Holomorphic Curves into the Product Space of the Riemann Spheres

By Yusaku Tiba

Abstract. We prove the second main theorem for the product space of the Riemann spheres by using the meromorphic connection. We deal with the divisors which are totally geodesic with respect to this meromorphic connection.

1. Introduction

The purpose of this paper is to prove the second main theorem for a holomorphic map from the complex plane $\mathbb{C}$ to the product space of the one-dimensional projective spaces $\mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$. Let $[X_0 : X_1]$ and $[Y_0 : Y_1]$ be the homogeneous coordinates in the first and second factors of the product space of the $\mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$. Let $m', n', m'', n''$ be positive integers. We define the effective divisors $D', D''$ on $\mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$ by the polynomials $X_0^m Y_0^n - X_1^m Y_1^n$, $X_0^m Y_1^n - X_1^m Y_0^n$. We prove the second main theorem for divisors $D'$ and $D''$.

We state our main theorem precisely. Let $H_{1,0}$, $H_{1,1}$, $H_{2,0}$ and $H_{2,1}$ be the hyperplanes in $\mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$ which are defined by the monomials $X_0$, $X_1$, $Y_0$ and $Y_1$. Put $Z_0 = \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$. Then there exists the sequence of the blowing-up

$$\pi_{1,0} : Z_1 \rightarrow Z_0,$$
$$\pi_{2,1} : Z_2 \rightarrow Z_1,$$
$$\vdots$$
$$\pi_{k,k-1} : Z_k \rightarrow Z_{k-1}.$$

which satisfies the following condition (*):

Put $\tilde{\pi}_{j,i} = \pi_{i+1,i} \circ \cdots \circ \pi_{j,j-1}$ for $i < j$. Let $\tilde{D}'$, $\tilde{D}''$ and $\tilde{H}_{i,j}$, $1 \leq i \leq 2$, $0 \leq j \leq 1$ be the proper transform of $D'$, $D''$, and $H_{i,j}$ under $\pi_{k,0}$. Let
$E_i$, $1 \leq i \leq k$ be the exceptional divisor of the blowing-up $\pi_{i-1}$, and let $\tilde{E}_i$ be the proper transform of $E_i$ under $\pi_{k,i}$. Then

\[(*) \quad \tilde{D}' + \tilde{D}'' + \sum_{i=1}^{2} \sum_{j=0}^{1} \tilde{H}_{i,j} + \sum_{i=1}^{k} \tilde{E}_i \quad \text{is simple normal crossing in } Z_k.
\]

Our goal is the following theorem.

**Theorem 1 (Main Theorem).** Let $f : \mathbb{C} \to \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$ be a non-constant holomorphic map. Let $\tilde{f} : \mathbb{C} \to Z_k$ be the lift of $f$. Assume that $f(\mathbb{C}) \not\subset \{(X_0 : X_1), (Y_0 : Y_1) \in \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C}) | C_0X_0^rY_0^s - C_1X_1^rY_1^s = 0\}$, for all $(r_1, r_2) \in \mathbb{Z} \times \mathbb{Z} \setminus \{(0, 0)\}$ and all $(C_0, C_1) \in \mathbb{C} \times \mathbb{C} \setminus \{(0, 0)\}$, and assume that there exist no holomorphic functions $g_1, g_2$ on $\mathbb{C}$ and no $(a, b) \in \mathbb{C} \times \mathbb{C} \setminus \{(0, 0)\}$ such that

\[f = (\exp g_1, \exp g_2), \quad ag_1 + bg_2 = (\text{constant}),\]

on $\mathbb{C}$. Then it follows that

\[T_f(r, [\tilde{D}' + \tilde{D}'']) \leq N_2(r, \tilde{f}^*\tilde{D}') + N_2(r, \tilde{f}^*\tilde{D}'') + 2 \sum_{i=1}^{2} \sum_{j=0}^{1} N_1(r, \tilde{f}^*\tilde{H}_{i,j}) + 2 \sum_{i=1}^{k} N_1(r, \tilde{f}^*\tilde{E}_i) + S_f(r),\]

where $S_f(r) = O(\log^+ T_f(r) + \log^+ r)\|$. Here “\|” means that the inequality holds for all $r \in (0, +\infty)$ possibly except for subset with finite Lebesgue measure.

In Section 4, we prove that $\tilde{D}' + \tilde{D}''$ is a big divisor on $Z_k$. Hence we can compute defects for holomorphic curves $f : \mathbb{C} \to \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$.

The second main theorem for hypersurfaces in $\mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$ is first obtained in Noguchi [3]. In Section 5 of Noguchi [3], there are some additional conditions for maps from $\mathbb{C}$ to $\mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$. In our paper, we do not assume these conditions.
In Siu [7], a meromorphic connection is used to prove the second main theorem. Because $\mathbb{C}^* \times \mathbb{C}^*$ is a Lie group, there exists the canonical connection on $\mathbb{C}^* \times \mathbb{C}^*$. We extend this connection to the meromorphic connection $\nabla$ on $\mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$. This meromorphic connection $\nabla$ is used in Section 5 of Noguchi [3]. We also use $\nabla$ to prove our main theorem. This connection does not “vary” under the blowing-up, and plays an important role in our arguments.

Let $i : \mathbb{C}^* \times \mathbb{C}^* \rightarrow \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$ be the inclusion map where $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$. Then $Z_k$ is the compactification of semi-Abelian variety $\mathbb{C}^* \times \mathbb{C}^*$. If holomorphic map $f : \mathbb{C} \rightarrow \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$ does not intersect $H_{1,0}, H_{1,1}, H_{2,0}, H_{2,1}$, then $f$ is a holomorphic map from $\mathbb{C}$ to semi-Abelian variety $\mathbb{C}^* \times \mathbb{C}^*$. In Noguchi, Winkelmann and Yamanoi [5], [6], the second main theorem for holomorphic map $f$ from $\mathbb{C}$ to a semi-Abelian variety $A$ with $D$ is proved, where $D$ is an effective reduced divisor on $A$.

**Theorem 2 ([6])**. Let $f : \mathbb{C} \rightarrow A$ be a holomorphic map such that the image of $f$ is Zariski dense in $A$. There is the compactification of $A$ such that $\overline{A}$ is smooth, equivalent with respect to the $A$-action, independent of $f$, and it follows that

$$T_f(r, |D|) \leq N_1(r, f^*D) + \varepsilon T_f(r, |D|) \|\varepsilon,$$

for all $\varepsilon > 0$, where $D$ is the closure of $D$ in $\overline{A}$.

If $A = \mathbb{C}^* \times \mathbb{C}^*$, and $D = D' + D''$ in Theorem 2, our main theorem deals with the holomorphic curves into $\overline{A}$.

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**2. Notation and Preliminaries**

We introduce some functions which play an important role in the Nevanlinna theory. Let $E$ be an effective divisor on $\mathbb{C}$. We write $E = \sum m_j P_j$, where
where \( \{ P_j \} \) is a set of discrete points in \( \mathbb{C} \) and \( m_j \) are positive integers. Put

\[
n_k(r, E) = \sum_{|P_j| < r} \min\{k, m_j\}.
\]

We define the counting function of \( E \) by

\[
N_k(r, E) = \int_1^r \frac{n_k(t, E)}{t} dt.
\]

Let \( X \) be a complex projective algebraic manifold, and let \( D \) be a divisor on \( X \). Let \([D]\) be the holomorphic line bundle on \( X \) which is defined by the divisor \( D \), and let supp \( D \) be the support of \( D \). Let \( \sigma \) be a holomorphic section of \([D]\) such that the zero divisor of \( \sigma \) is \( D \). Let \( f : \mathbb{C} \to X \) be a non-constant holomorphic map. We define the proximity function of \( D \) by

\[
m_f(r, D) = \int_0^{2\pi} \log \frac{1}{\|\sigma(f(re^{i\theta}))\|} \frac{d\theta}{2\pi},
\]

where \( \| \cdot \| \) is a Hermitian metric in \( L \). Let \( R(L, \| \cdot \|) \) be the curvature form of the metrized line bundle \( (L, \| \cdot \|) \) representing the first Chern class. Then we define the characteristic function of \( L \) by

\[
T_f(r, L) = \int_1^r \frac{dt}{t} \int_{\Delta(t)} f^*R(L, \| \cdot \|) + O(1),
\]

where \( \Delta(t) = \{ z \in \mathbb{C} \mid |z| < t \} \). We set \( T_f(r) = T_f(r, L) \) if \( L \) is an ample line bundle on \( X \). The equation

\[
T_f(r, L) = N(r, f^*D) + m_f(r, D) + O(1)
\]

is called the First Main Theorem (cf. Noguchi and Ochiai [4], Chapter V, §2). If \( X = \mathbb{P}^1(\mathbb{C}) \), \( f \) is a meromorphic function on \( \mathbb{C} \). Then we have the lemma on logarithmic derivative (cf. Noguchi and Ochiai [4], Chapter VI, §1)

\[
\int_0^{2\pi} \log^+ \left| \frac{f'(re^{i\theta})}{f(re^{i\theta})} \right| d\theta \leq S_f(r),
\]

where \( \log^+ r = \max\{0, \log r\} \), and \( S_f(r) = O(\log^+ T_f(r) + \log^+ r) \). Here “\( \| \)” means that the inequality holds for all \( r \in (0, +\infty) \) possibly except for a subset with finite Lebesgue measure.

The following lemma is also fundamental in Nevanlinna theory.
Lemma 1. Let $h(r) > 0$ be a monotone increasing function in $r \geq 1$. Then, for arbitrary $\delta > 0$, we have
\[ \frac{dh(r)}{dr} \leq (h(r))^{1+\delta}. \]

Proof. See Noguchi-Ochiai [4], Chapter V, §5. □

Let $X$ be a complex projective algebraic manifold, and let $Y$ be a smooth hypersurface of $X$. Let $s$ be the holomorphic function on an open subset $U \subset X$ such that $Y \cap U = \{ x \in U | s(x) = 0 \}$. Let $\nabla$ be a holomorphic connection on $U$. We write
\[ \nabla^{(m)} = \underbrace{\nabla \circ \cdots \circ \nabla}_{m \text{-times}}. \]

Then the following lemma holds (see the proof of Lemma 11.13. of J.-P. Demailly [1]).

