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# *Solvability of Difference Riccati Equations by Elementary Operations*

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**Abstract.** We generalize Franke's generalized Liouvillian extension and Karr's $\Pi\Sigma$-extension, and study solvability of difference Riccati equations. We define the difference field extensions of valuation ring type and prove the following. If a difference Riccati equation which does not turn out to be linear by iterations has a solution in some difference field extension of valuation ring type, then one of the iterated Riccati equations has an algebraic solution. Applying this theorem, we conclude unsolvability of the $q$-Airy equation and the $q$-Bessel equation.

## 1. Introduction

It is well-known that the Airy equation and the Bessel equation with the parameter $\nu$ satisfying $\nu - \frac{1}{2} \notin \mathbb{Z}$ are unsolvable. The $q$-analogues of them, $q$-Airy equation and $q$-Bessel equation respectively, are defined, but their unsolvability has not been investigated. In this paper, we obtain the following results: the $q$-Airy equation and $q$-Bessel equation with the parameter $\nu \in \mathbb{Q}$ are unsolvable.

*Notation.* Throughout the paper every field is of characteristic zero. When $K$ is a field and $\tau$ is an isomorphism of $K$ into itself, namely an injective endomorphism, the pair $\mathcal{K} = (K, \tau)$ is called a difference field. For $a \in K$ and $n \in \mathbb{Z}$, the element $\tau^n a \in K$ is called the $n$-th transform of $a$ and is denoted by $a_n$ if it exists. If $\tau K = K$, we say that $\mathcal{K}$ is inversive. For difference fields $\mathcal{K} = (K, \tau)$ and $\mathcal{K}' = (K', \tau')$, $\mathcal{K}'/\mathcal{K}$ is called a difference field extension if $K'/K$ is a field extension and $\tau'|_K = \tau$. In this case, $\mathcal{K}'$ is called a difference overfield of $\mathcal{K}$ and $\mathcal{K}$ a difference subfield of $\mathcal{K}'$. A solution of a difference equation over $\mathcal{K}$ is defined to be an element of some difference overfield of $\mathcal{K}$ which satisfies the equation. There exists a

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difference overfield $\overline{\mathcal{K}} = (\overline{K}, \overline{\tau})$ of $\mathcal{K} = (K, \tau)$ such that $\overline{K}$ is an algebraic closure of $K$. We call $\overline{\mathcal{K}}$ an algebraic closure of $\mathcal{K}$ (cf. [2, 9]).

In [3, 4] Franke studied the solvability of linear homogeneous difference equations by elementary operations using the notion of $q$LE. A difference field extension $\mathcal{N}/\mathcal{K}$ is called a $q$LE ($q \in \mathbb{Z}_{>0}$) if there exists a chain of inversive difference fields,

$$\mathcal{K} = \mathcal{K}_0 \subset \mathcal{K}_1 \subset \cdots \subset \mathcal{K}_n = \mathcal{N} = (N, \tau), \quad K_i = K_{i-1}(\{\tau^k a_i \mid k \in \mathbb{Z}\}),$$

where $a_i$ satisfies one of the following.

(i) $\tau^q a_i = a_i + \beta$ for some $\beta \in K_{i-1}$.

(ii) $\tau^q a_i = \alpha a_i$ for some $\alpha \in K_{i-1}$.

(iii) $a_i$ is algebraic over $K_{i-1}$.

When $q = 1$, $q$LE is called a generalized Liouvillian extension (GLE). For any $q$LE $(N, \tau)/(K, \tau)$, the extension $(N, \tau^q)/(K, \tau^q)$ is a GLE (see [4]).

In [8] Karr defined $\Pi\Sigma$-extensions, and obtained results on the computation of symbolic solutions to first order linear difference equations and an analogue to Liouville's theorem on elementary integrals. Any $\Pi\Sigma$-extension is a difference subfield of a GLE.

Here we introduce a new notion of difference field extension.

Definition 1 (difference field extensions of valuation ring type). Let $\mathcal{N}/\mathcal{K}$ be a difference field extension, and $\mathcal{N} = (N, \tau)$. We say $\mathcal{N}/\mathcal{K}$ is a *difference field extension of valuation ring type* if there is a chain of difference fields,

$$\mathcal{K} = \mathcal{K}_0 \subset \mathcal{K}_1 \subset \cdots \subset \mathcal{K}_{n-1} \subset \mathcal{K}_n = \mathcal{N},$$

such that for each $1 \leq i \leq n$ the extension $\mathcal{K}_i/\mathcal{K}_{i-1}$ satisfies one of the following.

(i) The extension $K_i/K_{i-1}$ is algebraic.

(ii) $\mathcal{K}_i$ and $\mathcal{K}_{i-1}$ are inversive, $K_i/K_{i-1}$ is an algebraic function field of one variable, and there is a valuation ring $\mathcal{O}$ of $K_i/K_{i-1}$ such that $\tau^j \mathcal{O} \subset \mathcal{O}$ for some $j \in \mathbb{Z}_{>0}$.

The idea to use valuation rings for investigating differential equations originated with Rosenlicht (cf. [10]). For algebraic function fields of one variable, refer to [7, 11], for example. In section 3 we prove that any GLE is of valuation ring type.

If a difference equation has no solution in any $q$LE of $\mathcal{K}$, then we say that it is unsolvable over $\mathcal{K}$. Since $q$LE is of valuation ring type for $\tau^q$, roughly speaking, nonexistence of solutions in a difference field extension of valuation ring type implies unsolvability of the difference equation.

In section 2 we prove

THEOREM 2. *Let $\mathcal{K} = (K, \tau_K)$ be a difference field, and $a, b, c, d \in K$. Define*

$$A = A_1 = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \textit{and} \quad A_i = (\tau_K A_{i-1})A = \begin{pmatrix} a^{(i)} & b^{(i)} \\ c^{(i)} & d^{(i)} \end{pmatrix}, \quad i \geq 2.$$

*Suppose $b^{(i)} \neq 0$ and $c^{(i)} \neq 0$ for all $i \geq 1$. Let $k \geq 1$, and suppose the equation over $\mathcal{K}$, $y_k(c^{(k)}y + d^{(k)}) = a^{(k)}y + b^{(k)}$ has a solution in a difference field extension $\mathcal{N}/\mathcal{K}$ of valuation ring type. Let $\overline{\mathcal{N}}$ be an algebraic closure of $\mathcal{N}$ and $\overline{\mathcal{K}}$ the algebraic closure of $\mathcal{K}$ in $\overline{\mathcal{N}}$. Then there exists $i \geq 1$ such that the equation over $\mathcal{K}$, $y_{ki}(c^{(ki)}y + d^{(ki)}) = a^{(ki)}y + b^{(ki)}$, has a solution in $\overline{\mathcal{K}}$.*

REMARK. We call equations of the form, $y_1(cy+d) = ay+b$, difference Riccati equations.

In section 3 we prove that the $q$-Airy equation and the $q$-Bessel equation with the parameter $\nu \in \mathbb{Q}$ have no algebraic solutions. Then, applying the theorem, we obtain unsolvability of these equations.

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## 2. Proof of Theorem

The following lemma is easily proved by induction.

