Smooth Right Quasigroup Structures on 1-Manifolds

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Abstract. Smooth loop structures on one-manifolds for which the groups topologically generated by right translations are locally compact, are known. In this article we study smooth right loop structures on one-manifolds.

1. Introduction

The smooth loop (quasigroup) structures on one dimensional manifolds are completely classified. One dimensional manifolds are (i) \( \mathbb{R} \), (ii) \( S^1 \) and (iii) Alexandroff half line [1, 2], [3], page 235. But on the Alexandroff half line, no smooth quasigroup structure can be defined. However on \( \mathbb{R} \) and \( S^1 \) we have smooth quasigroup structures [2, 3] which are obtained by looking at copies of \( S^1 \) appearing as transversals to 2-dimensional subgroups of \( PSL(2, \mathbb{R}) \) and that on \( \mathbb{R} \) appear as transversals to special type of subgroups in the universal cover of \( PSL(2, \mathbb{R}) \). This paper is devoted mainly to study smooth right quasigroup structures on \( \mathbb{R} \) and \( S^1 \) obtained by deformation of their group structures.

2. Preliminaries

Let \( S \) be a nonempty set and \( \circ \) be a binary operation on \( S \). Then the groupoid \( (S, \circ) \) is called a right quasigroup if for all \( x, y \in S \), the equation \( X \circ x = y \), where \( X \) is unknown in the equation, has a unique solution in \( S \). If there exists \( e \in S \) such that \( e \circ x = x = x \circ e \) for all \( x \in S \), then \( (S, \circ) \) is called a right quasigroup with identity also called a right loop. A right quasigroup (right loop) \( (S, \circ) \) is called a quasigroup (loop) if the equation \( x \circ X = y \), where \( X \) is unknown in the equation and \( x, y \in S \), has a unique solution in \( S \). Throughout the paper a right quasigroup will always be assumed to contain the identity.

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Let $S$ be a fixed right transversal to a subgroup $H$ in a group $G$. Then every right transversal to $H$ in $G$ determines and is determined uniquely by a map $g : S \rightarrow H$ such that $g(e) = e$, the identity of $G$ (see, [5]). The right transversal $S_g$ determined by a map $g : S \rightarrow H$ is given by $S_g = \{g(x)x \mid x \in S\}$. $S$ and $S_g$ are right quasigroups with identities with respect to the operation $o$ on $S$ and $o'$ on $S_g$ given by

$$\{xoy\} = Hxy \cap S$$

and

$$\{g(x)xo'g(y)y\} = S_g \cap Hg(x)xg(y)y$$

respectively. Further, $H$ acts on $S$ from right through an action $\theta$ given by $\{x\theta h\} = Hxh \cap S, \forall x \in S, h \in H$

**Proposition 2.1.** [5] The right quasigroup $(S_g, o')$ is isomorphic to the right quasigroup $(S, o_g)$ where $o_g$ is the operation given by $x o_g y = x\theta g(y) o y$.

The operation $o_g$ will be termed as the deformation of $o$ through the map $g : S \rightarrow H$.

**Definition 2.2.** Let $H$ be a subgroup of a group $G$. Then a map $s : G/H \rightarrow G$ is called a section if $s(H) = e$, the identity of group $G$ and $\nu s = 1$ where $\nu$ denotes the quotient map given by $\nu(x) = Hx$.

If $s$ is a section then the image $s(G/H)$ is a right transversal to $H$ in $G$. Conversely every right transversal $S$ determines a section $s$ given by $\{s(Hx)\} = S \cap Hx$.

**Proposition 2.3.** [5] Let $H$ be a closed subgroup of a topological group $G$ and $S$ a right transversal to $H$ in $G$. Suppose that the section $s$ from the quotient space $G/H$ (the set of right cosets of $H$ in $G$) to $G$ given by $\{s(Hg)\} = S \cap Hg$ is continuous. Then the binary operation $o$ on $S$ given by $\{xoy\} = S \cap Hxy$ and the map $\chi : S \times S \rightarrow S$ given by $\chi(x, y) o x = y$ are continuous (Here $S$ is given the subspace topology).
Proposition 2.4. [5] Let $S$ be a right transversal to a closed subgroup $H$ of a topological group $G$ which is the image of a continuous section $s : G/\tau H \to G$. Then the right transversal $S_g = \{g(x)x \mid x \in S\}$ corresponding to the map $g : S \to H$ with $g(e) = e$ is the image of a continuous section if and only if $g$ is continuous from $S$ (with subspace topology) to $H$.

