Uniform Estimate for Distributions of the Sum of i.i.d. Random Variables with Fat Tail

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Abstract. The research on asymptotic behavior of distributions of the sum of i.i.d random variables has a long history and a lot of facts are known. The authors consider the case where the distribution of a random variable has the second moment but has a fat tail, and they show a new limit theorem for large deviations.

1. Introduction

Let \((\Omega, \mathcal{F}, P)\) be a probability space and \(X_n, n = 1, 2, \ldots\), be independent identically distributed random variables with the same probability law \(\mu\).

In the present paper we assume that

(A-1) \(E[X_1^2] = 1\) and \(E[X_1] = 0\).

Let \(F : \mathbb{R} \to [0, 1]\) and \(\bar{F} : \mathbb{R} \to [0, 1]\) be given by

\[
F(x) = \mu((-\infty, x]) = P(X_1 \leq x) \quad \text{and} \quad \bar{F}(x) = \mu((x, \infty)) = P(X_1 > x), \quad x \in \mathbb{R}.
\]

We also assume the following.

(A-2) \(\bar{F}(x)\) is a regularly varying function of index \(-\alpha\) for some \(\alpha > 2\), as \(x \to \infty\), i.e., if we let

\[
L(x) = x^\alpha \bar{F}(x), \quad x \geq 1,
\]

then \(L(x) > 0\) for any \(x \geq 1\), and for any \(a > 0\)

\[
\frac{L(ax)}{L(x)} \to 1, \quad x \to \infty.
\]

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(A-3) $|x|^\alpha + 2 F(x) \to 0, \quad x \to -\infty$.

Recently people in finance are interested in computing the quantile of the distribution of $\sum_{k=1}^n X_k$ for the purpose of measuring market risk.

There are many works on this topic. In particular, there are many results on large deviation results (e.g. Borovkov-Borovkov [1], also see books, Borovkov-Borovkov [2] and Petrov [7]). However, there are not so many results on uniform estimates. Nagaev [5] and [6] proved the following theorem (also see Linnik [4]), and this is the best result so far to our best knowledge.

**Theorem 1 (Nagaev).** Assume (A-1)-(A-3). Then we have

$$\sup_{s \in [1, \infty)} \left| \frac{P(\sum_{k=1}^n X_k > sn^{1/2})}{\Phi_0(s) + nF(sn^{1/2})} - 1 \right| \to 0, \quad n \to \infty.$$  

Here $\Phi_0 : \mathbb{R} \to \mathbb{R}$ is given by

$$\Phi_0(x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty \exp\left(-\frac{y^2}{2}\right) dy, \quad x \in \mathbb{R}.$$  

In this paper, we show two theorems. Combining them, we can improve Nagaev’s result a little bit.

Let us explain our results. We assume the following assumption furthermore.

(A-4) The probability law $\mu$ is absolutely continuous and has a density function $\rho : \mathbb{R} \to [0, \infty)$ which is right continuous and has a finite total variation.

To state our theorem (Theorem 2), we need some preparations.

Let $K$ be an integer such that $K - 1 < \alpha \leq K$. Then $K \geq 3$. From the assumptions (A-2) and (A-3), we see that the probability law $\mu$ has $(K - 1)$-th moment. So let $\eta_k, k = 1, \ldots, K - 1$, be given by

$$\eta_k = \int_\mathbb{R} x^k \mu(dx).$$

Then we see that $\eta_1 = 0$ and $\eta_2 = 1$. Also, let us define $\Phi_k : \mathbb{R} \to \mathbb{R}, \quad k = 1, 2, \ldots$, by

$$\Phi_1(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) = -\frac{d}{dx}\Phi_0(x),$$

$$\Phi_k(x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty \exp\left(-\frac{y^2}{2}\right) dy, \quad x \in \mathbb{R}.$$
and

\[ \Phi_k(x) = (-1)^{k-1} \frac{d^{k-1}}{dx^{k-1}} \Phi_1(x), \quad k = 2, 3, \ldots. \]

**Theorem 2.** Assume (A-1)-(A-4). Then there are \( \delta > 0 \) and \( C > 0 \) such that

\[ \sup_{s \in [1, \infty)} |P(\sum_{k=1}^{n} X_k > sn^{1/2}) - G(n, s)| \leq Cn^{-(\alpha-2)/2-\delta}, \quad n = 3, 4, \ldots. \]

Here

\[ G(n, s) \]

\[ = \Phi_0(s) + \int_{-\infty}^{s} \bar{F}((s-x)n^{1/2}) \Phi_1(x) dx - \sum_{k=1}^{K-1} \frac{n^{-(k-2)/2}}{k!} \Phi_k(s) \int_{0}^{\infty} x^k \mu(dx) \]

\[ + \frac{n^{-(K-2)/2}}{K!} \Phi_K(s) \int_{-\infty}^{0} x^K \mu(dx) + \sum_{k=3}^{K-1} n^{-k/6} q_k(n^{-1/3}, \eta_2, \ldots, \eta_k) \Phi_k(s) \]

\[ + \sum_{k=K}^{3(K-1)} n^{-k/6} q_k(n^{-1/3}, \eta_2, \ldots, \eta_{K-1}, 0, \ldots, 0) \Phi_k(s), \]

and \( q_k \)'s are polynomials defined in the next section.

For the next theorem we assume the following also.

(A-5) There is an \( x_0 > 0 \) such that \( \bar{F} \) is twice continuously differentiable on \((x_0, \infty)\) and that

\[ x^2 \frac{d^2}{dx^2} \log \bar{F}(x) \to \alpha, \quad x \to \infty. \]

Then we have the following.

**Theorem 3.** Assume the assumptions (A-1)-(A-5) and let \( \beta : N \to (0, \infty) \) be such that

\[ \frac{\beta(n)}{(\log n)^{1/2}} \to \infty, \quad n \to \infty. \]
Then we have
\[
\sup_{s \geq n^{1/2} \beta(n)} \frac{s^2}{n} \left| \frac{P(\sum_{k=1}^{n} X_k > s)}{nF(s)} - (1 + \frac{\alpha(\alpha + 1)n}{2s^2}) \right| \to 0, \quad n \to \infty.
\]

Let
\[
H(n, s) = \Phi_0(s) + n \int_{-\infty}^{s} \tilde{F}((s-x)n^{1/2}) \Phi_1(x) dx - \sum_{k=1}^{2} \frac{n^{-(k-2)/2}}{k!} \Phi_k(s) \int_{0}^{\infty} x^k \mu(dx)
\]
for \( s \geq 1 \), and \( n \geq 1 \).

Then we also show the following.

**Theorem 4.** Assume (A-1)-(A-5). Then there exist a \( C > 0 \), \( \delta > 0 \) and \( n_0 \geq 1 \) such that
\[
\sup_{s \in [1, \infty)} \left| \frac{P(\sum_{k=1}^{n} X_k > sn^{1/2})}{H(n, s)} - 1 \right| \leq Cn^{-\delta}, \quad n \geq n_0.
\]

Note that by Theorem 3, we see that
\[
2(\log n)^2 \left( \frac{P(\sum_{k=1}^{n} X_k > (\log n)n^{1/2})}{\Phi_0(\log n) + n\tilde{F}((\log n)n^{1/2})} - 1 \right) \to \alpha(\alpha + 1) \quad n \to \infty.
\]

Therefore we see that \( H(n, s) \) is a better approximation for \( P(\sum_{k=1}^{n} X_k > sn^{1/2}) \) than \( \Phi_0(s) + n\tilde{F}(sn^{1/2}) \).

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2. **Algebraic Preparation**

In this section, we think of formal power series in \( z \). First, we think of the following formal power series in \( z \).

(1) \[
\log(1 + \sum_{k=2}^{\infty} \frac{a_k}{k!} z^k) = \sum_{\ell=1}^{\infty} \frac{(-1)^{\ell-1}}{\ell} \left( \sum_{k=2}^{\infty} \frac{a_k}{k!} z^k \right)^{\ell} = \sum_{\ell=2}^{\infty} c_\ell(a_2, \ldots, a_\ell) z^{\ell-1} \frac{z^\ell}{\ell!}
\]

Then we see that \( c_\ell(a_2, \ldots, a_\ell) \), \( \ell \geq 2 \), are polynomials in \( a_2, \ldots, a_\ell \), and
\[
c_\ell(t^2a_2, \ldots, t^\ell a_\ell) = t^\ell c_\ell(a_1, \ldots, a_\ell)
\]
for any $t, a_1, \ldots, a_\ell \in \mathbb{R}$. Moreover, we see that

$$c_2(a_2) = a_2 \quad \text{and} \quad c_\ell(a_2, \ldots, a_{\ell-1}, a_\ell) = c_\ell(a_2, \ldots, a_{\ell-1}, 0) + a_\ell, \quad \ell \geq 2.$$  

We also think of the following formal power series in $z$.

$$\exp(y^{-3} \sum_{\ell=3}^{\infty} c_\ell(a_2, \ldots, a_\ell) \frac{(yz)^\ell}{\ell!})$$

(2) \quad = 1 + \sum_{k=1}^{\infty} \frac{1}{k!} \left( \sum_{\ell=3}^{\infty} c_\ell(a_2, \ldots, a_\ell) \frac{y^{\ell-3} z^\ell}{\ell!} \right)^k = 1 + \sum_{k=3}^{\infty} q_k(y, a_2, \ldots, a_k) z^k.$$

Then we see that $q_k(y, a_2, \ldots, a_k), \quad k \geq 3,$ are polynomials in $y, a_2, \ldots, a_\ell$. Note that

$$q_k(y, t^2 a_2 \ldots, t^k a_k) = t^k q_k(y, a_2, \ldots, a_k)$$

and that

$$q_k(y, a_2, \ldots, a_k) = q_k(y, a_2, \ldots, a_{k-1}, 0) + \frac{y^{k-3}}{k!} a_k, \quad k \geq 3.$$  

Also we have

$$\exp(y^{-6} \sum_{\ell=3}^{\infty} c_\ell(a_2, \ldots, a_\ell) \frac{(y^3 z)^\ell}{\ell!})$$

$$= \exp((y^2)^{-3} \sum_{\ell=3}^{\infty} c_\ell(y^2 a_2, \ldots, y^\ell a_\ell) \frac{(y^2 z)^\ell}{\ell!})$$

(3) \quad = 1 + \sum_{k=3}^{\infty} q_k(y^2, y^2 a_2, \ldots, y^k a_k) z^k = 1 + \sum_{k=3}^{\infty} y^k q_k(y^2, a_2, \ldots, a_k) z^k$$

as a formal power series in $z$. 
3. Property of the Function $L$

**Proposition 5.** We have

$$\sup_{1/2 \leq a \leq 2} \frac{L(ax)}{L(x)} \to 1, \quad x \to \infty,$$

and

$$\inf_{1/2 \leq a \leq 2} \frac{L(ax)}{L(x)} \to 1, \quad x \to \infty.$$

**Proof.** Since the proof is similar, we prove the first equation only. If not, there are $\varepsilon > 0$, $\{a_n\}_{n=1}^{\infty}$ and $\{x_n\}_{n=1}^{\infty}$ such that $1/2 \leq a_n \leq 2$, $x_n \geq 1$, $n = 1, 2, \ldots$, $x_n \to \infty$, $n \to \infty$, and that

$$\frac{L(a_n x_n)}{L(x_n)} > 1 + \varepsilon, \quad n = 1, 2, \ldots.$$

Then taking a subsequence if necessary, we may assume that there is an $a \in [1/2, 2]$ such that $a_n \to a$, $n \to \infty$. Then we see that for any $m \geq 3$ there is a $n(m) \geq 1$ such that

$$(a - \frac{1}{m})^{-\alpha} L((a - \frac{1}{m})x_n) = \bar{F}((a - \frac{1}{m})x_n) \geq \bar{F}(a_n x_n)$$

$$= a_n^{-\alpha} L(a_n x_n), \quad n \geq n(m).$$

So we have

$$(1 - \frac{1}{ma})^{-\alpha} \geq \lim_{n \to \infty} \frac{L(a_n x_n)}{L((a - 1/m)x_n)} \geq 1 + \varepsilon, \quad m \geq 3.$$