Lemma 2. Let $X$ and $U$ be as above. Let $f : \mathbb{C} \to X$ be a holomorphic function. Assume that $Y$ is totally geodesic with respect to $\nabla$ on $U$. Then there exist holomorphic functions $h_0, h_1, \ldots, h_m$ on $U$ such that
\[ ds \cdot \nabla^{(m)}_f f'(z) = h_0(f(z))s \circ f(z) \]
\[ + \sum_{i=1}^{m} h_i(f(z)) \frac{d^i(s \circ f)}{dz^i}(z) + \frac{d^{m+1}(s \circ f)}{dz^{m+1}}(z), \]
for $z \in f^{-1}(U)$.

Let $X$ and $\tilde{X}$ be $n$-dimensional complex projective algebraic manifolds. Let $\pi : \tilde{X} \to X$ be a surjective holomorphic map. Then there exists a proper subvariety $S$ of $X$ such that $\tilde{X} \setminus \pi^{-1}(S)$ and $X \setminus S$ are locally biholomorphic. Let $\nabla$ be a meromorphic connection on $X$. Let $V$ be a small neighborhood of $p \in \tilde{X}$, and let $u, v$ be holomorphic vector fields on a small neighborhood $V$ of $p$. Then $V \setminus \pi^{-1}(S)$ is locally biholomorphic with $\pi(V) \setminus S$. We define the meromorphic connection $\pi^*\nabla$ on $\tilde{X} \setminus \pi^{-1}(S)$ by
\[ (\pi^*\nabla)_u v|_{V \setminus \pi^{-1}(S)} = (\pi_*|_{V \setminus \pi^{-1}S})^{-1} \nabla_{\pi_*u \pi_*v}. \]
Then the meromorphic vector field $(\pi^*\nabla)_u v$ on $V \setminus \pi^{-1}S$ is uniquely extended to the meromorphic vector field $(\pi^*\nabla_u v)$ on $V$. In this way, we define the meromorphic connection $\pi^*\nabla$ on $\tilde{X}$. 

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3. Meromorphic Connection and Blowing-Up

Let $D'$, $D''$ be divisors on $\mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$ defined by the polynomials $X^m_0 Y^n_0 - X^m_1 Y^n_1$, $X^{m''}_0 Y^{n''}_1 - X^{m''}_1 Y^{n''}_0$. Let $i : \mathbb{C}^* \times \mathbb{C}^* \to \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$ be the inclusion map. Then $\text{supp} \, i^* D'$ is a subgroup of $\mathbb{C}^* \times \mathbb{C}^*$. Therefore there exists the canonical connection $\nabla$ on $\mathbb{C}^* \times \mathbb{C}^*$ such that $\text{supp} \, i^* D'$ is totally geodesic with respect to $\nabla$. This connection is extended to the meromorphic connection on $\mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$. We also denote this extended connection by $\nabla$.

Let $U_{i,j} = \{([X_0 : X_1], [Y_0 : Y_1]) \in \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C}) | X_i \neq 0, Y_j \neq 0\}$, $0 \leq i, j \leq 1$. Take the canonical local coordinate system $(z, w)$ on $U_{i,j} \simeq \mathbb{C} \times \mathbb{C}$. Then, the meromorphic connection $\nabla$ is written by

$$d + \left( \begin{array}{cc} -\frac{dz}{z} & 0 \\ 0 & -\frac{dw}{w} \end{array} \right),$$

on $U_{i,j}$. It is easy to see that $\text{supp} \, i^* D''$ is also totally geodesic with respect to $\nabla$.

The universal covering space of $\mathbb{C}^* \times \mathbb{C}^*$ is $\mathbb{C} \times \mathbb{C}$. The connection on $\mathbb{C} \times \mathbb{C}$ which is induced by $\nabla$ is the flat connection $d$ on $\mathbb{C} \times \mathbb{C}$. Let $f : \mathbb{C} \to \mathbb{C}^* \times \mathbb{C}^*$ be a non-constant holomorphic map, and let $F : \mathbb{C} \to \mathbb{C} \times \mathbb{C}$ be the lift of $f$. Then $f' \wedge \nabla f' f' \equiv 0$ if and only if $F$ is a translation of a linear map.

**Lemma 3.** Let $f : \mathbb{C} \to \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$ be a non-constant holomorphic map such that $f(\mathbb{C})$ is not contained in the support of $H_{i,j}$, $i = 1, 2$, $j = 0, 1$. Then $f$ satisfies

$$f' \wedge \nabla f' f' \equiv 0,$$

if and only if $f$ satisfies the following condition (i) or (ii):

(i) $f(\mathbb{C}) \subset \{([X_0 : X_1], [Y_0 : Y_1]) \in \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C}) | X_0^{r_1} Y_0^{r_2} - C X_1^{r_1} Y_1^{r_2} = 0\}$, for some $(r_1, r_2) \in \mathbb{Z} \times \mathbb{Z} \setminus \{(0, 0)\}$ and some $C \in \mathbb{C} \setminus \{0\}$.

(ii) There exist holomorphic functions $g_1, g_2$ on $\mathbb{C}$ and $(a, b) \in \mathbb{C} \times \mathbb{C} \setminus \{(0, 0)\}$ such that

$$f = (\exp g_1, \exp g_2) : \mathbb{C} \to \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C}),$$
\[ ag_1 + bg_2 = \text{(constant)}, \]

on \( \mathbb{C} \).

**Proof.** Without loss of generality, we may assume that \( f(0) \in \mathbb{C}^* \times \mathbb{C}^* \). The holomorphic map

\[
(\exp(2\pi \sqrt{-1} \cdot), \exp(2\pi \sqrt{-1} \cdot)) : \mathbb{C} \times \mathbb{C} \rightarrow \mathbb{C}^* \times \mathbb{C}^*,
\]

is the universal covering of \( \mathbb{C}^* \times \mathbb{C}^* \). The induced connection on the covering space \( \mathbb{C} \times \mathbb{C} \) by \( \nabla \) is the flat connection \( d \). We put \( f = (f_1, f_2) \) where \( f_1 \) and \( f_2 \) are meromorphic functions on \( \mathbb{C} \). Let

\[
h_i = \frac{1}{2\pi \sqrt{-1}} \log f_i, \quad i = 1, 2.
\]

Assume that \( f' \wedge \nabla f' \equiv 0 \). Then there exists a meromorphic function \( h \) on \( \mathbb{C} \) such that

\[
\begin{pmatrix}
  h''_1(z) \\
  h''_2(z)
\end{pmatrix} = h(z) \begin{pmatrix}
  h'_1(z) \\
  h'_2(z)
\end{pmatrix},
\]

on \( \mathbb{C} \).

This means that

\[
h'_i(z) = h'_i(0) \exp H(z), \quad i = 1, 2,
\]

in a simple connected neighborhood \( U \) of \( 0 \in \mathbb{C} \). Here

\[
H(z) = \int_0^z h(t)dt.
\]

If \( h'_i(0) = 0 \), it follows that \( h_i \) is a constant function. Hence \( (h'_1(0), h'_2(0)) \in \mathbb{C} \times \mathbb{C} \setminus \{(0,0)\} \). It holds that

\[
h_i(z) = h'_i(0) \int_0^z \exp H(t) dt + h_i(0), \quad i = 1, 2.
\]

It follows that

\[
h'_2(0)h_1(z) - h'_1(0)h_2(z) = h'_2(0)h_1(0) - h'_1(0)h_2(0).
\]
Conversely, assume that there exists \((a, b) \in \mathbb{C} \times \mathbb{C} \setminus \{(0, 0)\}\) such that
\[ ah_1(z) + bh_2(z) = \text{(constant)}, \]
on \mathbb{C}. Then \(ah'_1(z) + bh'_2(z) = 0, ah''_1(z) + bh''_2(z) = 0\). Hence it follows that
\[ f' \wedge \nabla f' f' \equiv 0. \]
Therefore \(f' \wedge \nabla f' f' \equiv 0\) if and only if there exists \((a, b) \in \mathbb{C} \times \mathbb{C} \setminus \{(0, 0)\}\) such that
\[ a \log f_1(z) + b \log f_2(z) = \text{(constant)}, \]
on \mathbb{C}.
Assume that
\[ a \log f_1(z) + b \log f_2(z) = c, \tag{1} \]
for some \((a, b) \in \mathbb{C} \times \mathbb{C} \setminus \{(0, 0)\}, c \in \mathbb{C}\). Without loss of generality, we may assume that \(a = 1\). For \(x \in \mathbb{C}\), we put
\[ f_i(z) = (z - x)^{r_i} h_i(z), \quad i = 1, 2, \]
where \(r_i \in \mathbb{Z}\), \(h_i(z)\) is a holomorphic function on an open neighborhood of \(x\) such that \(h_i(x) \neq 0\). Then, by (1), we have \(r_1 + br_2 = 0\). When \(r_2 \neq 0\) for some \(x \in \mathbb{C}\), it follows that
\[ \log(f_1(z))^{r_2} + \log(f_2(z))^{-r_1} = r_2 c. \]
Then it holds that the meromorphic function \((f_1(z))^{r_2}(f_2(z))^{-r_1}\) is a constant function on \(\mathbb{C}\). This means that
\[ f(\mathbb{C}) \subset \{(X_0 : X_1, Y_0 : Y_1) \in \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C}) | X_1^{r_2}Y_0^{r_1} - CX_0^{r_2}Y_1^{r_1} = 0\}, \]
for \(r_1 \in \mathbb{Z}\), \(r_2 \in \mathbb{Z} \setminus \{0\}\), \(C \in \mathbb{C}\). When \(r_2 = 0\) for all \(x \in \mathbb{C}\), we have \(r_1 = 0\) for all \(x \in \mathbb{C}\). This means that there exist holomorphic functions \(g_1, g_2\) on \(\mathbb{C}\) such that \(f_i = \exp g_i, \quad i = 1, 2\). Then \(g_1 + bg_2 = c\).

