The Maximal Number of Singular Points on Log del Pezzo Surfaces

By Grigory Belousov

Abstract. We prove that a del Pezzo surface with Picard number one has at most four singular points.

1. Introduction

A log del Pezzo surface is a projective algebraic surface $X$ with only quotient singularities and ample anticanonical divisor $-K_X$.

Del Pezzo surfaces naturally appear in the log minimal model program (see, e. g., [7]). The most interesting class of del Pezzo surfaces is the class of surfaces with Picard number 1. It is known that a log del Pezzo surface of Picard number one has at most five singular points (see [8]). In [1] the author proved there is no log del Pezzo surfaces of Picard number one with five singular points. In this paper we give another, simpler proof.

Theorem 1.1. Let $X$ be a log del Pezzo surface and Picard number is 1. Then $X$ has at most four singular points.

Recall that a normal complex projective surface is called a rational homology projective plane if it has the same Betti numbers as the projective plane $\mathbb{P}^2$. J. Kollár [9] posed the problem to classify rational homology $\mathbb{P}^2$’s with quotient singularities having five singular points. In [4] this problem is solved for the case of numerically effective $K_X$. Our main theorem solves Kollár’s problem in the case where $-K_X$ is ample.

The author is grateful to Professor Y. G. Prokhorov for suggesting this problem and for his help. The author also would like to thank the referee for useful comments.

2000 Mathematics Subject Classification. 14J26, 14J45, 14J50.
The work was partially supported by grant N.Sh.-1987.2008.1.
2. Preliminary Results

We work over complex number field \(\mathbb{C}\). We employ the following notation:

- \((-n)\)-curve is a smooth rational curve with self intersection number \(-n\).
- \(K_X\): the canonical divisor on \(X\).
- \(\rho(X)\): the Picard number of \(X\).

**Theorem 2.1** (see [8, Corollary 9.2]). Let \(X\) be a rational surface with log terminal singularities and \(\rho(X) = 1\). Then

\[
\sum_{P \in X} \frac{m_P - 1}{m_P} \leq 3,
\]

where \(m_P\) is the order of the local fundamental group \(\pi_1(U_P - \{P\})\) (\(U_P\) is a sufficiently small neighborhood of \(P\)).

So, every rational surface \(X\) with log terminal singularities and Picard number one has at most six singular points. Assume that \(X\) has exactly six singular points. Then by (*) all singularities are Du Val. This contradicts the classification of del Pezzo surfaces with Du Val singularities (see, e.g., [3], [10]).

**2.2.** Thus to prove Theorem 1.1 it is sufficient to show that there is no log del Pezzo surfaces with five singular points and Picard number one. Assume the contrary: there is log del Pezzo surfaces with five singular points and Picard number one. Let \(P_1, \ldots, P_5 \in X\) be singular points and \(U_{P_i} \ni P_i\) small analytic neighborhood. By Theorem 2.1 the collection of orders of groups \(\pi_1(U_{P_1} - P_1), \ldots, \pi_1(U_{P_5} - P_5)\) up to permutations is one of the following:

**2.2.1.** \((2, 2, 3, 3), (2, 2, 2, 4, 4), (2, 2, 2, 3, n'), n' = 3, 4, 5, 6,\)

**2.2.2.** \((2, 2, 2, n'), n' \geq 2,\)
Remark 2.3. According to the classification of del Pezzo surfaces with Du Val singularities we may assume that there is a non-Du Val singular point. The case 2.2.1 is discussed in [4, Remark 4.2 and Section 6]. Thus it is sufficient to consider case 2.2.2.

2.4. Notation and assumptions. Let $X$ be a del Pezzo surface with log terminal singularities and Picard number $\rho(X) = 1$. We assume that we are in case 2.2.2, i.e. the singular locus of $X$ consists of four points $P_1, P_2, P_3, P_4$ of type $A_1$ and one non Du Val singular point $P_5$ with $|\pi_1(U_{P_5} - P_5)| = n' \geq 3$. Let $\pi: \bar{X} \to X$ be the minimal resolution and let $D = \sum_{i=1}^{n'} D_i$ be the reduced exceptional divisor, where the $D_i$ are irreducible components. Then there exists a uniquely defined effective $\mathbb{Q}$-divisor $D^\sharp = \sum_{i=1}^{n'} \alpha_i D_i$ such that $\pi^*(K_X) \equiv D^\sharp + K_{\bar{X}}$.