Since $m$ is arbitrary, this implies a contradiction. □

**Proposition 6.** For any $\varepsilon \in (0, 1)$, there is an $M \geq 1$ such that

$$M^{-1} y^{-\varepsilon} \leq \frac{L(yx)}{L(x)} \leq M y^\varepsilon \quad x, y \geq 1.$$

**Proof.** For any $\varepsilon \in (0, 1)$ there is an $m \geq 1$ such that

$$|\frac{L(ex)}{L(x)} - 1| \leq \varepsilon \quad x \geq e^m.$$
Let

\[ C = \sup_{x \in [1, e^m]} \left( \frac{L(ex)}{L(x)} + \frac{L(x)}{L(ex)} \right) < \infty. \]

Then we have

\[ C^{-m}(1 - \varepsilon)^n \leq \frac{L(e^n x)}{L(x)} \leq C^m (1 + \varepsilon)^n, \quad x \geq 1, \ n \geq 0. \]

For any \( y \geq 1 \), there is an \( n \geq 1 \) such that \( e^{n-1} \leq y \leq e^n \). Then we have

\[ \bar{F}(e^{n-1} x) \geq \bar{F}(y x) \geq \bar{F}(e^n x). \]

So we have for any \( x, y \geq 1 \)

\[ (e^{-1} y x)^{-\alpha} L(e^{n-1} x) \geq (e^{n-1} x)^{-\alpha} L(e^{n-1} x) \geq (y x)^{-\alpha} L(y x) \]

\[ \geq (e^n x)^{-\alpha} L(e^n x) \geq (e y x)^{-\alpha} L(e^n x), \]

which implies

\[ C^{-m} e^{-\alpha} (1 - \varepsilon)^n \leq \frac{L(y x)}{L(x)} \leq C^m e^{\alpha} (1 + \varepsilon)^{n-1}. \]

Therefore we have

\[ C^{-m} e^{-\alpha} (1 - \varepsilon)^{y \log(1 - \varepsilon)} \leq \frac{L(y x)}{L(x)} \leq C^m e^{\alpha} y^{\log(1 + \varepsilon)}, \quad x \geq 1, \ y \geq 1. \]

This implies our assertion. □

The following is known as Karamata’s theorem (c.f.[3] Appendix ), but we give a proof.

**PROPOSITION 7.** (1) For any \( \beta < -1 \),

\[ \frac{1}{t^{\beta+1} L(t)} \int_t^\infty x^\beta L(x) dx \to -\frac{1}{\beta + 1}, \quad t \to \infty. \]

(2) For any \( \beta > -1 \),

\[ \frac{1}{t^{\beta+1} L(t)} \int_1^t x^\beta L(x) dx \to \frac{1}{\beta + 1}, \quad t \to \infty. \]
(3) Let \( f : [1, \infty) \to (0, \infty) \) be given by
\[
f(t) = \int_1^t x^{-1} L(x) dx \quad t \geq 1.
\]
Then \( f \) is slowly varying.

**Proof.** Note that for \( t > 1 \)
\[
\frac{1}{t^{\beta+1} L(t)} \int_t^\infty x^\beta L(x) dx = \int_1^\infty x^\beta \frac{L(tx)}{L(t)} dx, \text{ if } \beta < -1
\]
and
\[
\frac{1}{t^{\beta+1} L(t)} \int_1^t x^\beta L(x) dx = \int_1^1 x^\beta \left( \frac{L(t)}{L(tx)} \right)^{-1} dx \text{ if } \beta > -1.
\]
Then the assertions (1) and (2) follow from this equation and Proposition 5.

Let us prove (3). If \( \lim_{t \to \infty} f(t) < \infty \), the assertion is obvious. So we assume that \( \lim_{t \to \infty} f(t) = \infty \). Then for any \( a > 0 \) and \( t_0 > 1 \)
\[
f(at) = \int_{1/a}^t x^{-1} L(ax) dx = \int_{1/a}^{t_0} x^{-1} L(ax) dx + \int_{t_0}^t x^{-1} L(x) \frac{L(ax)}{L(x)} dx.
\]
So we have
\[
\inf_{x \geq t_0} \frac{L(ax)}{L(x)} \leq \lim_{t \to \infty} \frac{f(at)}{f(t)} \leq \lim_{t \to \infty} \frac{f(at)}{f(t)} \leq \sup_{x \geq t_0} \frac{L(ax)}{L(x)}.
\]
Therefore by Proposition 6 and Lebesgue’s convergence theorem, we have our assertion. \( \square \)

4. **Estimate for Moments and Characteristic Functions**

Remind that \( K \) is an integer such that \( K - 1 < \alpha \leq K \) and
\[
\eta_k = \int_{-\infty}^\infty x^k \mu(dx), \quad k = 1, 2, \ldots, K - 1.
\]
Then by the assumption (A4) we have \( \eta_1 = 0 \) and \( \eta_2 = 1 \). Note that
\[
1 - \bar{F}(t) \geq 1 - \int_2^\infty \frac{x^2}{4} \mu(dx) \geq \frac{3}{4}.
\]
for any $t \geq 2$. Let 
\[ \eta_k(t) = \int_{(-\infty,t]} x^k \mu(dx), \quad t > 0, \; k = 1, 2, \ldots, K + 1, \]
and 
\[ \bar{\eta}_k(t) = \int_{(t,\infty)} x^k \mu(dx), \quad t > 0, \; k = 0, 1, 2, \ldots, K - 1. \]
Then we have
\[ \eta_k(t) = \int_{(-\infty,0)} x^k \mu(dx) + k \int_0^t x^{k-1} \bar{F}(x)dx - t^k \bar{F}(t), \quad t > 0, \; k = 1, 2, \ldots, K + 1, \]
and
\[ \bar{\eta}_k(t) = k \int_t^\infty x^{k-1} \bar{F}(x)dx + t^k \bar{F}(t) \quad t > 0, \; k = 0, 1, 2, \ldots, K - 1. \]
Then by Propositions 6 and 7 we have the following.

**Proposition 8.** For any $\varepsilon > 0$, there is a $C(\varepsilon) > 0$ such that
\[ L(t) \leq C(\varepsilon)t^\varepsilon, \]
\[ |\eta_K(t)| \leq C(\varepsilon)t^{-\alpha + K + \varepsilon}, \]
\[ |\bar{\eta}_k(t)| \leq C(\varepsilon)t^{-\alpha + k + \varepsilon}, \quad k = 0, 1, 2, \ldots, K - 1, \]
and
\[ \int_{(-\infty,t]} |x|^{K+1} \mu(dx) \leq C(\varepsilon)t^{-\alpha + K + 1 + \varepsilon} \]
for any $t \geq 1$.

The following is well known.

**Proposition 9.** (1) For any $m \geq 0$, let $r_{e,m} : \mathbb{R} \to \mathbb{C}$ be given by
\[ r_{e,m}(t) = \exp(it) - (1 + \sum_{k=1}^m \frac{(it)^k}{k!}), \quad t \in \mathbb{R}. \]
Then we have
\[ |r_{c,m}(t)| \leq \frac{|t|^{m+1}}{(m+1)!} \quad t \in \mathbb{R}. \]

(2) For any \( m \geq 1 \), let \( r_{l,m} : \{ z \in \mathbb{C}; |z| \leq 1/2 \} \to \mathbb{C} \) be given by
\[ r_{l,m}(z) = \log(1 + z) - \sum_{k=1}^{m} \frac{(-1)^{k-1}}{k} z^k, \quad z \in \mathbb{C}, \ |z| \leq 1/2. \]

Then we have
\[ |r_{l,m}(z)| \leq 2|z|^{m+1}, \quad z \in \mathbb{C}, \ |z| \leq 1/2. \]

Let \( \mu(t), t > 0 \), be a probability measure on \( (\mathbb{R}, \mathcal{B}(\mathbb{R})) \) given by
\[ \mu(t)(A) = (1 - \bar{F}(t))^{-1} \mu(A \cap (-\infty, t]), \]
for any \( A \in \mathcal{B}(\mathbb{R}) \).

Let \( \varphi(\cdot; \mu(t)), t > 0 \), be the characteristic function of the probability measure \( \mu(t) \), i.e.,
\[ \varphi(\xi; \mu(t)) = \int_{\mathbb{R}} \exp(ix\xi) \mu(t)(dx), \quad \xi \in \mathbb{R}. \]

By the assumption (A3), we see that the density function \( \rho(x) \to 0 \) as \( |x| \to \infty \). Also we see that the probability measure \( \mu(t), t \geq 2 \), is absolutely continuous and its density function is \( (1 - \bar{F}(t))^{-1} \rho(x) 1_{(-\infty,t]}(x) \), whose total variation is dominated by twice of that of \( \rho \).

Therefore we have the following.

**Proposition 10.** (1) For any \( t \geq 2 \) and \( \xi \in \mathbb{R} \),
\[ i\xi \varphi(\xi; \mu(t)) = (1 - \bar{F}(t))^{-1} \int_{\mathbb{R}} i\xi e^{ix} \rho(x) 1_{(-\infty,t]}(x) dx \]
\[ = -(1 - \bar{F}(t))^{-1} \int_{\mathbb{R}} e^{ix} d(\rho(x) 1_{(-\infty,t]}(x)). \]

(2) There is a \( C > 0 \) such that
\[ |\varphi(\xi, \mu(t))| \leq C(1 + |\xi|)^{-1} \quad \text{for any } t \geq 2 \text{ and } \xi \in \mathbb{R}. \]
Then we have the following.

**Proposition 11.** (1) There is a $c_0 > 0$ such that

$$|\varphi(\xi, \mu(t))| \leq (1 + c_0|\xi|^2)^{-1/4} \text{ for any } t \geq 2 \text{ and } \xi \in \mathbb{R}.$$  

(2) For any $t \geq 2$, $\xi \in \mathbb{R}$, and integers $n, m$ with $n \geq m$,

$$|\varphi(n^{-1/2}\xi, \mu(t))|^n \leq (1 + \frac{c_0}{m}|\xi|^2)^{-m/4}.$$

**Proof.** Let $g(x) = \rho(x)1_{(-2,2)}(x), x \in \mathbb{R}$. Then we have

$$p = \int_{\mathbb{R}} g(x)dx \geq 1 - \int_{\mathbb{R}} \frac{x^2}{4} \rho(x)dx \geq 3/4.$$  

Note that

$$|\varphi(\xi, \mu(t))|^2$$

$$= (1 - \bar{F}(t))^{-2} \int_{\mathbb{R}} \int_{\mathbb{R}} \exp(i\xi(x - y))\rho(x)1_{(-\infty, t]}(x)\rho(y)1_{(\infty, t]}(y)dxdy$$

$$\leq (1 - p^2) + \int_{\mathbb{R}} \int_{\mathbb{R}} \exp(i\xi(x - y))g(x)g(y)dxdy = 1 - f(\xi),$$

where

$$f(\xi) = \int_{\mathbb{R}} \int_{\mathbb{R}} (1 - \cos(\xi(x - y)))g(x)g(y)dxdy.$$  

So we see that

$$\lim_{\xi \to 0} |\xi|^{-2} f(\xi) = \frac{1}{2} \int_{\mathbb{R}} \int_{\mathbb{R}} (x - y)^2 g(x)g(y)dxdy > 0.$$  

Also, it is easy to see that $f(\xi) > 0$, for all $\xi \in \mathbb{R} \setminus \{0\}$, and so we see that

$$a(r) = \inf_{|\xi| \leq r} |\xi|^{-2} f(\xi) > 0 \quad \text{for all } r > 0.$$  

Therefore we see that

$$|\varphi(\xi, \mu(t))| \leq (1 - a(r)|\xi|^2)^{1/2} \leq (1 + a(r)|\xi|^2)^{-1/4}, \quad |\xi| \leq r.$$
Also by Proposition 10(2), we see that there is an \( r_0 > 0 \) such that
\[
|\varphi(\xi, \mu(t))| \leq (1 + |\xi|^2)^{-1/4}, \quad |\xi| \geq r_0
\]
So we have the assertion (1).