Conversely, if \(f\) satisfies the condition \((i)\) or \((ii)\). It is easy to see that there exists \((a, b) \in \mathbb{C} \times \mathbb{C} \setminus \{(0, 0)\}\) such that
\[ a \log f_1(z) + b \log f_2(z) = \text{(constant)}, \]
on \mathbb{C}. □
Remark 1. The condition of \((b)\) in Lemma 3 does not mean the algebraical degeneracy of \(f(\mathbb{C})\). For example, take
\[
f(z) = (\exp z, \exp \sqrt{-1} z) : \mathbb{C} \to \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C}).
\]

The divisor
\[
D' + D'' + \sum_{i=1}^{2} \sum_{j=0}^{1} H_{i,j},
\]
is not simple normal crossing at \([(0 : 1), (0 : 1)], [(0 : 1), (1 : 0)], [(1 : 0), (0 : 1)], [(1 : 0), (1 : 0)]\} \subset \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$. Put \(Z_0 = \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})\). Let \(\pi_{1,0} : Z_1 \to Z_0\) be the blowing-up of \(Z_0\) at the center \([(0 : 1), (0 : 1)], [(0 : 1), (1 : 0)], [(1 : 0), (0 : 1)], [(1 : 0), (1 : 0)]\}. Let \(D'_1, D''_1, H_{i,j,1}\) be the strict transform of \(D', D'', H_{i,j}\) under \(\pi_{1,0}\), and let \(E_1\) be the exceptional divisor of \(\pi_{1,0}\).

If the divisor
\[
D'_1 + D''_1 + \sum_{i=1}^{2} \sum_{j=0}^{1} H_{i,j,1} + E_1,
\]
is not simple normal crossing in \(Z_1\), we blow up \(Z_1\) at the points where this divisor is not simple normal crossing. We repeat this process for several times. We put the \(l\)-th blowing-up \(\pi_{l,l-1} : Z_l \to Z_{l-1}\). Let \(E_l\) be the exceptional divisor of \(\pi_{l,l-1}\). We define
\[
\pi_{j,i} = \pi_{i+1,i} \circ \pi_{i+2,i+1} \circ \cdots \circ \pi_{j,j-1},
\]
for \(i \leq j\) (we define \(\pi_{i,i} = \text{Id}\)). Let \(D'_l, D''_l, H_{i,j,l}\) be the strict transform of \(D', D'', H_{i,j}\) under \(\pi_{l,0}\), and let \(E_{i,l}, 1 \leq i \leq l\), be the strict transform of \(E_i\) under \(\pi_{l,i}\).

Then there exists a positive integer \(k\) such that
\[
D'_k + D''_k + \sum_{i=1}^{2} \sum_{j=0}^{1} H_{i,j,k} + \sum_{i=1}^{k} E_{i,k},
\]
is simple normal crossing. We put \(\tilde{D}' = D'_k, \tilde{D}'' = D''_k, \tilde{H}_{i,j} = H_{i,j,k},\) and \(\tilde{E}_i = E_{i,k}\).
Example 1. Let $D', D''$ be the divisor which are defined by the polynomials

$$X_0^2Y_1 - X_1^2Y_0, \quad X_0^3Y_1^2 - X_1^3Y_0^2.$$ 

Let $\pi_{1,0} : Z_1 \to Z_0$ be the blowing-up as above. Then $D'_1 + D''_1 + \sum_{i=1}^2 \sum_{j=0}^1 H_{i,j,1} + E_1$ is not simple normal crossing at four points in $Z_1$. Then $\pi_{2,1} : Z_2 \to Z_1$ is the blowing-up at these four points. We see that $D'_2 + D''_2 + \sum_{i=1}^2 \sum_{j=0}^1 H_{i,j,2} + \sum_{i=1}^2 E_{i,2}$ is not simple normal crossing at two points in $Z_2$. Then $\pi_{3,2} : Z_3 \to Z_2$ is the blowing-up at these two points. Then $D'_2 + D''_2 + \sum_{i=1}^2 \sum_{j=0}^1 H_{i,j,2} + \sum_{i=1}^2 E_{i,2}$ is normal crossing.

Let $E'_i$ and $E''_i$ be unions of irreducible components of $E_i$ such that $\pi_{i,0}(\text{supp } E'_i) \subset \text{supp } D'$ and $\pi_{i,0}(\text{supp } E''_i) \subset \text{supp } D''$. Then $E_1 = E'_1 + E''_1$, $E_2 = E'_2 + E''_2$ and $E_3 = E''_3$. Let $\tilde{E}'_i$ and $\tilde{E}''_i$ be the proper transform of $E'_i$ and $E''_i$. Then it follows that

$$\pi^*_{3,0} D' = \tilde{D}' + \tilde{E}'_1 + 2\tilde{E}'_2,$$

and

$$\pi^*_{3,0} D'' = \tilde{D}'' + 2\tilde{E}''_1 + 3\tilde{E}''_2 + 6\tilde{E}''_3.$$ 

Lemma 4. There exist affine open coverings $\{U^l_s\}_{1 \leq s \leq N_l}$ of $Z_l$, for $0 \leq l \leq k$, such that every $U^l_s$ satisfies the following five conditions:

(i)

$$U^l_s \simeq \mathbb{C} \times \mathbb{C}.$$ 

Let $(z, w)$ be the canonical local coordinate system of $U^l_s$.

(ii)

$$\sum_{i=1}^2 \sum_{j=0}^1 H_{i,j,l}|_{U^l_s} + \sum_{1 \leq i \leq l} E_{i,l}|_{U^l_s} = (z) + (w),$$

on $U^l_s$.

(iii)

$$D^l_{i}|_{U^l_s} = (z^{p'_i} - w^{q'_i}) \quad \text{or} \quad (1 - z^{p'_i} w^{q'_i}) \quad \text{respectively},$$
on $U_s^l$, where $p'$ and $q'$ are non-negative integers ($p', q'$ may depend on $l$ and $s$).

(iv) 
\[ D''_{l}|_{U_s^l} = (1 - z^{p''} w^{q''}) \quad (\text{or} \quad z^{p''} - w^{q''}) \quad \text{respectively}, \]
on $U_s^l$, where $p''$ and $q''$ are non-negative integers ($p'', q''$ may depend on $l$ and $s$).

(v) 
\[ \pi^*_{l,0} \nabla|_{U_s^l} = d + \left( \begin{array}{cc} -\frac{dz}{z} & 0 \\ 0 & -\frac{dw}{w} \end{array} \right), \]
on $U_s^l$.

**Proof.** We take affine open coverings \( \{U_s^l\}_{1 \leq s \leq N_l} \) by induction over $l$. For $l = 0$, we put \( \{U_s^0\}_{1 \leq s \leq 4} = \{U_{i,j}\}_{0 \leq i,j \leq 1} \). Here \( U_{i,j} = \{[X_0 : X_1], [Y_0 : Y_1] \in \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C}) | X_i \neq 0, Y_j \neq 0\} \). Then \( \{U_s^0\}_{1 \leq s \leq 4} \) satisfies above five conditions. Assume that we take the affine open covering \( \{U_{s-1}\}_{1 \leq s \leq N_l-1} \) of $Z_{l-1}$ for $l \leq k$ which satisfies the above five conditions. Let \( U_{l-1}^l \in \{U_{s-1}\}_{1 \leq s \leq N_l-1} \). Take the canonical local coordinate system $(z, w)$ of $U_{l-1}^l \simeq \mathbb{C} \times \mathbb{C}$.

If \( D'_{l-1}|_{U_{l-1}^l} = (z^{p'} - w^{q'}) \) for some positive integers $p', q'$. Then 
\[ D''_{l-1}|_{U_{l-1}^l} = (1 - z^{p''} w^{q''}), \]
for some non-negative integers $p'', q''$. The divisor 
\[ D'_{l-1} + D''_{l-1} + \sum_{i=1}^2 \sum_{j=0}^1 H_{i,j,l-1} + \sum_{1 \leq i \leq l-1} E_{i,l-1}, \]
in $Z_{l-1}$, is not normal crossing at $(0,0) \in U_{l-1}^l$. Then $(0,0)$ is contained in the center of the blowing-up $\pi_{l,l-1}$. We have 
\[ \pi^{-1}_{l,l-1}(U_{l-1}^l) = \{((z, w), [W_0 : W_1]) \in U_{l-1}^l \times \mathbb{P}^1(\mathbb{C}) | z W_1 = w W_0\}. \]
Let \( V_i = \{((z, w), [W_0 : W_1]) \in \pi^{-1}_{l,l-1}(U_{l-1}^l) | W_i \neq 0\}, i = 0, 1 \). Then \( \{V_0, V_1\} \) is an affine open covering of $\pi^{-1}_{l,l-1}(U_{l-1}^l)$. We show that affine open
sets $V_0$ and $V_1$ satisfy the five conditions of lemma. Let $u = W_1/W_0$ be the holomorphic function on $V_0$. Then $(z, u)$ is the local coordinate system of $V_0$. It is easy to verify that $V_0$ satisfies (i), (ii), (iii) and (iv). Since

$$\pi^*_{l,l-1} z = z, \quad \pi^*_{l,l-1} w = zu,$$

we have

$$\pi^*_{l,l-1} \left( \frac{\partial}{\partial z} \frac{\partial}{\partial u} \right) = \left( \frac{\partial}{\partial z} \frac{\partial}{\partial w} \right) \left( \begin{array}{cc} 1 & 0 \\ u & z \end{array} \right).$$