Lemma 2.5 (see, e.g., [13, Lemma 1.5]). Under the condition of 2.4, let $\Phi: \bar{X} \to \mathbb{P}^1$ be a generically $\mathbb{P}^1$-fibration. Let $m$ be the number of irreducible components of $D$ not contained in any fiber of $\Phi$ and let $d_f$ be the number of $(-1)$-curves contained in a fiber $f$. Then

1. $m = 1 + \sum_f (d_f - 1)$, where $f$ run only over the fibers with $d_f \geq 1$.
2. If $d_f = 1$ and $E$ is the only $(-1)$-curve in $f$, then its coefficient in $f$ is at least two.

The following lemma is a consequence of the Cone Theorem.

Lemma 2.6 (see, e.g., [13, Lemma 1.3]). Under the condition of 2.4, every curve on $\bar{X}$ with negative selfintersection number is either $(-1)$-curve or a component of $D$.

Definition 2.7. Let $(Y, D)$ be a projective log surface. $(Y, D)$ is called the weak log del Pezzo surface if the pair $(Y, D)$ is klt and the divisor $-(K_Y + D)$ is nef and big.

For example, in the above notation, $(\bar{X}, D^\sharp)$ is a weak del Pezzo surface. Note that if $(Y, D)$ is a weak log del Pezzo surface with $\rho(Y) = 1$ then divisor $-(K_Y + D) = A$ is ample and $Y$ has only log terminal singularities. Hence, $Y$ is a log del Pezzo surface.
Lemma 2.8 (see, e. g., [1, Lemma 2.9]). Suppose \((Y, D)\) is a weak log del Pezzo surface. Let \(f: Y \to Y'\) be a birational contraction. Then \((Y', D' = f_*D)\) is also a weak log del Pezzo surface.

3. Proof of the Main Theorem: The Case where \(X\) has Cyclic Quotient Singularities

In this section we assume that \(X\) has only cyclic quotient singularities. The following lemma is very similar to that in [5]. For the convenience of the reader we give a complete proof.

Lemma 3.1. Under the condition of 2.4, suppose that \(P_5\) is a cyclic quotient singularity. Then there exists a generically \(\mathbb{P}^1\)-fibration \(\Phi: \tilde{X} \to \mathbb{P}^1\) such that \(f \cdot D \leq 2\), where \(f\) is a fiber of \(\Phi\).

Proof. Let \(\nu: \hat{X} \to X\) be the minimal resolution of the non Du Val singularity and let \(E = \sum E_i\) be the exceptional divisor. By [12, Corollary 1.3] or [8, Lemma 10.4] we have \(|-K_X| \neq \emptyset\). Take \(B \in |-K_X|\). Then we can write
\[K_{\hat{X}} + \hat{B} = \nu^*(K_X + B) \sim 0,\]
where \(\hat{B}\) is an effective integral divisor. We obviously have \(\hat{B} \geq E\).

Run the MMP on \(\hat{X}\). We obtain a birational morphism \(\phi: \hat{X} \to \tilde{X}\) such that \(\tilde{X}\) has only Du Val singularities and either \(\rho(\tilde{X}) = 2\) and there is a generically \(\mathbb{P}^1\)-fibration \(\psi: \tilde{X} \to \mathbb{P}^1\) or \(\rho(\tilde{X}) = 1\). Moreover, \(\phi\) is a composition
\[\tilde{X} = X_1 \xrightarrow{\phi_1} X_2 \xrightarrow{\phi_2} \cdots \xrightarrow{\phi_n} X_{n+1} = \tilde{X},\]
where \(\phi_i\) is a weighted blowup of a smooth point of \(X_{i+1}\) with weights \((1, n_i)\) (see [11]).

Assume that \(\rho(\tilde{X}) = 1\), then every singular point on \(\tilde{X}\) is of type \(A_1\). By the classification of del Pezzo surfaces with Du Val singularities and Picard number one (see, e. g., [3], [10]) we have \(\tilde{X} = \mathbb{P}^2\) or \(\tilde{X} = \mathbb{P}(1,1,2)\).

Assume that \(\rho(\tilde{X}) = 1\) and \(\tilde{X} = \mathbb{P}(1,1,2)\). Note that \(\phi\) contracts \(\rho(\tilde{X}) - 1 = \#E\) curves, where \(\#E\) number of irreducible component of \(E\). Since \(\phi_*(\tilde{B})\) has at most two components and \(\tilde{B} \geq E\), we see that \(\phi\) contracts at most two curves \(K_1\) and \(K_2\) that are not components of \(E\).
Since $X$ has four singular points of type $A_1$, we see that $\tilde{X}$ has at least two singular points, a contradiction.