It is easy to check that \( (1 + x/\beta)^\beta \geq 1 + x \) for any \( \beta \geq 1 \) and \( x \geq 0 \). Therefore if \( n \geq m \), we have
\[
(1 + c_0 n^{-1/2}|\xi|^2)^{n/m} \geq 1 + \frac{c_0}{m}|\xi|^2.
\]
This implies the assertion (2). \( \Box \)

5. Asymptotic Expansion of Characteristic Functions

Let
\[
\varphi_1(\xi, t) = -\sum_{k=1}^{K-1} \frac{(i\xi)^k}{k!} \tilde{\eta}_k(t) + \frac{(i\xi)^K}{K!} \eta_K(t)
\]
and
\[
\psi_0(n, \xi) = \sum_{k=3}^{K-1} n^{-k/6} q_k(n^{-1/3}, \eta_2, \ldots, \eta_k)(i\xi)^k
\]
\[
+ \sum_{k=K}^{3(K-1)} n^{-k/6} q_k(n^{-1/3}, \eta_2, \ldots, \eta_{K-1}, 0, \ldots, 0)(i\xi)^k
\]
for \( t \geq 2, n \geq 1 \) and \( \xi \in \mathbb{R} \). Let \( \delta = ((\alpha-2)\wedge 1)/(4(K+2)) \), \( \delta' = \delta/(4(K+2)) \), and \( t_n = n^{1/2-\delta}, n = 1, 2, 3, \ldots \). Then \( t_n \geq 2 \) for any \( n \geq 8 \).

In this section, we prove the following.

**Lemma 12.** Let
\[
R_{n,0}(\xi) = \exp\left(\frac{1}{2} \xi^2\right) \varphi(n^{-1/2}\xi, \mu(t_n))^n - (1 + \psi_0(n, \xi) + n\varphi_1(n^{-1/2}\xi, t_n))
\]
\[
R_{n,1}(\xi) = \exp\left(\frac{1}{2} \xi^2\right) \varphi(n^{-1/2}\xi, \mu(t_n))^n - 1
\]
\[
R_{n,2}(\xi) = \exp\left(\frac{1}{2} \xi^2\right) \varphi(n^{-1/2}\xi, \mu(t_n))^{n-1} - 1
\]
Then there is a $C > 0$ such that
\[ |R_{n,0}(\xi)| \leq C n^{-(\alpha-2)/2-\delta/4} |\xi| \]
and
\[ |R_{n,1}(\xi)| + |R_{n,2}(\xi)| \leq C n^{-2K\delta} |\xi| \]
for any $n \geq 8$ and $\xi \in \mathbb{R}$ with $|\xi| \leq n^{\delta'}$.

We make some preparations to prove this lemma. First we prove the following.

**Proposition 13.** Let
\[ \varphi_0(\xi) = \sum_{k=2}^{K-1} \frac{(i\xi)^k}{k!} \eta_k, \]
and
\[ R_0(\xi, t) = \varphi(\xi; \mu(t)) - (1 + \varphi_0(\xi) + \varphi_1(\xi, t)). \]

Then we have for any $n \geq 8$, and $\xi \in \mathbb{R}$ with $|\xi| \leq n^{\delta'}$,
\[ |\varphi(n^{-1/2} \xi; \mu(t_n)) - 1| \leq \frac{2\sqrt{3}}{3} n^{-1/2} |\xi|, \]
\[ |\varphi_1(n^{-1/2} \xi, t_n)| \leq K C(\delta) n^{-\alpha/2+(K+1)\delta} |\xi| \]
and
\[ |R_0(n^{-1/2} \xi, t_n)| \leq 3C(\delta) n^{-\alpha/2-\delta/4} |\xi|. \]

Here $C(\delta)$ is as in Proposition 8.

**Proof.** We can easily see that
\[ \varphi(\xi; \mu(t)) = \int_{\mathbb{R}} \exp(ix\xi) \mu(t)(dx) \]
\[ = 1 + \sum_{k=1}^{K} \frac{(i\xi)^k}{k!} \eta_k(t) + \int_{(-\infty, t]} r_{e,K}(x\xi) \mu(dx) \]
\[ + \tilde{F}(t)(1 - \tilde{F}(t))^{-1} \int_{(-\infty, 0]} r_{e,0}(x\xi) \mu(dx) \]
So we see that

\[ R_0(\xi, t) = \bar{F}(t)(1 - \bar{F}(t))^{-1} \int_{(-\infty,t]} r_{e,0}(x\xi)\mu(dx) + \int_{(-\infty,t]} r_{e,K}(x\xi)\mu(dx). \]

By Propositions 8 and 9 we have

\[ |\varphi_1(\xi, t)| \leq C(\delta) \sum_{k=1}^{K} \frac{|\xi|^k}{k!} t^{-\alpha+k+\delta}, \quad \xi \in \mathbb{R}, \ t \geq 2, \]

and

\[ |R_0(\xi, t)| \leq C(\delta)|\xi|^{t-\alpha+\delta} \int_{\mathbb{R}} |x|\mu(t)(dx) + C(\delta)|\xi|^K t^{-\alpha+K+1+\delta}, \quad \xi \in \mathbb{R}, \ t \geq 2. \]

Also, we have

\[ |\varphi(\xi; \mu(t)) - 1| \leq |\xi| \int_{\mathbb{R}} |x|\mu(t)(dx) \leq (1 - \bar{F}(t))^{-1/2} |\xi| \leq \frac{2\sqrt{3}}{3} |\xi|, \quad \xi \in \mathbb{R}, \ t \geq 2. \]

Note that

\[ (n^{-1/2+\delta'})^k (n^{1/2-\delta})^{-\alpha+k+\delta} = n^{-\alpha/2+(\alpha+1/2)\delta - k(\delta-\delta')-\delta^2}. \]

So we have our assertion. □

**Proposition 14.** Let

\[ \psi_1(\xi) = \sum_{k=3}^{K} \frac{(i\xi)^k}{k!} c_k(\eta_2, \ldots, \eta_k) + \frac{(i\xi)^K}{K!} c_K(\eta_2, \ldots, \eta_K-1, 0), \quad \xi \in \mathbb{R}. \]

Also, for any \( n \geq 8, \) and \( \xi \in \mathbb{R} \) with \( |\xi| \leq n^{\delta'}, \) let

\[ R_1(n, \xi) = \log(\varphi(n^{-1/2}\xi, \mu(t_n))) - \left\{ -\frac{1}{2n} \xi^2 + \psi_1(n^{-1/2}\xi) + \varphi_1(n^{-1/2}\xi, t_n) \right\}. \]

Then there is a constant \( C > 0 \) such that

\[ |R_1(n, \xi)| \leq C n^{-\alpha/2-\delta/4} |\xi|. \]
for any $n \geq 8$, and $\xi \in \mathbb{R}$ with $|\xi| \leq n^{\delta'}$.

**Proof.** Let

$$R_{1,1}(\xi) = \sum_{k=1}^{K} \frac{(-1)^{k-1}}{k} (\varphi_0(\xi))^{k} + \frac{1}{2}\xi^2 - \psi_1(\xi).$$

Note that

$$\log(1 + \sum_{k=2}^{K-1} \eta_k z^k) = \sum_{k=2}^{K-1} c_k(\eta_2, \ldots, \eta_k) z^k + \sum_{k=K}^{\infty} c_k(\eta_2, \ldots, \eta_{K-1}, 0, \ldots, 0) z^k$$

as a formal power series of $z$. So we see that there is a constant $C > 0$ such that

$$|R_{1,1}(\xi)| \leq C|\xi|^{K+1}$$

for any $\xi \in \mathbb{R}$ with $|\xi| \leq 1$.

We can easily see that

$$R_1(n, \xi) = \log(1 + \varphi_0(n^{-1/2}\xi) + \varphi_1(n^{-1/2}\xi, t_n) + R_0(n^{-1/2}\xi, t_n))$$

$$-\{-\frac{1}{2n}\xi^2 + \psi_1(n^{-1/2}\xi) + \varphi_1(n^{-1/2}\xi, t_n)\}$$

$$= R_{1,1}(n^{-1/2}\xi) + r_{l,K}(\varphi(n^{-1/2}\xi, \mu(t_n)) - 1) + R_0(n^{-1/2}\xi, t_n)$$

$$+ \sum_{k=2}^{K} (-1)^{k-1}(\varphi_0(n^{-1/2}\xi))^{k-1}(\varphi_1(n^{-1/2}\xi, t_n) + R_0(n^{-1/2}\xi, t_n))$$

$$+ \sum_{k=1}^{K} \frac{(-1)^{k-1}}{k} \sum_{j=2}^{k} \binom{k}{j} (\varphi_0(n^{-1/2}\xi))^{k-j}(\varphi_1(n^{-1/2}\xi, t_n) + R_0(n^{-1/2}\xi, t_n))^j.$$ 

Then we have our assertion from Equation (4) and Proposition 13.

**Proposition 15.** Let

$$R_2(n, \xi) = \exp(n\psi_1(n^{-1/2}\xi)) - (1 + \psi_0(n, \xi)).$$
Then there is a constant $C > 0$ such that
\[ |R_2(n, \xi)| \leq C n^{-(\alpha-2)/2-1/4} |\xi| \]
for any $n \geq 8$, and $\xi \in \mathbb{R}$ with $|\xi| \leq n^{\delta'}$.

**Proof.** Note that
\[
\exp\left(y^{-6} \left( \sum_{k=3}^{K-1} \frac{y^3 z^k}{k!} c_k(\eta_2, \ldots, \eta_k) + \sum_{k=K}^{\infty} \frac{(y^3 z)^k}{k!} c_k(\eta_2, \ldots, \eta_{K-1}, 0, \ldots, 0) \right) \right)
= 1 + \sum_{k=3}^{K-1} y^k q_k(y^2, \eta_2, \ldots, \eta_k) z^k + \sum_{k=K}^{\infty} y^k q_k(\eta_2, \ldots, \eta_{K-1}, 0, \ldots, 0)) z^k
\]
as a formal power series in $z$. This implies our assertion. □

Now let us prove Lemma 12.
Note that for any $n \geq 8$, and $\xi \in \mathbb{R}$ with $|\xi| \leq n^{\delta'}$,
\[
\exp\left(\frac{1}{2} \xi^2 \varphi(n^{-1/2} \xi; \mu(t_n))\right) = \exp(n \varphi_1(n^{-1/2} \xi, t_n) + n \psi_1(n^{-1/2} \xi) + n R_1(n, \xi))
= (1 + n \varphi_1(n^{-1/2} \xi, t_n) + r_{e,1}(n \varphi_1(n^{-1/2} \xi, t_n)))
\times (1 + \psi_0(n, \xi) + R_2(n, \xi))(1 + r_{e,0}(n R_1(n, \xi))).
\]
So we see that
\[
R_{n,0}(n, \xi) = r_{e,0}(n R_1(n, \xi)) \exp(n \varphi_1(n^{-1/2} \xi, t_n) + n \psi_1(n^{-1/2} \xi))
+ R_2(n, \xi) \exp(n \varphi_1(n^{-1/2} \xi, t_n))
+ r_{e,1}(n \varphi_1(n^{-1/2} \xi, t_n)) + \psi_0(n, \xi)(n \varphi_1(n^{-1/2} \xi, t_n) + r_{e,1}(n \varphi_1(n^{-1/2} \xi, t_n))).
\]
Thus we have the first equation from Propositions 13, 14, 15.