Let $\Gamma$ be the connection form of $\pi^*_{l,l-1} \nabla|_{V_0}$ with respect to the frame $\partial/\partial z$, $\partial/\partial u$. Then it follows that

$$\Gamma = \left( \begin{array}{cc} 1 & 0 \\ u & z \end{array} \right)^{-1} d \left( \begin{array}{cc} 1 & 0 \\ u & z \end{array} \right)$$

$$+ \left( \begin{array}{cc} 1 & 0 \\ u & z \end{array} \right)^{-1} \pi^*_{l,l-1} \left( \begin{array}{cc} 0 & -\frac{du}{w} \\ -\frac{dz}{z} & 0 \end{array} \right) \left( \begin{array}{cc} 1 & 0 \\ u & z \end{array} \right).$$

Since

$$\left( \begin{array}{cc} 1 & 0 \\ u & z \end{array} \right)^{-1} = \left( \begin{array}{cc} 1 & 0 \\ -\frac{1}{z} & \frac{1}{u} \end{array} \right),$$

we have

$$\Gamma = \left( \begin{array}{cc} 0 & 0 \\ \frac{du}{z} & \frac{dz}{z} \end{array} \right) + \left( \begin{array}{cc} 1 & 0 \\ -\frac{1}{z} & \frac{1}{z} \end{array} \right) \left( \begin{array}{cc} \frac{dz}{z} & 0 \\ 0 & -\frac{dz}{z} - \frac{du}{u} \end{array} \right) \left( \begin{array}{cc} 1 & 0 \\ u & z \end{array} \right)$$

$$= \left( \begin{array}{cc} 0 & 0 \\ \frac{du}{z} & \frac{dz}{z} \end{array} \right) + \left( \begin{array}{cc} \frac{dz}{z} & 0 \\ 0 & -\frac{dz}{z} - \frac{du}{u} \end{array} \right) \left( \begin{array}{cc} 1 & 0 \\ u & z \end{array} \right)$$

$$= \left( \begin{array}{cc} 0 & 0 \\ \frac{du}{z} & \frac{dz}{z} \end{array} \right) + \left( \begin{array}{cc} \frac{dz}{z} - \frac{du}{z} & 0 \\ -\frac{du}{z} & -\frac{dz}{z} - \frac{du}{u} \end{array} \right)$$

$$= \left( \begin{array}{cc} \frac{dz}{z} & 0 \\ 0 & -\frac{du}{u} \end{array} \right).$$

Hence $V_0$ satisfies (v). In the same way, we can show that $V_1$ satisfies the conditions of the lemma.
If

\[ D'_l | U_{l-1}^{l-1} = (1 - z^{p'} w^{q'}) , \quad D''_l | U_{l-1}^{l-1} = (z^{p''} - w^{q''}) , \]

for some non-negative integers \( p', q' \) and some positive integers \( p'', q'' \). In the same way as above, we can take the affine open sets in \( \pi_{l,l-1}^{-1}(U_{l-1}^{l-1}) \) which satisfy the five conditions of the lemma.

If

\[ D'_l | U_{l-1}^{l-1} = (1 - z^{p'} w^{q'}) , \quad D''_l | U_{l-1}^{l-1} = (1 - z^{p''} w^{q''}) , \]

for some non-negative integers \( p', q', p'', q'' \). Then \( U_{l-1}^{l-1} \simeq \pi_{l,l-1}^{-1}(U_{l-1}^{l-1}) \) because \( U_{l-1}^{l-1} \) does not contain the center of the blowing-up \( \pi_{l,l-1} \). By the assumption of induction, the affine open subset \( \pi_{l,l-1}^{-1}(U_{l-1}^{l-1}) \) satisfies the five conditions of the lemma. This completes the proof. \( \square \)

4. Proof of the Bigness of \( \tilde{D}' + \tilde{D}'' \)

In this section, we show that \( \tilde{D}' + \tilde{D}'' \) is big in \( Z_k \). We note that there exists the proof of the bigness for more general cases in Proposition 3.9. of [6].

To prove the bigness of the line bundle \( \tilde{D}' + \tilde{D}'' \), it is sufficient to show the following lemma (cf. Theorem 2.2.16. of R. Lazarsfeld [2]).

**Lemma 5.** The divisor \( \tilde{D}' + \tilde{D}'' \) is nef and \( (\tilde{D}' + \tilde{D}'')^2 > 0 \).

**Proof.** Because

\[(\tilde{D}' + \tilde{D}'')^2 = (\tilde{D}')^2 + 2\tilde{D}' \cdot \tilde{D}'' + (\tilde{D}'')^2,\]

it is enough to show that \( (\tilde{D}')^2 = (\tilde{D}'')^2 = 0 \) and \( \tilde{D}' \) and \( \tilde{D}'' \) are nef. Without loss of generality, we may assume that \( m' \leq n' \). Let \( E'_1 \) be the reduced divisor on \( Z_1 \) such that

\[ \pi_{1,0}^* D' = D'_1 + m' E'_1. \]

Let \( F' \) be the divisor on \( Z_k \) such that

\[ \pi_{k,1}^* D'_1 = \tilde{D}' + F'. \]
It follows that
\[
(\tilde{D}')^2 = (\pi_{k,0}^* D' - F' - m' \pi_{k,1}^* E'_1)^2
\]
\[
= (\pi_{k,0}^* D')^2 + (F')^2 + m'^2 (\pi_{k,1}^* E'_1)^2 - 2\pi_{k,0}^* D' \cdot F'
+ 2m' F' \cdot \pi_{k,1}^* E'_1 - 2m' \pi_{k,1}^* E'_1 \cdot \pi_{k,0}^* D'
\]
\[
= 2m' n' + (F')^2 - 2m'^2 - 2(\tilde{D}' + F' + m' \pi_{k,1}^* E'_1) \cdot F'
+ 2m' F' \cdot \pi_{k,1}^* E'_1 - 2m' \pi_{k,1}^* E'_1 \cdot (\pi_{k,1}^* D'_1 + m' \pi_{k,1}^* E'_1)
\]
\[
= 2m' n' - 2m'^2 - (F')^2 - 2\tilde{D}' \cdot F' - 2m' D'_1 \cdot E'_1 - 2m'^2 (E'_1)^2.
\]

Because \( D'_1 \cdot E'_1 = 2m' \), we have
\[
(2) \quad (\tilde{D}')^2 = 2m' n' - 2m'^2 - (F')^2 - 2\tilde{D}' \cdot F'.
\]

If \( m' = n' \), then \( \tilde{D}' = D'_1 \), \( F' = 0 \) and we have \((\tilde{D}')^2 = 0\).

Now we prove \((\tilde{D}')^2 = 0\) by the induction over the positive integer \( m' + n' \). Let \( E'_i, i = 2, \cdots k \) be reduced effective divisors on \( Z \) such that
\[
supp(\pi_{t,i-1}^* D'_{i-1} - D'_i) = supp E'_i.
\]

Let \( \tilde{E}'_i \) be the strict transform of \( E'_i \) under \( \pi_{k,i} \). There exist non-negative integers \( a_2, a_3, \cdots, a_k \) such that
\[
F = a_2 \tilde{E}'_2 + a_3 \tilde{E}'_3 + \cdots + a_k \tilde{E}'_k.
\]

Now we take another divisor \( A' \) on \( \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C}) \) which is defined by the polynomial
\[
X_0^{m'} Y_0^{n'-m'} - X_1^{m'} Y_1^{n'-m'}.
\]

There is, as in Section 3, the sequence of the blowing-up
\[
\sigma_{1,0} : W_1 \to \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C}),
\]
\[
\vdots
\]
\[
\sigma_{k-1,k-2} : W_{k-1} \to W_{k-2},
\]

such that the following condition (**) satisfies:
Let $S$ be the reduced divisor such that
\[
\text{supp} \left( \sigma_{k-1,0}^* \left( A' + \sum_{i=1}^2 \sum_{j=0}^1 H_{i,j} \right) \right) = \text{supp} S,
\]
where $\sigma_{k-1,1} = \sigma_{1,0} \circ \cdots \circ \sigma_{k-1,k-2}$. Then
\[(**) \quad S \text{ is normal crossing in } W_{k-1}.\]

Let $B_i'$ be the exceptional divisor of $\sigma_{i,i-1}$, and let $\tilde{B}_i'$ be the strict transform of $B_i'$ under $\sigma_{i+1,i} \circ \cdots \circ \sigma_{k-1,k-2}$. Let $\tilde{A}'$ be the strict transform of $A'$ under $\sigma_{1,0} \circ \cdots \circ \sigma_{k-1,k-2}$. It follows that
\[
(\sigma_{1,0} \circ \cdots \circ \sigma_{k-1,k-2})^* A' = \tilde{A}' + a_2 \tilde{B}_1' + a_3 \tilde{B}_2' + \cdots + a_k \tilde{B}_{k-1}',
\]
and
\[
\tilde{E}'_i \cdot \tilde{E}'_j = \tilde{B}_{i-1}' \cdot \tilde{B}_{j-1}', \quad \tilde{D}' \cdot \tilde{E}'_i = \tilde{A}' \cdot \tilde{B}_{i-1}',
\]
for all $2 \leq i, j \leq k$. Put $G' = a_2 \tilde{B}_1' + a_3 \tilde{B}_2' + \cdots + a_k \tilde{B}_{k-1}'$. We have
\[(3) \quad (F')^2 = (G')^2, \quad \tilde{D}' \cdot F' = \tilde{A}' \cdot G'.\]

It follows that
\[
(\tilde{A}')^2 = (\sigma_{k-1,0}^* A' - G')^2 = 2m'(n' - m') - 2\sigma_{k-1,0}^* A' \cdot G' + (G')^2 = 2m'(n' - m') - 2(\tilde{A}' + G') \cdot G' + (G')^2 = 2m'(n' - m') - (G')^2 - 2\tilde{A}' \cdot G'.
\]

By the assumption of the induction, we have
\[(4) \quad (G')^2 + 2\tilde{A}' \cdot G' - 2m'(n' - m') = 0.\]

By (2), (3) and (4), it follows that
\[
(\tilde{D}')^2 = 2m'n' - 2m'^2 - (F')^2 - 2\tilde{D}' \cdot F' = 2m'(n' - m') - (G')^2 - 2\tilde{A}' \cdot G' = 0.
\]

Then we complete the induction. By the same way, we can show that $\tilde{(D'')}^2 = 0$. 

