{1+s}C_0 M_i^s], \]

but

\[ |y| - |y^\lambda| > |y| - \lambda > 0, \text{ and } |y^\lambda| = \frac{\lambda^2}{|y|}, \]

thus,

\[ w_\lambda(y) + h_\lambda(y) > 0 \text{ if } \lambda < \lambda_0^i, \lambda_0^i \text{ (small)}, \text{ and } 0 < \lambda < |y| < r_0. \]

**Case 2.** If \( r_0 \leq |y| \leq R_i M_i \)

\[ v_i \geq \min v_i = C_i^1, v_i^\lambda(y) \leq C_1(\lambda_1, r_0), \text{ if } r_0 \leq |y| \leq R_i M_i. \]

Thus, in \( r_0 \leq |y| \leq R_i M_i \) and \( \lambda \leq \lambda_1 \), we have,

\[ w_\lambda + h_\lambda \geq C_i - 4 \log \lambda + 4 \log r_0 - C' \lambda_1^{2+s} \]

then, if \( \lambda \to 0, -\log \lambda \to +\infty, \) and

\[ w_\lambda + h_\lambda > 0, \text{ if } \lambda < \lambda_1^i, \lambda_1^i \text{ (small)}, \text{ and } r_0 < |y| \leq R_i M_i. \]

As in Li-Zhang paper, see [19], by the maximum principle and the Hopf boundary lemma, we have
**Lemma 2.** Let $\tilde{\lambda}_i$ be a positive number such that

$$
\tilde{\lambda}_i = \sup\{\lambda < \lambda_1, \ w_{\lambda} + h_{\lambda} > 0 \text{ in } \Sigma_{\lambda}\}.
$$

Then

$$
\tilde{\lambda}_i = \lambda_1.
$$

**Proof of the Lemma 2.**

The blow-up analysis gives the following inequality for the boundary condition.

For $y = |y| \theta = R_i M_i \theta$ we have

$$
\begin{align*}
    w_{\lambda_i}(|y| = R_i M_i) + h_{\lambda_i}(|y| = R_i M_i) &= u_i(x_i + R_i \theta) - v_i R_i M_i - 4 \log \lambda + 4 \log (R_i M_i) + \\
    &+ C(s, \lambda_1) M_i^{-s} \lambda^{2+s}[1 - \left(\frac{\lambda}{R_i M_i}\right)^{2-s}],
\end{align*}
$$

because

$$
4 \log R_i + u_i(x_i) + \inf_{B_{2R_i}(0)} u_i \to +\infty,
$$

which we can write

$$
\begin{align*}
    w_{\lambda_i}(|y| = R_i M_i) + h_{\lambda_i}(|y| = R_i M_i) &\geq \\
    &\geq \min_{B_{2R_i}(0)} u_i + u_i(x_i) + 4 \log R_i - C(s, \lambda_1) \to +\infty,
\end{align*}
$$

because, $0 < \lambda \leq \lambda_1$.

Finally, we have

$$
w_{\lambda_i}(y) + h_{\lambda_i}(y) > 0 \ \forall \ |y| = R_i M_i,
$$

Now, we have

$$
\Delta w_{\lambda} + \xi V_i w_{\lambda} = E_{\lambda} \text{ in } \Sigma_{\lambda},
$$

where $\xi$ stays between $v_i$ and $v_i^\lambda$, and

$$
E_{\lambda} = -(V_i - V_i^\lambda)e^{\tilde{\nu}_i^\lambda}.
$$

Thus to prove that

$$
(\Delta + \xi V_i)(w_{\lambda} + h_{\lambda}) \leq 0 \text{ in } \Sigma_{\lambda},
$$
it suffices to prove that
\[ \Delta h_\lambda + (\xi V_i)h_\lambda + E_\lambda \leq 0 \text{ in } \Sigma_\lambda. \]

But we have
\[ h_\lambda < 0, \]
\[ |E_\lambda| \leq C_1 \lambda^4 M_i^{-s} |y|^{-4+s} \text{ in } \Sigma_\lambda, \]
and
\[ \Delta h_\lambda = -C_1 \lambda^4 M_i^{-s} |y|^{-4+s} \text{ in } \Sigma_\lambda. \]

We can use the maximum principle and the Hopf lemma to have
\[ w_{\tilde{\lambda}_i} + h_{\tilde{\lambda}_i} > 0, \text{ in } \Sigma_\lambda, \]
and
\[ \frac{\partial}{\partial \nu}(w_{\tilde{\lambda}_i} + h_{\tilde{\lambda}_i}) > 0, \text{ in } \partial B(0, \tilde{\lambda}_i). \]

From above we conclude that \( \tilde{\lambda}_i = \lambda_1 \) and lemma 2 is proved. \( \square \)

**Conclusion**

As in [19], we have
\[ \forall \lambda_1 > 0, \ v(y) \geq v^\lambda(y), \ \forall |y| \geq \lambda, \ \forall 0 < \lambda < \lambda_1. \]

And the same argument may be used to have
\[ \forall \lambda_1 > 0, \ v(y) \geq v^{\lambda,x}(y), \ \forall x, y \ |y - x| \geq \lambda, \ \forall 0 < \lambda < \lambda_1, \]
where
\[ v^{\lambda,x}(y) = v_i \left( x + \frac{\lambda^2 (y-x)}{|y-x|^2} \right) + 4 \log \frac{\lambda}{|y-x|}. \]

This implies that \( v \) is a constant, and because \( v(0) = 0, \ v \equiv 0 \) contradicting the fact that
\[ -\Delta v = V(0)e^v. \]

**References**

Bahoura, S. S., Majorations du type sup $u \times \inf u \leq c$ pour l’équation de la courbure scalaire sur un ouvert de $\mathbb{R}^n$, $n \geq 3$, J. Math. Pures. Appl. (9) 83 (2004), no. 9, 1109–1150.


(Received February 23, 2011)
(Revised October 31, 2011)

Equipe d’Analyse Complexe et Geometrie
Universite Pierre et Marie Curie
4 place Jussieu, 75005, Paris
France
E-mail: samybahoura@yahoo.fr
samybahoura@gmail.com