Also, we have
\[
R_{n,1}(n, \xi) = \exp(n \varphi_1(n^{-1/2} \xi, t_n) + n \psi_1(n^{-1/2} \xi) + n R_1(n, \xi))) - 1,
\]
and
\[
R_{n,2}(n, \xi) = \exp((n - 1) \varphi_1(n^{-1/2} \xi, t_n) + (n - 1) \psi_1(n^{-1/2} \xi)
+ (n - 1) R_1(n, \xi)) - \frac{\xi^2}{n}) - 1.
\]
So, again from Propositions 13, 14, 15 we have the second equation.
6. Proof of Theorem 2

First, we prove the following.

**Lemma 16.** Let \( \nu \) be a probability measure on \((\mathbb{R}, \mathcal{B}(\mathbb{R}))\) such that \( \int_{\mathbb{R}} x^2 \nu(dx) < \infty \). Also, assume that there is a constant \( C > 0 \) such that the characteristic function \( \varphi(\cdot, \nu) : \mathbb{R} \to \mathbb{C} \) satisfies
\[
|\varphi(\xi; \nu)| \leq C(1 + |\xi|)^{-2}, \quad \xi \in \mathbb{R}.
\]

Then for any \( x \in \mathbb{R} \)
\[
\nu((x, \infty)) = \Phi_0(x) + \frac{1}{2\pi} \int_{\mathbb{R}} \frac{e^{-ix\xi}}{i\xi} (\varphi(\xi, \nu) - \exp(-\frac{\xi^2}{2})) d\xi.
\]

**Proof.** From the assumption, \( \nu \) has a continuous density function \( \beta \) and
\[
\beta(x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-ix\xi} \varphi(\xi, \nu) d\xi.
\]
So we have
\[
\nu((x, x+n]) = \Phi_0(x) - \Phi_0(x+n) + \frac{1}{2\pi} \int_{\mathbb{R}} \left( \int_x^{x+n} e^{-iz\xi} dz \right) (\varphi(\xi, \nu) - \exp(-\frac{\xi^2}{2})) d\xi.
\]
\[
= \Phi_0(x) - \Phi_0(x+n) + \frac{1}{2\pi} \int_{\mathbb{R}} \frac{e^{-ix\xi} - e^{-i(x+n)\xi}}{i\xi} (\varphi(\xi, \nu) - \exp(-\frac{\xi^2}{2})) d\xi.
\]
Since
\[
\int_{\mathbb{R}} \frac{1}{|\xi|} |\varphi(\xi, \nu) - \exp(-\frac{\xi^2}{2})| d\xi < \infty,
\]
letting \( n \to \infty \), we have the assertion. \( \Box \)

Note that
\[
P(\sum_{k=1}^{n} X_k > sn^{1/2}) = \sum_{m=0}^{n} I_m(n, s),
\]
where
\[
I_m(n, s) = P(\sum_{k=1}^{n} X_k > sn^{1/2}, \sum_{k=1}^{n} 1\{X_k > t_n\} = m), \quad m = 0, 1, \ldots, n.
\]
Then we have

\[ I_m(n, s) = \binom{n}{m} P(\sum_{k=1}^{n} X_k > sn^{1/2}, X_i > t_n, i = 1, \ldots, m, \]

\[ X_j \leq t_n, j = m + 1, \ldots, n), \]

for \( m = 0, 1, \ldots, n \).

**Proposition 17.** There is a \( C > 0 \) such that

\[ \sum_{m=2}^{n} I_m(n, s) \leq C n^{-(\alpha-2)/2-\delta} \]

for any \( s \geq 1 \) and \( n \geq 8 \).

**Proof.** We see that by Proposition 8

\[
\sum_{m=2}^{n} I_m(n, s) \leq \sum_{m=2}^{n} \frac{n(n-1)}{m(m-1)} \left( \frac{n-2}{m-2} \right) \tilde{F}(t_n)^m (1 - \tilde{F}(t_n))^{n-m} \\
\leq \frac{n(n-1)}{2} \tilde{F}(t_n)^2 \leq C(\delta)^2 n^{-(\alpha-2)/2-\delta}.
\]

This implies our assertion. \( \square \)

**Proposition 18.** There is a \( C > 0 \) such that

\[
\sup_{s \in [1, \log n]} |I_0(n, s) - \{(1 - nF(t_n))\Phi_0(s) - \sum_{k=1}^{K-1} \frac{n^{-(k-2)/2}}{k!} \eta_k(t_n)\Phi_k(s) \\
+ \frac{n^{1/2}K-2}{K!} \eta_K(t_n)\Phi_K(s) + g(n, s)}| \leq C n^{-(\alpha-2)/2-\delta/4}
\]

for any \( n \geq 8 \). Here

\[
g(n, s) = \sum_{k=3}^{K-1} n^{-k/6} q_k(n^{-1/3}, \eta_2, \ldots, \eta_k)\Phi_k(s) \\
+ \sum_{k=K}^{3(K-1)} n^{-k/6} q_k(n^{-1/3}, \eta_2, \ldots, \eta_{K-1}, 0, \ldots, 0)\Phi_k(s).
\]
Proof. Note that
\[ I_0(n, s) = (1 - \bar{F}(t_n))^n \mu(t_n)^n((sn^{1/2}, \infty)) \]
\[ = I_{0,0}(n, s) + I_{0,1}(n, s) + I_{0,2}(n, s), \]
where
\[ I_{0,0}(n, s) = \mu(t_n)^n((sn^{1/2}, \infty)), \]
\[ I_{0,1}(n, s) = -n\bar{F}(t_n)\mu(t_n)^n((sn^{1/2}, \infty)), \]
\[ I_{0,2}(n, s) = ((1 - \bar{F}(t_n))^n - 1 + n\bar{F}(t_n))\mu(t_n)^n((sn^{1/2}, \infty)). \]

We remark that
\[ \Phi_k(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} (i\xi)^{k-1} \exp(-i\xi x - \frac{\xi^2}{2}) d\xi, \quad k = 1, 2, \ldots. \]

By Proposition 11 and Lemma 16, we have
\[ I_{0,0}(n, s) \]
\[ = \Phi_0(s) + \frac{1}{2\pi} \int_{\mathbb{R}} e^{-is\xi} \left( \varphi(n^{-1/2}\xi, \mu(t_n))^n - \exp\left(-\frac{\xi^2}{2}\right) \right) d\xi. \]

Let
\[ \tilde{R}_{0,0}(n, s) = I_{0,0}(n, s) \]
\[ - \{ \Phi_0(s) + \frac{1}{2\pi} \int_{\mathbb{R}} e^{-is\xi} \left( \psi_0(n, \xi) + n\varphi_1(n^{-1/2}\xi, t_n) \right) e^{-\xi^2/2} d\xi \} \]

Then by Lemma 12 we have
\[ |\tilde{R}_{0,0}(n, s)| \]
\[ \leq \int_{|\xi| \leq n^{\sigma'}} \frac{|R_{n,0}(\xi)|}{|\xi|} \exp\left(-\frac{\xi^2}{2}\right) d\xi \]
\[ + \int_{|\xi| > n^{\sigma'}} \frac{1}{|\xi|} \left( |\varphi(n^{-1/2}\xi, \mu(t_n))|^n + \exp\left(-\frac{\xi^2}{2}\right) \right) d\xi \]
\[ + \int_{|\xi| > n^{\sigma'}} \left( |\psi_0(n, \xi)| + n|\varphi_1(n^{-1/2}\xi, t_n)| \right) e^{-\xi^2/2} d\xi \]
So by Proposition 11 and Lemma 12, we see that there is a $C_0 > 0$ such that

$$(5) \quad \left| \tilde{R}_{0,0}(n, s) \right| \leq C_0 n^{-(\alpha - 2)/2 - \delta/4}, \quad n \geq 8, \ s \geq 1.$$ 

Also, we see that

$$\frac{1}{2\pi} \int_{\mathbb{R}} \frac{e^{-is\xi}}{i\xi} n\varphi_1(n^{-1/2} \xi, t_n) \exp\left(-\frac{1}{2} \xi^2\right) d\xi$$

$$= -\sum_{k=1}^{K-1} \frac{(n^{-1/2})^{k-2}}{k!} \eta_k(t_n) \Phi_k(s) + \frac{(n^{-1/2})^{K-2}}{K!} \eta_K(t_n) \Phi_K(s),$$

and

$$\frac{1}{2\pi} \int_{\mathbb{R}} \frac{e^{-is\xi}}{i\xi} \psi_0(n, \xi) \exp\left(-\frac{1}{2} \xi^2\right) d\xi = g(n, s).$$

Similarly by Lemma 12, we see that there is a $C_1 > 0$ such that

$$(6) \quad \sup_{s \in [1, \log n]} \left| I_{0,1}(n, s) - n \bar{F}(t_n) \Phi_0(s) \right| \leq C_1 n^{-(\alpha - 2)/2 - \delta}, \quad n \geq 8.$$ 

Note that $|(1 - x)^n - (1 - nx)| \leq n^2 x^2$ for any $x \in [0, 1], \ n \geq 1$. So we have

$$\left| I_{0,2}(n, s) \right| \leq n^2 \bar{F}(t_n)^2 \leq C(\delta)^2 n^{-(\alpha - 2)/2 - \delta}.$$ 

This and Equations 5, 6 imply our assertion. □

**Proposition 19.** There is a $C > 0$ such that

$$\sup_{s \in [1, \log n]} \left| I_1(n, s) - \{n \int_{-\infty}^{s} \bar{F}((s-x)n^{1/2}) \Phi_1(x) dx + n \bar{F}(t_n) \Phi_0(s) \right.$$ 

$$- \sum_{k=1}^{K} \frac{n^{-(k-2)/2}}{k!} \Phi_k(s) \int_{0}^{t_n} x^k \mu(dx) \left| \right| \leq C n^{-(\alpha - 2)/2 - \delta/4}.$$ 

**Proof.** We see that

$$I_1(n, s) = n(1 - \bar{F}(t_n))^{n-1} \int_{\mathbb{R}} P(X_1 + x > sn^{1/2}, \ X_1 > t_n) \mu(t_n)^{(n-1)}(dx)$$
= n(1 - \bar{F}(t_n))^{n-1} \int_{-\infty}^{\infty} \bar{F}((sn^{1/2} - x) \vee t_n) \mu(t_n)^{*(n-1)}(dx)
= nJ_0(n, s) + nJ_1(n, s) + nJ_2(n, s),
where

(7) J_0(n, s) = \int_{-\infty}^{\infty} \bar{F}((s - x)n^{1/2} \vee t_n)\Phi_1(x)dx,

(8) J_1(n, s) = \int_{-\infty}^{\infty} \bar{F}((sn^{1/2} - x) \vee t_n)(\mu(t_n)^{*(n-1)}(dx) - n^{-1/2}\Phi_1(xn^{1/2})dx),

and

(9) J_2(n, s) = -(1 - (1 - \bar{F}(t_n))^{n-1})I_1(n, s).

Note that

J_0(n, s) = J_{0,0}(n, s) + J_{0,1}(n, s) + J_{0,2}(n, s),

where

J_{0,0}(n, s) = \int_{-\infty}^{s} \bar{F}((s - x)n^{1/2})\Phi_1(x)dx,

J_{0,1}(n, s) = -\int_{s-n^{-\delta}}^{s} \bar{F}((s - x)n^{1/2})\Phi_1(x)dx
= -\int_{0}^{n^{-\delta}} \bar{F}(xn^{1/2})\Phi_1(s - x)dx,

and

J_{0,2}(n, s) = \bar{F}(t_n) \int_{s-n^{-\delta}}^{\infty} \Phi_1(x)dx = \bar{F}(t_n)\Phi_0(s - n^{-\delta}).