\textit{Curves in the Product Space of the Riemann Spheres}
Now we show that \( \tilde{D}' \) is nef. Let \( m' = dp, n' = dq \), where \( d \) is the greatest common divisor of \( m' \) and \( n' \). Then it follows that

\[
X_0^{m'}Y_0^{n'} - X_1^{m'}Y_1^{n'} = \prod_{i=0}^{d-1} \left( X_0^{p}Y_0^{q} - (\varepsilon_d)^i X_1^{p}Y_1^{q} \right),
\]

where \( \varepsilon_d = \exp((2\pi \sqrt{-1})/d) \). Let \( C_i \) be the irreducible divisor on \( \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C}) \) which is defined by the polynomial \( X_0^{p}Y_0^{q} - (\varepsilon_d)^i X_1^{p}Y_1^{q} \), and let \( \tilde{C}_i \) be the strict transform of \( C_i \) under \( \pi_{k,0} \). By the above arguments, we have \( (\tilde{C}_0)^2 = 0 \). Because \( \tilde{C}_0 \) and \( \tilde{C}_i, 1 \leq i \leq d-1 \), are linearly equivalent, we have

\[
\tilde{C}_i \cdot \tilde{D}' = (\tilde{C}_i)^2 = (\tilde{C}_0)^2 = 0.
\]

Therefore \( \tilde{D}' \) is nef. By the same way, we can show that \( \tilde{D}'' \) is nef. \( \square \)

5. Proof of the Main Theorem

Let \( f : \mathbb{C} \to \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C}) \) be the holomorphic map, let \( \tilde{f} : \mathbb{C} \to Z_k \) be the lift of \( f \), and let \( \tilde{\nabla} = \pi_{k,0}^* \nabla \).

\[
\begin{array}{cccccccc}
Z_0 & \xleftarrow{\pi_{1,0}} & Z_1 & \xleftarrow{\pi_{2,1}} & Z_2 & \xleftarrow{\pi_{3,2}} & \cdots & \xleftarrow{\pi_{k,k-1}} & Z_k \\
\downarrow f & & & & & & & & \downarrow f \\
\mathbb{C} & & & & & & & & \\
\end{array}
\]

Let \( \tilde{\sigma}' \in \Gamma(Z_k, [\tilde{D}']), \tilde{\sigma}'' \in \Gamma(Z_k, [\tilde{D}'']), \tilde{h}_{i,j} \in \Gamma(Z_k, [\tilde{H}_{i,j}]), \tilde{e}_i \in \Gamma(Z_k, \tilde{E}_i) \) be the holomorphic section such that

\[
(\tilde{\sigma}') = \tilde{D}', \quad (\tilde{\sigma}'') = \tilde{D}'', \quad (\tilde{h}_{i,j}) = \tilde{H}_{i,j}, \quad (\tilde{e}_i) = \tilde{E}_i.
\]

**Lemma 6.** Assume that

\[
f(\mathbb{C}) \not\subset \text{supp} \left( D' + D'' + \sum_{i=1}^{2} \sum_{j=0}^{1} H_{i,j} \right),
\]
and assume that
\[ f' \land \nabla f' \neq 0. \]

Then it follows that
\[
\int_{|z|=r} \log^+ \frac{\|\tilde{f}' \land \tilde{\nabla} f'(z)\|_{\Lambda^2 T Z_k}}{\|\tilde{\sigma}'(\tilde{f})\|_{[D']\|\tilde{\sigma}''(\tilde{f})\|_{[D''}} \prod_{i=1}^2 \prod_{j=0}^1 \|\tilde{h}_{i,j}(\tilde{f})\|_{[H_{i,j}]}} \prod_{i=1}^k \|\tilde{e}_i(\tilde{f})\|_{[E_i]} \frac{d\theta}{2\pi} \leq S_f(r).
\]

**Proof.** For the convenience of the notation, we assume that \( D' \) and \( D'' \) are irreducible. Put
\[
A = \tilde{D}' + \tilde{D}'' + \sum_{i=1}^2 \sum_{j=0}^1 \tilde{H}_{i,j} + \sum_{i=1}^k \tilde{E}_i,
\]
and put
\[
\xi(z) = \frac{\|\tilde{f}' \land \tilde{\nabla} f'(z)\|_{\Lambda^2 T Z_k}}{\|\tilde{\sigma}'(\tilde{f})\|_{[D']\|\tilde{\sigma}''(\tilde{f})\|_{[D'\|D''\|\tilde{h}_{i,j}(\tilde{f})\|_{[H_{i,j}]}} \prod_{i=1}^k \|\tilde{e}_i(\tilde{f})\|_{[E_i]} \}}.
\]

Note that \( A \) is simple normal crossing in \( Z_k \).

Let
\[
x \in \bigcup_{i=1}^2 \bigcup_{j=0}^1 \text{supp} \tilde{H}_{i,j} \cap \bigcup_{i=1}^k \text{supp} \tilde{E}_i.
\]

By Lemma 4, there exists an affine open neighborhood \( U_x \) of \( x \) and local coordinate system \( z_x, w_x \) on \( U_x \) which satisfy the five conditions of Lemma 4. We put
\[
V_x = U_x \setminus \text{supp} (\tilde{D}' + \tilde{D}''),
\]
and put
\[
\tilde{f}_1 = z_x \circ \tilde{f}, \quad \tilde{f}_2 = w_x \circ \tilde{f},
\]
on $\tilde{f}^{-1}(V_x)$. It follows that
\[
\tilde{f}' \wedge \nabla_{\tilde{f}', \tilde{f}'} = \left( \tilde{f}'_1 \tilde{f}'^2 - \tilde{f}'_1 \tilde{f}'_2 + \tilde{f}'_1 \tilde{f}'_2 \frac{\tilde{f}'_1}{\tilde{f}'_1} - \tilde{f}'_1 \tilde{f}'_2 \frac{\tilde{f}'_2}{\tilde{f}'_2} \right) \frac{\partial}{\partial z} \wedge \frac{\partial}{\partial w}.
\]
on $\tilde{f}^{-1}(V_x)$. Then it follows that
\[
(5) \quad \xi(z) = \left( \frac{\tilde{f}'_1}{\tilde{f}'_1} \frac{\tilde{f}'_2}{\tilde{f}'_2} - \frac{\tilde{f}'_1}{\tilde{f}'_1} \frac{\tilde{f}'_2}{\tilde{f}'_2} + \left( \frac{\tilde{f}'_1}{\tilde{f}'_1} \right)^2 \frac{\tilde{f}'_2}{\tilde{f}'_2} - \frac{\tilde{f}'_1}{\tilde{f}'_1} \left( \frac{\tilde{f}'_2}{\tilde{f}'_2} \right)^2 \right) \Phi_x(f(z)),
\]
on $\tilde{f}^{-1}(V_x)$, where $\Phi_x$ is a smooth function on $V_x$.

Let
\[x \in \text{supp } \tilde{D}' \cap \left( \bigcup_{i=1}^2 \bigcup_{j=0}^1 \supp \tilde{H}_{i,j} \cup \bigcup_{i=1}^k \supp \tilde{E}_i \right)\]
\[\left( \text{or } x \in \text{supp } \tilde{D}' \cap \left( \bigcup_{i=1}^2 \bigcup_{j=0}^1 \supp \tilde{H}_{i,j} \cup \bigcup_{i=1}^k \supp \tilde{E}_i \right) \right), \text{ respectively}\]

By Lemma 4, there exists an affine open neighborhood $U_x$ of $x$ and local coordinate system $z_x, w_x$ on $U_x$ which satisfy the five condition of Lemma 4. Because $D'$ and $D''$ are irreducible, it follows that
\[D'|_{U_x} = (z_x - 1) \quad \text{(or } D''|_{U_x} = (z_x - 1), \text{ respectively}),\]
and $z_x(x) = 1, w_x(x) = 0$. We take $z'_x = z_x - 1$. Let $V_x$ be an affine open subset of $U_x$ such that
\[A|_{V_x} = (z'_x) + (w_x),\]
and
\[\nabla|_{V_x} = d + \left( \begin{array}{cc} -(d z'_x)/(z'_x + 1) & 0 \\ 0 & -(d w_x)/w_x \end{array} \right).\]
We note that $z'_x(x) + 1 \neq 0$ on $\tilde{f}^{-1}(V_x)$. We put
\[\tilde{f}_1 = z'_x \circ \tilde{f}, \quad \tilde{f}_2 = w_x \circ \tilde{f},\]
on $\tilde{f}^{-1}(V_x)$. It follows that

$$\xi(z) = \left( \frac{\tilde{f}_1}{f_1} \frac{\tilde{f}_2''}{f_2} - \frac{\tilde{f}_1''}{f_1} \frac{\tilde{f}_2'}{f_2} - \frac{\tilde{f}_1'}{f_1} \left( \frac{\tilde{f}_2'}{f_2} \right)^2 \right) \Phi_x(f(z))$$

$$+ \frac{\tilde{f}_1'}{f_1} \frac{\tilde{f}_2'}{f_2} \Psi_x(f(z)),$$

on $\tilde{f}^{-1}(V_x)$, where $\Phi_x$ and $\Psi_x$ are smooth functions on $V_x$.