3. Main Results

The main results of the paper are:

Theorem 3.1. The circle group $\mathbb{S}^1 = \left\{ \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \right\} 0 \leq t < \pi$ is a right transversal to the subgroup $H = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \right\} |a > 0, b \in \mathbb{R}$ in the projective special linear group $PSL(2, \mathbb{R})$. Let $g$ be a map from $\mathbb{S}^1$ to $H$ given by

\[ g \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} = \begin{pmatrix} u(t) & v(t) \\ 0 & (u(t))^{-1} \end{pmatrix} \]

where $u : [0, \pi) \to \mathbb{R} \setminus \{0\}$ and $v : [0, \pi) \to \mathbb{R}$ are smooth maps with $u(0) = 1$, $v(0) = 0$. Then $(\mathbb{S}^1, o_g)$ is a smooth right quasigroup with identity where $o_g$ is given by

\[ \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} o_g \begin{pmatrix} \cos s & \sin s \\ -\sin s & \cos s \end{pmatrix} = \begin{pmatrix} \cos(t_1 + s) & \sin(t_1 + s) \\ -\sin(t_1 + s) & \cos(t_1 + s) \end{pmatrix} \]

and

\[ t_1 = \tan^{-1} \left( \frac{u(s) \sin t}{u(s)^{-1} \cos t - v(s) \sin t} \right). \]
Proof. A two dimensional subgroup of $PSL(2, \mathbb{R})$ is a conjugate of the subgroup

$$H = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \mid a > 0, b \in \mathbb{R} \right\}.$$ 

The circle group $S^1$ is given by $S^1 = \left\{ \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \mid 0 \leq t < 2\pi \right\}$. Clearly $S^1 \subseteq SL(2, \mathbb{R})$. The subgroup $S^1/\{I, -I\}$ of the group $PSL(2, \mathbb{R}) = SL(2, \mathbb{R})/\{I, -I\}$ is again a circle group which is given by

$$S^1 = \left\{ \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \mid 0 \leq t < \pi \right\}.$$ 

We claim that $S^1$ is a right transversal to the subgroup $H$ in the group $PSL(2, \mathbb{R})$. To prove this it is sufficient to prove that $PSL(2, \mathbb{R}) = HS^1$ and $H \cap S^1 = \{\bar{I}\}$.

Let $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL(2, \mathbb{R})$. Then $ad - bc = 1$. Thus $c$ and $d$ will not be zero simultaneously. If $c \neq 0$. Without loss we may assume that $c > 0$. Then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} u(t) & v(t) \\ 0 & (u(t))^{-1} \end{pmatrix} \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}$$

where $u(t) = a \cos t + b \sin t$, $v(t) = -a \sin t + b \cos t$ and $t = \tan^{-1} \left( \frac{-c}{d} \right)$. If $c = 0$ then $d = a^{-1}$. In this case,

$$\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} = \begin{cases} \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \text{if } a > 0 \\ \begin{pmatrix} -a & -b \\ 0 & -a^{-1} \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} & \text{if } a < 0 \end{cases}$$

Thus $PSL(2, \mathbb{R}) = HS^1$. Clearly $H \cap S^1 = \{\bar{I}\}$. This shows that $S^1$
is a right transversal to $H$ in $PSL(2, \mathbb{R})$. The induced right quasigroup structure $o$ on $S^1$ is given by

\[
\begin{pmatrix}
\cos t & \sin t \\
-\sin t & \cos t
\end{pmatrix}
\begin{pmatrix}
\cos s & \sin s \\
-\sin s & \cos s
\end{pmatrix}
\begin{pmatrix}
\cos(t+s) & \sin(t+s) \\
-\sin(t+s) & \cos(t+s)
\end{pmatrix}
= \begin{pmatrix}
\cos(t+s) & \sin(t+s) \\
-\sin(t+s) & \cos(t+s)
\end{pmatrix}
\]

and the action $\theta$ of $H$ on $S^1$ is given by

\[
\begin{pmatrix}
\cos t & \sin t \\
-\sin t & \cos t
\end{pmatrix}
\begin{pmatrix}
u(s) & v(s) \\
0 & (u(s))^{-1}
\end{pmatrix}
\begin{pmatrix}
\cos t_1 & \sin t_1 \\
-\sin t_1 & \cos t_1
\end{pmatrix}
\]

where $t_1$ is given by Eq.(3).