We see that

J_{0,1}(n, s) = -\sum_{k=1}^{K} \frac{1}{(k-1)!} \Phi_k(s) \int_{0}^{n^{-\delta}} \bar{F}(xn^{1/2})x^{k-1}dx + R_{J,1}(n, s),

where

R_{J,1}(n, s) = -\int_{0}^{n^{-\delta}} \bar{F}(xn^{1/2})(\Phi_1(s - x) - \sum_{k=1}^{K} \frac{x^{k-1}}{(k-1)!} \Phi_k(s))dx.
Then

\[ |R_{J,1}(n, s)| \]

\[ \leq \sup_{x \in [0, n^{-\delta}]} |\Phi_{K+1}(s-x)| \left( \int_{n^{-1/2}}^{n^{-\delta}} x^K (xn^{1/2})^{-\alpha} L(xn^{1/2}) \, dx + \int_0^{n^{-1/2}} x^K \, dx \right). \]

\[ \leq \sup_{x \in \mathbb{R}} |\Phi_{K+1}(x)| (C(\delta)n^{-\alpha/2+\delta/2} \int_0^{n^{-\delta}} x^{\delta+(K-\alpha)} \, dx + n^{-(K+1)/2}) \]

(10)

\[ \leq \sup_{x \in \mathbb{R}} |\Phi_{K+1}(x)| (C(\delta) + 1)n^{-\alpha/2-\delta/2}. \]

Also, we see that

\[ J_{0,2}(n, s) \]

\[ = \bar{F}(t_n) \Phi_0(s) + \sum_{k=1}^{K} \bar{F}(t_n) \frac{(n^{-\delta})^k}{k!} \Phi_k(s) + R_{J,2}(n, s), \]

where

\[ R_{J,2}(n, s) = \bar{F}(t_n)(\Phi_0(s-n^{-\delta}) - \sum_{k=0}^{K} \frac{(-n^{-\delta})^k}{k!} \frac{d^k \Phi_0}{dx^k}(s)). \]

We see that

\[ |R_{J,2}(n, s)| \leq \bar{F}(t_n)n^{-(K+1)\delta} \sup_{x \in \mathbb{R}} |\Phi_{K+1}(x)| \]

(11)

\[ \leq C(\delta) \sup_{x \in \mathbb{R}} |\Phi_{K+1}(x)| n^{-\alpha/2-\delta/4}. \]

It is easy to see that

\[ \int_0^{n^{-\delta}} \bar{F}(xn^{1/2})x^{k-1} \, dx = n^{-k/2} \int_0^{t_n} \bar{F}(x)x^{k-1} \, dx \]

\[ = n^{-k/2} \left( -\frac{1}{k} \int_0^{t_n} x^k \mu(dx) + \frac{n^{\delta k}}{k} \bar{F}(t_n) \right), \quad k = 1, \ldots, K. \]

So we have

\[ J_{0,1}(n, s) + J_{0,2}(n, s) = \bar{F}(t_n) \Phi_0(s) \]
\[ -\sum_{k=1}^{K} \frac{n^{-k/2}}{k!} \Phi_k(s) \int_{0}^{t_n} x^k \mu(dx) + R_{J,1}(n,s) + R_{J,2}(n,s) \]

Also, we have
\[ J_1(n,s) = J_{1,1}(n,s) + J_{1,2}(n,s) \]

where
\[ J_{1,1}(n,s) = \bar{F}(t_n)(\mu(t_n)^{(n-1)}((s-n^{-\delta})n^{1/2}, \infty)) - \Phi_0(s-n^{-\delta}) \]

and
\[ J_{1,2}(n,s) = \int_{-\infty}^{s-n^{-\delta}} dx \bar{F}((s-x)n^{1/2}) \]
\[ \times \frac{1}{2\pi} \int_{\mathbb{R}} e^{-ix\xi}(\varphi(n^{-1/2}\xi; \mu(t_n)))^{n-1} - \exp(-\frac{\xi^2}{2})d\xi \]

By Proposition 11 and Lemma 16, we see that there is a \( C_1 > 0 \) such that
\[ |\mu(t_n)^{(n-1)}((xn^{1/2}, \infty)) - \Phi_0(x)| \]
\[ \leq |\int_{-\infty}^{\infty} \frac{e^{-ix\xi}}{\xi} \varphi(n^{-1/2}\xi; \mu(t_n))^{n-1} - \exp(-\frac{\xi^2}{2})d\xi| \]
\[ \\
\leq \int_{|\xi| > n^{\delta'}} \frac{1}{|\xi|} |\varphi(\xi; \mu(t_n))|^{n-1} + \exp(-\frac{\xi^2}{2})d\xi \]
\[ + \int_{|\xi| < n^{\delta'}} \frac{1}{|\xi|} |R_{n,2}(\xi)| \exp(-\frac{\xi^2}{2})d\xi \]
\[ \leq C_1 n^{-2K\delta}, \text{ for any } x \in \mathbb{R} \text{ and } n \geq 8. \]

Therefore we have
\[ |J_{1,1}(n,s)| \leq C_1 \bar{F}(t_n)n^{-2K\delta} \leq C(\delta)C_1 n^{-\alpha/2-\delta}. \]

Similarly by Lemma 16, we see that there is a \( C_2 > 0 \) such that
\[ |\int_{\mathbb{R}} e^{-ix\xi}(\varphi(n^{-1/2}\xi; \mu(t_n))^{n-1} - \exp(-\frac{\xi^2}{2})d\xi| \]
\[ \leq C_2 n^{-2K\delta}, \text{ for any } x \in \mathbb{R} \text{ and } n \geq 8. \]
Then we have
\[ |J_{1,2}(n, s)| \leq C_2 n^{-2K\delta} C(\delta) \int_{n^{-\delta}}^\infty (xn^{1/2})^{-\alpha+\delta} dx \leq C_2 C(\delta)n^{-\alpha/2-\delta} \]

So we see that there is a \( C > 0 \) such that
\[\sup_{s \in [1, \log n]} |J_1(n, s)| \leq Cn^{-(\alpha-2)/2-\delta} \quad \text{(13)}\]

Note that
\[ |J_2(n, s)| \leq n^2 \bar{F}(t_n)^2 \quad \text{(14)} \]

So Equations (7) - (14) imply our assertion. □

**Proposition 20.** Then there is a \( C > 0 \) such that
\[\sup_{s \in [1, \log n]} |P(\sum_{k=1}^n X_k > sn^{1/2}) - G(n, s)| \leq Cn^{-(\alpha-2)/2-\delta/4}, \quad n = 3, 4, \ldots . \]

**Proof.** Note that
\[ \bar{\eta}_k(t_n) + \int_0^{t_n} x^k \mu(dx) = \int_0^{\infty} x^k \mu(dx), \quad k = 1, 2, \ldots, K-1, \]
and
\[ \eta_K(t_n) - \int_0^{t_n} x^K \mu(dx) = \int_{-\infty}^0 x^K \mu(dx) . \]

So our assertion is an easy consequence of Propositions 17, 18, 19. □

**Proposition 21.** There is a \( C > 0 \) such that
\[\sup_{s \in [\log n, \infty)} I_0(n, s) \leq Cn^{-(\alpha-2)/2-\delta/4}, \quad n = 3, 4, \ldots . \]

**Proof.** We have
\[ I_0(n, s) \leq \mu(t_n)^*n((sn^{1/2}, \infty)) \]
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\[ = \Phi_0(s) + \frac{1}{2\pi} \int_{\mathbb{R}} \frac{e^{-is\xi}}{i\xi} (\varphi(n^{-1/2} \xi, \mu(t_n))^n - \exp(-\frac{\xi^2}{2})) d\xi. \]

\[ = \Phi_0(s) + \frac{1}{2\pi} \int_{\mathbb{R}} \frac{e^{-is\xi}}{i\xi} (\varphi(n^{-1/2} \xi, \mu(t_n))^n - \exp(-\frac{\xi^2}{2})) d\xi. \]

\[ \leq \Phi_0(s) + \frac{1}{2\pi} \int_{\mathbb{R}} \frac{e^{-is\xi}}{i\xi} (\psi_0(n, \xi) + n\varphi_1(n^{-1/2} \xi, t_n) \exp(-\frac{\xi^2}{2})) d\xi. \]

\[ + \frac{1}{2\pi} \int_{\mathbb{R}} \frac{e^{-is\xi}}{i\xi} R_{n,0}(\xi) \exp(-\frac{\xi^2}{2})) d\xi. \]

Since \( \sup_{s \geq \log n} |\Phi_k(s)| \) is of \( O(n^{-M}) \) for any \( M \geq 1 \), we have our assertion similar to the proof of Proposition 18.

**Proposition 22.** There is a \( C > 0 \) such that

\[ \sup_{s \in [\log n, \infty)} |I_1(n, s) - n \int_{-\infty}^{s} \tilde{F}((s - x)n^{1/2}) \Phi_1(x) dx| \leq Cn^{-(\alpha-2)/2-\delta/4}. \]

**Proof.** Remind that

\[ I_1(n, s) = n(1 - \tilde{F}(t_n))^{n-1} \int_{-\infty}^{\infty} \tilde{F}((sn^{1/2} - x) \vee t_n) \mu(t_n)^{n-1}(dx) \]

\[ = n(1 - \tilde{F}(t_n))^{n-1} \left\{ \int_{-\infty}^{\infty} \tilde{F}(((s - x)n^{1/2}) \vee t_n) \Phi_1(x) dx \]

\[ + \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{F}(((s - x)n^{1/2}) \vee t_n) \]

\[ \times \left( \int_{\mathbb{R}} \frac{e^{-is\xi}}{i\xi} (\varphi(n^{-1/2} \xi, \mu(t_n))^{n-1} - \exp(-\frac{\xi^2}{2})) d\xi \right). \]

Then similarly to the proof of Propositions 19 and 21, we have our assertion. \( \square \)

Now Theorem 2 is a consequence of Propositions 20 and 22.
7. Preliminary for Theorem 3

**Proposition 23.** Let $Y$ be a random variable, and assume that
\[ E[|Y|^2] < \infty \quad \text{and} \quad E[Y] = 0. \]
Then for any $s \in \mathbb{R} \setminus \{0\}$ and $b > 0$
\[ E[\exp(sY1_{\{|Y| \leq b\}})] \leq 1 + s^2(1 + \frac{1}{|s|b})\exp(|s|b)E[|Y|^2]. \]

**Proof.** First, note that
\[ |\exp(x) - 1| \leq 1 \lor \exp(x), \]
and
\[ |\exp(x) - 1| = \left| \int_{0}^{x} e^y dy \right| \leq |x|(1 \lor \exp(x)), \quad x \in \mathbb{R}. \]
So we have
\[ |\exp(x) - (1 + x)| = \left| \int_{0}^{x} (e^y - 1) dy \right| \leq (|x| \land |x|^2)(1 \lor \exp(x)) \]
for any $x \in \mathbb{R}$. Therefore we see that
\[ |\exp(x) - (1 + x)| \leq |x|^2 \exp(|x|), \quad x \in \mathbb{R}. \]
This implies that
\[ |E[\exp(sY1_{\{|Y| \leq b\}})] - (1 + E[sY1_{\{|Y| \leq b\}}])| \leq s^2 \exp(|s|b)E[|Y|^2]. \]
Since
\[ |E[sY1_{\{|Y| \leq b\}}]| = |sE[Y, |Y| > b]| \leq |s|b^{-1}E[|Y|^2], \]
we have our assertion. \( \square \)

**Proposition 24.** Let $X$ be a random variable and assume that
\[ E[|X|^2] < \infty \quad \text{and} \quad E[X] = 0. \]
Then for any \( t > 0 \) and \( n \geq 1 \)

\[
n \log E[\exp(\pm \frac{1}{tn^{1/2}} X_1 \mathbb{1}_{|X| \leq tn^{1/2}})] \leq \frac{6}{t^2} E[|X|^2].
\]

**Proof.** Let \( Y = (1/t)X \), \( s = \pm n^{-1/2} \), \( b = n^{1/2} \), and apply Proposition 23. Since \( \log(1 + x) \leq x \), \( x \geq 0 \), we have our assertion. \( \square \)

Now let \( X_n, n = 1, 2, \ldots \), be independent identically distributed random variables. Throughout this section we assume that

\[
E[|X_1|^2] < \infty \text{ and } E[X_1] = 0.
\]

**Proposition 25.** For any \( s, t > 0 \) and \( \varepsilon > 0 \)

\[
P(\left| \sum_{k=1}^{n} X_k \mathbb{1}_{|X_k| \leq tn^{1/2}} \right| \geq sn^{1/2}) \leq 2 \exp(\frac{6}{t^2} E[|X_1|^2]) \exp(-\frac{s}{t}).
\]

**Proof.** We see that

\[
P(\pm \sum_{k=1}^{n} X_k \mathbb{1}_{|X_k| \leq tn^{1/2}} \geq sn^{1/2})
\]

\[
\leq \exp(-\frac{s}{t}) E[\exp(\frac{1}{tn^{1/2}} \sum_{k=1}^{n} X_k \mathbb{1}_{|X_k| \leq tn^{1/2}})]
\]

\[
\leq \exp(-\frac{s}{t}) E[\exp(\frac{1}{tn^{1/2}} X_1 \mathbb{1}_{|X| \leq tn^{1/2}})]^n.
\]

Then by Proposition 24 we have our assertion.