Let $x \in \text{supp} \tilde{D}' \cap \text{supp} \tilde{D}''$. There exists an affine open neighborhood $V_x$ of $x$ and holomorphic functions $z_x, w_x$ on $V_x$ such that

$$\tilde{D}'|_{V_x} = (z_x), \quad \tilde{D}''|_{V_x} = (w_x),$$

$$A|_{V_x} = (z_x) + (w_x),$$

on $V_x$. It follows that $dz_x$ and $dw_x$ are linearly independent on $V_x$. We put

$$\tilde{f}_1 = z_x \circ \tilde{f}, \quad \tilde{f}_2 = w_x \circ \tilde{f}.$$ 

By Lemma 2, there exist holomorphic functions $g_0, g_1, h_0, h_1$ on $V_x$ such that

$$dz_x \cdot \nabla_{\partial z} \tilde{f}'(\gamma) = g_0(\tilde{f}(\gamma)) \tilde{f}_1(\gamma) + g_1(\tilde{f}(\gamma)) \tilde{f}_1'(\gamma) + \tilde{f}_1''(\gamma),$$

for all $\gamma \in \tilde{f}^{-1}(V_x)$, and

$$dw_x \cdot \nabla_{\partial w} \tilde{f}'(\gamma) = h_0(\tilde{f}(\gamma)) \tilde{f}_2(\gamma) + h_1(\tilde{f}(\gamma)) \tilde{f}_2'(\gamma) + \tilde{f}_2''(\gamma),$$

for all $\gamma \in \tilde{f}^{-1}(V_x)$. It follows that

$$\tilde{f}' \wedge \nabla_{\partial z} \tilde{f}' = \left[ \tilde{f}_1 \left( g_0(\tilde{f}) \tilde{f}_2 + h_1(\tilde{f}) \tilde{f}_2' + \tilde{f}_2'' \right) \
- \tilde{f}_2 \left( g_0(\tilde{f}) \tilde{f}_1 + g_1(\tilde{f}) \tilde{f}_1' + \tilde{f}_1'' \right) \right] \frac{\partial}{\partial z_x} \wedge \frac{\partial}{\partial w_x}.$$ 

Then it follows that

$$\xi(z) = \Phi_{x,1}(\tilde{f}) \frac{\tilde{f}_1}{f_1} + \Phi_{x,2}(\tilde{f}) \frac{\tilde{f}_2}{f_2}$$

$$+ \Phi_{x,3}(\tilde{f}) \frac{\tilde{f}_1'}{f_1} \frac{\tilde{f}_2'}{f_2} + \Phi_{x,4}(\tilde{f}) \frac{\tilde{f}_1'}{f_1} \frac{\tilde{f}_2''}{f_2} + \Phi_{x,5}(\tilde{f}) \frac{\tilde{f}_1'}{f_1} \frac{\tilde{f}_2'}{f_2},$$

where $\Phi_{x,i}(\tilde{f})$ are smooth functions on $V_x$. The integrals $\int_{\tilde{f}^{-1}(V_x)}$ are taken over $\tilde{f}^{-1}(V_x)$. The specific form of $\Phi_{x,i}(\tilde{f})$ depends on the properties of the functions $g_0, g_1, h_0, h_1$. The expressions for $\Phi_{x,i}(\tilde{f})$ are determined by the conditions that

1. $\Phi_{x,i}(\tilde{f})$ are smooth functions on $V_x$.
2. $\Phi_{x,i}(\tilde{f})$ are linearly independent on $V_x$.
3. $\Phi_{x,i}(\tilde{f})$ are chosen such that

$$\int_{\tilde{f}^{-1}(V_x)} \xi(z) \, dz_x \wedge dw_x = 0,$$

for all $V_x$.
on $\tilde{f}^{-1}(V_x)$, where $\Phi_{x,1}, \ldots, \Phi_{x,5}$ are smooth functions on $V_x$.

Let $R = \{ x \in Z_k \mid x \text{ is contained in two irreducible components of } A \}$. Note that $R$ is a finite subset of $Z_k$. For $x \in R$, we take an affine open subset $V_x$ and holomorphic functions $z_x, w_x$ as above arguments. Then $\{V_x\}_{x \in R}$ is an open covering of $Z_k$. We take an open covering $\{V'_x\}_{x \in R}$ of $Z_k$ such that $V'_x \subset V_x$ and $V'_x$ is relatively compact in $V_x$. We take a partition of unity $\{\phi_x\}_{x \in R}$ which is subordinate to the covering $\{V'_x\}_{x \in R}$. Fix $x \in R$. Let $\tilde{f}_1 = z_x \circ \tilde{f}$, $\tilde{f}_2 = w_x \circ \tilde{f}$ be a holomorphic function on $\tilde{f}^{-1}(V_x)$. Then $\tilde{f}_1$ and $\tilde{f}_2$ are extended to meromorphic functions on $\mathbb{C}$. By (5), (6) and (7), we have

$$
\int_{|z|=r} \phi_i(\tilde{f}(z)) \log^+ \xi(z) \frac{d\theta}{2\pi}
\leq \int_{|z|=r} \Gamma(\tilde{f}(z)) \frac{d\theta}{2\pi} + 4 \sum_{i=1}^2 \int_{|z|=r} \log^+ \left| \frac{\tilde{f}_i'(z)}{\tilde{f}_i(z)} \right| \frac{d\theta}{2\pi}
+ \sum_{i=1}^2 \int_{|z|=r} \log^+ \left| \frac{\tilde{f}_i''(z)}{\tilde{f}_i(z)} \right| \frac{d\theta}{2\pi} + \int_{|z|=r} \phi_i(\tilde{f}(z)) \log^+ |\tilde{f}_1'(z)| \frac{d\theta}{2\pi},
$$

where $\Gamma$ is a bounded smooth function on $Z_k$. By using the lemma on logarithmic derivative, it follows that

$$
\int_{|\gamma|=r} \log^+ \left| \frac{\tilde{f}_i'(\gamma)}{\tilde{f}_i(\gamma)} \right| \frac{d\theta}{2\pi} \leq S_f(r).
$$

It holds that

$$
\int_{|z|=r} \phi_i(\tilde{f}(z)) \log^+ |\tilde{f}_1'(z)| \frac{d\theta}{2\pi} = \frac{1}{2} \int_{|z|=r} \phi_i(\tilde{f}(z)) \log^+ |\tilde{f}_1'(z)|^2 \frac{d\theta}{2\pi}
\leq \frac{1}{2} \int_{|z|=r} \log^+ ||\tilde{f}'(z)||^2_{TZ_k} \frac{d\theta}{2\pi} + O(1),
$$

where $|| \cdot ||_{TZ_k}$ is a hermitian metric of $TZ_k$. By Lemma 1 and the concavity
of log, we have that
\[
\frac{1}{2} \int_{|z|=r} \log^+ \| \tilde{f}'(z) \|^2_{TZ_k} \frac{d\theta}{2\pi} \geq \frac{1}{2} \int_{|z|=r} \log \{ \| \tilde{f}'(z) \|^2_{TZ_k} + 1 \} \frac{d\theta}{2\pi} \geq \frac{1}{2} \log \left( 1 + \int_{|z|=r} \| \tilde{f}'(z) \|^2_{TZ_k} \frac{d\theta}{2\pi} \right) + O(1)
\]
\[
\leq \frac{1}{2} \log \left( 1 + \frac{1}{2\pi r} \frac{d}{dr} \int_{|z|=r} \| \tilde{f}'(z) \|^2_{TZ_k} \frac{\sqrt{-1}}{2} d\zeta \right) + O(1)
\]
\[
\leq \frac{1}{2} \log \left( 1 + \frac{1}{2\pi r} \left( \int_{|z|\leq r} \| \tilde{f}'(z) \|^2_{TZ_k} \frac{\sqrt{-1}}{2} d\zeta \right)^{1+\delta} \right) + O(1)
\]
\[
= \frac{1}{2} \log \left( 1 + \frac{r^\delta}{2\pi} \left( \frac{d}{dr} \int_{1}^{r} \frac{dt}{t} \int_{|z|\leq r} \| \tilde{f}'(z) \|^2_{TZ_k} \frac{\sqrt{-1}}{2} d\zeta \right)^{(1+\delta)^2} \right) + O(1)
\]
\[
\leq \frac{1}{2} \log \left( 1 + \frac{r^\delta}{2\pi} \left( \int_{1}^{r} \frac{dt}{t} \int_{|z|\leq r} \| \tilde{f}'(z) \|^2_{TZ_k} \frac{\sqrt{-1}}{2} d\zeta \right)^{(1+\delta)^2} \right) + O(1)
\]
\[
\leq S_f(r),
\]
where \(\delta\) is any positive number.

Because \(\sum_{x \in R} \phi_x(\tilde{f}) = 1\) on \(C\), it follows that
\[
\int_{|z|=r} \log^+ \xi(z) \frac{d\theta}{2\pi} = \sum_{x \in R} \int_{|z|=r} \phi_x(\tilde{f}(z)) \log^+ \xi(z) \frac{d\theta}{2\pi} \leq S_f(r).
\]

The following lemma is useful.

**Lemma 7.** It follows that
\[
\sum_{i=1}^{k} \sum_{j=0}^{1} \pi_{k,0} H_{i,j} = \sum_{i=1}^{k} \sum_{j=0}^{1} \tilde{H}_{i,j} + \sum_{i=1}^{k} \pi_{k,i} E_i + \sum_{i=1}^{k} \tilde{E}_i.
\]