LEMMA 3. *Let $\mathcal{L}/\mathcal{K}$ be a difference field extension, $\mathcal{L} = (L, \tau)$, and $a, b, c, d \in K$. Define the matrices $A_i$ as in Theorem 2. Let $k \geq 1$. Then we*

*have the following.*
(a) $A_i = (\tau^{i-1}A)(\tau^{i-2}A)\dots(\tau A)A$.
(b) *Define the matrices* $B = B_1 = A_k$, $B_i = (\tau^k B_{i-1})B$ $(i \geq 2)$. *Then* $B_i = A_{ki}$.
(c) *Let* $f \in \mathcal{L}$ *be a solution of* $y_k(c^{(k)}y + d^{(k)}) = a^{(k)}y + b^{(k)}$. *Then* $f \in \mathcal{L}$ *is a solution of* $y_{ki}(c^{(ki)}y + d^{(ki)}) = a^{(ki)}y + b^{(ki)}$ *for all* $i \geq 1$.

LEMMA 4. *Let* $\mathcal{L}/\mathcal{K}$ *be a difference field extension, both* $\mathcal{L} = (L, \tau_L)$ *and* $\mathcal{K}$ *inversive, and* $L/K$ *an algebraic function field of one variable. Suppose there exists a valuation ring* $\mathcal{O}$ *of* $L/K$ *such that* $\tau_L^j \mathcal{O} \subset \mathcal{O}$ *for some* $j \in \mathbb{Z}_{>0}$. *Let* $\overline{\mathcal{L}} = (\overline{L}, \tau)$ *be an algebraic closure of* $\mathcal{L}$ *and* $\overline{\mathcal{K}}$ *the algebraic closure of* $\mathcal{K}$ *in* $\overline{\mathcal{L}}$. *Let* $a, b, c, d \in K$, *and define the matrices* $A_i$ *as in Lemma 3. Suppose* $b^{(i)} \neq 0$ *and* $c^{(i)} \neq 0$ *for all* $i \geq 1$, *and the equation over* $\mathcal{K}$, $y_1(cy+d) = ay+b$, *has a solution* $f$ *in* $\overline{\mathcal{L}}$. *Then for some* $i \geq 1$ *the equation over* $\mathcal{K}$, $y_i(c^{(i)}y + d^{(i)}) = a^{(i)}y + b^{(i)}$, *has a solution in* $\overline{\mathcal{K}}$.

PROOF. It is enough to prove this for $f \notin \overline{K}$. The additional assumption implies $cf + d \neq 0$, and so we obtain

$$f_1 = \frac{af + b}{cf + d}.$$

Put $\mathcal{M} = \mathcal{L}\langle f \rangle \subset \overline{\mathcal{L}}$, where the field of $\mathcal{L}\langle f \rangle$ is $L(f, f_1, f_2, \dots)$. We find $\mathcal{M}$ is inversive. In fact, since $cf_1 - a = 0$ implies $f = \tau^{-1}(a/c) \in K$, we have

$$f = -\frac{df_1 - b}{cf_1 - a} = \tau\left(-\frac{\tau^{-1}(d)f - \tau^{-1}b}{\tau^{-1}(c)f - \tau^{-1}a}\right) \in \tau M.$$

As a field, $M = L(f)$ is an algebraic function field of one variable over $K$, and so $M\overline{K}$ is an algebraic function field of one variable over $\overline{K}$.

Choose $j \in \mathbb{Z}_{>0}$ such that $\tau^j \mathcal{O} \subset \mathcal{O}$, and choose valuation ring $\mathcal{O}'$ of $M\overline{K}/\overline{K}$ such that $\mathcal{O}' \cap L = \mathcal{O}$. Note that $\tau^j \mathcal{O} \subset \mathcal{O}$ implies $\tau^j \mathcal{O} = \mathcal{O}$. Therefore for any $i \geq 0$ the following holds.

$$\tau^{ij}\mathcal{O}' \cap L = \tau^{ij}(\mathcal{O}' \cap L) = \tau^{ij}\mathcal{O} = \mathcal{O}.$$

From this we obtain $\#\{\tau^{ij}\mathcal{O}' \mid i \geq 0\} < \infty$, which implies $\tau^k \mathcal{O}' = \mathcal{O}'$ for some $k \geq 1$. Let $v$ be the normalized discrete valuation associated with

$\mathcal{O}'$, and $t \in M\overline{K}$ a prime element of $\mathcal{O}'$. Then we have $v(\tau^k t) = 1$, and so $v(\tau^k x) = v(x)$ for any $x \in M\overline{K}$.

By Lemma 3 we find that $f$ satisfies

$$f_k(c^{(k)}f + d^{(k)}) = a^{(k)}f + b^{(k)}, \tag{1}$$

which yields $v(f) = 0$. In fact, firstly assume $v(f) > 0$. Then we have $v(f_k) = v(f) > 0$. This contradicts $v(f_k) = -v(c^{(k)}f + d^{(k)}) \leq 0$ obtained from the above equation (1). Secondly assume $v(f) < 0$. Then $v(f_k) = v(f) < 0$ contradicts

$$v(f_k) = v(a^{(k)}f + b^{(k)}) - v(f) \geq 0.$$

Let $\phi$ be the embedding of $M\overline{K}$ into $\overline{K}((t))$, and express $f$ and $\tau^k t$ as

$$\phi(f) = \sum_{i=0}^{\infty} h_i t^i, \quad h_i \in \overline{K},\ h_0 \neq 0,$$
$$\phi(\tau^k t) = \sum_{i=1}^{\infty} e_i t^i, \quad e_i \in \overline{K},\ e_1 \neq 0.$$

Then

$$\phi(f_k) = \sum_{i=0}^{\infty} \tau^k(h_i) \left( \sum_{l=1}^{\infty} e_l t^l \right)^i.$$

Note that $\phi$ is a difference isomorphism of $(M\overline{K}, (\tau|_{M\overline{K}})^k)$ into $(\overline{K}((t)), \sigma)$, where $\sigma$ sends $\sum_{i=0}^{\infty} \alpha_i t^i$ to $\sum_{i=0}^{\infty} \tau^k(\alpha_i)(\sum_{l=1}^{\infty} e_l t^l)^i$. Comparing the coefficients of $t^0$ of the equation (1), we obtain

$$\tau^k(h_0)(c^{(k)}h_0 + d^{(k)}) = a^{(k)}h_0 + b^{(k)}.$$

Therefore $h_0 \in \overline{\mathcal{K}}$ is a solution of the equation, $y_k(c^{(k)}y + d^{(k)}) = a^{(k)}y + b^{(k)}$. □

Proof of Theorem 2. We prove this by induction on $\operatorname{tr.deg} N/K$. When $\operatorname{tr.deg} N/K = 0$, the equation, $y_k(c^{(k)}y + d^{(k)}) = a^{(k)}y + b^{(k)}$, has a solution in $\overline{\mathcal{K}}$. Suppose $\operatorname{tr.deg} N/K \geq 1$, and the theorem is true for the transcendence degree $< \operatorname{tr.deg} N/K$.

Let $\overline{\mathcal{N}} = (\overline{N}, \tau)$. There is a chain of difference fields,

$$\mathcal{K} = \mathcal{K}_0 \subset \mathcal{K}_1 \subset \cdots \subset \mathcal{K}_{n-1} \subset \mathcal{K}_n = \mathcal{N}, \quad n \geq 1,$$

such that for each $1 \leq i \leq n$ the extension $\mathcal{K}_i/\mathcal{K}_{i-1}$ satisfies one of the conditions (i), (ii) in Definition 1. Put

$$n_0 = \min\{0 \leq i \leq n \mid K_n/K_i \text{ is algebraic}\}.$$

We find $n_0 \geq 1$, and that the extension $\mathcal{K}_{n_0}/\mathcal{K}_{n_0-1}$ satisfies the condition (ii). Choose a valuation ring $\mathcal{O}$ of $K_{n_0}/K_{n_0-1}$ such that $\tau^j\mathcal{O} \subset \mathcal{O}$ for some $j \in \mathbb{Z}_{>0}$. We have $(\tau^k)^j\mathcal{O} \subset \mathcal{O}$.