Let \( F : \mathbb{R} \rightarrow [0, 1] \) and \( \bar{F} : \mathbb{R} \rightarrow [0, 1] \) be given by

\[
F(x) = P(X_1 \leq x), \quad x \in \mathbb{R}
\]

and

\[
\bar{F}(x) = P(X_1 > x), \quad x \in \mathbb{R}.
\]
Then we have the following.

**Proposition 26.** (1) For any $t, s > 0$, and $n \geq 2$,

$$P\left(\left|\sum_{k=2}^{n} X_k 1_{\{|X_k| \leq tn^{1/2}\}}\right| > sn^{1/2}\right) \leq 2 \exp\left(\frac{6}{t^2} E[|X_1|^2]\right) \exp\left(-\frac{s}{t}\right).$$

(2) For any $s, t > 0$, $\varepsilon \in (0, 1)$ with $t < (1 - \varepsilon)s$,

$$|P\left(\sum_{k=1}^{n} X_k > sn^{1/2}\right) - nP(X_1 + \sum_{k=2}^{n} X_k 1_{\{|X_k| \leq tn^{1/2}\}} > sn^{1/2},$$

$$\left|\sum_{k=2}^{n} X_k 1_{\{|X_k| \leq tn^{1/2}\}}\right| \leq \varepsilon sn^{1/2})| \leq 2n(n-1)(F(-tn^{1/2}) + \bar{F}(tn^{1/2}))^2 + 2 \exp\left(\frac{6}{t^2} E[|X_1|^2]\right) \exp\left(-\frac{s}{t}\right)$$

$$+ 2n(F(-tn^{1/2}) + \bar{F}(tn^{1/2})) \exp\left(\frac{6}{t^2} E[|X_1|^2]\right) \exp\left(-\frac{\varepsilon s}{2t}\right).$$

**Proof.** Note that

$$P\left(\left|\sum_{k=2}^{n} X_k 1_{\{|X_k| \leq tn^{1/2}\}}\right| > sn^{1/2}\right)$$

$$= P\left(\left|\sum_{k=1}^{n-1} X_k 1_{\{|X_k| \leq \tilde{t}(n-1)^{1/2}\}}\right| > \tilde{s}(n-1)^{1/2}\right),$$

where

$$\tilde{t} = t\left(\frac{n}{n-1}\right)^{1/2}, \quad \tilde{s} = s\left(\frac{n}{n-1}\right)^{1/2}. $$

So we have the assertion (1) from Proposition 25.

Let us denote

$$\tilde{F}(x) = P(|X_1| > x) \leq F(-x) + \bar{F}(x), \quad x > 0.$$ 

Note that

$$P\left(\sum_{k=1}^{n} X_k > sn^{1/2}\right) = \sum_{m=0}^{n} I_m,$$
where

\[ I_m = P(\sum_{k=1}^{n} X_k > sn^{1/2}, \sum_{k=1}^{\min\{m, n\}} 1_{|X_k| > t} = m), \quad m = 0, 1, \ldots, n. \]

Then we have

\[ I_m = \binom{n}{m} P(\sum_{k=1}^{n} X_k > sn^{1/2}, |X_i| > t, i = 1, \ldots, m, |X_j| \leq t, j = m + 1, \ldots, n), \]

for \( m = 0, 1, \ldots, n \). So we see that

\[
\sum_{m=2}^{n} I_m \leq \sum_{m=2}^{n} \frac{n(n-1)}{m(m-1)} \left( \frac{n-2}{m-2} \right) \tilde{F}(tn^{1/2})^m (1 - \tilde{F}(tn^{1/2}))^{n-m} \leq \frac{n(n-1)}{2} \tilde{F}(tn^{1/2})^2.
\]

(15)

Also, by Proposition 25, we have

\[
I_0 \leq 2 \exp\left(-\frac{s}{t}\right) \exp\left(\frac{6}{t^2} E[|X_1|^2]\right).
\]

(16)

Let

\[
A_1 = \{|X_1| > tn^{1/2}\}, \quad A_2 = \{|X_k| \leq tn^{1/2}, k = 2, 3, \ldots, n\},
\]

\[
B_1 = \{X_1 + \sum_{k=2}^{\min\{m, n\}} X_k 1_{|X_k| \leq tn^{1/2}} > sn^{1/2}\},
\]

and

\[
B_2 = \{|\sum_{k=2}^{\min\{m, n\}} X_k 1_{|X_k| \leq tn^{1/2}}| \leq \varepsilon sn^{1/2}\}.
\]

Note that \( B_1 \cap B_2 \subset A_1 \), since \( t < (1 - \varepsilon)s \). So we see that

\[
|P(B_1 \cap A_1 \cap A_2) - P(B_1 \cap B_2)| \leq P(B_1 \cap B_2^c \cap A_1 \cap A_2) + P(B_1 \cap B_2 \cap A_1 \cap A_2^c)
\]

(17)

\[
\leq P(A_1) P(B_2^c) + P(A_1) P(A_2^c).
\]
Note that
\[ P(A_2^c) \leq \sum_{k=2}^{n} P(|X_k| > tn^{1/2}) = (n - 1)\bar{F}(tn^{1/2}). \]

Also, by the assertion (1) we have
\[ P(B_2^c) \leq 2 \exp\left(\frac{6t^2 \mathbb{E}[|X_1|^2]}{\varepsilon s} \right) \exp\left(-\frac{\varepsilon s}{2t}\right). \]

Since \( I_1 = nP(B_1 \cap A_1 \cap A_2) \), we have the assertion from Equations (15), (16) and (17).

This completes the proof. □

8. Some Estimates

In this section, we assume that (A-1) and (A-5).

Let \( g : (x_0, \infty) \to \mathbb{R} \), \( H : [-1/2, 1/2] \times (2x_0, \infty) \to (0, \infty) \) and \( R : [-1/2, 1/2] \times (2x_0, \infty) \to (0, \infty) \) be given by
\[
g(x) = x^2 \frac{d^2}{dx^2} \left( \log \bar{F}(x) \right) - \alpha, \quad x > x_0,
\]
\[
H(y; x) = \frac{\bar{F}(x(1 + y))}{\bar{F}(x)}, \quad y \in [-1/2, 1/2], \quad x > 2x_0,
\]
and
\[
R(y; x) = H(y; x) - \left\{ 1 - \alpha y + \frac{\alpha(\alpha + 1)y^2}{2} \right\}, \quad y \in [-1/2, 1/2], \quad x > 2x_0,
\]

We prove the following in this section.

**Proposition 27.** There are functions \( a : (2x_0, \infty) \to \mathbb{R} \), \( c : (2x_0, \infty) \to [0, \infty) \) and a constant \( C > 0 \) such that \( a(x) \to 0 \) and \( c(x) \to 0 \), as \( x \to \infty \), and that
\[ |R(y; x) - a(x)y| \leq C(c(x)y^2 + |y|^3), \quad y \in [-1/2, 1/2], \quad x > 2x_0. \]

First we prove the following.
Proposition 28. (1) For any $x > x_0$,
\[
\frac{d}{dx} \log(x^\alpha \bar{F}(x)) = -\int_x^\infty \frac{g(z)}{z^2} dz.
\]
(2) For any $y \in [-1/2, 1/2]$ and $x > 2x_0$,
\[
H(y; x) = (1 + y)^{-\alpha} \exp\left(-\int_{1+y}^y dy' \int_{1+y}^\infty \frac{g(xz)}{z^2} dz\right).
\]
Proof. Note that
\[
g(x) = x^2 \frac{d^2}{dx^2} (\log(x^\alpha \bar{F}(x)))
\]
and $g(x) \to 0$ as $x \to \infty$. Then we see that
\[
\frac{d}{dy} (\log(y^\alpha \bar{F}(y))) - \frac{d}{dx} (\log(x^\alpha \bar{F}(x))) = \int_x^y \frac{g(z)}{z^2} dz,
\]
and so we see that
\[
c_0 = \lim_{y \to \infty} \frac{d}{dy} (\log(y^\alpha \bar{F}(y)))
\]
even exists. Note that
\[
\exp\left(\int_x^{2x} \frac{d}{dy} (\log(y^\alpha \bar{F}(y)))dy\right) = \frac{L(2x)}{L(x)} \to 1, \quad x \to \infty.
\]
So we see that $c_0 = 0$. Therefore letting $y \to \infty$ in Equation (18) we have the assertion (1).

By the assertion (1), we have
\[
\frac{d}{dy} \log((1 + y)^\alpha H(y; x)) = -x \int_{x(1+y)}^\infty \frac{g(z)}{z^2} dz = -\int_{1+y}^\infty \frac{g(xz)}{z^2} dz.
\]
Since $H(0; x) = 1$, we have the assertion (2). \(\square\)

Proposition 29. Let $\tilde{a} : (2x_0, \infty) \to \mathbb{R}$ and $\tilde{c} : (2x_0, \infty) \to \mathbb{R}$ be given by
\[
\tilde{a}(x) = \frac{d}{dy}((1 + y)^\alpha H(y, x))|_{y=0},
\]
and

\[
\tilde{c}(x) = \sup_{y \in [-1/2, 1/2]} \left| \frac{d^2}{dy^2}((1 + y)^\alpha H(y, x)) \right|.
\]

Then \(\tilde{a}(x) \to 0\) and \(\tilde{c}(x) \to 0\), as \(x \to \infty\), and that

\[
|H(y; x) - (1 + y)^{-\alpha} - \tilde{a}(x)y(1 + y)^{-\alpha}| \leq 2^\alpha \tilde{c}(x)y^2,
\]

\(y \in [-1/2, 1/2], \ x > 2x_0\).