Assume that $\rho(\tilde{X}) = 1$ and $\tilde{X} = \mathbb{P}^2$. Since $\phi_* (\hat{B})$ has at most three components, as above, we see that $\phi$ contracts at most three curves $K_1$, $K_2$ and $K_3$ that are not components of $E$. Since $X$ has four singular points of type $A_1$, we see that $\tilde{X}$ has at least one singular point, a contradiction.

Therefore, $\rho(\tilde{X}) = 2$ and there is a generically $\mathbb{P}^1$-fibration $\psi : \tilde{X} \to \mathbb{P}^1$.

3.2. Let $f$ be a fiber of $\Phi$. By Lemma 3.1 we have the following cases:

3.2.1. $f$ meets exactly one irreducible component $D_0$ of $D$ and $f \cdot D_0 = 1$.

Let $L$ be a singular fiber of $\Phi$. By Lemma 2.5 (1) the fiber $L$ contains exactly one $(-1)$-curve $F$. By Lemma 2.5 (2) $F$ does not meet $D_0$. Then $F$ meets at most two components of $D$. Blowup one of the intersection points of $F$ and $D$. We obtain a surface $Y$. Let $h : Y \to Y'$ be a contraction of all curves with selfintersection number at most $-2$. Note that $Y'$ has only log terminal singularities but not of type 2.2.2, a contradiction.

3.2.2. $f$ meets exactly two irreducible components $D_1, D_2$ of $D$ and $D_1 \cdot f = D_2 \cdot f = 1$.

By Lemma 2.5 (1) there exists a unique singular fiber $L$ such that $L$ has two $(-1)$-curves $F_1$ and $F_2$. Note that one of these curves, say $F_1$, meets $D$ at one or two points. Blowup one the intersection points of $F_1$ and $D$. We obtain a surface $Y$. Let $h : Y \to Y'$ be a contraction of all curves with selfintersection number at most $-2$. Note that $Y'$ has only log terminal singularities but not of type 2.2.2, a contradiction.

3.2.3. $f$ meets exactly one irreducible component $D_0$ of $D$ and $f \cdot D_0 = 2$.

Let $A$ be a connected component of $D$ containing $D_0$.

By Lemma 2.5 (1) every singular fiber of $\Phi$ contains exactly one $(-1)$-curve. Note that every singular fiber of $\Phi$ either contains two connected components of $A - D_0$ or the coefficient of a unique $(-1)$-curve in this fiber is equal to two. If a singular fiber $L$ contains exactly one $(-1)$-curve with coefficient two, then the dual graph of $L$ is the following:

\[
\begin{array}{ccc}
-2 & -1 & -2 \\
\circ & \circ & \circ
\end{array}
\]
Since $X$ has five singular points with orders of local fundamental groups $(2, 2, 2, 2, n)$, we see that $\Phi$ has two singular fibers $L_1, L_2$ of type $(**)$ and possibly one more singular fiber $L_3$. Note that $L_3$ contains both connected component of $A - D_0$. Let $\mu : \tilde{X} \to \mathbb{F}_n$ be the contraction of all $(-1)$-curves in fibers of $\Phi$, where $\mathbb{F}_n$ is the Hirzebruch surface of degree $n$ (rational ruled surface) and $n = 0, 1$. Denote $\tilde{D}_0 := \mu_* D_0$. Note that $\tilde{D}_0 \sim 2M + kf$, where $M^2 = -n$ and $M \cdot f = 1$. Since we contract at most five curves that meet $D_0$, and $D_0^2 \leq -2$, we see that $0 < \tilde{D}_0^2 \leq 3$. Hence, $0 < -4n + 4k \leq 3$. This is impossible, a contradiction.

4. Proof of the Main Theorem: The Case where $X$ has a Non-Cyclic Quotient Singularity

Under the condition of 2.4, assume $X$ has a non-cyclic singular point, say $P$. Then there is a unique component $D_0$ of $D$ such that $D_0 \cdot (D - D_0) = 3$ (see [2]).

**Lemma 4.1.** There is a generically $\mathbb{P}^1$-fibration $\Phi : \tilde{X} \to \mathbb{P}^1$ such that $\Phi$ has a unique section $D_0$ in $D$ and $D_0 \cdot f \leq 3$, where $f$ is a fiber of $\Phi$.