Let $\overline{\mathcal{K}}_{n_0-1}$ be the algebraic closure of $\mathcal{K}_{n_0-1}$ in $\overline{\mathcal{N}}$, and put $\overline{\mathcal{N}}^{(k)} = (\overline{N}, \tau^k)$, $\mathcal{K}_{n_0}^{(k)} = (K_{n_0}, \tau^k|_{K_{n_0}})$, $\mathcal{K}_{n_0-1}^{(k)} = (K_{n_0-1}, \tau^k|_{K_{n_0-1}})$ and $\overline{\mathcal{K}}_{n_0-1}^{(k)} = (\overline{K}_{n_0-1}, \tau^k|_{\overline{K}_{n_0-1}})$. By the hypothesis we find that the equation over $\mathcal{K}_{n_0}^{(k)}$, $y_1(c^{(k)}y + d^{(k)}) = a^{(k)}y + b^{(k)}$, has a solution in $\mathcal{N}^{(k)}$.

Define the matrices $B = B_1 = A_k$, $B_i = (\tau^k B_{i-1})B$ $(i \geq 2)$. By Lemma 3 we obtain $B_i = A_{ki}$. Therefore by Lemma 4 we find that there exists $i_0 \geq 1$ such that the equation over $\mathcal{K}_{n_0-1}^{(k)}$, $y_{i_0}(c^{(ki_0)}y + d^{(ki_0)}) = a^{(ki_0)}y + b^{(ki_0)}$, has a solution in $\overline{\mathcal{K}}_{n_0-1}^{(k)}$. Let $f \in \overline{K}_{n_0-1}$ be such a solution. It satisfies

$$\tau^{ki_0}(f)(c^{(ki_0)}f + d^{(ki_0)}) = a^{(ki_0)}f + b^{(ki_0)},$$

which implies that the equation over $\mathcal{K}$, $y_{ki_0}(c^{(ki_0)}y + d^{(ki_0)}) = a^{(ki_0)}y + b^{(ki_0)}$, has a solution in $\overline{\mathcal{K}}_{n_0-1}$.

Since $\overline{\mathcal{K}}_{n_0-1}/\mathcal{K}$ is a difference field extension of valuation ring type whose transcendence degree is less than $\operatorname{tr.deg} N/K$, we find by the induction hypothesis that there exists $i_1 \geq 1$ such that the equation over $\mathcal{K}$, $y_{ki_0i_1}(c^{(ki_0i_1)}y + d^{(ki_0i_1)}) = a^{(ki_0i_1)}y + b^{(ki_0i_1)}$, has a solution in $\overline{\mathcal{K}}$. □

The following is concerned with the case that the matrix turns out to be triangular by iterations.

PROPOSITION 5. *Let $\mathcal{K}$ be an inversive difference field, and $a, b, c, d \in K$ satisfy $ad - bc \neq 0$. Define the matrices $A_i$ as in Lemma 3, and suppose $b^{(k)} = 0$ or $c^{(k)} = 0$ for some $k \geq 1$. Let $f$ be a solution transcendental over $K$ of the equation over $\mathcal{K}$, $y_1(cy + d) = ay + b$, and put $\mathcal{L} = \mathcal{K}\langle f \rangle$. Then the following hold.*

(i) $\mathcal{L}$ *is inversive.*

(ii) $L/K$ *is an algebraic function field of one variable.*

(iii) *There is a valuation ring* $\mathcal{O}$ *of* $L/K$ *such that* $\tau^k\mathcal{O} \subset \mathcal{O}$.

(iv) $\mathcal{L}/\mathcal{K}$ *is of valuation ring type.*

PROOF. Let $\mathcal{L} = (L, \tau)$. Since $cf_1 - a = 0$ implies $f = \tau^{-1}(a/c) \in K$, we obtain

$$f = -\frac{df_1 - b}{cf_1 - a} = \tau\left(-\frac{\tau^{-1}(d)f - \tau^{-1}b}{\tau^{-1}(c)f - \tau^{-1}a}\right) \in \tau L.$$

Therefore $\mathcal{L}$ is inversive, which is the result (i). Since $cf + d = 0$ implies $f = -d/c \in K$, we obtain $f_1 \in K(f)$, which yields $L = K(f)$. This proves (ii).

By Lemma 3 we have $f_k(c^{(k)}f + d^{(k)}) = a^{(k)}f + b^{(k)}$. Put

$$g = \begin{cases} f & \text{if } c^{(k)} = 0, \\ 1/f & \text{if } c^{(k)} \neq 0. \end{cases}$$

We find that $g_k = \alpha g + \beta$ for some $\alpha, \beta \in K$, $\alpha \neq 0$. In fact, if $c^{(k)} = 0$, we have

$$g_k = f_k = \frac{a^{(k)}}{d^{(k)}}f + \frac{b^{(k)}}{d^{(k)}}.$$

Note that we obtain $\det A_k \neq 0$ from $\det A \neq 0$ by Lemma 3. If $c^{(k)} \neq 0$, we have $b^{(k)} = 0$ and

$$g_k = \frac{1}{f_k} = \frac{d^{(k)}}{a^{(k)}} \cdot \frac{1}{f} + \frac{c^{(k)}}{a^{(k)}}.$$

For the algebraic function field $L = K(g)$ of one variable over $K$, let $\mathcal{O}$ be the following valuation ring.

$$\mathcal{O} = \{p/q \in L \mid p, q \in K[g], \deg q - \deg p \geq 0\}.$$

For any $p \in K[g]$, the $k$-th transform $\tau^k p$ has the same degree as $p$. Therefore we obtain $\tau^k\mathcal{O} \subset \mathcal{O}$, which is the result (iii).

(i),(ii) and (iii) yield (iv). □

As a corollary of this proposition, we find that if a difference Riccati equation turns out to be linear by iterations, then any solution is an element of a certain difference field extension of valuation ring type.

## 3. Application to Solvability

In this section $C$ denotes an algebraically closed field.

### 3.1. Preliminaries

Lemma 6. *If $\mathcal{L}/\mathcal{K}$ is a GLE, then $\mathcal{L}/\mathcal{K}$ is of valuation ring type.*

Proof. We prove this by induction on the transcendence degree of $\mathcal{L}/\mathcal{K}$. There is nothing to prove in case $\operatorname{tr.deg} L/K = 0$. Suppose $\operatorname{tr.deg} L/K > 0$, and the lemma is true for the transcendence degree $<$ $\operatorname{tr.deg} L/K$. Let $\mathcal{L} = (L, \tau)$. There is a chain of inversive difference fields,

$$\mathcal{K} = \mathcal{K}_0 \subset \mathcal{K}_1 \subset \cdots \subset \mathcal{K}_n = \mathcal{L}, \quad K_i = K_{i-1}(\{\tau^k a_i \mid k \in \mathbb{Z}\}),$$

such that $a_i$ satisfies one of the following.

(i) $\tau a_i = a_i + \beta$ for some $\beta \in K_{i-1}$.

(ii) $\tau a_i = \alpha a_i$ for some $\alpha \in K_{i-1}$.

(iii) $a_i$ is algebraic over $K_{i-1}$.

Put $m = \min\{1 \leq i \leq n \mid \operatorname{tr.deg} K_i/K > 0\}$. The chain $\mathcal{K}_m \subset \cdots \subset \mathcal{K}_n = \mathcal{L}$ is a GLE and satisfies $\operatorname{tr.deg} L/K_m < \operatorname{tr.deg} L/K$. Therefore by the induction hypothesis we find that $\mathcal{L}/\mathcal{K}_m$ is of valuation ring type.