Let $g : S^1 \to H$ be a map given by Eq.(1). Since $u, v$ are smooth, therefore $g$ is smooth. By Eq.(4), Eq.(5) and Proposition 2.1, the right quasigroup structure $o_g$ on $S^1$ is given by

\[
\begin{pmatrix}
\cos t & \sin t \\
-\sin t & \cos t
\end{pmatrix}
\begin{pmatrix}
\cos s & \sin s \\
-\sin s & \cos s
\end{pmatrix}
\begin{pmatrix}
\cos(t_1+s) & \sin(t_1+s) \\
-\sin(t_1+s) & \cos(t_1+s)
\end{pmatrix}
\]

where $t_1$ is given by Eq.(3). By Proposition 2.4, the operation $o_g$ is smooth. □

Remark 3.2. In general $(S^1, o_g)$ need not be a loop and it may not have right inverse property. For example, if we take $u$ and $v$ given by $u(t) = \sin \frac{t}{2} + \cos \frac{t}{2}$ and $v(t) = \sin t \cos t$ then \[
\begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix}
\in S^1
\]
has no right
inverse. However for certain \( g \), for example

\[
g \left( \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \right) = \begin{pmatrix} 1 & \frac{1}{2}t \sin t \\ 0 & 1 \end{pmatrix},
\]

\((S^1, o_g)\) is indeed a loop.

Consider the additive group of real numbers. Since a differentiable function can be approximated by a polynomial function. In this light, we establish the following

**Proposition 3.3.** Let \( n \geq 3 \). Then a polynomial \( P_n(x, y) = \sum_{i=0}^{n} a_i y^i + x \sum_{j=0}^{n-1} b_j y^j \) of degree \( n \) in two variables \( x \) and \( y \) over the field \( \mathbb{R} \) determines a smooth right quasigroup structure \( o \) with identity 0 given by \( xoy = P_n(x, y) \) if and only if \( P_n(x, y) = x + y + x\phi(y) \) where \( \phi(y) \) is a polynomial in \( y \) of degree \( n - 1 \) such that \( \phi(a) \neq -1 \) for every \( a \in \mathbb{R} \) and \( \phi(0) = 0 \). In particular \( P_{2n+1}(x, y) = x + y + xy^{2n} \) where \( n \in \mathbb{N} \) determines a smooth right quasigroup structure on \( \mathbb{R} \).

**Proof.** Suppose that \( n \geq 3 \) and \( P_n(x, y) = \sum_{i=0}^{n} a_i y^i + x \sum_{j=0}^{n-1} b_j y^j \). Suppose that the structure \( o \) given by \( xoy = P_n(x, y) \) is a right quasigroup structure with identity 0. Then \( 0o0 = 0 \) and \( xo0 = x = 0ox \) implies that

\[
a_0 = 0, \quad a_1 = b_0 = 1, \quad a_i = 0, \; \forall \; i \geq 2
\]

Thus \( P_n(x, y) = x + y + x\phi(y) \), where \( \phi(y) = \sum_{j=1}^{n-1} b_j y^j \) is a polynomial of degree \( n - 1 \) such that \( \phi(0) = 0 \). Consider the equation \( Xoa = 0 \) where \( a \in \mathbb{R} \) and \( X \) is unknown in the equation. Then the existence of solution to the equation implies that \( \phi(a) \neq -1 \).

Conversely, suppose that \( P_n(x, y) = x + y + x\phi(y) \) and \( xoy = P_n(x, y) \), where \( \phi(y) \) is a polynomial of degree \( n - 1 \) such that \( \phi(a) \neq -1 \) for every \( a \in \mathbb{R} \) and \( \phi(0) = 0 \). Let \( a, b \in \mathbb{R} \). Then \( U = \frac{b-a}{1+\phi(a)} \) is the solution to the equation \( Xoa = b \), where \( X \) is unknown in the equation. Also \( xo0 = x = 0ox \). The smoothness of polynomial function implies the smoothness of \( o \). Thus \((\mathbb{R}, o)\) is a smooth right quasigroup with identity. In particular
if \( \phi(y) = y^{2n}, n \in \mathbb{N} \) then \( \phi(a) = a^{2n} \geq 0 \) for every \( a \in \mathbb{R} \). Thus the result follows. □

More generally we have the following proposition:

**Proposition 3.4.** Let \( \phi(y) \) be a differentiable function on \( \mathbb{R} \) such that \( \phi(y) \neq -1 \), for all \( y \in \mathbb{R} \) and \( \phi(0) = 0 \). Then \( (\mathbb{R}, o) \) is a smooth right quasigroup with identity \( 0 \), where \( xoy = x + y + x\phi(y) \).