**Proof.** By Proposition 11 We have

\[
\frac{d}{dy}((1 + y)^\alpha H(y; x)) = -(1 + y)^\alpha H(y; x) \int_{1+y}^\infty \frac{g(xz)}{z^2} dz
\]

and so

\[
\tilde{a}(x) = -\int_1^\infty \frac{g(xz)}{z^2} dz
\]

Similarly, we have

\[
\frac{d^2}{dy^2}((1 + y)^\alpha H(y; x)) = (1 + y)^\alpha H(y; x) \{ (\int_{1+y}^\infty \frac{g(xz)}{z^2} \, dz)^2 - (1 + y)^{-2} g(x(1 + y)) \}
\]

Therefore we have

\[
\tilde{c}(x) \leq 2^\alpha \{(\int_{1/2}^\infty \frac{|g(xz)|}{z^2} \, dz)^2 + 4 \sup_{y \in [-1/2, 1/2]} |g(x(1 + y))| \}
\]

\times \exp(\int_{1/2}^\infty \frac{|g(xz)|}{z^2} \, dz)
\]

These imply that \(\tilde{a}(x) \to 0, \) \(\tilde{c}(x) \to 0,\) as \(x \to \infty.\) Also we have

\[
|(1 + y)^\alpha H(y; x) - (1 + \tilde{a}(x))| \leq \tilde{c}(x)y^2, \quad x \geq 2x_0, \ y \in [-1/2, 1/2].
\]

This implies our assertion. \(\square\)

Now Proposition 27 is an easy corollary to Proposition 29.

In this section, we assume that $X_n$, $n = 1, 2, \ldots$, are i.i.d. random variables, $\alpha > 2$ and (A-1) - (A-5) are satisfied. Let $p = (\alpha + 2)/2$ and $\beta = (\alpha + p)/2$. Then we see that $E[|X_1|^p] < \infty$ and there is a $C_0 > 1$ such that

$$F(-x) + \bar{F}(x) \leq C_0 x^{-\beta}, \quad x \geq 1.$$ 

Proposition 30. Let $b(x) = E[X_1, |X_1| \leq x] = -E[X_1, |X_1| > x]$, $x > 0$. Then we have the following.

(1) $|b(x)| \leq E[|X_1|^p]^{1/p}(F(-x) + \bar{F}(x))^{1-1/p} \leq C_0 x^{-\beta(p-1)/p} E[|X_1|^p]^{1/p}$, $x \geq 1$.

(2) There is a constant $C_1 > 1$ only dependent on $p$ such that

$$E[\sum_{k=1}^{n} X_k 1_{|X_k| \leq x}]^{p/2} \leq C_1 n^{1/2} (E[|X_1|^p]^{1/p} + |b(x)|) + n|b(x)|$$

$$\leq C_1 E[|X_1|^p]^{1/p} (1 + C_0)(n^{1/2} + nx^{-\beta(p-1)/p})$$

for any $n = 1, 2, \ldots$, and $x \geq 1$.

Proof. The assertion (1) is an easy consequence of Hölder’s inequality. So we prove the assertion (2). Since $E[X_k 1_{|X_k| \leq x} - b(x)] = 0$, $k = 1, 2, \ldots$, we see by Burkholder-Davis-Gundy’s theorem that there is a constant $C_1 > 0$ depending on $p$ only such that

$$E[\sum_{k=1}^{n} (X_k 1_{|X_k| \leq x} - b(x))]^{p/2} \leq C_1 E[\sum_{k=1}^{n} (X_k 1_{|X_k| \leq x} - b(x))]^{2/p}$$

Then by Hölder’s inequality, we have

$$E[\sum_{k=1}^{n} (X_k 1_{|X_k| \leq x} - b(x))]^{p/2} \leq C_1 E[n^{p/2-1} \sum_{k=1}^{n} |X_k 1_{|X_k| \leq x} - b(x)|^{p}]^{1/p}$$

$$= C_1 n^{1/2} E[|X_1 1_{|X_1| \leq x} - b(x)|^{p}]^{1/p} \leq C_1 n^{1/2} (E[|X_1 1_{|X_1| \leq x}|^p]^{1/p} + |b(x)|)$$

This implies our assertion. □
Let \( a : (2x_0, \infty) \to \mathbb{R} \) and \( c : (2x_0, \infty) \to [0, \infty) \) be as in Proposition 27. Also, let
\[
Y_n(t) = \sum_{k=1}^{n} X_k 1_{|X_k| \leq tn^{1/2}}, \quad n \geq 2, \ t > 0.
\]

Then we have the following.

**Proposition 31.** Let \( r \in ((\alpha + 2)/(2\alpha), 1) \). Then for any \( \varepsilon \in (0, 1/2) \)
\[
\lim_{n \to \infty} \sup \{ s^2 E[|H(-\frac{1}{sn^{1/2}} Y_n(t), sn^{1/2})|, |Y_n(t)| \leq \varepsilon sn^{1/2}] - (1 + \frac{\alpha(\alpha + 1)}{2s^2})
\]
\[
\geq (\log n)^{1/2}, \ t \geq (\log n)^{-1}s^{(1+r)/2} \} = 0.
\]

**Proof.** Let \( s \geq (\log n)^{1/2}, \ t \geq (\log n)^{-1}s^{(1+r)/2}, \) and \( n \geq 3 \). Then \( tn^{1/2} \geq 1 \). Note that
\[
r\beta(p-1)/p > 1 + \frac{3(\alpha - 2)}{8} > 1.
\]

We see that
\[
E[H(-\frac{1}{sn^{1/2}} Y_n(t), sn^{1/2})|, |Y_n(t)| \leq \varepsilon sn^{1/2}] - (1 + \frac{\alpha(\alpha + 1)}{2s^2})
\]
\[
= a(sn^{1/2} - \alpha) E[Y_n(t)] + \frac{\alpha(\alpha + 1)}{2s^2 n} (E[Y_n(t)^2] - n)
\]
\[
- E[1 + \frac{a(sn^{1/2} - \alpha)}{sn^{1/2}} Y_n(t) + \frac{\alpha(\alpha + 1)}{2s^2 n} Y_n(t)^2, |(sn^{1/2})^{-1} Y_n(t)| \geq \varepsilon]
\]
\[
+ E[R(-\frac{1}{sn^{1/2}} Y_n(t), sn^{1/2}), |(sn^{1/2})^{-1} Y_n(t)| \leq \varepsilon].
\]

Note that
\[
s|E[Y_n(t)]| = ns|b(tn^{1/2})| \leq C_0 E[|X_1|^p]^{1/p} s(tn^{1/2})^{-\beta(p-1)/p}
\]
\[
\leq C_0 E[|X_1|^p]^{1/p} (sn^{1/2})^{-\beta(p-1)/p} s^{-1}(1+r)\beta(p-1)/2p,
\]
\[
E[Y_n(t)^2] - n = n(E[(X_1 |X_1| \leq tn^{1/2})^2] - b(tn^{1/2})^2) + E[Y_n(t)]^2 - n
\]
Then we have the following:

\[ -nE[X_1^2, |X_1| > tn^{1/2}] + n(n - 1)b(tn^{1/2})^2, \]

and

\[ n^{-p/2}E[|Y_n(t)|^p] \leq C_1^p(1 + C_0^pE[|X_1|^p](1 + t^{-\beta(p-1)/p}n^{1/2(1-\beta(p-1)/p)})^p. \]

So we see that

\[
\frac{1}{n}|E[Y_k(t)^2] - n| \leq E[X_1^2, |X_1| > tn^{1/2}] \\
+ C_0(n - 1)n^{-\beta(p-1)/p}(\log n)^{2\beta(p-1)/p}E[|X_1|^p]^{2/p},
\]

\[
s^2(sn^{1/2})^{-k}E[|Y_n(t)|^k, |(sn^{1/2})^{-1}Y_n(t)| > \varepsilon] \\
\leq s^{2-p}e^{-p+k}n^{-p/2}E[|Y_n(t)|^p], \quad k = 0, 1, 2,
\]

and

\[
s^2E[|R(-\frac{1}{sn^{1/2}}Y_n(t), sn^{1/2})|, |(sn^{1/2})^{-1}Y_n(t)| \leq \varepsilon],
\]

\[
\leq C_2(|c(sn^{1/2})n^{-1}E[Y_n(t)^2] + \varepsilon^{-p}s^{2-p}n^{-p/2}E[|Y_n(t)|^p]).
\]

Combining them, we have our assertion. \(\square\)

Now we prove Theorem 3. Let \(\beta : \mathbb{N} \to (0, \infty)\) be such that

\[
\frac{\beta(n)}{(\log n)^{1/2}} \to \infty, \quad n \to \infty.
\]

Assume that Theorem 3 is not valid. Then there is a sequence of positive numbers \(\{s'_n\}_{n=1}^\infty\) such that \(s'_n \geq n^{1/2}\beta(n), n = 1, 2, \ldots, \) and

\[
\lim_{n \to \infty} \left( s'_n \right)^2 \frac{P(\sum_{k=1}^n X_k > s'_n)}{nF(s'_n)} - (1 + \frac{\alpha(\alpha+1)\varepsilon n}{(s'_n)^2}) > 0.
\]

Let \(s_n = n^{-1/2}s'_n \geq \beta(n).\) Let us take an \(r \in ((\alpha + 2)/(2\alpha), 1)\) and fix it. Let \(t_n, n = 1, 2, \ldots,\) be a sequence of positive numbers given by

\[
t_n = (\log n)^{-1/2} + (\log n)^{-1}s_n^{(1+r)/2}, \quad n \geq 2.
\]

Then we have the following.

\[
t_n s_n \geq \frac{\beta(n)}{(\log n)^{1/2}} \to \infty, \quad n \to \infty,
\]
\[
\frac{2s_n}{t_n} \geq ((\log n)^{1/2}s_n) \land ((\log n)s_n^{(1-r)/2}), \quad n \geq 2,
\]
(20)

\[
\frac{2s_n}{(\log n)t_n} \geq \left( \frac{\beta(n)}{(\log n)^{1/2}} \right) \land (s_n^{(1-r)/2}) \to \infty, \quad n \to \infty,
\]
(21)

and

\[
\frac{(t_n^{1/2})^2}{(s_n^{1+r})} \geq \frac{(\log n)^{-2}s_n^{1+r}}{s_n^{1+r}n^{(1+r)/2}} \to \infty, \quad n \to \infty.
\]
(22)

Therefore by Equation (22), we have

\[
\frac{(F(-t_n^{1/2}) + \bar{F}(t_n^{1/2}))^2}{F(s_n^{2r})} \to 0, \quad n \to \infty.
\]

Since \(2r - 1 > 2/\alpha\), we have

\[
(ns_n')^m \exp\left(\frac{m}{t_n^2} - \frac{1}{m} \frac{s_n}{t_n}\right)
\]

\[
= \exp\left(\frac{m}{t_n^2} (1 - \frac{1}{3m^2} t_n s_n) \right) n^{3m/2} \exp\left(-\frac{1}{3m} \frac{s_n}{(\log n)t_n}\right)
\]

\[
\times (s_n)^m \exp\left(-\frac{1}{3m} \frac{s_n}{t_n}\right) \to 0, \quad n \to \infty.
\]

Also, by Equations (19), (20) and (21) we see that for any \(m \geq 1\)

\[
(n s_n')^m \exp\left(\frac{m}{t_n^2} - \frac{1}{m} \frac{s_n}{t_n}\right) \to 0, \quad n \to \infty.
\]

Note that

\[
(n + 1)P(X_{n+1} + Y_n(t_{n+1}) > s_{n+1}(n + 1)^{1/2}, \ |Y_n(t_{n+1})| \leq \varepsilon s_{n+1}(n + 1)^{1/2})
\]

\[
= E[H(-\frac{1}{s_{n+1}(n + 1)^{1/2}} Y_n(t_{n+1}), s_{n+1}(n + 1)^{1/2}), \ |Y_n(t_{n+1})| \leq \varepsilon s_{n+1}(n + 1)^{1/2}].
\]

From Proposition 31, we see that

\[
s_{n+1}^2 E[H(-\frac{1}{s_{n+1}(n + 1)^{1/2}} Y_n(t_{n+1}), s_{n+1}(n + 1)^{1/2}),
\]

\[
|Y_n(t_{n+1})| \leq \varepsilon s_{n+1}(n + 1)^{1/2}
\]

\[
- (1 + \frac{\alpha(\alpha + 1)(n + 1)}{s_{n+1}^2(n + 1)})] \to 0
\]
as $n \to \infty$, by letting $s = s_{n+1}(1 + 1/n)^{1/2}$, $t = t_{n+1}$. Then from this, Proposition 26(2), Equations (19), (20), (21), (22) and (23), we have

$$\frac{(s'_n)^2}{n} \left| \frac{P(\sum_{k=1}^n X_k > s'_n)}{nF(s'_n)} - (1 + \frac{\alpha(\alpha + 1)n}{s'^2_n}) \right| \to 0$$

as $n \to \infty$. This is a contradiction.