Since $a_m$ is transcendental over $K_{m-1}$ because of $\operatorname{tr.deg} K_{m-1}/K = 0$, there are $\alpha, \beta \in K_{m-1}$, $\alpha \neq 0$ such that $\tau a_m = \alpha a_m + \beta$. By Proposition 5 we find that $\mathcal{K}_{m-1}\langle a_m \rangle/\mathcal{K}_{m-1}$ is of valuation ring type. Note that we have $\mathcal{K}_m = \mathcal{K}_{m-1}\langle a_m \rangle$. Therefore the chain

$$\mathcal{K} \subset \mathcal{K}_{m-1} \subset \mathcal{K}_m \subset \mathcal{L}$$

implies $\mathcal{L}/\mathcal{K}$ is of valuation ring type. □

Proposition 7. *Let $\mathcal{K}$ be a inversive difference field, $a, b, c, d \in K$, and $q \in \mathbb{Z}_{>0}$. Define the matrices $A_i$ as in Lemma 3. Suppose $b^{(qi)} \neq 0$ and $c^{(qi)} \neq 0$ for all $i \geq 1$, and the equation over $\mathcal{K}$, $y_1(cy+d) = ay+b$, has a solution $f$ in a qLE $\mathcal{L}/\mathcal{K}$. Let $\overline{\mathcal{L}} = (\overline{L}, \tau)$ be an algebraic closure of $\mathcal{L}$, and $\overline{\mathcal{K}}$ be the algebraic closure of $\mathcal{K}$ in $\overline{\mathcal{L}}$. Then there exists $i \geq 1$ such that the equation over $\mathcal{K}$, $y_{qi}(c^{(qi)}y + d^{(qi)}) = a^{(qi)}y + b^{(qi)}$, has a solution in $\overline{\mathcal{K}}$.*

Proof. Put $\overline{\mathcal{L}}^{(q)} = (\overline{L}, \tau^q)$, $\mathcal{L}^{(q)} = (L, \tau^q|_L)$, $\overline{\mathcal{K}}^{(q)} = (\overline{K}, \tau^q|_{\overline{K}})$, and $\mathcal{K}^{(q)} = (K, \tau^q|_K)$. Since $\mathcal{L}/\mathcal{K}$ is a $q$LE, $\mathcal{L}^{(q)}/\mathcal{K}^{(q)}$ is a GLE. By Lemma 6 we find that $\mathcal{L}^{(q)}/\mathcal{K}^{(q)}$ is of valuation ring type.

Since we have $f_q(c^{(q)}f + d^{(q)}) = a^{(q)}f + b^{(q)}$ by Lemma 3, $f \in \overline{\mathcal{L}}^{(q)}$ is a solution of the equation over $\mathcal{K}^{(q)}$, $y_1(c^{(q)}y + d^{(q)}) = a^{(q)}y + b^{(q)}$. Therefore by Theorem 2 we conclude that there exists $i \geq 1$ such that the equation over $\mathcal{K}^{(q)}$, $y_i(c^{(qi)}y + d^{(qi)}) = a^{(qi)}y + b^{(qi)}$, has a solution $g$ in $\overline{\mathcal{K}}^{(q)}$, which implies $g \in \overline{\mathcal{K}}$ is a solution of the equation over $\mathcal{K}$, $y_{qi}(c^{(qi)}y + d^{(qi)}) = a^{(qi)}y + b^{(qi)}$. $\square$

Lemma 8. *Let $q \in C^\times$ be not a root of unity, $t$ transcendental over $C$, $F/C(t)$ a finite extension of degree $n$, and $\tau$ an isomorphism of $F$ into $F$ over $C$ sending $t$ to $qt$. Then $F = C(x)$, $x^n = t$.*

Proof. Put $\mathbb{P}$ and $\mathbb{P}'$ be the sets of all prime divisors of $C(t)/C$ and $F/C$ respectively. As in [11] we identify a prime divisor with the maximal ideal of the valuation ring associated with it. Define the following valuation rings of $C(t)/C$,

$$\mathcal{O}_\alpha = \{f/g \mid f, g \in C[t],\ t - \alpha \nmid g\} \quad \text{for each } \alpha \in C,$$
$$\mathcal{O}_\infty = \{f/g \mid f, g \in C[t],\ \deg g - \deg f \geq 0\},$$

and let $P_\alpha = \mathcal{O}_\alpha \setminus \mathcal{O}_\alpha^\times$ be the prime divisor associated with $\mathcal{O}_\alpha$ for each $\alpha \in C \cup \{\infty\}$.

We show that if $\alpha \in C^\times$ then $P_\alpha$ is unramified in $F/C(t)$. Let $\alpha \in C^\times$ and assume that $P_\alpha$ is ramified in $F/C(t)$. Then there is $P' \in \mathbb{P}'$ such that $e(P'|P_\alpha) > 1$, where $e(P'|P_\alpha)$ is the ramification index of $P'$ over $P_\alpha$. Let $\mathcal{O}'$ be the valuation ring associated with $P'$. We find that for any $i \in \mathbb{Z}_{\geq 0}$, $\tau^i P_\alpha = P_{\alpha/q^i} \in \mathbb{P}$ and $\tau^i P'$ is the prime divisor associated with the valuation ring $\tau^i \mathcal{O}'$ of $\tau^i F/C$. We also find that $e(\tau^i P'|\tau^i P_\alpha) > 1$ for all $i \geq 0$. For any $i \geq 0$ there is $Q_i \in \mathbb{P}'$ such that $Q_i \cap \tau^i F = \tau^i P'$, and we have

$$e(Q_i|\tau^i P_\alpha) = e(Q_i|\tau^i P')e(\tau^i P'|\tau^i P_\alpha) \geq e(\tau^i P'|\tau^i P_\alpha) > 1,$$

which implies $\tau^i P_\alpha = P_{\alpha/q^i}$ is ramified in $F/C(t)$ for any $i \geq 0$. Since $q \in C^\times$ is not a root of unity, the prime divisors $P_{\alpha/q^i}$ $(i \geq 0)$ are distinct, a contradiction. Therefore $P_\alpha$ is unramified in $F/C(t)$.

Let $g$ be the genus of $F/C$. By the Riemann-Hurwitz Genus Formula we obtain

$$\begin{aligned} 2g-2 &= -2n + \sum_{\alpha=0,\infty} \left( \sum_{P'\in\mathbb{P}',\, P'\cap C(t)=P_\alpha} (e(P'|P_\alpha)-1) \right) \\ &\leq -2n + 2(n-1) = -2, \end{aligned}$$

which implies $g = 0$. Therefore $F = C(y)$ for some $y \in F$.

Again by the Riemann-Hurwitz Genus Formula we obtain

$$\sum_{\alpha=0,\infty} \left( \sum_{P'\in\mathbb{P}',\, P'\cap C(t)=P_\alpha} (e(P'|P_\alpha)-1) \right) = 2(n-1),$$

which implies

$$\sum_{P'\in\mathbb{P}',\, P'\cap C(t)=P_\alpha} (e(P'|P_\alpha)-1) = n-1$$

for $\alpha = 0, \infty$. Therefore $P_\alpha$ $(\alpha = 0, \infty)$ has only one extension $P'_\alpha$ in $\mathbb{P}'$, which satisfies $e(P'_\alpha|P_\alpha) = n$.