**Proof.** Let the divisor \(H_{i,j,l}\) on \(Z_l\) be the strict transform of \(H_{i,j}\) under \(\pi_{l,0}\), and let \(E_{i,l}, i \leq l\), be the strict transform of \(E_i\) under \(\pi_{l,i}\), where \(E_{l,l} = E_l\).
We show
\[ \sum_{i=1}^{2} \sum_{j=0}^{1} \pi^*_i H_{i,j} = \sum_{i=1}^{2} \sum_{j=0}^{1} H_{i,j,l} + \sum_{i=1}^{l} \pi^*_i E_i + \sum_{i=1}^{l} E_i, \]
by induction over \( l \). If \( l = 1 \), we have
\[ \sum_{i=1}^{2} \sum_{j=0}^{1} \pi^*_i H_{i,j} = \sum_{i=1}^{2} \sum_{j=0}^{1} H_{i,j,1} + 2E_i. \]
Therefore the statement of the induction holds for \( l = 1 \). Assume that the statement holds for \( l-1 \), \( 1 < l \leq k \). Let \( C_i \) \( i = 1, 2, \ldots, r \) be irreducible divisor on \( Z_{l-1} \) such that
\[ \text{supp} \left( \sum_{i=1}^{2} \sum_{j=0}^{1} \pi^*_{l-1,0} H_{i,j} \right) = \bigcup_{i=1}^{r} \text{supp} C_i. \]
There exist positive integers \( a_1, a_2, \ldots, a_r \) such that
\[ \sum_{i=1}^{2} \sum_{j=0}^{1} \pi^*_{l-1,0} H_{i,j} = \sum_{i=1}^{r} a_i C_i. \]
By the assumption of the induction, we have
\[ \sum_{i=1}^{l} \pi^*_i E_i = \sum_{i=1}^{r} (a_i - 1) C_i. \]
Let \( x \in Z_{l-1} \) be one of the points of the center of \( \pi_{l,l-1} \), and let \( F_l \) be the irreducible component of \( E_l \) such that \( \pi_{l,l-1}(\text{supp} F_l) = x \). Assume that \( x \in \text{supp} C_p \cap \text{supp} C_q \) for \( 1 \leq p < q \leq r \). Then the coefficients of \( F_l \) in \( \sum_{i=1}^{2} \sum_{j=0}^{1} \pi^*_{l,0} H_{i,j} \) is \( a_p + a_q \), and the coefficient of \( F_l \) in \( \sum_{i=1}^{l-1} \pi^*_i E_i \) is \( a_p + a_q - 2 \). Therefore we have
\[ \sum_{i=1}^{2} \sum_{j=0}^{1} \pi^*_i H_{i,j} - \sum_{i=1}^{l} \pi^*_i E_i = \sum_{i=1}^{2} \sum_{j=0}^{1} H_{i,j,l} + \sum_{i=1}^{l} E_i. \]
This complete the induction, and the lemma follows. \( \square \)
Proof of the Main Theorem. We put \( W_\nabla(\tilde{f}) = \tilde{f}' \wedge \tilde{\nabla}' \tilde{f}' \). We denote by \( \text{ord}_z g \) the order of zero of \( g \) at \( z \), where \( g \) is a holomorphic section of a line bundle on a neighborhood of \( z \). By (5), (6) and (7) in Lemma 6, it follows that

\[
\text{ord}_z \left( \tilde{\sigma}'(\tilde{f}) \tilde{\sigma}''(\tilde{f}) \prod_{i=1}^2 \prod_{j=0}^1 \tilde{h}_{i,j}(\tilde{f}) \prod_{i=1}^k \tilde{e}_i(\tilde{f}) \right) - \text{ord}_z \left( W_\nabla(\tilde{f}) \right)
\leq \min\{\text{ord}_z \tilde{\sigma}'(\tilde{f}), 2\} + \min\{\text{ord}_z \tilde{\sigma}''(\tilde{f}), 2\}
+ 2 \sum_{i=1}^2 \sum_{j=0}^1 \min\{\text{ord}_z \tilde{h}_{i,j}(\tilde{f}), 1\} + 2 \sum_{i=1}^k \min\{\text{ord}_z \tilde{e}_i(\tilde{f}), 1\}.
\]

Therefore it follows that

\[
\begin{align*}
T_j(r, K_{Z_k}) + T_j(r, [\tilde{D}' + \tilde{D}'']) &+ \sum_{i=1}^2 \sum_{j=0}^1 T_j(r, \tilde{H}_{i,j}) + \sum_{i=1}^k T_j(r, \tilde{E}_i) \\
&\leq N_2(r, \tilde{f}' \tilde{D}') + N_2(r, \tilde{f}' \tilde{D}'') + 2 \sum_{i=1}^2 \sum_{j=0}^1 N_1(r, \tilde{f}' \tilde{H}_{i,j}) \\
&+ 2 \sum_{1 \leq i \leq k} N_1(r, \tilde{f}' \tilde{E}_i) + S_f(r),
\end{align*}
\]

where \( K_{Z_k} \) is the canonical line bundle of \( Z_k \). The canonical line bundle of \( Z_k \) is equal to

\[
\pi_{k,0}^* K_{P^1(\mathbb{C}) \times P^1(\mathbb{C})} + \pi_{k,1}^* E_1 + \pi_{k,2}^* E_2 + \cdots + E_k.
\]

By Lemma 7, it follows that

\[
\begin{align*}
-T_f(r, K_{P^1(\mathbb{C}) \times P^1(\mathbb{C})}) &= T_f(r, \mathcal{O}(2, 2)) = \sum_{i=1}^2 \sum_{j=0}^1 T_f(r, \pi_{k,0}^* H_{i,j}) \\
&= \sum_{i=1}^2 \sum_{j=0}^1 T_f(r, \tilde{H}_{i,j}) + \sum_{i=1}^k T_f(r, \pi_{k,i}^* E_i) \\
&+ \sum_{i=1}^k T_f(r, \tilde{E}_i)
\end{align*}
\]
By (8), (9), it follows that
\[
T_f(r, [\tilde{D}' + \tilde{D}'']) \leq N_2(r, \tilde{f}^*\tilde{D}') + N_2(r, \tilde{f}^*\tilde{D}'') + 2 \sum_{i=1}^{2} \sum_{j=0}^{1} N_1(r, \tilde{f}^*\tilde{H}_{i,j}) + 2 \sum_{i=1}^{k} N_1(r, \tilde{f}^*\tilde{E}_i) + S_f(r).
\]

By Lemma 3 and Lemma 5, our main theorem follows. □

**Corollary 1.** Let \( f : \mathbb{C} \to \mathbb{C}^* \times \mathbb{C}^* \subset \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C}) \) be a non-constant map. Assume that
\[
f(\mathbb{C}) \not\subset \{(X_0 : X_1, [Y_0 : Y_1]) \mid C_0X_0^{r_1}Y_0^{r_2} - C_1X_1^{r_1}Y_1^{r_2} = 0\},
\]
for all \((r_1, r_2) \in \mathbb{Z} \times \mathbb{Z} \setminus \{(0, 0)\}\) and all \((C_0, C_1) \in \mathbb{C} \times \mathbb{C} \setminus \{(0, 0)\}\), and assume that there exists no \((a, b) \in \mathbb{C} \times \mathbb{C} \setminus \{(0, 0)\}\) such that
\[
a \log f_1 + b \log f_2 = (\text{constant}),
\]
on \(\mathbb{C}\). Then it follows that
\[
T_f(r, [\tilde{D}]) \leq N_2(r, f^*D') + N_2(r, f^*D'') + S_f(r).
\]

**Proof.** Because \(N_2(r, \tilde{f}^*\tilde{H}_{i,j}) = 0\) and \(N_2(r, \tilde{f}^*\tilde{E}_i) = 0\), we have the corollary. □

**Example 2.** Let \(D', D''\) be the divisor which are defined by the polynomials
\[
X_0Y_0 - X_1Y_1, \quad X_0Y_1 - X_1Y_0.
\]

Then
\[
D'_1 + D''_1 + \sum_{i=1}^{2} \sum_{j=0}^{1} H_{i,j,1} + E_1,
\]
is normal crossing in \(Z_1\). Therefore \(\tilde{D}' = D'_1, \tilde{D}'' = D''_1\). Let \(E_{(0,0)}, E_{(0,\infty)}, E_{(\infty,0)}, E_{(\infty,\infty)}\) be irreducible components of \(E_1\) such that
\[
\pi_{1,0}(\text{supp } E_{(0,0)}) = ([0 : 1], [0 : 1]), \quad \pi_{1,0}(\text{supp } E_{(0,\infty)}) = ([0 : 1], [1 : 0]),
\]
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\[ \pi_{1,0}(\text{supp } E_{(\infty,0)}) = ([1 : 0], [0 : 1]), \quad \pi_{1,0}(\text{supp } E_{(\infty,\infty)}) = ([1 : 0], [1 : 0]). \]

Let \( f = (f_1, f_2) : \mathbb{C} \to Z_0 \) be a non-constant holomorphic map, and let \( \tilde{f} : \mathbb{C} \to Z_1 \) be the lift of \( f \). It follows that

\[ T_f(r, [\tilde{D}']) = T_f(r, [\pi_{1,0}^* D']) - T_f(r, [E_{(0,\infty)}]) - T_f(r, [E_{(\infty,0)}]), \]

and

\[ T_f(r, [\pi_{1,0}^* D']) = T_f(r, O(1, 1)) = T(r, f_1) + T(r, f_2), \]

where

\[ T(r, f_i) = \int_{|z|=r} \log^+ |f_i| \frac{d\theta}{2\pi} + N(r, (f_i)_\infty), \]

for \( i = 1, 2 \). By the first main theorem, we have

\[ T_f(r, E_{(0,\infty)}) = N(r, \tilde{f}^* E_{(0,\infty)}) + m_f(r, E_{(0,\infty)}), \]

\[ T_f(r, E_{(\infty,0)}) = N(r, \tilde{f}^* E_{(\infty,0)}) + m_f(r, E_{(\infty,0)}). \]

It holds that

\[ m_f(r, E_{(0,\infty)}) = \int_{|z|=r} \log^+ \frac{1}{\sqrt{|f_1|^2 + |f_2^{-1}|^2}} \frac{d\theta}{2\pi}, \]

and

\[ m_f(r, E_{(\infty,0)}) = \int_{|z|=r} \log^+ \frac{1}{\sqrt{|f_1^{-1}|^2 + |f_2|^2}} \frac{d\theta}{2\pi}. \]

By these equations, we have

\[ T_f(r, \tilde{D}') = N(r, (f_1)_\infty) + N(r, (f_2)_\infty) - N(r, \tilde{f}^* E_{(0,\infty)}) - N(r, \tilde{f}^* E_{(\infty,0)}) \]

\[ + \int_{|z|=r} (\log^+ |f_1| + \log^+ |f_2|) \frac{d\theta}{2\pi} \]

\[ - \int_{|z|=r} \left( \log^+ \frac{1}{\sqrt{|f_1|^2 + |f_2^{-1}|^2}} + \log^+ \frac{1}{\sqrt{|f_1^{-1}|^2 + |f_2|^2}} \right) \frac{d\theta}{2\pi}. \]
Let \( f_1 = P(z), f_2 = \exp z \), where \( P(z) \) is a polynomial of degree \( p \) on \( \mathbb{C} \). Then \( T(r, f_1) = p \log r + O(1) \), and \( T(r, f_2) = |r| + O(1) \). Because
\[
\log^+ \frac{1}{\sqrt{|f_1|^2 + |f_2^{-1}|^2}} \leq \log^+ \frac{1}{|f_1|},
\]
it follows that
\[
m_f(r, E_{(0,\infty)}) \leq T(r, f_1^{-1}) = T(r, f_1) + O(1) = p \log |r| + O(1).
\]
Therefore we have
\[
m_f(r, E_{(0,\infty)}) = o(r).
\]
By the same arguments, we have
\[
m_f(r, E_{(\infty,0)}) = o(r).
\]
Then it holds that
\[
T_f(r, \bar{D}') = r + o(r).
\]

Let \( D' \) and \( D'' \) be divisors on \( \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C}) \) which are defined by the polynomials
\[
X_0^n Y_0^m - X_1^n Y_1^m, \quad X_0^n Y_1^m - X_1^n Y_0^m.
\]
(i.e., \( m = m' = n'' \) and \( n = n' = m'' \).) We have the following theorem.