**Remark 3.5.** The right quasigroup \( (\mathbb{R}, o) \) where \( xoy = x + y + xy^{2n} \) does not have right inverse property for \( 1 \) has no right inverse.

Smooth right quasigroup structures on \( \mathbb{R} \) are obtained also in the following manner:

**Theorem 3.6.** Let \( \phi: \mathbb{R} \rightarrow \mathbb{R} \setminus \{0\} \) be a differentiable map such that \( \phi(0) = 1 \). Then \( (\mathbb{R}, \circ) \) is a smooth right quasigroup with identity \( 0 \), where \( \circ \) is given by \( x \circ y = y + x[\phi(y)]^{-2} \).

**Proof.** Consider the Borel subgroup \( B = \left\{ \begin{pmatrix} \alpha & \beta \\ 0 & \alpha^{-1} \end{pmatrix} \right\} \alpha, \beta \in \mathbb{R}, \alpha > 0 \right\} \) of the projective special linear group \( PSL(2, \mathbb{R}) \). Then \( B = HS, H \cap S = \{I_2\} \) where

\[
H = \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} \right\} \alpha \in \mathbb{R}, \alpha > 0 \right\} \quad \text{and} \quad S = \left\{ \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} \right\} \beta \in \mathbb{R} \right\}.
\]

Let \( A_\alpha = \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix} \), \( \forall \alpha \in \mathbb{R} \). Then \( (S, o) \) is a right quasigroup with identity \( \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \) where \( o \) is given by \( A_\alpha o A_\beta = A_{\alpha+\beta} \). Let \( g: S \rightarrow H \) be a deformation map given by \( g(A_\alpha) = \begin{pmatrix} \phi(\alpha) & 0 \\ 0 & \phi(\alpha)^{-1} \end{pmatrix} \) where \( \phi: \mathbb{R} \rightarrow \mathbb{R} \).
\( \mathbb{R} \setminus \{0\} \) is a differentiable map such that \( \phi(0) = 1 \). The induced right quasigroup structure \( o_g \) on \( S \) is given by

\[
A_{\alpha}o_g A_{\beta} = S \cap H[g(A_{\alpha})A_{\alpha}g(A_{\beta})A_{\beta}].
\]

Since,

\[
g(A_{\alpha})A_{\alpha}g(A_{\beta})A_{\beta} = g(A_{\alpha})g(A_{\beta}) \begin{pmatrix} 1 & \beta + \alpha\phi(\beta)^{-2} \\ 0 & 1 \end{pmatrix}.
\]

Therefore \( A_{\alpha}o_g A_{\beta} = A_{\beta + \alpha\phi(\beta)^{-2}} \). Clearly the map \( \psi : \mathbb{R} \rightarrow S \) defined by \( \psi(\alpha) = A_{\alpha} \) is bijective. This in turn induces the operation \( \circ \) on \( \mathbb{R} \) given by \( \alpha \circ \beta = \beta + \alpha\phi(\beta)^{-2} \) so that \( \psi \) is an isomorphism. Clearly the operation \( \circ \) on \( \mathbb{R} \) is smooth. \( \square \)

**Remark 3.7.** The right quasigroup structure \( o \) on \( \mathbb{R} \) given by \( xoy = x + y + x\phi(y) \), is determined by the structure \( \circ \) on \( \mathbb{R} \) given by \( x \circ y = y + x[\psi(y)]^{-2} \), where \( \psi(y) = \frac{1}{\sqrt{1 + \phi(y)}} \) and \( \phi \) is a differentiable map from \( \mathbb{R} \) into \((-1, \infty)\) such that \( \phi(0) = 0 \). Further, none of the right loops on \( \mathbb{R} \) are loops except the trivial one which is the additive group of real numbers.

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**References**


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