$t \in C(y)$ yields the expression,

$$t = c\prod_{i=1}^{m}(y-\alpha_i)^{k_i}, \quad c \in C^\times,\ m \in \mathbb{Z}_{\geq 1},\ \alpha_i \in C,\ k_i \in \mathbb{Z},$$

where $\alpha_i$ $(1 \leq i \leq m)$ are distinct. Let $Q'_i$ be the prime divisor of $C(y)/C$ associated with the prime element $y - \alpha_i$, and put $Q_i = Q'_i \cap C(t)$ for each $1 \leq i \leq m$. We obtain

$$k_i = v_{Q'_i}(t) = e(Q'_i|Q_i)v_{Q_i}(t) = \begin{cases} 0 & \text{if } Q_i = P_\alpha,\ \alpha \in C^\times, \\ n & \text{if } Q_i = P_0, \\ -n & \text{if } Q_i = P_\infty, \end{cases}$$

where $v_{Q_i'}$ and $v_{Q_i}$ are the normalized discrete valuations associated with $Q_i'$ and $Q_i$ respectively, which implies $n \mid k_i$ for all $1 \le i \le m$. Put $x = c^{1/n} \prod_{i=1}^{m} (y - \alpha_i)^{k_i/n} \in C(y)$. We have $x^n = t$, and so $[C(t,x) : C(t)] = n$, which implies $F = C(t,x) = C(x)$. □

### 3.2. $q$-Airy equation

In their [6], Hamamoto, Kajiwara and Witte introduced that each of the basic hypergeometric solutions of the $q$-difference equation, $y(qt) + ty(t) = y(t/q)$, has a limit to the Airy function. Let $f \in \mathcal{K}^\times$ be a solution of the equation over $(C(t), t \mapsto qt)$, $y_2 + qty_1 - y = 0$, and put $g = f_1/f$. Then $g \in \mathcal{K}$ is a solution of the equation over $(C(t), t \mapsto qt)$, $y_1 y + qty - 1 = 0$, the object here.

The outline of the proof of the unsolvability of the above equation is the following. *Step 1.* Define the matrices $A_i$ as in Lemma 3, and show that they are not triangular. *Step 2.* Prove that there is no algebraic solution of the equation associated with $A_i$ for all $i \ge 1$. *Step 3.* Apply Proposition 7.

PROPOSITION 9. *Let $q \in C$ be transcendental over $\mathbb{Q}$, and $t$ transcendental over $C$. Put $\mathcal{K} = (C(t), t \mapsto qt)$, and let $\overline{\mathcal{K}} = (\overline{C(t)}, \tau)$ be an algebraic closure of $\mathcal{K}$. Put $a = -qt$, $b = 1$, $c = 1$ and $d = 0$, and define the matrices $A_i$ as in Lemma 3. Then the following hold.*

(i) $b^{(i)} \neq 0$ *and* $c^{(i)} \neq 0$ *for all* $i \ge 1$.

(ii) *For any $i \ge 1$ the equation over $\mathcal{K}$, $y_i(c^{(i)} y + d^{(i)}) = a^{(i)} y + b^{(i)}$, has no solution in $\overline{\mathcal{K}}$.*

PROOF. We have

$$A = \begin{pmatrix} -qt & 1 \\ 1 & 0 \end{pmatrix}, \quad A_2 = (\tau A)A = \begin{pmatrix} q^3t^2 + 1 & -q^2 t \\ -qt & 1 \end{pmatrix},$$

and for any $i \ge 2$,

$$A_i = (\tau A_{i-1})A = \begin{pmatrix} -qta_1^{(i-1)} + b_1^{(i-1)} & a_1^{(i-1)} \\ -qtc_1^{(i-1)} + d_1^{(i-1)} & c_1^{(i-1)} \end{pmatrix},$$

$$A_i = (\tau^{i-1}A)A_{i-1} = \begin{pmatrix} -q^i ta^{(i-1)} + c^{(i-1)} & -q^i tb^{(i-1)} + d^{(i-1)} \\ a^{(i-1)} & b^{(i-1)} \end{pmatrix},$$

which imply $b^{(i)} = a_1^{(i-1)}$ and $c^{(i)} = a^{(i-1)}$ for all $i \geq 2$, and $d^{(i)} = a_1^{(i-2)}$ for all $i \geq 3$. From these we obtain

$$a^{(i)} = -q^i t a^{(i-1)} + c^{(i-1)} = -q^i t a^{(i-1)} + a^{(i-2)}, \quad \text{for any } i \geq 3.$$

Note $A_i \in M_2(C[t])$. We find

$$a^{(i)} = (-1)^i q^{\frac{i(i+1)}{2}} t^i + (\text{a polynomial of deg} \leq i-2) \tag{2}$$

by induction, and so $\deg a^{(i)} = i$. This implies $a^{(i)} \neq 0$, by which we conclude $b^{(i)} \neq 0$ and $c^{(i)} \neq 0$ for all $i \geq 1$, the result (i).

Assume that there exists $i_0 \geq 1$ such that the equation over $\mathcal{K}$, $y_{i_0}(c^{(i_0)}y + d^{(i_0)}) = a^{(i_0)}y + b^{(i_0)}$, has a solution $f$ in $\overline{\mathcal{K}}$. Put $k = 3i_0 \geq 3$. By Lemma 3, $f \in \overline{\mathcal{K}}$ is a solution of the equation over $\mathcal{K}$, $y_k(c^{(k)}y + d^{(k)}) = a^{(k)}y + b^{(k)}$. Put $\mathcal{L} = \mathcal{K}\langle f \rangle \subset \overline{\mathcal{K}}$. Since both of the assumptions, $c^{(k)}f_k - a^{(k)} = 0$ and $c^{(k)}f + d^{(k)} = 0$, yield $\det A_k = 0$, which contradicts $\det A = -1$ by Lemma 3, we find that $\mathcal{L}$ is inversive, and $L = C(t)(f, f_1, \ldots, f_{k-1})$. Put $n = [L : C(t)] < \infty$. Then from Lemma 8 we obtain $L = C(x)$ with $x^n = t$. Note that $x$ is transcendental over $C$, $f \in C(x)$, $A_i \in M_2(C[x^n])$, and $(\frac{\tau x}{x})^n = q \in C$, which implies $\frac{\tau x}{x} \in C$. Put $r = \frac{\tau x}{x} \in C^\times$.

Express $f = P/Q$, where $P, Q \in C[x]$ are relatively prime. The equation $f_k(c^{(k)}f + d^{(k)}) = a^{(k)}f + b^{(k)}$ yields

$$P_k(c^{(k)}P + d^{(k)}Q) = Q_k(a^{(k)}P + b^{(k)}Q) \quad (\neq 0), \tag{3}$$

where both sides of this are not equal to 0. We find by induction that $a^{(i)}P + b^{(i)}Q$ and $c^{(i)}P + d^{(i)}Q$ are relatively prime. In fact we obtain that $aP + bQ = -qtP + Q$ and $cP + dQ = P$ are relatively prime from the hypothesis, $P$ and $Q$ are relatively prime, the case $i = 1$. Let $i \geq 2$ and suppose the statement is true for $i - 1$. Since we have

$$\begin{aligned} a^{(i)}P + b^{(i)}Q &= (-q^i t a^{(i-1)} + c^{(i-1)})P + (-q^i t b^{(i-1)} + d^{(i-1)})Q \\ &= -q^i t(a^{(i-1)}P + b^{(i-1)}Q) + (c^{(i-1)}P + d^{(i-1)}Q) \end{aligned}$$

and $a^{(i-1)}P + b^{(i-1)}Q = c^{(i)}P + d^{(i)}Q$, we conclude that $a^{(i)}P + b^{(i)}Q$ and $c^{(i)}P + d^{(i)}Q$ are relatively prime by the induction hypothesis.