**Proof.** Recall that $P$ is not Du Val. Let $h : \tilde{X} \to \tilde{X}$ be a contraction of all curves in $D$ except $D_0$. Let $\tilde{D}_0 = h_* (D_0)$ then $\tilde{X}$ has seven singular points, $\rho(\tilde{X}) = 2$ and there is $\nu : \tilde{X} \to X$ such that $K_{\tilde{X}} + a\tilde{D}_0 = \nu^* K_X$. Note that $(\tilde{X}, aD_0)$ is a weak log del Pezzo. Let $R$ be the extremal rational curve different from $\tilde{D}$. Let $\phi : \tilde{X} \to \tilde{X}$ be the contraction of $R$.

4.2. There are two cases:

4.2.1. $\rho(\tilde{X}) = 1$. Then, by Lemma 2.8, $\tilde{X}$ is a del Pezzo surface. If the number of singular points of $\tilde{X}$ on $R$ is at most two, $\tilde{X}$ has at least five singular points and all points are cyclic quotients. Thus assume that there is at least three singular points of $\tilde{X}$ on $R$, say $P_1, P_2, P_3$. Let $R_1 = \sum_i R_{1i}$, $R_2 = \sum_i R_{2i}$ and $R_3 = \sum_i R_{3i}$ be the exceptional divisors on $\tilde{X}$ over $P_1, P_2$ and $P_3$, respectively. Let $\tilde{R}$ is the proper transformation of $R$ on $\tilde{X}$. Since $\tilde{R}$ is not component of $D$, we see that $\tilde{R}^2 \geq -1$. Indeed, this follows from Lemma 2.6. Note that matrix of intersection of component $\tilde{R} + R_1 + R_2 + R_3$ is not negative definite. Hence, $\tilde{R} + E_1 + E_2 + E_3$ can not be contracted, a contradiction.
4.2.2. \( \tilde{X} = \mathbb{P}^1 \). Let \( g : X \rightarrow \tilde{X} \) be the resolution of singularities. Then \( \Phi = \phi \circ g : X \rightarrow \mathbb{P}^1 \). Note that there is a unique horizontal curve \( D_0 \) in \( D \). Let \( f \) be a fiber of \( \Phi \). Denote coefficient of \( D_0 \) in \( D^\# \) by \( \alpha \). Then
\[
0 > (K_{\tilde{X}} + D^\#) \cdot f = -2 + \alpha(D_0 \cdot f).
\]
Hence, \( D_0 \cdot f < \frac{2}{\alpha} \). Since \( P \) is not Du Val, we see that \( \alpha \geq \frac{1}{2} \). Hence, \( D_0 \cdot f \leq 3 \). \( \square \)

By Lemma 2.5 (1) every singular fiber of \( \Phi \) contains exactly one \((-1)\)-curve. Let \( B \) be the exceptional divisor corresponding to the non-cyclic singular point. Note that \( B \) contains \( D_0 \).

4.3. Consider the following three cases.

4.3.1. \( D_0 \cdot f = 1 \). Then every singular fiber of \( \Phi \) contains exactly one connected component of \( B - D_0 \). On the other hand, \( B - D_0 \) contains three connected components. Hence \( X \) has at most four singular points, a contradiction.

4.3.2. \( D_0 \cdot f = 2 \). Let \( F_1, F_2, F_3 \) be a connected components of \( B - D_0 \). We may assume \( F_1 \) is \((-2)\)-curve (see [2]). Let \( L_1 \) be a singular fiber of \( \Phi \). Assume that \( L_1 \) contains \( F_1 \). Then \( L_1 \) is of type (***) and \( L_1 \) contain \( F_2 \). Hence, \( F_2 \) is a \((-2)\)-curve. Let \( L_2 \) be a singular fiber of \( \Phi \). Assume that \( L_2 \) contains \( F_3 \) and let \( E \) be a unique \((-1)\)-curve in \( L_2 \). By blowing up the intersection point of \( E \) and \( F_3 \), we obtain a surface \( Y \). Let \( h : Y \rightarrow Y' \) be a contraction of all curves with selfintersection number at most \(-2\). Note that \( Y' \) has only log terminal singularities but not of type 2.2.2, a contradiction.

4.3.3. \( D_0 \cdot f = 3 \). Since every component of \( D - B \) is a \((-2)\)-curve, we see that every singular fiber of \( \Phi \) contains a connected component of \( B - D_0 \). Note that \( B - D_0 \) contains three connected components. Hence \( X \) has at most four singular points, a contradiction.

This completes the proof of Theorem 1.1.
References


(Received November 2, 2008)

Department of Higher Algebra
Faculty of Mechanics and Mathematics
Lomonosov Moscow State University
Vorob’evy Gory, Main Building
MSU, Moscow 119899, Russia
E-mail: belousov_grigory@mail.ru