**Theorem 3.** Let \( f : \mathbb{C} \to \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C}) \) be a non-constant holomorphic map. Let \( \tilde{f} : \mathbb{C} \to \mathbb{Z}_k \) be the lift of \( f \). Assume that
\[
f(\mathbb{C}) \not\subset \{([X_0 : X_1], [Y_0 : Y_1]) \in \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C}) | C_0 X_0^{r_1} Y_0^{r_2} - C_1 X_1^{r_1} Y_1^{r_2} = 0 \},
\]
for all \( (r_1, r_2) \in \mathbb{Z} \times \mathbb{Z} \setminus \{(0,0)\} \) and all \( (C_0, C_1) \in \mathbb{C} \times \mathbb{C} \setminus \{(0,0)\}, \) and assume that there exist no holomorphic functions \( g_1, g_2 \) on \( \mathbb{C} \) and no \( (a, b) \in \mathbb{C} \times \mathbb{C} \setminus \{(0,0)\} \) such that
\[
f = (\exp g_1, \exp g_2) : \mathbb{C} \to \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C}),
\]
\[ ag_1 + bg_2 = \text{(constant)}, \]
on \mathbb{C}. Then it follows that

\[
\left(1 - \frac{4}{m+n}\right) T_f(r, \left[\widetilde{D}' + \widetilde{D}''\right]) \leq N_2(r, \widetilde{f}' \widetilde{D}') + N_2(r, \widetilde{f}' \widetilde{D}'') + S_f(r).
\]

**Proof.** Let \( a_1 = \min\{m, n\} \). It follows that

\[
\pi_{1,0}^*(D' + D'') = D'_1 + D''_1 + a_1 E_1,
\]
on \mathbb{Z}_1, where \( D'_1 \) and \( D''_1 \) are proper transforms of \( D' \) and \( D'' \) under \( \pi_{1,0} \).

Let \( a_2 = \min\{\max\{m, n\} - a_1, a_1\} \leq a_1 \). It follows that

\[
\pi_{2,0}^*(D' + D'') = D'_2 + D''_2 + a_2 E_2 + a_1 \pi_{2,1} E_1,
\]
on \mathbb{Z}_2, where \( D'_2 \) and \( D''_2 \) are proper transforms of \( D' \) and \( D'' \) under \( \pi_{2,0} \).

Repeating this process, there exist positive integers \( a_3, \ldots, a_k \) such that

\[
\pi_{k,0}^*(D' + D'') = \widetilde{D}' + \widetilde{D}'' + \sum_{i=1}^{k} a_i \pi_{k,i} E_i.
\]

Without loss of generality, we may assume that \( m \leq n \). Then it holds that \( m \geq a_1 \geq a_2 \geq \cdots \geq a_k \). It follows that

\[
T_f(r, \left[\widetilde{D}' + \widetilde{D}''\right]) \geq T_f(r, \pi_{k,0}^*\mathcal{O}(m + n, m + n)) - m \sum_{i=1}^{k} T_f(r, \pi_{k,i} E_i).
\]

By Lemma 7, we have

\[
T_f(r, \pi_{k,0}^*\mathcal{O}(2, 2)) = \sum_{i=1}^{2} \sum_{j=0}^{1} T_f(r, \tilde{H}_{i,j}) + \sum_{i=1}^{k} T_f(r, \pi_{k,i} E_i) + \sum_{i=1}^{k} T_f(r, \tilde{E}_i).\]

Then we have

\[
T_f(r, \left[\widetilde{D}' + \widetilde{D}''\right]) \\
\geq \frac{m + n}{2} \left( T_f(r, \pi_{k,0}^*\mathcal{O}(2, 2)) - \sum_{i=1}^{k} T_f(r, \pi_{k,i} E_i) \right).
\]
\begin{equation}
+ \left( \frac{m + n}{2} - m \right) \sum_{i=1}^{k} T_f(r, \pi_{k,i}^* E_i)
\end{equation}

\begin{equation}
\geq \frac{m + n}{2} \left( \sum_{i=1}^{2} \sum_{j=0}^{1} T_f(r, \tilde{H}_{i,j}) + \sum_{i=1}^{k} T_f(r, \tilde{E}_i) \right).
\end{equation}

By Theorem 1, it follows that

\begin{equation}
T_f(r, [\tilde{D}' + \tilde{D}'']) \leq N_2(r, f^* \tilde{D}') + N_2(r, f^* \tilde{D}'') + \frac{4}{m + n} T_f(r, [\tilde{D}' + \tilde{D}'']) + S_f(r).
\end{equation}

Then the theorem follows. □

**Corollary 2.** Assume the hypothesis of Theorem 3, and assume that

\[ f(\mathbb{C}) \subset \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C}) \setminus \text{supp}(D' + D''). \]

If \( m + n \geq 5 \), then it follows that \( f(\mathbb{C}) \subset \text{supp} H_{i,j} \) for \( i = 1, 2 \) and \( j = 0, 1 \).

**Proof.** Assume that \( f(\mathbb{C}) \) is not contained in the support of \( \sum_{i=1}^{2} \sum_{j=0}^{1} H_{i,j} \). By Theorem 3, \( f \) satisfies the following condition (i) or condition (ii):

(i) \( f(\mathbb{C}) \subset \{([X_0 : X_1], [Y_0 : Y_1]) \in \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C}) | X_0^{r_1}Y_0^{r_2} - C_1X_1^{r_1}Y_1^{r_2} = 0 \}, \)

for some \( (r_1, r_2) \in \mathbb{Z} \times \mathbb{Z} \setminus \{(0,0)\} \) and some \( C_1 \in \mathbb{C} \setminus \{(0)\} \).

(ii) There exist holomorphic functions \( g_1, g_2 \) on \( \mathbb{C} \) and \( (a, b) \in \mathbb{C} \times \mathbb{C} \setminus \{(0,0)\} \) such that

\[ f = (\exp g_1, \exp g_2) : \mathbb{C} \to \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C}), \]

\[ ag_1 + bg_2 = \text{(constant)}, \]

on \( \mathbb{C} \).

If \( f \) satisfies condition (i), without loss of generality, we may assume that \( r_1 > 0, r_2 \geq 0 \). Assume that \( r_2 > 0 \). Let \( R \) be an irreducible component
of \( \{X_0^{r_1}Y_0^{r_2} - CX_1^{r_1}Y_1^{r_2} = 0 \} \). Then \(([0 : 1],[1 : 0]),([1 : 0],[0 : 1]) \in \text{supp} R \cap \text{supp} D' \) and \( \text{supp} R \cap \text{supp} D'' \) contains at least one point which is not \(([0 : 1],[1 : 0]) \) nor \(([1 : 0],[0 : 1]) \). Therefore the holomorphic map

\[
    f : \mathbb{C} \to \text{supp} R \setminus \text{supp} (D' + D'')
\]

is a constant map.

Assume that \( r_2 = 0 \). We have

\[
    f(\mathbb{C}) \subset \{(X_0 : X_1), (Y_0 : Y_1) \in \mathbb{P}(\mathbb{C}) \times \mathbb{P}(\mathbb{C}) | X_0^{r_1} - CX_1^{r_1} = 0 \}.
\]

Let \( S \) be an irreducible component of \( \{X_0^{r_1} - CX_1^{r_1} = 0 \} \). Because \( m+n \geq 5 \), \( m \) or \( n \) is more than 2, it follows that \( \text{supp} S \cap \text{supp} D' \) or \( \text{supp} S \cap \text{supp} D'' \) contains at least three points. Then \( f \) is a constant map.

If \( f \) satisfies condition (ii), it is easy to see that \( f \) is a constant map. □

**Remark 2.** Let \( x_{1,0} = ([0 : 1],[1 : 1]), x_{1,1} = ([1 : 0],[1 : 1]), x_{2,0} = ([1 : 1],[0 : 1]), x_{2,1} = ([1 : 1],[1 : 0]) \in Z_0 = \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C}) \). Let \( W = Z_0 \setminus \text{supp} D' \cup \text{supp} D'' \), and let \( W^* = W \setminus \{x_{1,0}, x_{1,1}, x_{2,0}, x_{2,1}\} \). By Corollary 2, there exist no non-constant holomorphic maps from \( \mathbb{C} \) to \( W^* \).

Let \( i : W^* \to W \) be the inclusion map, and let \( d_{W^*}, d_W \) be the Kobayashi pseudo distance of \( W^*, W \) (see Noguchi-Ochiai [4]). By Proposition 1.3.14. of [4], we have \( i^*d_W = d_{W^*} \). Therefore \( W^* \) is Brody hyperbolic but not Kobayashi hyperbolic.

**References**


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Graduate School of Mathematical Sciences
University of Tokyo
3-8-1 Komaba, Meguro-ku
Tokyo 153-8914, Japan
E-mail: chiba@ms.u-tokyo.ac.jp