Therefore $a^{(k)}P + b^{(k)}Q$ and $c^{(k)}P + d^{(k)}Q$ are relatively prime. From the equation (3) we obtain $\deg_x(a^{(k)}P + b^{(k)}Q) = \deg_x P_k = \deg_x P$. Since

$\deg_x a^{(k)}P = nk + \deg_x P > \deg_x P$, we find $\deg_x a^{(k)}P = \deg_x b^{(k)}Q$, which implies $\deg_x Q - \deg_x P = n$.

Express

$$f = \sum_{i=n}^{\infty} e_i \left(\frac{1}{x}\right)^i, \quad e_i \in C,\, e_n \neq 0.$$

We will show $f \in C(t)$. Assume there exists $i \geq n$ such that $n \nmid i$ and $e_i \neq 0$, and put $ln + m$ $(0 < m < n)$ be the minimum number of them. Note

$$\deg_x a^{(k)} = kn, \quad \deg_x b^{(k)} = \deg_x c^{(k)} = (k-1)n, \quad \deg_x d^{(k)} = (k-2)n.$$

The first term of

$$\begin{aligned} &a^{(k)}f + b^{(k)} \\ &= a^{(k)}\left(e_n\left(\frac{1}{x}\right)^n + \cdots + e_{ln}\left(\frac{1}{x}\right)^{ln} + e_{ln+m}\left(\frac{1}{x}\right)^{ln+m} + \cdots\right) + b^{(k)} \end{aligned}$$

whose exponent is not divisible by $n$ has the exponent, $-kn+(ln+m)$. The first term of

$$\begin{aligned} &f_k(c^{(k)}f + d^{(k)}) \\ &= \left\{\frac{e_n}{r^{kn}}\left(\frac{1}{x}\right)^n + \cdots + \frac{e_{ln}}{r^{kln}}\left(\frac{1}{x}\right)^{ln} + \frac{e_{ln+m}}{r^{k(ln+m)}}\left(\frac{1}{x}\right)^{ln+m} + \cdots\right\} \\ &\times \left\{c^{(k)}\left(e_n\left(\frac{1}{x}\right)^n + \cdots + e_{ln}\left(\frac{1}{x}\right)^{ln} + e_{ln+m}\left(\frac{1}{x}\right)^{ln+m} + \cdots\right) + d^{(k)}\right\} \end{aligned}$$

whose exponent is not divisible by $n$ has the exponent $\geq (2-k)n+(ln+m)$, which is impossible. Therefore we obtain $f = \sum_{i=1}^{\infty} e_{ni}(1/x^n)^i$, and so $f \in C(1/x^n) = C(t)$.

Then we have $L = C(t)(f, f_1, \ldots, f_{k-1}) \subset C(t)$, which implies $n = [L : C(t)] = 1$, $x = t$ and $r = q$. We find $a^{(i)} \in \mathbb{Z}[q,t]$ by induction, and so $b^{(i)}, c^{(i)}, d^{(i)} \in \mathbb{Z}[q,t]$. We will show $e_j \in \mathbb{Z}[q, 1/q]$ for any $j \geq 1$ by induction. We have

$$f_k(c^{(k)}f + d^{(k)}) = \left(\sum_{i=1}^{\infty} \frac{e_i}{q^{ki}}\left(\frac{1}{t}\right)^i\right)\left(c^{(k)}\sum_{i=1}^{\infty} e_i\left(\frac{1}{t}\right)^i + d^{(k)}\right) \tag{4}$$

and

$$a^{(k)}f + b^{(k)} = a^{(k)}\sum_{i=1}^{\infty} e_i \left(\frac{1}{t}\right)^i + b^{(k)}. \tag{5}$$

Note that the equation (2) yields

$$a^{(k)} = (-1)^k q^{\frac{k(k+1)}{2}} t^k + (\text{a polynomial of deg} \le k-2),$$
$$b^{(k)} = a_1^{(k-1)} = (-1)^{k-1} q^{\frac{(k-1)(k+2)}{2}} t^{k-1} + (\text{a polynomial of deg} \le k-3).$$

Comparing the terms of exponent $-k+1$ of the equation (4) = (5), we obtain

$$0 = (-1)^k q^{\frac{k(k+1)}{2}} e_1 + (-1)^{k-1} q^{\frac{(k-1)(k+2)}{2}},$$

which implies $e_1 = q^{-1} \in \mathbb{Z}[q, 1/q]$.

Let $j \ge 2$ and suppose the statement is true for the numbers $\le j-1$. On the one hand the term of exponent $-k+j$ of (5) has the coefficient,

$$(-1)^k q^{\frac{k(k+1)}{2}} e_j + (\text{an element of } \mathbb{Z}[q][e_1, e_2, \dots, e_{j-1}])$$
$$\in (-1)^k q^{\frac{k(k+1)}{2}} e_j + \mathbb{Z}[q, 1/q].$$

On the other hand the term of exponent $-k+j$ of (4) is the same one of

$$\left(\sum_{i=1}^{j-1} \frac{e_i}{q^{ki}} \left(\frac{1}{t}\right)^i\right) \left(c^{(k)} \sum_{i=1}^{j-1} e_i \left(\frac{1}{t}\right)^i + d^{(k)}\right) \in \mathbb{Z}[q, 1/q]((1/t)) \subset C((1/t)).$$

Therefore we obtain

$$(-1)^k q^{\frac{k(k+1)}{2}} e_j \in \mathbb{Z}[q, 1/q],$$

which implies $e_j \in \mathbb{Z}[q, 1/q]$.

Let $\phi \colon \mathbb{Q}[q, 1/q] \mapsto \mathbb{Q}$ be a ring homomorphism sending $q$ to 1, and extend it to the ring homomorphism $\overline{\phi} \colon \mathbb{Q}[q, 1/q]((1/t)) \mapsto C((1/t))$ sending $\sum_{i=m}^{\infty} h_i (1/t)^i$ to $\sum_{i=m}^{\infty} \phi(h_i)(1/t)^i$. This $\overline{\phi}$ is a difference homomorphism of $(\mathbb{Q}[q, 1/q]((1/t)), t \mapsto qt)$ to $(C((1/t)), id)$, and so we obtain

$$\overline{\phi}(f)(\overline{\phi}(c^{(k)})\overline{\phi}(f) + \overline{\phi}(d^{(k)})) = \overline{\phi}(a^{(k)})\overline{\phi}(f) + \overline{\phi}(b^{(k)}).$$

We find $\overline{\phi}(f) \in C(t)$. In fact since $f \in C(1/t)$, there are $s \in \mathbb{Z}_{\geq 0}$ and $m_0 \in \mathbb{Z}_{\geq 0}$ such that $F_f(m, s) = 0$ for all $m \geq m_0$, where $F_f(m, s)$ is the Hankel determinant $\det(e_{m+i+j})_{0 \leq i,j \leq s}$ of $f$ (refer to [1] for the Hankel determinant). Therefore for any $m \geq m_0$ we obtain

$$\begin{aligned} F_{\overline{\phi}(f)}(m, s) &= \det(\phi(e_{m+i+j}))_{0 \leq i,j \leq s} = \phi(\det(e_{m+i+j})_{0 \leq i,j \leq s}) \\ &= \phi(F_f(m, s)) = 0, \end{aligned}$$

which implies $\overline{\phi}(f) \in C(1/t) = C(t)$.