This proves Theorem 3.

10. Proof of Theorem 4

Let $\hat{F}_n : [1, \infty) \to [0, 1], n \geq 1$, be given by

$$\hat{F}_n(s) = \int_{-\infty}^{s} \hat{F}((s - x)n^{1/2})\Phi_1(x)dx.$$  

Then we have the following.

**Proposition 32.** Let $\beta : \mathbb{N} \to (0, \infty)$ be such that

$$\frac{\beta(n)}{(\log n)^{1/2}} \to \infty, \quad n \to \infty.$$

Then

$$\sup_{s \geq \beta(n)} s^2 \left| \frac{\hat{F}_n(s)}{F(sn^{1/2})} - (1 + \frac{\alpha(\alpha + 1)n}{s^2}) \right| \to 0, \quad n \to \infty.$$

**Proof.** By Proposition 27, we see that

$$| \frac{\hat{F}_n(s)}{F(sn^{1/2})} - (1 + \frac{\alpha(\alpha + 1)n}{s^2}) |$$

$$\leq \int_{[-s/2,s/2]} |R(y/s; sn^{1/2}) - a(sn^{1/2})(y/s)|\Phi_1(y)dy$$

$$+ \int_{[-s/2,s/2]} 4(1 + (a(sn^{1/2}) + \alpha(\alpha + 1))(\frac{y}{s^2})\Phi_1(y)dy.$$

This and Proposition 27 imply our assertion. □
It is well known (e.g. Williams [8]) that there is a $C_0 > 0$ such that

\begin{align}
|\Phi_k(x)| &\leq C_0(1 + x)^{k-1}\Phi_1(x), \quad x \geq 0, \ k = 1, \ldots, 3K, \tag{24}
\end{align}

and

\begin{align}
C_0^{-1}\Phi_1(x) &\leq x\Phi_0(x) \leq C_0\Phi_1(x), \quad x \geq 1. \tag{25}
\end{align}

Let

\begin{align*}
H_0(n, s) &= \Phi_0(s) + n\tilde{F}_n(s), \\
A_1(n, s) &= \sum_{k=1}^{2} \frac{n^{-(k-2)/2}}{k!} \Phi_k(s) \int_{0}^{\infty} x^k \mu(dx),
\end{align*}

and

\begin{align}
A(n, s) &= n\tilde{F}_n(s) - A_1(n, s).
\end{align}

First we prove the following.

**Proposition 33.**

\[
\sup_{s \in [1, \log n]} \left| \frac{A(n, s) - n\tilde{F}(n^{1/2} s)}{H_0(n, s)} \right| \to 0, \quad n \to \infty.
\]

**Proof.** Let us take a $\gamma \in (0, (\alpha - 2)/(4\alpha))$ and fix it. Let $s \geq 0$ and $n \geq 3$. Note that

\[
\tilde{F}_n(s) = \sum_{k=1}^{4} I_k(n, s),
\]

where

\begin{align*}
I_1(n, s) &= \int_{s-n^{-\gamma}}^{s} \tilde{F}((s - x)n^{1/2})\Phi_1(x)dx, \\
I_2(n, s) &= \int_{-s}^{s-n^{-\gamma}} \tilde{F}((s - x)n^{1/2})\Phi_1(x)dx, \\
I_3(n, s) &= \int_{s-n^{-\gamma}}^{s} \tilde{F}((s - x)n^{1/2})\Phi_1(x)dx, \\
I_4(n, s) &= \int_{-\infty}^{-s} \tilde{F}((s - x)n^{1/2})\Phi_1(x)dx.
\end{align*}
It is easy to see that

\[
I_1(n, s) = n^{-1/2} \int_0^{n^{1/2 - \gamma}} \bar{F}(y) \Phi_1(s - n^{-1/2}y)dy.
\]

Let

\[
R(n, s, y) = \Phi_1(s - n^{-1/2}y) - (\Phi_1(s) + n^{-1/2}y \Phi_2(s))
\]

Then for \(y \in [0, sn^{1/2 - \gamma}]\)

\[
|R(n, s, y)| \leq n^{-1}y^2 \sup_{z \in [s - n^{1-\gamma}, s]} |\Phi_3(z)|
\]

\[
\leq C_0 n^{-1}y^2(1 + s)^2 \Phi_1(s - n^{-1}) = C_0 n^{-1}y^2(1 + s)^2 \Phi_1(s) \exp(sn^{-\gamma} - n^{-2\gamma}/2)
\]

\[
\leq C_0^2 n^{-1}y^2(1 + s)^3 \exp(n^{-\gamma}s) \Phi_0(s).
\]

So we see that

\[
n|I_1(n, s) - \sum_{k=1}^{2} \frac{n^{-k/2}}{k!} \Phi_k(s) \int_0^{\infty} x^k \mu(dx)|
\]

\[
\leq C_0^2(1 + s)^3 n^{-1/2} \exp(n^{-\gamma}s)\int_0^{n^{1/2 - \gamma}} y^2 \bar{F}(y)dy \Phi_0(s)
\]

\[
+ C_0^2 (1 + s)n^{1/2}(\int_{n^{1/2 - \gamma}}^{\infty} \bar{F}(y)dy) \Phi_0(s) + C_0^2(1 + s)^2(\int_{n^{1/2 - \gamma}}^{\infty} y \bar{F}(y)dy) \Phi_0(s)
\]

This implies that

\[
(26) \sup_{s \in [1, \log n]} \Phi_0(s)^{-1} |nI_1(n, s) - \sum_{k=1}^{2} \frac{n^{-(k-2)/2}}{k!} \Phi_k(s) \int_0^{\infty} x^k \mu(dx)| \to 0,
\]

\[
n \to \infty.
\]

Note that

\[
I_2(n, s) = \bar{F}(sn^{1/2}) \int_{-s}^{7s/8} (1 - \frac{x}{s})^{-\alpha} \frac{L((s - x)n^{1/2})}{L(sn^{1/2})} \Phi_1(x)dx
\]
It is easy to see that
\[
\sup_{s \in [(\log n)^{1/4}, \log n]} \left| \int_{-s}^{7s/8} (1 - \frac{x}{s})^{-\alpha} \frac{L((s-x)n^{1/2})}{L(sn^{1/2})} \Phi_1(x)dx - 1 \right| \to 0, \quad n \to \infty
\]

Also we see that
\[
n|I_2(n, s)| \leq n \tilde{F}(sn^{1/2})8^\alpha \int_{-s}^{7s/8} \frac{L((s-x)n^{1/2})}{L(sn^{1/2})} \Phi_1(x)dx
\]

Therefore we have
\[
\sup_{s \in [1, (\log n)^{1/4}]} \Phi_0(s)^{-1}(n|I_2(n, s)| + n \tilde{F}(sn^{1/2})) \to 0, \quad n \to \infty.
\]

Thus we have
\[
(27) \quad \sup_{s \in [1, \log n]} H_0(n, s)^{-1}|nI_2(n, s)| - n \tilde{F}(sn^{1/2})| \to 0, \quad n \to \infty.
\]

Note that \(\sqrt{3}/2 \leq 7/8\). Then we have
\[
\Phi_1(7s/8) \leq (\Phi_1(s))^{3/4},
\]
and so we have
\[
nI_3(n, s) \leq ns \tilde{F}(n^{1/2-\gamma})\Phi_1(7s/8)
\]
\[
\leq (n \tilde{F}(n^{1/2 \log n}))^{1/2}(s \Phi_1(s))^{3/4} \frac{n^{1/4} \tilde{F}(n^{1/2-\gamma})}{(n \tilde{F}(n^{1/2 \log n}))^{1/2}}.
\]

Since
\[
\sup_{n \geq 3} \sup_{s \in [1, \log n]} \frac{n^{1/4} \tilde{F}(n^{1/2-\gamma})}{(n \tilde{F}(n^{1/2 \log n}))^{1/2}} < \infty,
\]
we see that there is a constant \(C > 0\) such that
\[
nI_3(n, s) \leq C(n \tilde{F}(sn^{1/2}))^{1/2} \Phi_0(s)^{3/4} \leq C(n \tilde{F}(sn^{1/2}))^{1/4} H_0(n, s), \quad n \geq 3, \ s \in [1, \log n].
\]

So we have
\[
(28) \quad \sup_{s \in [1, \log n]} H_0(n, s)^{-1}|nI_3(n, s)| \to 0, \quad n \to \infty.
\]
Also we have
\[
n|I_4(n, s)| \leq n\bar{F}(2sn^{1/2})\Phi_0(s).
\]
So this equation, Equations (26) (27) and (28) imply our assertion. □

**Proposition 34.** (1) There is a \(C > 0\) such that
\[
\Phi_0(s) + |A_1(n, s)| \leq Cn^{-2}\bar{F}(n^{1/2}s), \quad n \geq 2, \ s \geq \log n.
\]

(2)
\[
\sup_{s \in [1, \infty)} \frac{|A(n, s) - n\bar{F}(n^{1/2}s)|}{H_0(n, s)} \to 0, \quad n \to \infty.
\]

**Proof.** The assertion (1) is obvious from Equations (24) and (25). To prove the assertion (2), because of Proposition 33, it is sufficient to prove
\[
\sup_{s \in [\log n, \infty)} \frac{|A(n, s) - n\bar{F}(n^{1/2}s)|}{H_0(n, s)} \to 0, \quad n \to \infty.
\]
However, by Theorem 3 and Proposition 32, we see that
\[
\sup_{s \in [\log n, \infty)} \left| \frac{n\hat{F}(n^{1/2}s)}{n\bar{F}(n^{1/2}s)} - 1 \right| \to 0, \quad n \to \infty.
\]
Therefore, combining this with the assertion (1), we have the assertion (2). □

Now let us prove Theorem 4.

By Proposition 34(2), we see that
\[
\sup_{s \in [1, \infty)} \left| \frac{H(n, s)}{H_0(n, s)} - 1 \right| \to 0, \quad n \to \infty.
\]
Therefore there is an \(n_0\) such that
\[
H(n, s) \geq \frac{1}{2}H_0(n, s), \quad n \geq n_0, \ s \geq 1.
\]

By Equation (24), we see that there is a \(C > 0\) such that
\[
\sup_{s \in [1, n^{1/12}]} |G(n, s) - H(n, s)| \leq Cn^{-1/12}, \quad n \geq 1
\]
Then combining this with Theorem 2, we see that there are $C > 0$ and \( \delta_0 \in (0, 1/12) \) such that

\[
\sup_{s \in [1,n^{\delta_0}]} \left| \frac{P(\sum_{k=1}^{n} X_k > n^{1/2} s)}{H(n,s)} - 1 \right| \leq C n^{-\delta_0}, \quad n \geq n_0.
\]

On the other hand, by Theorem 3 and Proposition 32 we see that there is a $C > 0$ such that

\[
\sup_{s \in [n^{\delta_0}, \infty)} \left| \frac{P(\sum_{k=1}^{n} X_k > n^{1/2} s)}{n F_n(s)} - 1 \right| \leq C n^{-2\delta_0}, \quad n \geq n_0.
\]

So we see by Proposition 34 that we see that there is a $C > 0$ such that

\[
\sup_{s \in [n^{\delta_0}, \infty)} \left| \frac{P(\sum_{k=1}^{n} X_k > n^{1/2} s)}{H(n,s)} - 1 \right| \leq C n^{-2\delta_0}, \quad n \geq n_0.
\]

These imply Theorem 4.

**References**


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