Express $\overline{\phi}(f) = P'/Q'$, where $P', Q' \in C[t]$ are relatively prime, and put $a' = \overline{\phi}(a^{(k)})$, $b' = \overline{\phi}(b^{(k)})$, $c' = \overline{\phi}(c^{(k)})$ and $d' = \overline{\phi}(d^{(k)})$. Note

$$\begin{aligned} c' &= \overline{\phi}(c^{(k)}) = \overline{\phi}(a^{(k-1)}) = \overline{\phi}(a_1^{(k-1)}) = \overline{\phi}(b^{(k)}) = b', \\ d' = \overline{\phi}(d^{(k)}) &= \overline{\phi}(a_1^{(k-2)}) = \overline{\phi}(a^{(k-2)}) = \overline{\phi}(a^{(k)} + q^k t a^{(k-1)}) = a' + tb', \end{aligned}$$

and $b' = (-1)^{k-1} t^{k-1} +$ (a polynomial of deg $\leq k-3) \neq 0$. Then we obtain the following from $P'(c'P' + d'Q') = Q'(a'P' + b'Q')$,

$$P'^2 + tP'Q' = Q'^2. \tag{6}$$

This equation yields $P' \mid Q'^2$ and $Q' \mid P'^2$, which imply $\deg P' = \deg Q' = 0$. Comparing the degree of the equation (6), we find $1 = 0$, a contradiction. Therefore we obtain (ii). □

Corollary 10. *Let $q \in C$ be transcendental over $\mathbb{Q}$, $t$ transcendental over $C$, $\mathcal{K} = (C(t), t \mapsto qt)$, and $k \in \mathbb{Z}_{>0}$. Then the equation over $\mathcal{K}$, $y_1 y + qty - 1 = 0$, has no solution in any $k$LE of $\mathcal{K}$.*

Proof. Assume the equation has a solution in a $k$LE $\mathcal{N}/\mathcal{K}$. Put $a = -qt$, $b = c = 1$ and $d = 0$. Define the matrices $A_i$ as in Lemma 3. By Proposition 9 we have $b^{(i)} \neq 0$ and $c^{(i)} \neq 0$ for all $i \geq 1$.

Let $\overline{\mathcal{N}}$ be an algebraic closure of $\mathcal{N}$, and $\overline{\mathcal{K}}$ the algebraic closure of $\mathcal{K}$ in $\overline{\mathcal{N}}$. By Proposition 7 we find that there exists $i \geq 1$ such that the equation over $\mathcal{K}$, $y_{ki}(c^{(ki)} y + d^{(ki)}) = a^{(ki)} y + b^{(ki)}$, has a solution in $\overline{\mathcal{K}}$, which contradicts Proposition 9. □

### 3.3. $q$-Bessel equation

Seeing [5], we find one of the $q$-Bessel functions, $J_\nu^{(3)}(x;q)$, and the equation,

$$g_\nu(qx) + (x^2/4 - q^\nu - q^{-\nu})g_\nu(x) + g_\nu(xq^{-1}) = 0,$$

where $g_\nu(x) = J_\nu^{(3)}(xq^{\nu/2};q^2)$. This section deals with the Riccati equation associated with it.

Proposition 11. *Let $q \in C$ be transcendental over $\mathbb{Q}$, and $t$ transcendental over $C$. Put $\mathcal{K} = (C(t), t \mapsto qt)$, and let $\overline{\mathcal{K}} = (\overline{C(t)}, \tau)$ be an algebraic closure of $\mathcal{K}$. Put $a = -(t^2/4 - q^\nu - q^{-\nu})$, $b = -1$, $c = 1$ and $d = 0$, where $\nu \in \mathbb{Q}$, and define the matrices $A_i$ as in Lemma 3. Then the following hold.*

(i) $b^{(i)} \neq 0$ *and* $c^{(i)} \neq 0$ *for all* $i \geq 1$.

(ii) *For any* $i \geq 1$ *the equation over* $\mathcal{K}$, $y_i(c^{(i)}y + d^{(i)}) = a^{(i)}y + b^{(i)}$, *has no solution in* $\overline{\mathcal{K}}$.

Proof. Put $p = q^\nu + q^{-\nu} \in C$. We have

$$A = \begin{pmatrix} a & -1 \\ 1 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} a_1 a - 1 & -a_1 \\ a & -1 \end{pmatrix},$$

and for any $i \geq 2$,

$$A_i = (\tau A_{i-1})A = \begin{pmatrix} aa_1^{(i-1)} + b_1^{(i-1)} & -a_1^{(i-1)} \\ ac_1^{(i-1)} + d_1^{(i-1)} & -c_1^{(i-1)} \end{pmatrix},$$

$$A_i = (\tau^{i-1}A)A_{i-1} = \begin{pmatrix} a_{i-1}a^{(i-1)} - c^{(i-1)} & a_{i-1}b^{(i-1)} - d^{(i-1)} \\ a^{(i-1)} & b^{(i-1)} \end{pmatrix},$$

which imply $b^{(i)} = -a_1^{(i-1)}$ and $c^{(i)} = a^{(i-1)}$ for all $i \geq 2$, and $d^{(i)} = -a_1^{(i-2)}$ for all $i \geq 3$. From these we obtain

$$a^{(i)} = a_{i-1}a^{(i-1)} - c^{(i-1)} = a_{i-1}a^{(i-1)} - a^{(i-2)}, \quad \text{for any } i \geq 3.$$

Note $A_i \in M_2(C[t])$. We find

$$a^{(i)} = (-1)^i \frac{q^{(i-1)i}}{4^i} t^{2i} + (\text{a polynomial of deg} \leq 2i-2) \tag{7}$$

by induction, and so $\deg a^{(i)} = 2i$. This implies $a^{(i)} \neq 0$, by which we conclude $b^{(i)} \neq 0$ and $c^{(i)} \neq 0$ for all $i \geq 1$, the result (i).

Assume that there exists $i_0 \geq 1$ such that the equation over $\mathcal{K}$, $y_{i_0}(c^{(i_0)}y + d^{(i_0)}) = a^{(i_0)}y + b^{(i_0)}$, has a solution $f$ in $\overline{\mathcal{K}}$. Put $k = 3i_0 \geq 3$. By Lemma 3, $f \in \overline{\mathcal{K}}$ is a solution of the equation over $\mathcal{K}$, $y_k(c^{(k)}y + d^{(k)}) = a^{(k)}y + b^{(k)}$. Put $\mathcal{L} = \mathcal{K}\langle f\rangle \subset \overline{\mathcal{K}}$. We find that $\mathcal{L}$ is inversive, and $L = C(t)(f, f_1, \ldots, f_{k-1})$. Put $n = [L : C(t)] < \infty$. Then from Lemma 8 we obtain $L = C(x)$ with $x^n = t$. Note that $x$ is transcendental over $C$, $f \in C(x)$, $A_i \in M_2(C[x^n])$, and $(\frac{\tau x}{x})^n = q \in C$, which implies $\frac{\tau x}{x} \in C$. Put $r = \frac{\tau x}{x} \in C^{\times}$.

Express $f = P/Q$, where $P, Q \in C[x]$ are relatively prime. The equation $f_k(c^{(k)}f + d^{(k)}) = a^{(k)}f + b^{(k)}$ yields

$$P_k(c^{(k)}P + d^{(k)}Q) = Q_k(a^{(k)}P + b^{(k)}Q) \quad (\neq 0). \tag{8}$$

We find by induction that $a^{(i)}P+b^{(i)}Q$ and $c^{(i)}P+d^{(i)}Q$ are relatively prime. In fact we obtain that $aP + bQ = aP - Q$ and $cP + dQ = P$ are relatively prime, the case $i = 1$. Let $i \geq 2$ and suppose the statement is true for $i - 1$. Since we have

$$\begin{aligned} a^{(i)}P + b^{(i)}Q &= (a_{i-1}a^{(i-1)} - c^{(i-1)})P + (a_{i-1}b^{(i-1)} - d^{(i-1)})Q \\ &= a_{i-1}(a^{(i-1)}P + b^{(i-1)}Q) - (c^{(i-1)}P + d^{(i-1)}Q) \end{aligned}$$

and $a^{(i-1)}P + b^{(i-1)}Q = c^{(i)}P + d^{(i)}Q$, we conclude that $a^{(i)}P + b^{(i)}Q$ and $c^{(i)}P + d^{(i)}Q$ are relatively prime by the induction hypothesis.

Therefore $a^{(k)}P + b^{(k)}Q$ and $c^{(k)}P + d^{(k)}Q$ are relatively prime. From the equation (8) we obtain $\deg_x(a^{(k)}P + b^{(k)}Q) = \deg_x P_k = \deg_x P$. Since $\deg_x a^{(k)}P = 2kn + \deg_x P > \deg_x P$, we find that $\deg_x a^{(k)}P = \deg_x b^{(k)}Q$, which implies $\deg_x Q - \deg_x P = 2n$.

Express

$$f = \sum_{i=2n}^{\infty} e_i \left(\frac{1}{x}\right)^i, \quad e_i \in C,\, e_{2n} \neq 0.$$

We obtain $f \in C(t)$ by the same way as in the proof of Proposition 9, and so $L = C(t)$, $n = 1$, $x = t$ and $r = q$. Note $a^{(i)}, b^{(i)}, c^{(i)}, d^{(i)} \in \mathbb{Q}[q, p, t]$. We will show $e_j \in \mathbb{Q}[q, 1/q, p]$ for any $j \geq 2$ by induction. We have

$$f_k(c^{(k)}f + d^{(k)}) = \left(\sum_{i=2}^{\infty} \frac{e_i}{q^{ki}} \left(\frac{1}{t}\right)^i\right) \left(c^{(k)} \sum_{i=2}^{\infty} e_i \left(\frac{1}{t}\right)^i + d^{(k)}\right) \tag{9}$$

and

$$a^{(k)}f + b^{(k)} = a^{(k)}\sum_{i=2}^{\infty} e_i\left(\frac{1}{t}\right)^i + b^{(k)}. \tag{10}$$

The equation (7) yields

$$\begin{aligned} a^{(k)} &= (-1)^k\frac{q^{(k-1)k}}{4^k}t^{2k} + (\text{a polynomial of deg} \le 2k-2),\\ b^{(k)} &= (-1)^k\frac{q^{(k-1)k}}{4^{k-1}}t^{2(k-1)} + (\text{a polynomial of deg} \le 2k-4). \end{aligned}$$

Comparing the terms of exponent $-2k+2$ of the equation (9) = (10), we obtain

$$0 = (-1)^k\frac{q^{(k-1)k}}{4^k}e_2 + (-1)^k\frac{q^{(k-1)k}}{4^{k-1}},$$

which implies $e_2 = -4$.

Let $j \ge 3$ and suppose the statement is true for the numbers $\le j-1$. On the one hand the term of exponent $-2k+j$ of (10) has the coefficient,

$$\begin{aligned} &(-1)^k\frac{q^{(k-1)k}}{4^k}e_j + (\text{an element of } \mathbb{Q}[q,p,e_2,e_3,\ldots,e_{j-1}])\\ &\in (-1)^k\frac{q^{(k-1)k}}{4^k}e_j + \mathbb{Q}[q,1/q,p]. \end{aligned}$$

On the other hand the term of exponent $-2k+j$ of (9) is the same one of

$$\left(\sum_{i=2}^{j-1}\frac{e_i}{q^{ki}}\left(\frac{1}{t}\right)^i\right)\left(c^{(k)}\sum_{i=2}^{j-1}e_i\left(\frac{1}{t}\right)^i + d^{(k)}\right) \in \mathbb{Q}[q,1/q,p]((1/t)) \subset C((1/t)).$$

Therefore we obtain

$$(-1)^k\frac{q^{(k-1)k}}{4^k}e_j \in \mathbb{Q}[q,1/q,p],$$

which implies $e_j \in \mathbb{Q}[q,1/q,p]$.

Let $\nu = \nu_1/\nu_2$, where $\nu_1 \in \mathbb{Z}$ and $\nu_2 \in \mathbb{Z}_{>0}$ are relatively prime. Then we have

$$\mathbb{Q}[q,1/q,p] \subset \mathbb{Q}[q^{\frac{1}{\nu_2}}, 1/q^{\frac{1}{\nu_2}}].$$

Let $\phi\colon \mathbb{Q}[q^{(1/\nu_2)}, 1/q^{(1/\nu_2)}] \mapsto \mathbb{Q}$ be a ring homomorphism sending $q^{(1/\nu_2)}$ to 1, and extend it to the ring homomorphism $\overline{\phi}\colon \mathbb{Q}[q^{(1/\nu_2)}, 1/q^{(1/\nu_2)}]((1/t)) \mapsto \mathbb{Q}((1/t))$ sending $\sum_{i=m}^{\infty} h_i(1/t)^i$ to $\sum_{i=m}^{\infty} \phi(h_i)(1/t)^i$. This $\overline{\phi}$ is a difference homomorphism of $(\mathbb{Q}[q^{(1/\nu_2)}, 1/q^{(1/\nu_2)}]((1/t)), t \mapsto qt)$ to $(\mathbb{Q}((1/t)), id)$, and so we obtain

$$\overline{\phi}(f)(\overline{\phi}(c^{(k)})\overline{\phi}(f) + \overline{\phi}(d^{(k)})) = \overline{\phi}(a^{(k)})\overline{\phi}(f) + \overline{\phi}(b^{(k)}).$$

We find $\overline{\phi}(f) \in C(t)$ by seeing the Hankel determinant. Express $\overline{\phi}(f) = P'/Q'$, where $P', Q' \in C[t]$ are relatively prime, and put $a' = \overline{\phi}(a^{(k)})$, $b' = \overline{\phi}(b^{(k)})$, $c' = \overline{\phi}(c^{(k)})$ and $d' = \overline{\phi}(d^{(k)})$. Note

$$c' = \overline{\phi}(c^{(k)}) = \overline{\phi}(a^{(k-1)}) = \overline{\phi}(a_1^{(k-1)}) = -\overline{\phi}(b^{(k)}) = -b',$$

$$\begin{aligned} d' &= \overline{\phi}(d^{(k)}) = \overline{\phi}(-a_1^{(k-2)}) = \overline{\phi}(-a^{(k-2)}) = \overline{\phi}(a^{(k)} - a_{k-1}a^{(k-1)}) \\ &= a' + \left(-\frac{t^2}{4} + 2\right) b', \end{aligned}$$

and $b' \neq 0$. Then we obtain the following from $P'(c'P' + d'Q') = Q'(a'P' + b'Q')$,

$$-P'^2 + \left(-\frac{t^2}{4} + 2\right) P'Q' = Q'^2. \tag{11}$$

This equation yields $P' \mid Q'^2$ and $Q' \mid P'^2$, which imply $\deg P' = \deg Q' = 0$. Comparing the degree of the equation (11), we find $2 = 0$, a contradiction. Therefore we obtain (ii). □

COROLLARY 12. *Let $q \in C$ be transcendental over $\mathbb{Q}$, $t$ transcendental over $C$, $\mathcal{K} = (C(t), t \mapsto qt)$, and $k \in \mathbb{Z}_{>0}$. Then the equation over $\mathcal{K}$, $y_1 y = -(t^2/4 - q^\nu - q^{-\nu})y - 1$, where $\nu \in \mathbb{Q}$, has no solution in any kLE of $\mathcal{K}$.*

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